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On the Spectrum of Riemannian Manifolds with Attached Thin Handles

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The behavior as $\varepsilon \to 0$ of the spectrum of the Laplace–Beltrami operator Δ^ε is studied on Riemannian manifolds depending on a small parameter ε. They consist of a fixed compact manifold with attached handles whose radii tend to zero as $\varepsilon \to 0$. We consider two cases: when the number of the handles is fixed and their lengthes are also fixed and when the number of the handles tend to infinity and their lengthes tend to zero as $\varepsilon \to 0$. For these cases we obtain the operators whose spectrum attracts the spectrum of Δ^ε as $\varepsilon \to 0$.

Key words: homogenization, Laplace–Beltrami operator, spectrum, Riemannian manifold.

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Introduction

The aim of the paper is to study the behavior as $\varepsilon \to 0$ of the spectrum of the Laplace–Beltrami operator Δ^ε on the Riemannian manifolds M^ε depending on a small parameter ε. We consider two different problems.

In Section 1 we consider a manifold M^ε that consists of a fixed two-dimensional compact Riemannian manifold without boundary Ω and an attached "thin" manifold Γ^ε. The last one consists of several tubes with fixed lengthes and radii ε (see Fig. 1 below). Thus Γ^ε "converges" to some graph Γ as $\varepsilon \to 0$.

Let Δ_Ω be the Laplace–Beltrami operator on Ω and $\mathbf{L}$ be the Laplace operator on Γ, i.e., $\mathbf{L}$ is defined by the operation $\frac{d^2}{ds^2}$ on the edges of Γ (s is a natural parameter on the edge), Dirichlet boundary conditions on the ends of Γ and Kirchhoff conditions on the vertices of Γ. We prove that the spectrum of Δ^ε

converges in some suitable sense to the union of the spectrum of Δ_Ω and the spectrum of $\mathbf{L}$. Also we study the behavior of corresponding eigenvalues.

These results generalize the results by C. Anne [1]. The behavior of spectrum is studied on a manifold with *one* attached handle having a fixed length and a vanishingly small radius in [1]. These results are extended to the case of the Laplacian acting on differential p-forms in [2]. The convergence of spectra on manifolds which collapse to a graph was studied in [6].

In Section 2 we consider the manifold M^ε whose topological genus increases as $\varepsilon \to 0$. It is constructed in the following way. Let Ω be a compact two-dimensional Riemannian manifold without boundary, and D_i^ε, $i = 1 \ldots N(\varepsilon) = 3N_1(\varepsilon)$ be a system of nonintersecting balls ("holes") in Ω depending on ε. Let $\Omega^\varepsilon = \Omega \backslash \bigcup\limits_{i=1}^{N(\varepsilon)} D_i^\varepsilon$. Suppose that the set $\{1 \ldots N(\varepsilon)\}$ is divided into subsets that consist of three elements. If the indexes i, j, k lie on one subset we connect the "holes" $D_i^\varepsilon, D_j^\varepsilon, D_k^\varepsilon$ by means of a manifold that consists of the tubes $G_i^\varepsilon, G_j^\varepsilon, G_k^\varepsilon$ and a truncated sphere B_{ijk}^ε (see Fig. 2 below). As a result, we obtain the manifold

$$M^\varepsilon = \Omega^\varepsilon \bigcup_{i,j,k} \left[G_i^\varepsilon \cup G_j^\varepsilon \cup G_k^\varepsilon \cup B_{ijk}^\varepsilon\right].$$

We suppose that the number of "holes" increases as $\varepsilon \to 0$, while their radii tend to 0. It is supposed that the radii of the "holes" are much smaller than the distances between them. We also suppose that, in contrast to the manifold Γ^ε in Sect. 1 and in contrast to [1], the metric is such that the lengthes of the tubes converge to 0.

We obtain the following result: if some conditions on a distribution of the "holes" and on the metrics on the tubes and the truncated spheres are hold, then the spectrum of the operator $-\Delta^\varepsilon$ converges in some suitable sense to the spectrum of the operator $\mathcal{L}$ defined by the formula

$$[\mathcal{L}u](x) = -\Delta_\Omega u(x) + \int\limits_\Omega W(x,y)(u(x) - u(y))dy.$$

Here $W(x,y)$ is a positive symmetric function. We present an example for which $W(x,y)$ is calculated explicitly.

The behavior of the spectrum of manifolds with complex microstructure was studied in [5, 8] for another type of manifolds. We note that the behavior of spectrum of manifold with the attached *one* handle, having a vanishingly small radius and (in contrast to [1]) a vanishingly small length, was studied in [4].

The proof of main results is based on the abstract scheme proposed in [7].

Throughout the paper, we will denote by C various constants independent from ε.

1. Riemannian Manifold with Attached "Graph"

1.1. Problem Setting and Main Result

Let Ω be a two-dimensional compact Riemannian manifold without boundary and with a metric g. By Δ_Ω we denote the corresponding Laplace–Beltrami operator. Let D_i^ε, $i = 1 \dots N$ be a system of balls in Ω with the centers $x_i \in \Omega$ and the radii ε. We consider the following domain with holes:

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{i=1}^{N} D_i^\varepsilon.$$

To Ω^ε we glue the manifold Γ^ε illustrated on Fig. 1 and constructed as follows.

Let Γ be a graph in $\mathbb{R}^3$. We denote the vertices of this graph by p_i, $i = 1 \dots m$ ($m > N$) and the edges of the same graph by γ_{ij}^ε. γ_{ij}^ε connects the vertices p_i and p_j. We introduce the symmetric matrix $\{A_{ij}\}_{i,j=1}^m$ such that $A_{ij} = 1$ if p_i^ε and p_j^ε are connected and $A_{ij} = 0$ otherwise. We suppose that for the first N vertices p_i, $i = 1 \dots N$ there is only one p_j such that $A_{ij} = 1$. These are the ends of the graph.

Let z_{ij} be the natural parameter on γ_{ij}, $z_{ij} \in [0, l_{ij}]$. We denote by $p(z_{ij})$ the point on γ_{ij} that corresponds to the natural parameter z_{ij}.

We denote by G_{ij}^ε the cylinder with the axis $\widehat{\gamma}_{ij} = \{p(z_{ij}) \in \gamma_{ij} : z_{ij} \in [\delta^\varepsilon, l_{ij} - \delta^\varepsilon], \delta^\varepsilon \geq 0\}$ and with the radius ε. The length of G_{ij}^ε is equal to $l_{ij}^\varepsilon = l_{ij} - 2\delta^\varepsilon$. We choose the standard cylindrical coordinates on G_{ij}^ε

$$G_{ij}^\varepsilon = \left\{ (\varphi_{ij}, z_{ij}) : \varphi_{ij} \in [0, 2\pi], z_{ij} \in [\delta^\varepsilon, l_{ij} - \delta^\varepsilon] \right\}.$$

Clearly, δ^ε can be chosen such that:

1. G_{ij}^ε are pairwise disjoint,
2. $|\delta^\varepsilon| \leq C \cdot \varepsilon$.

The boundary of G_{ij}^ε consists of two circles S_{ij}^ε and S_{ji}^ε. Here we suppose that S_{ij}^ε is closer to the vertex p_i, and S_{ji}^ε is closer to the vertex p_j.

For $i \in \{N+1 \dots m\}$, let $\mathcal{B}_i^\varepsilon$ be the sphere of the radius $b^\varepsilon = \sqrt{\varepsilon^2 + \delta^{\varepsilon 2}}$ with the center p_i. It is clear that $S_{ij}^\varepsilon \subset \mathcal{B}_i^\varepsilon$. Let $\mathcal{D}_{ij}^\varepsilon$ be a part of $\mathcal{B}_i^\varepsilon$ that lies inside the cylinder G_{ij}^ε, and let

$$B_i^\varepsilon = \mathcal{B}_i^\varepsilon \setminus \bigcup_{j:A_{ij}=1} \mathcal{D}_{ij}.$$

We obtain a two-dimensional manifold (see Fig. 1):

$$\Gamma^\varepsilon = \bigcup_{i=1}^{m} \left[\bigcup_{i,j:A_{ij}=1, i<j} G_{ij}^\varepsilon \right] \bigcup_{i=N+1}^{m} B_i^\varepsilon.$$

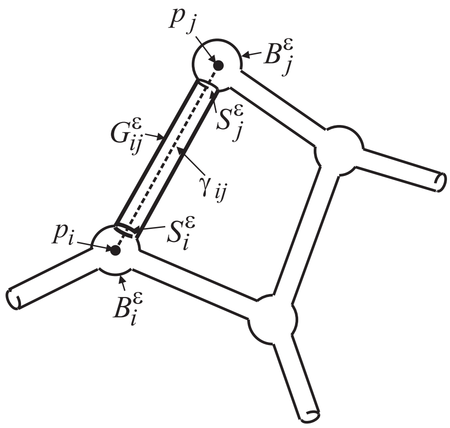

Fig. 1: Manifold Γ^ε.

The boundary of Γ^ε consists of S^ε_{ij}, i, $j : i = 1 \dots N$, $A_{ij} = 1$.

Now we suppose that S^ε_{ij}, i, $j : i = 1 \dots N$, $A_{ij} = 1$ are diffeomorphic to ∂D^ε_i. Using this diffeomorphisms, we glue Γ^ε to Ω^ε and obtain a manifold without boundary

$$M^\varepsilon = \Omega^\varepsilon \cup \Gamma^\varepsilon.$$

We denote by $\tilde{x}$ the points of this manifold. Clearly, M^ε can be covered by a system of charts and suitable local coordinates $\{x_1, x_2\}$ can be introduced.

It is supposed that M^ε is equipped with the metric g^ε that coincides with the metric g on Ω^ε and with the Euclidean metrics induced from $\mathbb{R}^3$ on Γ^ε. By $g^\varepsilon_{\alpha\beta}$, we denote the components of the metric tensor in local coordinates.

