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Multi-Term Asymptotic Representations of the Riesz Measure of Subharmonic Functions in the Plane

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In the paper, a multi-term asymptotic representation for distribution function of the Riesz measure of subharmonic function in the plane is considered. It is shown that the "smallness" of the reminder term of asymptotic representation does not guarantee the bounded variation with respect to the angle variable of all terms of this asymptotics, and the conditions for this property to be held are given.

Key words: Subharmonic function, the Riesz measure, multi-term asymptotic representation, function of bounded variation.

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One of the most important problems in the function theory is a question on the connection between the regularity in distribution of zeros (masses) of an entire (subharmonic) function and its behavior at infinity. A number of problems in the fields close to complex analysis, contiguous areas of mathematics, physics and radiophysics lead to this question.

In the 30s of the previous century B. Levin (Ukraine) and A. Pflüger (Switzerland) simultaneously and independently constructed the function theory of completely regular growth. The theory describes the connection between the distribution functions of zeros and the entire function in the terms of one-term asymptotic representations*.

But sometimes either the behavior of function or the growth of distribution function is given by multi-term asymptotic representation.

Let us recall these notions.

*In [6] there is an extensive bibliography.

Definition 1.⋆ *Let μ be a measure in the plane. Its distribution function $\mu(t,\theta)$ is equal to measure μ of sector $\{(r,\alpha) : 0 < r \le t; 0 \le \alpha < \theta\}$.*

Definition 2. *A multi-term (polynomial) asymptotic representation of function $f(t,\theta), t > 0, \theta \in [0, 2\pi)$, as $t \to \infty$, is*

$$f(t,\theta) = \Delta_1(\theta)t^{\rho_1} + \Delta_2(\theta)t^{\rho_2} + \ldots + \Delta_n(\theta)t^{\rho_n} + \varphi(t,\theta),$$

where $\Delta_j, j = 1, 2, \ldots, n$, are real functions; $0 \le [\rho_1] < \rho_n < \rho_{n-1} < \ldots < \rho_1$, and function $\varphi(t,\theta)$ is small in a certain sense compared to the previous term.

Let $\mu(t,\theta)$ be a distribution function of positive measure μ in the plane. We suppose that $\mu(t,\theta)$ has a multi-term asymptotics, i.e.,

$$\mu(t,\theta) = \Delta_1(\theta)t^{\rho_1} + \Delta_2(\theta)t^{\rho_2} + \ldots + \Delta_n(\theta)t^{\rho_n} + \varphi(t,\theta),\ t > 0,\ \theta \in [0, 2\pi),$$

where $\Delta_1(\theta) > 0$; $\Delta_j, j = 2, 3, \ldots, n$, are real functions; $0 \le [\rho_1] < \rho_n < \rho_{n-1} < \ldots < \rho_1$, and function $\varphi(t,\theta)$ is small in a certain sense compared to the previous term.

It is known that in the case of polynomial asymptotics ($n > 1$) the properties of the first term differ essentially from other terms of this asymptotics. By [1] and [2] the first term of asymptotics is a monotone nondecreasing function of θ for any fixed t. At the same time the second and the next terms of asymptotics may have unbounded variation. Thus it is natural to study the influence of the reminder term on the properties of the main terms of asymptotics. This problem is the central item of the paper.

The example below is taken from [2] wherein there is some inaccuracy.

E x a m p l e 1. Let $0 \le [\rho_1] < \rho_2 < \rho_1 < [\rho_1] + 1$;

$$\omega_j = \sum_{k=1}^{j} k^{-1-(\rho_1-\rho_2)};$$

$$c\omega_\infty = 2\pi;$$

$c_j = c\omega_j$, $j = 1, 2, \ldots$; $c_0 = 0$; $c'_j = c_{j-1} + \frac{c}{2} j^{-1-(\rho_1-\rho_2)}$, $j = 1, 2, \ldots$. Notice that c'_j is the middle of the interval (c_{j-1}, c_j).

For $\theta \in [0, 2\pi]$ define a continuous function Δ_2 as follows:

$$\Delta_2(c_j) = \Delta_2(2\pi) = 0,\ j = 0, 1, \ldots;$$
$$\Delta_2(c'_j) = \frac{1}{j},\ j = 1, 2, \ldots .$$

Let Δ_2 be a linear function on the other parts of interval $[0, 2\pi]$.

⋆For the case of discrete measures the analogous notion is in [4].

Evidently, the total variation $V_0^{2\pi}\{\Delta_2\}$ of Δ_2 is ∞.

For each fixed $t \geq 0$ let $\chi(t,\cdot)$ be a characteristic function of the segment $[c\omega_{[t]}, 2\pi]$.

We put

$$\varphi(t,\theta) := \begin{cases} 0, & 0 < t \leq 1, 0 \leq \theta < 2\pi, \\ -\Delta_2(\theta)t^{\rho_2}\chi(t,\theta), & t > 1, 0 \leq \theta < 2\pi. \end{cases} \tag{1}$$

Now we divide the set

$$\mathbb{C} \setminus \{(t,\theta) : t \leq 1, 0 \leq \theta < 2\pi\}$$

into "curvilinear" rectangulars in the following way. First, we represent the set as a union of annuli

$$\bigcup_{j=1}^{\infty}\{(t,\theta) : j < t \leq j+1, 0 \leq \theta < 2\pi\}.$$

Then we cut the $j-$ring into "curvilinear" rectangulars:

$$B_*(j,l) = \{j \leq t < j+1, c_l \leq \theta < c'_{l+1}\}$$

and

$$B^*(j,l) = \{j \leq t < j+1, c'_{l+1} \leq \theta < c_{l+1}\},$$
$$l = 0, 1, \dots .$$

Consider three measures in the plane defined by the following densities with respect to measure $dtd\theta$, respectively:

$$p_1(t,\theta) = \begin{cases} 0, & 0 < t \leq 1, 0 \leq \theta < 2\pi, \\ \rho_1 h t^{\rho_1 - 1}, & t > 1, 0 \leq \theta < 2\pi, \end{cases}$$

where the positive constant h will be chosen later;

$$p_2(t,\theta) = \begin{cases} 0, & 0 < t \leq 1, 0 \leq \theta < 2\pi, \\ \rho_2 \frac{2}{c} l^{\rho_1-\rho_2} t^{\rho_2 - 1}, & t > 1, c_{l-1} < \theta \leq c'_l, \\ -\rho_2 \frac{2}{c} l^{\rho_1-\rho_2} t^{\rho_2 - 1}, & t > 1, c'_l < \theta \leq c_l, \end{cases}$$
$$l = 1, 2, \dots .$$

$$p_3(t,\theta) = \begin{cases} 0, & 0 < t \leq 1, 0 \leq \theta < 2\pi, \\ 0, & j < t \leq j+1, 0 \leq \theta \leq c_{j-1}, \\ -\rho_2 \frac{2}{c} l^{\rho_1-\rho_2} t^{\rho_2 - 1}, & (t,\theta) \in B_*(j,l), \\ \rho_2 \frac{2}{c} l^{\rho_1-\rho_2} t^{\rho_2 - 1}, & (t,\theta) \in B^*(j,l), \end{cases}$$
$$j = 1, 2, \dots;\ l = j-1, j, \dots .$$

Consider the function

$$p = p_1 + p_2 + p_3.$$

Notice that on the set $\bigcup\limits_{k=j-1}^{\infty} (B_*(j,k) \bigcup B^*(j,k))$ the function p is equal to p_1 and on the set $\bigcup\limits_{k=1}^{j-2} (B_*(j,k) \bigcup B^*(j,k))$:

$$p = p_1 + p_2.$$

It is not difficult to show that p is a nonnegative function if $h > \frac{4}{c}$.

Let μ be a positive measure corresponding to density p with respect to measure $dtd\theta$. It is easy to see that the distribution function $\mu(t,\theta)$ of this measure has the form

$$\mu(t,\theta) = h\theta t^{\rho_1} + \Delta_2(\theta) t^{\rho_2} + \varphi(t,\theta), \tag{2}$$

where φ is defined by (1).

We have the following estimate for φ :

$$\varphi(t,\theta) = O(t^{\rho_1 - 1}),\ t \to \infty,$$

uniformly for $\theta \in [0, 2\pi]$.

So, we have constructed the distribution function of the Riesz measure of subharmonic function in the plane with the two-term asymptotic representation. The second main term of this asymptotics Δ_2 has the infinite variation on $[0, 2\pi]$.