Let $L_2(B)$ be a Hilbert space of the real-valued functions on $B \subset M^\varepsilon$ with the scalar product and the norm

$$(u, v)_{L_2(B)} = \int\limits_B u(\tilde{x}) \cdot v(\tilde{x}) d\tilde{x},\ \|u\|^2_{L_2(B)} = \int\limits_B \big(u(\tilde{x})\big)^2 d\tilde{x},$$

where $d\tilde{x} = \sqrt{\det g^\varepsilon_{\alpha\beta}} dx_1 dx_2$ is the volume form.

We denote $\mathcal{H}^\varepsilon := L_2(M^\varepsilon)$, $\mathcal{H}_0 := L_2(\Omega) \times L_2(\Gamma)$.

Let Δ^ε be a Laplace–Beltrami operator on M^ε. It is well known that the spectrum of the operator $-\Delta^\varepsilon$ is purely discrete. Let $0 = \lambda_1^\varepsilon < \lambda_2^\varepsilon \le \lambda_3^\varepsilon \le \dots \le \lambda_k^\varepsilon \underset{k\to\infty}{\to} \infty$ be the eigenvalues of $-\Delta^\varepsilon$ written with account of their multiplicity, $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon \dots$ be the corresponding eigenvectors normalized by the condition $(u_i^\varepsilon, u_j^\varepsilon)_{\mathcal{H}^\varepsilon} = \delta_{ij}$.

In this section we study the behavior of λ_k^ε as $\varepsilon \to 0$.

Let $\mathbf{L} : L_2(\Gamma) \to L_2(\Gamma)$ be a Laplace operator on the graph Γ with Dirichlet boundary conditions, i.e., $\mathbf{L}$ is defined by the operation

$$[\mathbf{L}u](x) = -\frac{d^2u}{dz_{ij}}(x), \; x = p(z_{ij}) \in \gamma_{ij}$$

and by a definitional domain consisting of the functions $u \in H^2(\gamma_{ij})$ $\forall i,j$ and such that if we denote by u_{ij} the restriction of u on γ_{ij}, then

$$\text{for } i = \overline{1,N}: \; u(x_i) = 0,$$

$$\text{for } i = \overline{N+1,m}: \; \begin{cases} u_{ij}(p_i) \text{ are equivalent for all } j : A_{ij} = 1, \\ \displaystyle\sum_{j:A_{ij}=1} \frac{\partial u_{ij}}{\partial \nu}(p_i) = 0, \end{cases}$$

where $\dfrac{\partial}{\partial \nu}$ means the derivative in the direction outward to γ_{ij}. In short, u is a continuous function on Γ that satisfies the Dirichlet conditions on the ends of the graph as well as Kirchhoff conditions in the vertices (for more precise description of differential operators on the graphs and its properties see, e.g., [6]).

To describe the behavior of eigenfunctions we introduce the operator $R^\varepsilon : \mathcal{H}_0 \to \mathcal{H}^\varepsilon$:

$$[R^\varepsilon f](\tilde{x}) = \begin{cases} f_0(\tilde{x}), \tilde{x} \in \Omega^\varepsilon, \\ f_{ij}(z_{ij})\varepsilon^{-1/2}, \tilde{x} = (z_{ij}, \varphi_{ij}) \in G^\varepsilon_{ij}, \\ 0, \tilde{x} \in B^\varepsilon_i, \end{cases}$$

$$f = (f_0, f_{ij}, i, j : A_{ij} = 1) \in L_2(\Omega) \times L_2(\Gamma).$$

Let $\mathcal{L} : \mathcal{H}_0 \to \mathcal{H}_0$:

$$\mathcal{L} = \begin{pmatrix} -\Delta_\Omega & 0 \\ 0 & \mathbf{L} \end{pmatrix},$$

and let $\lambda_0, \lambda_1, \lambda_2 ...$ be the eigenvalues of $\mathcal{L}$ written with account of their multiplicity. It is clear that *the spectrum of $\mathcal{L}$ is the union of the eigenvalues of the operator $-\Delta_\Omega$ and the eigenvalues of the operator $\mathbf{L}$ that are taken with account of their multiplicity.*

Theorem 1.1. *For any k=1,2,3. . .*

$$\lambda_k^\varepsilon \to \lambda_k, \; \varepsilon \to 0.$$

Theorem 1.2. *Let $\lambda_k < \lambda_{k+1} = \lambda_{k+2} = \ldots = \lambda_{k+m} < \lambda_{k+m+1}$ (i.e., the multiplicity of λ_{k+1} is equal to m). Let $N(\lambda_{k+1})$ be the eigenspace of the* *eigenvalue* λ_{k+1}. *Then for any* $w \in N(\lambda_{k+1})$ *there exists a linear combination* $\bar{u}^\varepsilon$ *of the eigenfunctions* $u^\varepsilon_{k+1} \dots u^\varepsilon_{k+m}$ *such that*

$$\|\bar{u}^\varepsilon - R^\varepsilon w\|_{\mathcal{H}^\varepsilon} \to 0,\ \varepsilon \to 0. \tag{1.1.1}$$

1.2. Proof of Theorems 1.1 and 1.2

We prove Theorems 1.1 and 1.2 for the case $N = 3, m = 4$, i.e., Γ^ε consists of three tubes $G^\varepsilon_{14}, G^\varepsilon_{24}, G^\varepsilon_{34}$ and the truncated sphere B^ε_4 that connects these tubes. For the general case the theorems are proved in a similar way. We introduce new notations:

$$l^\varepsilon_i := l^\varepsilon_{i4},\ z_i := z_{i4},\ \varphi_i := \varphi_{i4},\ G^\varepsilon_i := G^\varepsilon_{i4},$$
$$S^\varepsilon_j := S^\varepsilon_{4j},\ C^\varepsilon_i := S^\varepsilon_{i4},\ \mathcal{B}^\varepsilon := \mathcal{B}^\varepsilon_4,\ B^\varepsilon := B^\varepsilon_4,\ i,j = 1,2,3,$$

(i.e. $\partial\Gamma^\varepsilon = \bigcup\limits_{i=1,2,3} C^\varepsilon_i$).

For simplicity we suppose that the metric g is Euclidean in some neighbourhood of the holes D^ε_i (and thus g^ε is continuous). For the general case the proof needs small modifications.

We denote by $\mathcal{A}^\varepsilon$ and $\mathcal{A}_0$ the operators inverse to $-\Delta^\varepsilon+\mathrm{I}$ and $\mathcal{L}+\mathrm{I}$, respectively (I is an identical operator).

Now we study the behavior of $\mathcal{A}^\varepsilon$ as $\varepsilon \to 0$.

Theorem 1.3. *The following conditions are fulfilled:*

C1. *For any* $f \in \mathcal{H}_0$

$$\|R^\varepsilon f\|_{\mathcal{H}^\varepsilon} \to \|f\|_{\mathcal{H}_0}, \varepsilon \to 0. \tag{1.2.1}$$

C2. *The operators* $\mathcal{A}^\varepsilon, \mathcal{A}_0$ *are positive, compact, self-adjoint and bounded in* $\mathcal{L}(\mathcal{H}^\varepsilon)$ *uniformly with respect to* ε.

C3. *For any* $f \in \mathcal{H}_0$

$$\|\mathcal{A}^\varepsilon R^\varepsilon f - R^\varepsilon \mathcal{A}_0 f\|_{\mathcal{H}^\varepsilon} \to 0, \varepsilon \to 0. \tag{1.2.2}$$

C4. *For any sequence* $f^\varepsilon \in \mathcal{H}^\varepsilon$ *such that* $\sup \|f^\varepsilon\|_{\mathcal{H}^\varepsilon} < \infty$ *there exists the subsequence* ε' *and* $w \in \mathcal{H}_0$ *such that*

$$\|\mathcal{A}^\varepsilon f^\varepsilon - R^\varepsilon w\|_{\mathcal{H}^\varepsilon} \to 0, \varepsilon = \varepsilon' \to 0. \tag{1.2.3}$$

P r o o f **1.** The condition C1 follows directly from the definition of the operator R^ε.

2. The condition C2 follows easily from the properties of the resolvent, namely the following estimate is valid

$$\|\mathcal{A}^\varepsilon\|_{\mathcal{L}(H^\varepsilon)} \le 1.$$

3. Let $f \in \mathcal{H}$. We denote $u^\varepsilon = \mathcal{A}^\varepsilon R^\varepsilon f$, $f^\varepsilon = R^\varepsilon f$. To describe the behavior of u^ε on Ω^ε we introduce the operator $\Pi_0^\varepsilon : H^1(M^\varepsilon) \to H^1(\Omega)$ with the following properties:

1) $\|\Pi^\varepsilon u^\varepsilon\|_{\mathcal{H}_0} + \|\nabla^\varepsilon \Pi^\varepsilon u^\varepsilon\|_{\mathcal{H}_0} \le C\Big\{\|u^\varepsilon\|_{\mathcal{H}^\varepsilon} + \|\nabla^\varepsilon u^\varepsilon\|_{\mathcal{H}^\varepsilon}\Big\}$, $C > 0$,

2) $\Pi_0^\varepsilon u^\varepsilon(\tilde{x}) = u^\varepsilon(\tilde{x})$ on Ω^ε.

(Here $\|\nabla^\varepsilon u^\varepsilon\|_{\mathcal{H}^\varepsilon} := \int\limits_{M^\varepsilon} \sum\limits_{\alpha,\beta=1}^{2} g_\varepsilon^{\alpha\beta} \frac{\partial u}{\partial x_\alpha} \frac{\partial u}{\partial x_\beta} d\tilde{x}$, where $g_\varepsilon^{\alpha\beta}$ are the components of the tensor inverse to g^ε). This operator exists, see, e.g, [3].