R e m a r k 1. It is easy to see that essential circumstance in the construction of this example is the following fact. The slope of Δ_2 is not less than $-\frac{2}{c} j^{\rho_1 - \rho_2}$ on the interval (c_{j-1}, c_j).

Let us modify this example a little. Put

$$\Delta_2(c'_n) = \gamma_n,\ n = 1, 2, \ldots,$$

where $0 < \gamma_n < \frac{1}{n}$ and

$$\sum_{n=1}^{\infty} \gamma_n = \infty.$$

If we repeat the construction of Ex. 1 with these data, then we again obtain a distribution function of the Riesz measure of subharmonic function in the plane. This distribution function has a two-term asymptotic representation with the analogous conclusions for function Δ_2. The reminder term of this asymptotics satisfies the estimate

$$|\varphi(t,\theta)| = O(j^{\rho_1} \gamma_j),\ j \le t < j+1,\ j \to \infty,$$

uniformly for $\theta \in [0, 2\pi]$.

Now we will show that it is possible to reduce essentially the growth of the reminder term $\varphi(t,\theta)$ of a multi-term asymptotic representation, nevertheless, the second main term of this asymptotics will still have the infinite variation with respect to the angle variable.

E x a m p l e 2. Consider the convergent series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

Further we will preserve the notations of Example 1.

Divide each set

$$i_j := (c_{j-1}, c_j),\ j = 1, 2, \dots,$$

into intervals by points:

$$c_{j,m} := c_{j-1} + cj^{-(\rho_1-\rho_2)} \sum_{k=j}^{m} \frac{1}{k(k+1)},\ m = j,\ j = 1, \dots.$$

On the segment $[0, 2\pi]$ we define a continuous function Δ_2 in the following way:

$$\Delta_2(0) = \Delta_2(2\pi) = \Delta_2(c_j) = \Delta_2(c_{j,m}) = 0,$$
$$j = 1, 2, \dots;\ m = j, j+1, \dots;$$
$$\Delta_2(c'_{j,m}) = \frac{1}{(m+1)(m+2)},$$
$$j = 1, 2, \dots;\ m = j, j+1, \dots,$$

where $c'_{j,m}$ is the middle of the interval $(c_{j,m}, c_{j,m+1})$.

Δ_2 is taken to be a linear function on the rest of the parts of segment $[0, 2\pi]$.

The maximum value of Δ_2 is equal to $1/j(j+1)$ on segment (c_{j-1}, c_j), $j = 1, 2, \dots$.

Simple calculations show that the variation of Δ_2 is equal to $1/j$ on segment (c_{j-1}, c_j), $j = 1, 2, \dots$. So, the function Δ_2 has the infinite variation on $[0, 2\pi]$.

Let us define the functions $\varphi(t,\theta)$ and $\mu(t,\theta)$ by formulas (1) and (2), respectively.

On each interval i_j there is a sequence of intervals on which $\Delta_2(\theta)$ is a decreasing linear function. Notice* that on these intervals the slope of Δ_2 is equal to $-\frac{2}{c} j^{\rho_1-\rho_2}$.

Now we carry out the construction to Example 1. It is clear how to choose three densities of measures in the plane so that their sum is a density of non-negative measure μ with respect to measure $dtd\theta$ in the plane. It is easy to see

*See Remark 1 .

that $\mu(t,\theta)$ is the distribution function of this measure. The reminder term φ of this asymptotics satisfies the estimate

$$|\varphi(t,\theta)| = O(j^{\rho_1 - 2}),\ j \to \infty,$$

uniformly for $\theta \in [0, 2\pi]$.

Moreover, the analysis of the constructions in Ex. 1 and Ex. 2 shows that it is possible to reduce the growth of the remainder term and to obtain the same conclusion about the behavior of the main terms of asymptotics.

We have demonstrated that the "smallness" of the reminder term of asymptotic representation does not guarantee the bounded variation with respect to the angle variable of all terms of this asymptotics.

Moreover, the above examples show that if the distribution function of the Riesz measure and the first main term of asymptotics satisfy the Lipschitz condition* with respect to the angle variable at some point, then this condition does not necessarily hold for other terms of asymptotics. In fact, it is easy to see that in our examples this effect appears at point $\theta = 2\pi$.