Due to C1–C2 we have $\|u^\varepsilon\|_{\mathcal{H}^\varepsilon} \le \|f^\varepsilon\|_{\mathcal{H}^\varepsilon} \to \|f\|_{\mathcal{H}_0}$. Moreover, using variational methods, we obtain

$$\|\nabla^\varepsilon u^\varepsilon\|^2_{\mathcal{H}^\varepsilon} \le 2\|f^\varepsilon\|_{\mathcal{H}^\varepsilon} \cdot \|u^\varepsilon\|_{\mathcal{H}^\varepsilon}.$$

Using these inequalities and the properties of the operator Π_0^ε, we conclude that $\Pi_0^\varepsilon u^\varepsilon$ is bounded in $H^1(\Omega)$ uniformly with respect to ε, and therefore there exists a subsequence (still denoted by ε) such that

$$\Pi_0^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u_0 \in H^1(\Omega) \text{ weakly in } H^1(\Omega) \text{ and strongly in } L_2(\Omega). \tag{1.2.4}$$

To describe the behavior of u^ε on the tubes G_i^ε, $i = 1, 2, 3$, we represent u^ε in the form

$$u^\varepsilon(\varphi_i, z_i) = P_i^\varepsilon u^\varepsilon(z_i) + Q_i^\varepsilon u^\varepsilon(\varphi_i, z_i), \tag{1.2.5}$$

where

$$P_i^\varepsilon u^\varepsilon(z_i) = \frac{1}{2\pi} \int\limits_0^{2\pi} u^\varepsilon(\varphi_i, z_i) d\varphi_i.$$

Let $\Pi_i^\varepsilon : H^1(G_i^\varepsilon) \to H^1([0, l_i])$ that is defined by the formula

$$\Pi_i^\varepsilon u^\varepsilon(z_i) = \begin{cases} \sqrt{\varepsilon} P_i^\varepsilon(z_i), \ z_i \in [\delta^\varepsilon, l_i - \delta^\varepsilon], \\ \sqrt{\varepsilon} P_i^\varepsilon(\delta^\varepsilon), \ z_i \in [0, \delta^\varepsilon), \\ \sqrt{\varepsilon} P_i^\varepsilon(l_i - \delta^\varepsilon), \ z_i \in (l_i - \delta^\varepsilon, l_i]. \end{cases}$$

We have the following estimates:

$$\left\|\frac{d}{dz_i}\Pi_i^\varepsilon u^\varepsilon\right\|^2_{L_2[0,l_i]} = \int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\left(\frac{\partial}{\partial z_i}\frac{1}{2\pi}\int\limits_0^{2\pi}u^\varepsilon(\varphi_i,z_i)\sqrt{\varepsilon}d\varphi_i\right)^2 dz_i$$

$$\leq \frac{1}{2\pi}\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\int\limits_0^{2\pi}\left(\frac{\partial}{\partial z_i}u^\varepsilon(\varphi_i,z_i)\right)^2\varepsilon d\varphi_i dz_i \leq (2\pi)^{-1}\|\nabla^\varepsilon u^\varepsilon\|^2_{0\varepsilon}, \tag{1.2.6}$$

$$\|\Pi_i^\varepsilon u^\varepsilon\|^2_{L_2[0,l_i]} \leq \delta^\varepsilon\left[(\Pi_i^\varepsilon u^\varepsilon(\delta^\varepsilon))^2 + (\Pi_i^\varepsilon u^\varepsilon(l_i-\delta^\varepsilon))^2\right] + (2\pi)^{-1}\|u^\varepsilon\|^2_{L_2(G_i^\varepsilon)}. \tag{1.2.7}$$

Further,

$$(\Pi_i^\varepsilon u^\varepsilon(\delta^\varepsilon))^2 \leq 2\left((\Pi_i^\varepsilon u^\varepsilon(z_i))^2 + l_i\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\left|\frac{\partial}{\partial z_i}\Pi_i^\varepsilon u^\varepsilon(\varphi_i,z_i)\right|^2 dz_i\right).$$

Integrating this estimate on z_i from δ^ε to $l_i-\delta^\varepsilon$ one has

$$(\Pi_i^\varepsilon u^\varepsilon(\delta^\varepsilon))^2 \leq C\left(\|\Pi_i^\varepsilon u^\varepsilon\|^2_{L_2[\delta^\varepsilon,l_i-\delta^\varepsilon]} + \left\|\frac{d}{dz_i}\Pi_i^\varepsilon u^\varepsilon\right\|^2_{L_2[\delta^\varepsilon,l_i-\delta^\varepsilon]}\right) \tag{1.2.8}$$

and, similarly,

$$(\Pi_i^\varepsilon u^\varepsilon(l_i-\delta^\varepsilon))^2 \leq C\left(\|\Pi_i^\varepsilon u^\varepsilon\|^2_{L_2[\delta^\varepsilon,l_i-\delta^\varepsilon]} + \left\|\frac{d}{dz_i}\Pi_i^\varepsilon u^\varepsilon\right\|^2_{L_2[\delta^\varepsilon,l_i-\delta^\varepsilon]}\right). \tag{1.2.9}$$

It follows from (1.2.6)-(1.2.9) that $\Pi_i u^\varepsilon$ is bounded in $H^1([0,l_i])$, and therefore for $i=1,2,3$ there exists a subsequence (still denoted by ε) such that

$$\Pi_i^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u_i \in H^1([0,l_i]) \text{ weakly in } H^1([0,l_i]) \text{ and strongly in } L_2([0,l_i]). \tag{1.2.10}$$

The following lemma says that u^ε is vanishingly small in B^ε.

Lemma 1.1. *Let $u^\varepsilon \in H^1(M^\varepsilon)$. Then*

$$\|u^\varepsilon\|^2_{L_2(B^\varepsilon)}$$
$$\leq C\left(\varepsilon^2\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(B^\varepsilon)} + \varepsilon\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(\cup_i G_i^\varepsilon)} + \varepsilon^2|\ln\varepsilon|\left(\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} + \|u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)}\right)\right).$$

P r o o f. At first we note that u^ε can be extended to the whole ball $\mathcal{B}^\varepsilon$ in such a way that $\|\nabla^\varepsilon u^\varepsilon\|_{L_2(\mathcal{B}^\varepsilon)} \leq C\|\nabla^\varepsilon u^\varepsilon\|_{L_2(B^\varepsilon)}$ (see [9, p. 118, Ex. 4.10]). Let us fix i

from $\{1,2,3\}$. We introduce the spherical coordinates $\varphi \in [0,2\pi], \theta \in [0,\pi]$ on $\mathcal{B}^\varepsilon$ such that the points of S_i^ε have the coordinates $\varphi \in [0,2\pi], \theta = \arcsin(\varepsilon/b^\varepsilon) =: \theta^\varepsilon$.

So, we extend u^ε to $\mathcal{B}^\varepsilon$ and have

$$u^\varepsilon(\varphi,\theta) = u^\varepsilon(\varphi,\theta^\varepsilon) + \int\limits_{\theta^\varepsilon}^{\theta} \frac{\partial u^\varepsilon(\varphi,\psi)}{\partial \psi} d\psi.$$

Further,

$$\int\limits_0^{2\pi}\int\limits_{\theta^\varepsilon}^{\pi-\theta^\varepsilon} (u^\varepsilon(\varphi,\theta))^2 b^{\varepsilon 2} \sin\theta d\theta d\varphi$$

$$\le C\varepsilon^2 \left(\int\limits_{\theta^\varepsilon}^{\pi-\theta^\varepsilon} \left(\frac{\partial u^\varepsilon(\varphi,\psi)}{\partial\psi}\right)^2 \sin\psi d\psi d\varphi \cdot \int\limits_{\theta^\varepsilon}^{\pi-\theta^\varepsilon} (\sin\psi)^{-1} d\psi + \int\limits_0^{2\pi} \left(u^\varepsilon(\varphi,\theta^\varepsilon)\right)^2 d\varphi \right). \tag{1.2.11}$$

Since $C_1 \le \theta^\varepsilon \le C_2$, the first term is estimated by $C\varepsilon^2\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(B^\varepsilon)}$. Now we estimate the second term. Representing the corresponding integral in the cylindrical coordinates, one has

$$u^\varepsilon(\varphi,\theta^\varepsilon) \equiv u^\varepsilon(\varphi_i,l_i) = u^\varepsilon(\varphi_i,0) + \int\limits_0^{l_i} \frac{\partial u^\varepsilon(\varphi_i,z_i)}{\partial z_i} dz_i.$$

Let D^ε and R^ε be the balls in Ω with the radii d^ε and r^ε ($r^\varepsilon > d^\varepsilon$). Then for any $u \in H^1(R^\varepsilon \setminus D^\varepsilon)$ the following estimate is valid (see [1]):

$$\|u\|^2_{L_2(\partial D^\varepsilon)} \le C d^\varepsilon \left[|\ln d^\varepsilon| \cdot \|\nabla u\|^2_{L_2(R^\varepsilon\setminus D^\varepsilon)} + \frac{1}{(r^\varepsilon)^2}\|u\|^2_{L_2(R^\varepsilon\setminus D^\varepsilon)} \right]. \tag{1.2.12}$$

Using (1.2.12), we have

$$\varepsilon^2 \int\limits_0^{2\pi} \left(u^\varepsilon(\varphi_i,l_i)\right)^2 d\varphi_i \le C\varepsilon^2 \left[\left(\int\limits_0^{2\pi} u^\varepsilon(\varphi_i,0) d\varphi_i \right)^2 + \int\limits_0^{2\pi}\int\limits_0^{l_i} \left(\frac{\partial u^\varepsilon(\varphi_i,z_i)}{\partial z_i}\right)^2 dz_i \right]$$

$$\le C\left[\varepsilon^2 |\ln\varepsilon| \left(\|u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} + \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} \right) + \varepsilon \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(G_i^\varepsilon)} \right].$$

We denote $\mathcal{D}_i^{*\varepsilon} = \{(\varphi,\theta) \in \mathcal{B}^\varepsilon : \theta \in [\pi-\theta^\varepsilon, \theta^\varepsilon]\}$. It follows from (1.2.11) that for $i = 1,2,3$:

$$\|u^\varepsilon\|^2_{L_2(\mathcal{B}^\varepsilon\setminus(D_i^\varepsilon\cup D_i^{*\varepsilon}))}$$

$$\le C\Big(\varepsilon^2 \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(B^\varepsilon)} + \varepsilon\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(G_i^\varepsilon)} + \varepsilon^2|\ln\varepsilon| \big(\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} + \|u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} \big) \Big).$$

The lemma is proved since $\bigcup\limits_{i=1,2,3} [\mathcal{B}^\varepsilon \setminus (D_i^\varepsilon \cup D_i^{*\varepsilon})] = \mathcal{B}^\varepsilon$.

We return to the proof of Theorem 1.3. We denote $u := (u_0, u_1, u_2, u_3)$. Let us prove that $u = \mathcal{A}_0 f$, what is equal to the fulfilment of the following conditions:

$$\textbf{I}.\ u_i(0) = 0, i = 1,2,3, \tag{1.2.13}$$

$$\textbf{II}.\ u_1(l_1) = u_2(l_2) = u_1(l_3), \tag{1.2.14}$$

$$\textbf{III}.\ (\nabla u_0, \nabla w)_{\mathcal{H}_0} + (u_0, w)_{\mathcal{H}_0} = (f_0, w)_{\mathcal{H}_0},\ \forall w \in H^1(\Omega), \tag{1.2.15}$$

$$\textbf{IV}.\ \sum_{i=1}^{3} \int_0^{l_i} (u_i(z))'(w_i(z))' dz + \sum_{i=1}^{3} \int_0^{l_i} u_i(z) w_i(z) dz$$

$$= \sum_{i=1}^{3} \int_0^{l_i} f_i(z) w_i(z) dz,\ \forall w_i \in H^1[0, l_i]. \tag{1.2.16}$$

Let us verify the fulfilment of these conditions.

I. Using the trace theorem, for $i = 1,2,3$ we have

$$u_i(0) = \lim_{\varepsilon\to 0} \Pi_i^\varepsilon u^\varepsilon(0) = \lim_{\varepsilon\to 0} \Pi_i^\varepsilon u^\varepsilon(\delta^\varepsilon) = \sqrt{\varepsilon} \lim_{\varepsilon\to 0} \bar{u}_i^\varepsilon, \tag{1.2.17}$$

where $\bar{u}_i^\varepsilon$ is the mean value of u^ε over ∂D_i^ε. It follows from (1.2.12), (1.2.17) that (1.2.13) is valid.

II. For $i, j = 1,2,3$, one has

$$|u_i(l_i) - u_j(l_j)| = \lim_{\varepsilon\to 0} \sqrt{\varepsilon} |\hat{u}_i^\varepsilon - \hat{u}_j^\varepsilon|, \tag{1.2.18}$$

where $\hat{u}_i^\varepsilon$ is the average value of u^ε over the circle S_i^ε.

We denote $v^\varepsilon(\tilde{x}) := u^\varepsilon(\tilde{x}) - U^\varepsilon$, where U^ε is the average value of u^ε over B^ε, and by $\hat{v}_i^\varepsilon$ we denote the average value of v^ε over the circle S_i^ε.

Using the inequality of the type (1.2.12) and Poincare inequality, one has

$$|\hat{v}_i^\varepsilon|^2 \le C \left[\left| \ln \tan \frac{\theta^\varepsilon}{2} \right| \cdot \|\nabla^\varepsilon v^\varepsilon\|^2_{L_2(B^\varepsilon)} + \frac{1}{(b_i^\varepsilon)^2} \|v^\varepsilon\|^2_{L_2(B^\varepsilon)} \right]$$

$$\le C \left| \ln \tan \frac{\theta^\varepsilon}{2} \right| \cdot \|\nabla^\varepsilon v^\varepsilon\|^2_{L_2(B^\varepsilon)}. \tag{1.2.19}$$

Using (1.2.19), we have

$$|\hat{u}_i^\varepsilon - \hat{u}_j^\varepsilon| = |\hat{v}_i^\varepsilon - \hat{v}_j^\varepsilon| \le |\hat{v}_i^\varepsilon| + |\hat{v}_j^\varepsilon| \le C \sqrt{\left| \ln \tan \frac{\theta^\varepsilon}{2} \right|} \cdot \|\nabla^\varepsilon v^\varepsilon\|_{L_2(B^\varepsilon)}. \tag{1.2.20}$$

Since $\left|\ln\tan\frac{\theta^\varepsilon}{2}\right| < C$, it follows from (1.2.18), (1.2.20) that (1.2.14) is fulfilled.

III. Clearly, it is sufficient to prove (1.2.15) for w such that

$$\exists\delta>0\ \forall i=1\ldots 3:\ \rho_g(\mathrm{supp}(w),x_i^\varepsilon)\geq\delta,$$

where ρ_g is the distance on Ω generated by the metric g (because the set of these w is dense in $H^1(\Omega)$). Then for these w and for sufficiently small ε $\mathrm{supp}(w)\subset\Omega^\varepsilon$. Let $w^\varepsilon(\tilde{x})\in L_2(M^\varepsilon)$: $w^\varepsilon(\tilde{x})=w(\tilde{x})$ in Ω^ε and $w^\varepsilon=0$ in $M^\varepsilon\setminus\Omega^\varepsilon$. Clearly, $w^\varepsilon\in H^2(M^\varepsilon)$ for ε being small enough.

We have

$$\begin{aligned}
0&=\lim_{\varepsilon\to 0}\Big((\nabla^\varepsilon u^\varepsilon,\nabla^\varepsilon w^\varepsilon)_{\mathcal{H}^\varepsilon}+(u^\varepsilon,w^\varepsilon)_{\mathcal{H}^\varepsilon}-(f^\varepsilon,w^\varepsilon)_{\mathcal{H}^\varepsilon}\Big)\\
&=\lim_{\varepsilon\to 0}\Big((\nabla\Pi^\varepsilon u^\varepsilon,\nabla w)_{L_2(\Omega)}+(\Pi^\varepsilon u^\varepsilon,w)_{L_2(\Omega)}-(f,w)_{L_2(\Omega)}\Big)\\
&=(\nabla u_0,\nabla w)_{\mathcal{H}_0}+(u_0,w)_{\mathcal{H}_0}-(f_0,w)_{\mathcal{H}_0},
\end{aligned}$$

and (1.2.15) is valid.

IV. It is sufficient to prove (1.2.16) for such w_i that

$$\exists\delta>0\ \forall z\in[0,\delta]:\ w_i(z)=0 \text{ and } \forall z\in[l_i-\delta,l_i]:\ w_i(z)=w_i(l_i),$$

because the set of these w_i is dense in the set of test functions mentioned above. For sufficiently small ε $\delta^\varepsilon\leq\delta$, and therefore these w_i satisfy the conditions: $\forall z\in[0,\delta^\varepsilon]:\ w_i(z)=0$ and $\forall z\in[l_i-\delta^\varepsilon,l_i]:\ w_i(z)=w_i(l_i)$.

At first, let us estimate the reminder $Q^\varepsilon u^\varepsilon$ on G_i^ε. Using Poincare inequality, we have

$$\begin{aligned}
\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\int\limits_0^{2\pi}(Q_i^\varepsilon u^\varepsilon(\varphi_i,z_i))^2 d\varphi_i dz_i &\leq C\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\int\limits_0^{2\pi}\left(\frac{\partial Q_i^\varepsilon u^\varepsilon}{\partial\varphi_i}\right)^2 d\varphi_i dz_i\\
&=C\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\int\limits_0^{2\pi}\left(\frac{\partial u^\varepsilon}{\partial\varphi_i}\right)^2 d\varphi dz_i\leq C\varepsilon\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(G_i^\varepsilon)}. \qquad (1.2.21)
\end{aligned}$$

Using the above and the representation (1.2.5), we have

$$\sum_{i=1}^3\int\limits_0^{l_i}(u_i(z_i))'(w_i(z_i))'dz_i=\sum_{i=1}^3\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\frac{\partial\Pi_i^\varepsilon u^\varepsilon}{\partial z_i}\frac{\partial w_i^\varepsilon}{\partial z_i}dz_i d\varphi_i$$

$$=\sum_{i=1}^3\lim_{\varepsilon\to 0}\frac{1}{2\pi}\left[\int\limits_0^{2\pi}\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\frac{\partial u^\varepsilon}{\partial z_i}\cdot\frac{\partial}{\partial z_i}\left(\frac{w_i}{\sqrt{\varepsilon}}\right)\varepsilon dz_i d\varphi_i+\int\limits_0^{2\pi}\int\limits_{\delta^\varepsilon}^{l_i-\delta^\varepsilon}\sqrt{\varepsilon}Q_i^\varepsilon u^\varepsilon\cdot\frac{\partial^2 w_i}{\partial z_i^2}dz_i d\varphi_i\right].$$

In view of (1.2.21) the second integral tends to zero. Therefore we have

$$\sum_{i=1}^{3}\int\limits_{0}^{l_i}(u_i(z))'(w_i(z))'dz = \frac{1}{2\pi}\lim_{\varepsilon\to 0}(\nabla^\varepsilon u^\varepsilon, \nabla w^\varepsilon)_{\mathcal{H}^\varepsilon},$$

where $w^\varepsilon \in H^1(M^\varepsilon)$

$$w^\varepsilon(x) = \begin{cases} w_i(z_i)\varepsilon^{-1/2}, \ \tilde{x} = (z_i, \varphi_i) \in G_i, \\ 0, \ \tilde{x} \in \Omega^\varepsilon, \\ w_i(l_i - \delta^\varepsilon)\varepsilon^{-1/2}, \ \tilde{x} \in B^\varepsilon. \end{cases}$$

In the same way using Lemma 1.1, we obtain

$$\sum_{i=1}^{3}\left(\int\limits_{0}^{l_i}u_i(z)w_i(z)dz - \int\limits_{0}^{l_i}f_i(z)w_i(z)dz\right) = \frac{1}{2\pi}\lim_{\varepsilon\to 0}\Big((u^\varepsilon, w^\varepsilon)_{\mathcal{H}^\varepsilon} - (f^\varepsilon, w^\varepsilon)_{\mathcal{H}^\varepsilon}\Big).$$

The last two equalities imply the condition (1.2.16).

Thus we prove that $u = \mathcal{A}_0 f$.

It is easy to see that (1.2.2) follows from (1.2.4), (1.2.10), (1.2.21), and Lemma 1.1. The condition C3 is fulfilled.