There is a special situation when the boundedness of variation can be claimed for all terms. This is the case

$$\varphi(t,\theta) = \varphi_1(t)\varphi_2(\theta).$$

Theorem 1. *Let a distribution function of measure μ have the representation*

$$\mu(t,\theta) = \sum_{j=1}^{n} \Delta_j(\theta) t^{\rho_j} + \varphi(t,\theta), t > 0, \theta \in [0, 2\pi], \tag{3}$$

where Δ_1 is a monotone nondecreasing function, and $\varphi(t,\theta) = \varphi_1(t)\varphi_2(\theta)$ such that for some $q \geq 1$

$$\int_{T}^{2T} |\varphi_1(t)|^q dt = o(T^{\rho_n q + 1}),\ T \to \infty. \tag{4}$$

Then each of asymptotic representation (3) is a function of bounded variation.

To prove this theorem we will use the following auxiliary statements about the determinants of a specific type.

*Recall that function $f(x)$ satisfies the Lipschitz condition in some point x_o if there are such positive numbers A and δ that

$$|f(x_o) - f(y)| \leq A|x_o - y|,$$

for $|x_o - y| < \delta$.

Lemma 1. ([5, vol. 2, V, probl. 76]) *Let* $0 < \rho_n < \rho_{n-1} < \ldots < \rho_2 < \rho_1$ *and* $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$. *Then the determinant*

$$\begin{vmatrix} \alpha_1^{\rho_n} & \alpha_1^{\rho_{n-1}} & \ldots & \alpha_1^{\rho_1} \\ \alpha_2^{\rho_n} & \alpha_2^{\rho_{n-1}} & \ldots & \alpha_2^{\rho_1} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_n^{\rho_n} & \alpha_n^{\rho_{n-1}} & \ldots & \alpha_n^{\rho_1} \end{vmatrix}$$

is positive.

Lemma 2. *Let* $0 < \rho_{n-1} < \ldots < \rho_2 < \rho_1$ *and* $\alpha_k > 0, \alpha_k \to +\infty$. *If a function* $\gamma(t)$ *satisfies the estimate*

$$\gamma(t) = o(t^{\rho_{n-1}}),\ t \to \infty,$$

then it is possible to choose n *numbers* α_{k_j}, $j = 1, 2, \ldots, n$, *from the sequence* $\{a_k\}$ *such that the determinant*

$$A = \begin{vmatrix} \gamma(\alpha_{k_1}) & \alpha_{k_1}^{\rho_{n-1}} & \ldots & \alpha_{k_1}^{\rho_1} \\ \gamma(\alpha_{k_2}) & \alpha_{k_2}^{\rho_{n-1}} & \ldots & \alpha_{k_2}^{\rho_1} \\ \ldots & \ldots & \ldots & \ldots \\ \gamma(\alpha_{k_n}) & \alpha_{k_n}^{\rho_{n-1}} & \ldots & \alpha_{k_n}^{\rho_1} \end{vmatrix} \neq 0.$$

P r o o f. Without loss of generality, one may suppose that $|\gamma(t)|/t^{\rho_{n-1}}$ tends to zero monotonically as $t \to \infty$.

We will use the induction for the proof of this lemma. We may choose two numbers α_{k_1} and α_{k_2} such that the determinant

$$\begin{vmatrix} \gamma(\alpha_{k_1}) & \alpha_{k_1}^{\rho_{n-1}} \\ \gamma(\alpha_{k_2}) & \alpha_{k_2}^{\rho_{n-1}} \end{vmatrix}$$

does not equal zero. It follows from the conditions for numbers α_k, $k = 1, 2$, and the function $\gamma(t)$.

Assume this lemma is true for the determinants of order at most $n - 1$. Let us use the Laplace expansion of determinant A along the last column. In virtue of the assumption of induction the last element of this column $\alpha_{k_n}^{\rho_1}$ is multiplied by nonzero minor. Taking into account the inequalities for the orders ρ_j, $j = 1, 2, \ldots, n-1$, we can conclude that in the sequence $\{\alpha_k\}$ there is such a sufficiently large number α_n that the determinant $A \neq 0$. The lemma is proved.

Now we return to our theorem.