4. It remains to verify the fulfilment of the condition C4. Let $f^\varepsilon \in \mathcal{H}^\varepsilon$ be such that $\sup \|f^\varepsilon\|_{\mathcal{H}^\varepsilon} < \infty$. We denote $u^\varepsilon = \mathcal{A}^\varepsilon f^\varepsilon$. It is clear that the norms $\|u^\varepsilon\|^2_{L_2(M^\varepsilon)} + \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(M^\varepsilon)}$ are uniformly bounded with respect to ε. In the same way as in item **3** one can prove that there exists a subsequence (still denoted by ε) such that the following limits exist

$$w_0 = \lim_{\varepsilon\to 0}\Pi_0^\varepsilon u^\varepsilon \in H^1(\Omega) \text{ strongly in } L_2(\Omega), \tag{1.2.22}$$

$$w_i = \lim_{\varepsilon\to 0}\Pi_i^\varepsilon u^\varepsilon \in H^1[0, l_i], i = 1, 2, 3 \text{ strongly in } L_2[0, l_i]. \tag{1.2.23}$$

By means Lemma 1.1 we have

$$\|u^\varepsilon\|^2_{L_2(B^\varepsilon)} \to 0, \varepsilon \to 0. \tag{1.2.24}$$

The fulfilment of the condition C4 (with $w = (w_0, w_1, w_2, w_3)$) follows easily from (1.2.21)–(1.2.24).

Theorem 1.3 is proved.

We continue the proves of Theorems 1.1 and 1.2. Let $\mu_1^\varepsilon \geq \mu_2^\varepsilon \geq \mu_3^\varepsilon \geq \ldots \geq \mu_k^\varepsilon \underset{k\to\infty}{\to} 0$ be the eigenvalues of $\mathcal{A}^\varepsilon$ written with account of their multiplicity and let $f_1^\varepsilon, f_2^\varepsilon \ldots$ be the corresponding eigenvectors normalized by the condition $(f_i^\varepsilon, f_j^\varepsilon)_{\mathcal{H}^\varepsilon} = \delta_{ij}$. Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots \geq \mu_k \underset{k\to\infty}{\to} 0$ be the eigenvalues of $\mathcal{A}$.

It is proved in [7] that the conditions C1–C4 imply

$$\mu_k^\varepsilon \to \mu_k,\ \varepsilon \to 0,\ k = 1, 2, 3 \dots$$

and, moreover, if $\mu_k \geq \mu_{k+1} = \mu_{k+2} = \ldots = \mu_{k+m} > \mu_{k+m+1}$, then for any $w \in N(\mu_{k+1})$ there exists a linear combination $\bar{f}^\varepsilon$ of the eigenfunctions $f_{k+1}^\varepsilon \cdots f_{k+m}^\varepsilon$ such that

$$\|\bar{f}^\varepsilon - R^\varepsilon w\|_{\mathcal{H}^\varepsilon} \to 0,\ \varepsilon \to 0.$$

Since $\lambda_k^\varepsilon = \frac{1}{\mu_k^\varepsilon} - 1$, $\lambda_k = \frac{1}{\mu_k} - 1$, $u_k^\varepsilon = f_k^\varepsilon$ (and so $N(\lambda_k) = N(\mu_k)$), it follows that Theorems 1.1 and 1.2 are proved.

2. Riemannian Manifold of Increasing Genus

2.1. Setting of the Problem and Main Result

Let Ω be a two-dimensional compact Riemannian manifold without boundary and with a metric g. By Δ_Ω we denote the corresponding Laplace–Beltrami operator. Let D_i^ε, $i = 1 \dots N(\varepsilon) = 3N_1(\varepsilon)$ be a system of the balls in Ω with centers $x_i^\varepsilon \in \Omega$ and radii d^ε. We consider the following domain with holes:

$$\Omega^\varepsilon = \Omega \backslash \bigcup_{i=1}^{N(\varepsilon)} D_i^\varepsilon.$$

Let G_i^ε, $i = 1 \dots N(\varepsilon)$, be a set of tubes

$$G_i^\varepsilon = \{\tilde{x} = (\varphi_i, z_i) : \varphi_i \in [0, 2\pi], z_i \in [0, 1]\}.$$

We suppose that

$$C_i^\varepsilon = \{\tilde{x} = (\varphi_i, z_i) \in G_i^\varepsilon : z_i = 0\} \subset \partial G_i^\varepsilon$$

is diffeomorphic to ∂D_i^ε. Using this diffeomorphism, we glue G_i^ε, $i = 1 \dots N(\varepsilon)$ to Ω^ε. By S_i^ε we denote the "ends" of G_i^ε

$$S_i^\varepsilon = \{\tilde{x} = (\varphi_i, z_i) \in G_i^\varepsilon : z_i = 1\} \subset \partial G_i^\varepsilon.$$

We divide the set $\{1 \dots N(\varepsilon)\}$ into subsets, each consisting of three elements. For any three indexes i, j, k we introduce the number A_{ijk}, and set $A_{ijk} = 1$ if i, j, k belong to the same subset, and we set $A_{ijk} = 0$ otherwise. If $A_{ijk} = 1$, we say that the corresponding holes $D_i^\varepsilon, D_j^\varepsilon, D_k^\varepsilon$ are connected.

For any $i,j,k : A_{ijk} = 1$ we consider the sphere $\mathcal{B}^\varepsilon_{ijk} \subset \mathbb{R}^3$ with the radius b^ε. Let $\mathcal{D}^\varepsilon_i, \mathcal{D}^\varepsilon_j, \mathcal{D}^\varepsilon_k$ be the geodesic balls on $\mathcal{B}^\varepsilon_{ijk}$ with the radii $b^\varepsilon \arcsin\left(\frac{d^\varepsilon}{b^\varepsilon}\right)$. It is clear that the radii of the circles $\partial\mathcal{D}^\varepsilon_i, \partial\mathcal{D}^\varepsilon_j, \partial\mathcal{D}^\varepsilon_k$ are equal to d^ε. Let

$$B^\varepsilon_{ijk} = \mathcal{B}^\varepsilon_{ijk} \backslash (\mathcal{D}^\varepsilon_i \cup \mathcal{D}^\varepsilon_j \cup \mathcal{D}^\varepsilon_k).$$

One can see that $\partial\mathcal{D}^\varepsilon_i, \partial\mathcal{D}^\varepsilon_j, \partial\mathcal{D}^\varepsilon_k$ are diffeomorphic to $S^\varepsilon_i, S^\varepsilon_j, S^\varepsilon_k$, respectively. Using these diffeomorphisms, we glue B^ε_{ijk} to $G^\varepsilon_i \cup G^\varepsilon_j \cup G^\varepsilon_k$. Thus we obtain the manifold (see Fig. 2)

$$M^\varepsilon = \Omega^\varepsilon \cup \left[\bigcup_{i,j,k: A_{ijk}=1} (B^\varepsilon_{ijk} \cup G^\varepsilon_i \cup G^\varepsilon_j \cup G^\varepsilon_k)\right].$$

We denote the points of the manifold by $\tilde{x}$. Clearly, M^ε can be covered by a system of charts, and suitable local coordinates $\{x_1, x_2\}$ can be introduced. It is supposed that M^ε is equipped with the metric g^ε that coincides with the metric g on Ω^ε and with the metrics induced from $\mathbb{R}^3$ on B^ε_{ijk}. On G^ε_i the metric is defined by the formula for the square of the element of length:

$$ds^2 = q^\varepsilon_i dz_i^2 + (d^\varepsilon)^2 d\varphi_i^2,\ q^\varepsilon_i > 0.$$

By $g^\varepsilon_{\alpha\beta}$, we denote the components of metric tensor in local coordinates.

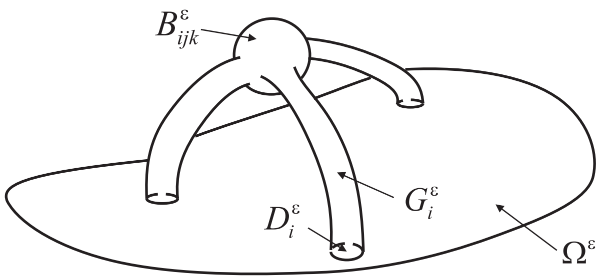

Fig. 2: Manifold M^ε.

We denote $r^\varepsilon_i = \min\limits_j \rho_g(x^\varepsilon_i, x^\varepsilon_j)$, where ρ_g is a distance on Ω generated by metric g. It is supposed that the following properties are valid:

(i) $|\ln d^\varepsilon|^{-1} \le C(r^\varepsilon_i)^2$, $r^\varepsilon_i = O(\varepsilon)$, $0 < C_1 \le \varepsilon^2 N(\varepsilon) \le C_2$, $\varepsilon \to 0$;

(ii) $q^\varepsilon_i \le q^\varepsilon \to 0,\ \varepsilon \to 0$, i.e., the lengthes of the cylinders G^ε_i tend to zero;

(iii) $(b^\varepsilon)^2 \left(|\ln d^\varepsilon| + \left|\ln\tan\frac{\theta^\varepsilon}{2}\right| + \frac{\sqrt{q^\varepsilon}}{d^\varepsilon}\right) \to 0,\ \varepsilon \to 0,\ \theta^\varepsilon = \arcsin\frac{d^\varepsilon}{b^\varepsilon}$.

Let Δ^ε be a Laplace–Beltrami operator on M^ε. Let $0 = \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \lambda_3^\varepsilon \leq \ldots \leq \lambda_k^\varepsilon \underset{k\to\infty}{\to} \infty$ be the eigenvalues of $-\Delta^\varepsilon$ written with account of their multiplicity, and $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon \ldots$ be the corresponding eigenvectors normalized by the condition $(u_i^\varepsilon, u_j^\varepsilon)_{\mathcal{H}^\varepsilon} = \delta_{ij}$.

To describe the behavior of λ_k^ε as $\varepsilon \to 0$ we introduce the notations:

$$R_i^\varepsilon = \{\tilde{x} \in \Omega^\varepsilon : d^\varepsilon \leq \rho_g(\tilde{x}, x_i^\varepsilon) \leq r_i^\varepsilon/2\},\ \widehat{C}_i^\varepsilon = \{\tilde{x} \in \Omega^\varepsilon : \rho_g(\tilde{x}, x_i^\varepsilon) = r_i^\varepsilon/2\},$$
$$\Gamma_{ijk}^\varepsilon = G_i^\varepsilon \cup G_j^\varepsilon \cup G_k^\varepsilon \cup B_{ijk}^\varepsilon,\ \widehat{\Gamma}_{ijk}^\varepsilon = R_i^\varepsilon \cup R_j^\varepsilon \cup R_k^\varepsilon \cup \Gamma_{ijk}^\varepsilon.$$

For $i, j, k : A_{ijk} = 1$, we consider the problem

$$\Delta^\varepsilon v = 0 \text{ in } \widehat{\Gamma}_{ijk}^\varepsilon, \quad v = 1 \text{ on } \widehat{C}_i^\varepsilon \text{ and } v = 0 \text{ on } \widehat{C}_j^\varepsilon \cup \widehat{C}_k^\varepsilon. \tag{2.1.1}$$

The solution of (2.1.1) we denote by v_{ijk}^ε. It is clear that $v_{ijk}^\varepsilon = v_{ikj}^\varepsilon$.

For $i, j, k : A_{ijk} = 1$, we denote

$$W_{ijk}^\varepsilon = -\int\limits_{\widehat{\Gamma}_{ijk}^\varepsilon} (\nabla^\varepsilon v_{ijk}^\varepsilon, \nabla^\varepsilon v_{jik}^\varepsilon) d\tilde{x},$$

(here $(\nabla^\varepsilon u, \nabla^\varepsilon v) := \sum\limits_{\alpha,\beta=1}^{2} g_\varepsilon^{\alpha\beta} \frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial x_\beta}$), otherwise we set $W_{ijk}^\varepsilon = 0$ (i.e., $W_{ijk}^\varepsilon = -(\nabla^\varepsilon v_{ijk}^\varepsilon, \nabla^\varepsilon v_{jik}^\varepsilon)_{L_2(\widehat{\Gamma}_{ijk}^\varepsilon)}$).

We introduce the generalized function

$$W^\varepsilon(x, y) = \sum_{i,j,k=1\ldots N(\varepsilon)} W_{ijk}^\varepsilon \delta(x - x_i^\varepsilon)\delta(y - x_j^\varepsilon) \in \mathcal{D}'(\Omega \times \Omega).$$

The limit

(iv) $\exists \lim\limits_{\varepsilon\to 0} W^\varepsilon(x, y) = W(x, y) \in L_\infty(\Omega \times \Omega)$ - positive symmetric function, is supposed to exist.

We denote $\mathcal{H}^\varepsilon := L_2(M^\varepsilon)$, $\mathcal{H}_0 := L_2(\Omega)$.

Theorem 2.1. *For any* $k = 1, 2, 3 \ldots$

$$\lambda_k^\varepsilon \to \lambda_k,\ \varepsilon \to 0,$$

where $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ *are the eigenvalues of the operator* $\mathcal{L} : L_2(\Omega) \to L_2(\Omega)$*:*

$$[\mathcal{L}u](x) = -\Delta_\Omega u(x) + \int\limits_\Omega W(x, y)(u(x) - u(y)) dy.$$

Theorem 2.2. *Let $R^\varepsilon : \mathcal{H}_0 \to \mathcal{H}^\varepsilon$:*

$$[R^\varepsilon f](\tilde{x}) = \begin{cases} f(\tilde{x}), & \tilde{x} \in \Omega^\varepsilon, \\ 0, & \tilde{x} \in \bigcup\limits_{i,j,k: A_{ijk}=1} \Gamma^\varepsilon_{ijk}. \end{cases}$$

Then the eigenfunctions of $-\Delta^\varepsilon$ converge in the sense (1.1.1) to the eigenfunctions of the operator $\mathcal{L}$.

2.2. Proof of Theorems 2.1 and 2.2

We denote by $\mathcal{A}^\varepsilon$ and $\mathcal{A}$ the operators inverse to $-\Delta^\varepsilon+\mathrm{I}$ and $\mathcal{L}+\mathrm{I}$, respectively. Analogously as in the previous section, Theorems 2.1 and 2.2 follow from

Theorem 2.3. *The conditions C1–C4 are fulfilled.*

P r o o f. The conditions C1–C2 are trivial. Let us check the condition C3.

Let $f \in \mathcal{H}$. We denote $u^\varepsilon = \mathcal{A}^\varepsilon R^\varepsilon f$, $f^\varepsilon = R^\varepsilon f$, $u_0 = \mathcal{A}_0 f$. Notice that the following estimates are valid:

$$\|u^\varepsilon\|_{L_2(M^\varepsilon)} \le \|f^\varepsilon\|_{L_2(M^\varepsilon)}, \ \|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(M^\varepsilon)} \le 2\|f^\varepsilon\|_{L_2(M^\varepsilon)} \cdot \|u^\varepsilon\|_{L_2(M^\varepsilon)}. \quad (2.2.1)$$

It is well known that u^ε minimizes the functional

$$J^\varepsilon[u^\varepsilon] = \int\limits_{M^\varepsilon} \left(|\nabla^\varepsilon u^\varepsilon|^2 + (u^\varepsilon)^2 - 2f^\varepsilon u^\varepsilon\right) d\tilde{x} \quad (2.2.2)$$

in the class of functions $H^1(M^\varepsilon)$, while u_0 minimizes the functional

$$J_0[u] = \int\limits_\Omega \left(|\nabla u|^2 + u^2 - 2fu\right) dx + \int\limits_\Omega \int\limits_\Omega \frac{1}{2} W(x,y)\,(u(x)-u(y))^2\, dxdy \quad (2.2.3)$$

in the class of functions $H^1(\Omega)$. The converse assertions are also true.

In order to prove that u^ε converges to u_0, we consider the following abstract scheme.

Let H^ε be a Hilbert space depending on the parameter $\varepsilon > 0$, $(u^\varepsilon, v^\varepsilon)_\varepsilon, \|u^\varepsilon\|_\varepsilon$ be a scalar product and norm in this space, and F^ε be the continuous linear functionals in H^ε which are uniformly bounded with respect to ε. Let H be a Hilbert space with the scalar product (u, v) and the norm $\|u\|$, and F be a continuous linear functional in H.

Consider the following two problems of minimization:

$$\|u^\varepsilon\|^2_\varepsilon + F^\varepsilon[u^\varepsilon] \to \inf, \quad u^\varepsilon \in H^\varepsilon, \quad (2.2.4)$$

$$\|u\|^2 + F[u] \to \inf, \quad u \in H. \quad (2.2.5)$$

Let u^ε and u_0 be the minimizants of the problems (2.2.4) and (2.2.5). The following theorem is proved in [3].

Theorem 2.4. *Let M be a dense subset of H, let $\Pi^\varepsilon : H^\varepsilon \to H$, and $P^\varepsilon : M \to H^\varepsilon$ be the operators satisfying the following conditions:*

(a) $\|\Pi^\varepsilon w^\varepsilon\| \le C\|w^\varepsilon\|, \forall w^\varepsilon \in H^\varepsilon$;

(b_1) $\Pi^\varepsilon P^\varepsilon w \to w$ *weakly in* H *as* $\varepsilon \to 0, \forall w \in M$;

(b_2) $\lim\limits_{\varepsilon\to 0} \|P^\varepsilon w\|_\varepsilon = \|w\|, \forall w \in M$;

(b_3) *for any sequence* $\gamma^\varepsilon \in H^\varepsilon$ *such that* $\Pi^\varepsilon\gamma^\varepsilon \to \gamma$ *weakly as* $\varepsilon \to 0$, *for any* $w \in M$ *one has*

$$\lim_{\varepsilon\to 0} |(P^\varepsilon w, \gamma^\varepsilon)_\varepsilon| \le C\|w\|\|\gamma\|;$$

(c) *for any sequence* $\gamma^\varepsilon \in H^\varepsilon$, *such that* $\Pi^\varepsilon\gamma^\varepsilon \to \gamma$ *weakly, as* $\varepsilon \to 0$, *one has*

$$\lim_{\varepsilon\to 0} F^\varepsilon[\gamma^\varepsilon] = F[\gamma].$$

Then

$$\Pi^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u_0 \text{ weakly in } H.$$

Notice that Theorem 2.4 holds true if the conditions (b$_3$) and (c) hold only for such sequences γ^ε that the norms $\|\gamma^\varepsilon\|_\varepsilon$ are uniformly bounded with respect to ε because in the proof of Theorem 2.4 the conditions (b$_3$) and (c) are used only with these sequences.

Now we apply our abstract scheme. Let H^ε be the Hilbert space $H^1(M^\varepsilon)$ of the functions on M^ε with the scalar product

$$(u^\varepsilon, v^\varepsilon)_\varepsilon = \int\limits_{M^\varepsilon} [(\nabla^\varepsilon u^\varepsilon, \nabla^\varepsilon v^\varepsilon) + u^\varepsilon v^\varepsilon]\, d\tilde{x},$$

and let F^ε be a linear functional defined by the formula

$$F^\varepsilon[u^\varepsilon] = \int\limits_{M^\varepsilon} -2f^\varepsilon u^\varepsilon d\tilde{x}.$$

Let H be the Hilbert space $H^1(\Omega)$ with the scalar product

$$(u, v) = \int\limits_\Omega [(\nabla u, \nabla v) + uv]\, dx + \int\limits_\Omega\int\limits_\Omega \frac{1}{2}W(x,y)(u(x) - u(y))(v(x) - v(y))dxdy,$$

and f be a linear functional on it defined by the formula

$$F[u] = \int\limits_\Omega -2fudx.$$

Obviously, the functionals F^ε are uniformly bounded with respect to ε.

Now we introduce the operators Π^ε and P^ε satisfying the conditions (a)–(c) of Theorem 2.4.