From (4) we get such a sequence of points $\{s_k\}_{k=1}^{\infty}$ that $\lim\limits_{k\to\infty} s_k = +\infty$ and

$$\varphi_1(s_k) = o(s_k^{\rho_n}),\ k \to \infty.$$

In virtue of Lemma 2 we can choose n points $s_1, \ldots, s_n$ from this sequence such that the determinant

$$\begin{vmatrix} s_1^{\rho_2} & s_1^{\rho_3} & \ldots & s_1^{\rho_n} & \varphi_1(s_1) \\ s_2^{\rho_2} & s_2^{\rho_3} & \ldots & s_2^{\rho_n} & \varphi_1(s_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ s_n^{\rho_2} & s_n^{\rho_3} & \ldots & s_n^{\rho_n} & \varphi_1(s_n) \end{vmatrix} \neq 0.$$

Substituting these points s_k, $k = 1, \ldots, n$, in (3) we obtain the system of the linear equations with non-vanishing determinant. Consequently, every term of asymptotics (3) is the function of bounded variation with respect to variable θ. The theorem is proved.

Now we consider the case of the remainder term of general form.

Theorem 2. *Let a distribution function of measure μ have representation (3), where Δ_1 is a monotone nondecreasing function, and there are $t_1 < t_2 < \ldots < t_{n-1}$ such that the remainder term $\varphi(t_j, \theta)$, $j = 1, \ldots, n-1$, is a function of bounded variation.*

Then all terms of asymptotic representation (3) are the functions of bounded variation.

P r o o f. Substituting the values t_j, $j = 1, \ldots, n-1$, in (3) we obtain the system of the linear equations

$$\sum_{j=2}^{n} \Delta_j(\theta) t_k^{\rho_j} = \mu(t_k, \theta) - \Delta_1(\theta) t_k^{\rho_1}, \ k = 1, \ldots, n-1.$$

In view of Lemma 1 the determinant of this system

$$\begin{vmatrix} t_1^{\rho_2} & t_1^{\rho_3} & \ldots & t_1^{\rho_n} \\ t_2^{\rho_2} & t_2^{\rho_3} & \ldots & t_2^{\rho_n} \\ \ldots & \ldots & \ldots & \ldots \\ t_n^{\rho_2} & t_n^{\rho_3} & \ldots & t_n^{\rho_n} \end{vmatrix}$$

is not zero. So, it is easy to see that the bounded variation of the functions

$$\mu(t_j, \theta) - \Delta_1(\theta) t_j^{\rho_1}, \ j = 1, \ldots, n-1,$$

implies the bounded variation of the functions Δ_k, $k = 2, \ldots, n$. Hence the remainder term $\varphi(t, \theta)$ is also the function of bounded variation with respect to θ for any t. The theorem is proved.

R e m a r k 2. Notice that the above examples show that any "smallness" of the remainder term does not retain differential properties of the functions $\mu(t, \theta)$ and Δ_1 for other terms of asymptotics, even the Lipschitz condition. At the same time, the fulfilment of conditions of these theorems guarantees that the functions $\Delta_j, j = 2, 3, \ldots, n$, and $\varphi(t, \theta)$ are differentiable with respect to θ at those points, where the functions Δ_1 and $\mu(t, \theta)$ are differentiable.

Thus for the asymptotic representations of measure distribution functions we have found the sufficient conditions on the remainder term that guarantee the boundedness of variation and the differentiability with respect to the angle variable of all terms of this asymptotics.

Consider now the measure μ that satisfies the conditions of Theorems 1 or 2. It is known [1] that outside of any exceptional set the subharmonic function $u(re^{i\theta})$ corresponding to μ has the following asymptotics:

$$u(re^{i\theta}) = \sum_{j=1}^{n} H_j(\theta) r^{\rho_j} + \psi(re^{i\theta}),$$

where

$$H_j(\theta) = \frac{\pi}{\sin \pi\rho_j} \int\limits_{\theta-2\pi}^{\theta} \cos \rho_j(\theta - \alpha - \pi) d\Delta_j(\alpha), \ j = 1, 2, \ldots, n.$$

Obviously, from our theorems we obtain that every term of this asymptotics, starting from the second one, is a δ-subharmonic function.

This special case has been considered recently in the paper [3].

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