The existence of the operator $\Pi^\varepsilon : H^1(M^\varepsilon) \to H^1(\Omega)$ that has the properties

$$\|\nabla \Pi^\varepsilon u^\varepsilon\|^2_{L_2(\Omega)} + \|\Pi^\varepsilon u^\varepsilon\|^2_{L_2(\Omega)} \le C\Big(\|\nabla u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)} + \|u^\varepsilon\|^2_{L_2(\Omega^\varepsilon)}\Big), \tag{2.2.6}$$

$$\Pi^\varepsilon u^\varepsilon(\tilde{x}) = u^\varepsilon(\tilde{x}),\ \tilde{x} \in \Omega^\varepsilon \tag{2.2.7}$$

is proved in [9, p. 118, Ex. 4.10]).

Clearly, (a) follows from (2.2.6).

We introduce the operator P^ε. Let $\varphi(r)$ be a twice continuously differentiable non-negative function on the half-line $[0,\infty)$ equal to 1 for $r \in [0, 1/4]$ and equal to 0 for $r \ge 1/2$. We set

$$\varphi_i^\varepsilon(x) = \varphi\left(\frac{\rho_g(x, x_i^\varepsilon)}{r_i^\varepsilon}\right), \varphi_{0i}^\varepsilon(x) = \varphi\left(\frac{\rho_g(x, x_i^\varepsilon)}{d_0^\varepsilon}\right),$$

where $d_{0i}^\varepsilon = \exp(-|\ln d^\varepsilon|^{1/2})$.

Let $M = C^2(\Omega)$, M is dense in $H^1(\Omega)$ and let $w \in M$. We define the operator P^ε by the equality

$$[P^\varepsilon w](\tilde{x}) = \begin{cases} w(\tilde{x}) + (w_i^\varepsilon - w(\tilde{x}))\varphi_{0i}^\varepsilon(\tilde{x}) + \Big((v_{ijk}^\varepsilon(\tilde{x}) - 1)w_i^\varepsilon \\ \quad + v_{jik}^\varepsilon(\tilde{x})w_j^\varepsilon + v_{kij}^\varepsilon(\tilde{x})w_k^\varepsilon\Big)\varphi_i^\varepsilon(\tilde{x}),\ \tilde{x} \in R_i^\varepsilon,\ j,k : A_{ijk} = 1, \\ v_{ijk}^\varepsilon(\tilde{x})w_i^\varepsilon + v_{jik}^\varepsilon(\tilde{x})w_j^\varepsilon + v_{kij}^\varepsilon(\tilde{x})w_k^\varepsilon,\ \tilde{x} \in \Gamma_{ijk}^\varepsilon, \end{cases}$$

where $w_i^\varepsilon = w(x_i^\varepsilon)$.

To see that the conditions (b_1)–(b_3) hold, we use the following estimates of the solution v_{ijk}^ε of (2.1.1).

Lemma 2.1 *Let $R_{0q}^\varepsilon = \{\tilde{x} \in \Omega^\varepsilon : d_{0q}^\varepsilon \le \rho_g(x, x_q^\varepsilon) \le r_q^\varepsilon/2\}$. Then for $i,j,k : A_{ijk} = 1$ and $q \in \{i,j,k\}$:*

$$|D^\alpha(v_{ijk}^\varepsilon(\tilde{x}) - \delta_{iq})| \le C\left|\frac{D^\alpha(\ln \rho_g(x_q^\varepsilon, \tilde{x}))}{\ln d^\varepsilon}\right|,\ \tilde{x} \in R_{0q}^\varepsilon\ |\alpha| = 0, 1.$$

The proof of the lemma is carried out in the same way as that of Lemma 2.4 in [9, p. 44] using the inequality $0 \le v_{ijk}^\varepsilon \le 1$ which follows from the maximum principle.

Lemma 2.2 *Let $u^\varepsilon \in H^1(M^\varepsilon)$. Then for any $i,j,k : A_{ijk} = 1$*

$$\|u^\varepsilon\|^2_{B^\varepsilon_{ijk}} \le C\left[(b^\varepsilon)^2\left|\ln\tan\frac{\theta^\varepsilon}{2}\right|\cdot\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(B^\varepsilon_{ijk})} + \frac{\sqrt{q^\varepsilon}(b^\varepsilon)^2}{d^\varepsilon}\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(G^\varepsilon_i\cup G^\varepsilon_j\cup G^\varepsilon_k)}\right.$$
$$\left.+(b^\varepsilon)^2\left(|\ln d^\varepsilon|\cdot\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(R^\varepsilon_i\cup R^\varepsilon_j\cup R^\varepsilon_k)} + \frac{1}{r^{\varepsilon\,2}_i}\|u^\varepsilon\|^2_{L_2(R^\varepsilon_i\cup R^\varepsilon_j\cup R^\varepsilon_k)}\right)\right],$$
$$\|u^\varepsilon\|^2_{G^\varepsilon_i} \le C\left[q^\varepsilon\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(G_i)} + d^\varepsilon\sqrt{q^\varepsilon}\big(|\ln d^\varepsilon|\cdot\|\nabla^\varepsilon u^\varepsilon\|^2_{L_2(R^\varepsilon_i)} + \frac{1}{r^{\varepsilon\,2}_i}\|u^\varepsilon\|^2_{L_2(R^\varepsilon_i)}\big)\right].$$

The p r o o f of this lemma is carried out in the same way as the proof of Lemma 1.1.

We verify that the condition (b$_2$) holds. We denote $\widehat{R}^\varepsilon_i = \{\tilde{x}\in\Omega^\varepsilon : r^\varepsilon_i/4 \le \rho_g(\tilde{x}, x^\varepsilon_i) \le r^\varepsilon_i/2\}$. Let $w \in M$. Then

$$\|P^\varepsilon w\|^2_\varepsilon = \int\limits_{\Omega^\varepsilon}\left[|\nabla w|^2 + w^2\right]dx$$
$$+\sum_{i<j<k:A_{ijk}=1}\ \int\limits_{\widehat{\Gamma}^\varepsilon_{ijk}}\left[w^{\varepsilon\,2}_i|\nabla v^\varepsilon_{ijk}|^2 + w^{\varepsilon\,2}_j|\nabla v^\varepsilon_{jik}|^2 + w^{\varepsilon\,2}_k|\nabla v^\varepsilon_{kij}|^2\right.$$
$$\left.+2w^\varepsilon_i w^\varepsilon_j(\nabla v^\varepsilon_{ijk}, \nabla v^\varepsilon_{jik}) + 2w^\varepsilon_j w^\varepsilon_k(\nabla v^\varepsilon_{jik}, \nabla v^\varepsilon_{kij}) + 2w^\varepsilon_i w^\varepsilon_k(\nabla v^\varepsilon_{ijk}, \nabla v^\varepsilon_{kij})\right]d\tilde{x} + \delta(\varepsilon). \tag{2.2.8}$$

Here $\delta(\varepsilon)$ are the remaining integrals estimated as follow*:

$$|\delta(\varepsilon)| \le C(w)\sum_{i,j,k:A_{ijk}=1}\left[J^\varepsilon_{ijk} + E^\varepsilon_{ijk} + I^\varepsilon_{ijk} + Y^\varepsilon_{ijk} + (d^\varepsilon_0)^2\right],$$

where

$$J^\varepsilon_{ijk} = \int\limits_{\widehat{R}^\varepsilon_i\cup\widehat{R}^\varepsilon_j\cup\widehat{R}^\varepsilon_k}\left(|\nabla^\varepsilon v^\varepsilon_{ijk}|^2 + \frac{1}{r^{\varepsilon\,2}_i}|v^\varepsilon_{ijk}|^2\right)d\tilde{x},$$
$$E^\varepsilon_{ijk} = \int\limits_{R^\varepsilon_{0i}\cup R^\varepsilon_{0j}\cup R^\varepsilon_{0k}}\left(|\nabla^\varepsilon v^\varepsilon_{ijk}| + \frac{1}{r^\varepsilon_i}|v^\varepsilon_{ijk}|\right)d\tilde{x},$$
$$I^\varepsilon_{ijk} = \int\limits_{R^\varepsilon_i}|v^\varepsilon_{ijk} - 1|^2 d\tilde{x} + \int\limits_{R^\varepsilon_j\cup R^\varepsilon_k}|v^\varepsilon_{ijk}|^2 d\tilde{x},\ \ Y^\varepsilon_{ijk} = \int\limits_{\Gamma^\varepsilon_{ijk}}|v^\varepsilon_{ijk}|^2 d\tilde{x}.$$

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*The sum $\sum\limits_{i<j<k:A_{ijk}=1}$ means that any three indexes $\{i,j,k\}$ appear only ones in this sum.

Using Lemma 2.1 and maximum principle for v^{ε}_{ijk}, we have

$$J^{\varepsilon}_{ijk} \le C|\ln d^{\varepsilon}|^{-2}(1+|\ln r^{\varepsilon}_i|^2), \tag{2.2.9}$$

$$E^{\varepsilon}_{ijk} \le C|\ln d^{\varepsilon}|^{-1}\big(r^{\varepsilon}_i|\ln r^{\varepsilon}_i| + d^{\varepsilon}_0|\ln d^{\varepsilon}_0|\big), \tag{2.2.10}$$

$$I^{\varepsilon}_{ijk} \le C\big[(d^{\varepsilon}_0)^2(1+|\ln d^{\varepsilon}|^{-1}) + (r^{\varepsilon}_i \ln r^{\varepsilon}_i/\ln d^{\varepsilon})^2\big], \tag{2.2.11}$$

$$Y^{\varepsilon}_{ijk} \le C\cdot|\Gamma^{\varepsilon}_{ijk}| \le C(d^{\varepsilon}\sqrt{q^{\varepsilon}} + (b^{\varepsilon})^2), \tag{2.2.12}$$

Using (i)–(iii), we conclude that

$$\delta(\varepsilon)\to 0,\ \varepsilon\to 0. \tag{2.2.13}$$

We denote $V^{\varepsilon}_{ijk} = \int\limits_{\widehat{\Gamma}^{\varepsilon}_{ijk}} |\nabla^{\varepsilon} v^{\varepsilon}_{ijk}|^2 d\tilde{x}$, where $i,j,k : A_{ijk=1}$. Since $v^{\varepsilon}_{ijk} + v^{\varepsilon}_{jik} + v^{\varepsilon}_{kij} = 1$ for any $i,j,k : A_{ijk} = 1$, we have $V^{\varepsilon}_{ijk} = W^{\varepsilon}_{ijk} + W^{\varepsilon}_{ikj}$. Therefore (2.2.8) can be rewritten in the form

$$\|P^{\varepsilon}w\|^2 = \int\limits_{\Omega}\Big(|\nabla w|^2 + w^2\Big)dx + \sum_{i,j,k=1\ldots N(\varepsilon)} W^{\varepsilon}_{ijk}\Big((w(x^{\varepsilon}_i))^2 - w(x^{\varepsilon}_i)w(x^{\varepsilon}_j)\Big) + \delta(\varepsilon)$$

$$= \int\limits_{\Omega}\Big(|\nabla w|^2 + w^2\Big)dx + \frac{1}{2}\sum_{i,j,k=1\ldots N(\varepsilon)} W^{\varepsilon}_{ijk}\Big(w(x^{\varepsilon}_i) - w(x^{\varepsilon}_j)\Big)^2 + \delta(\varepsilon). \tag{2.2.14}$$

It is easy to see that (b_2) follows from (iv), (2.2.13) and (2.2.14).

We verify the condition (b_1). Let $w\in M$. In view of the conditions (a) and (b_2), the norms $\|\Pi^{\varepsilon}P^{\varepsilon}w\|_{\varepsilon}$ are uniformly bounded with respect to ε, and in the same way as in (b_2) one can prove that $\Pi^{\varepsilon}P^{\varepsilon}w\to w$ strongly in $L^2(\Omega)$. Thus the condition (b_1) also holds.

We verify the condition (b_3). Let $w\in M$, and the sequence $\gamma^{\varepsilon}\in H^{\varepsilon}$ is such that the norms $\|\gamma^{\varepsilon}\|_{\varepsilon}$ are uniformly bounded with respect to ε, and $\Pi^{\varepsilon}\gamma^{\varepsilon}\to\gamma$ weakly in H as $\varepsilon\to 0$. Integrating by parts, we have

$$(P^{\varepsilon}w,\gamma^{\varepsilon})_{\varepsilon} = (-\Delta_{\Omega}w + w, \Pi^{\varepsilon}\gamma^{\varepsilon})_{L_2(\Omega)} + \delta(\varepsilon), \tag{2.2.15}$$

where $\delta(\varepsilon)$ are the remaining integrals. Using Lemma 2.1, in the same way as in (b_2) we obtain the estimate

$$\lim_{\varepsilon\to 0}|\delta(\varepsilon)| \le C\lim_{\varepsilon\to 0}\left\{\left(\sum_{i=1}^{N(\varepsilon)}(w^{\varepsilon}_i)^2|R^{\varepsilon}_i|\right)^{1/2}\|\Pi^{\varepsilon}\gamma^{\varepsilon}\|_{L_2(\Omega)}\right\}.$$

Since $\Pi^{\varepsilon}\gamma^{\varepsilon}$ converges weakly to γ in H, then $\Pi^{\varepsilon}\gamma^{\varepsilon}$ converges strongly to γ in $L_2(\Omega)$, and therefore we have

$$\lim_{\varepsilon\to 0}|\delta(\varepsilon)| \le C\|w\|_{L_2(M^{\varepsilon})}\cdot\|\gamma\|_{L_2(\Omega)} \le C\|w\|\cdot\|\gamma\|. \tag{2.2.16}$$

It follows from (2.2.15)–(2.2.16) that (b_3) holds.

Further, we verify that the condition (c) holds. Let the sequence $\gamma^\varepsilon \in H^\varepsilon$ be such that $\Pi^\varepsilon\gamma^\varepsilon \to \gamma$ weakly in H. Then $\Pi^\varepsilon\gamma^\varepsilon \to \gamma$ strongly in $L_2(\Omega)$. We have

$$|F^\varepsilon[\gamma^\varepsilon] - F[\gamma]| = \left|\int\limits_{\Omega^\varepsilon} f\cdot(\Pi^\varepsilon\gamma^\varepsilon - \gamma)d\tilde{x}\right| + \left|\int\limits_{\Omega\setminus\Omega^\varepsilon} f\gamma d\tilde{x}\right| \to 0, \varepsilon \to 0,$$

and so the condition (c) holds.

Thus all the conditions of Theorem 2.4 hold. Hence $\Pi^\varepsilon u^\varepsilon \to u_0$ weakly in H. Therefore, by the embedding theorem, $\Pi^\varepsilon u^\varepsilon \to u_0$ strongly in $L^2(\Omega)$. Finally, we have

$$\|\mathcal{A}^\varepsilon R^\varepsilon f - R^\varepsilon\mathcal{A}_0 f\|^2_{\mathcal{H}^\varepsilon} = \|u^\varepsilon\|^2_{L_2(\cup\Gamma^\varepsilon_{ijk})} + \|\Pi^\varepsilon u^\varepsilon - u_0\|^2_{L_2(\Omega^\varepsilon)}.$$

In view of Lemma 2.2, (i–iii) and (2.2.1) $\|u^\varepsilon\|^2_{L_2(\cup\Gamma^\varepsilon_{ijk})} \to 0, \varepsilon \to 0$. Thus C3 is proved.

And finally, we verify the fulfilment of the condition C4. Let $f^\varepsilon \in \mathcal{H}^\varepsilon$ be such that $\sup\|f^\varepsilon\|_{\mathcal{H}^\varepsilon} < \infty$. Let $u^\varepsilon = \mathcal{A}^\varepsilon f^\varepsilon$. In view of (2.2.1), $\Pi^\varepsilon u^\varepsilon$ is weakly compact in $H^1(\Omega)$ and so there exists the subsequence ε' and $w \in H^1(\Omega)$ such that $\Pi^\varepsilon u^\varepsilon \to w$ strongly in $L_2(\Omega)$. This and Lemma 2.2 imply C4.

Theorem 2.3 and therefore Theorems 2.1 and 2.2 are proved.

2.3. Example

We consider an example of the manifold M^ε and calculate the function $W(x,y)$ explicitly.

Let Ω contain the subset K, which is a flat square with the side equal to l. Let $\varepsilon > 0$ and let $n^\varepsilon = \left[\frac{1}{\varepsilon}\right]^{1/3}$.

We divide K into the squares K^ε_α, $\alpha = 1\dots n^{\varepsilon 2}$ with the side length l/n^ε. Within each square K^ε_α we cut out $n^{\varepsilon 4}$ holes D^ε_i with the radius $d^\varepsilon = \exp\left(-n^{\varepsilon 6}/l^6\right)$ and such that their centers form a periodic lattice with the period $\frac{l}{n^{\varepsilon 3}}$. It is clear that $|\ln d^\varepsilon|^{-1} = l^4(r^\varepsilon_i)^2$. The total number of D^ε_i is equal to $N(\varepsilon) = n^{\varepsilon 6}$.

For each hole D^ε_i we denote the number of square K^ε_α containing this hole by $\alpha(i)$. Since the number of holes within the square K^ε_α is equal to $n^{\varepsilon 2}\cdot n^{\varepsilon 2}$, we can assign to each hole $D^\varepsilon_i \subset K^\varepsilon_\alpha$ the pair $(\beta(i),\gamma(i))$, $\beta(i),\gamma(i) \in \{1\dots n^{\varepsilon 2}\}$. So, each hole D^ε_i is characterized by $(\alpha(i),\beta(i),\gamma(i))$.

If $\alpha(i) = \beta(j) = \gamma(k), \alpha(j) = \beta(k) = \gamma(i), \alpha(k) = \beta(i) = \gamma(j)$ and only in this case, then we join the boundaries of the holes $D^\varepsilon_i, D^\varepsilon_k, D^\varepsilon_j$ by means of the manifold $\Gamma^\varepsilon_{ijk} = G^\varepsilon_i \cup G^\varepsilon_j \cup G^\varepsilon_k \cup B^\varepsilon_{ijk}$.

We set

$$q_i^\varepsilon = [q \cdot |\ln d^\varepsilon| \cdot d^\varepsilon]^2,\ q > 0$$

and choose such b^ε that (iii) is valid and

$$\ln\left(\tan\frac{\theta^\varepsilon}{2}\right)/\ln d^\varepsilon \to 0,\ \varepsilon \to 0,\ \theta^\varepsilon = \arcsin\frac{d^\varepsilon}{b^\varepsilon}$$

(for example $d^\varepsilon \sim Cb^\varepsilon$).

In order to calculate $W(x,y)$ we find a suitable approximation for the solution v_{ijk}^ε to (2.1.1). Namely, we represent it in the form $v_{ijk}^\varepsilon = \widehat{v}_{ijk}^\varepsilon + w_{ijk}^\varepsilon$, where

$$\widehat{v}_{ijk}^\varepsilon(\tilde{x}) = \begin{cases} a_i^\varepsilon \ln|\tilde{x} - x_i^\varepsilon| + b_i^\varepsilon,\ \tilde{x} \in R_i^\varepsilon, \\ A_i^\varepsilon z + B_i^\varepsilon,\ \tilde{x} = (z_i, \varphi_i) \in G_i^\varepsilon, \\ a_j^\varepsilon \ln|\tilde{x} - x_j^\varepsilon| + b_j^\varepsilon,\ \tilde{x} \in R_j^\varepsilon, \\ A_j^\varepsilon z + B_j^\varepsilon,\ \tilde{x} = (z_j, \varphi_j) \in G_j^\varepsilon, \\ a_k^\varepsilon \ln|\tilde{x} - x_k^\varepsilon| + b_k^\varepsilon,\ \tilde{x} \in R_k^\varepsilon, \\ A_k^\varepsilon z + B_k^\varepsilon,\ \tilde{x} = (z_k, \varphi_k) \in G_k^\varepsilon, \\ C_{ijk}^\varepsilon,\ \tilde{x} \in B_{ijk}^\varepsilon. \end{cases}$$

We chose the constants $a_i^\varepsilon, b_i^\varepsilon \dots A_k^\varepsilon, B_k^\varepsilon, C_{ijk}^\varepsilon$ such that:

1) $\widehat{v}_{ijk}^\varepsilon$ is a harmonic function in $G_i^\varepsilon \cup R_i^\varepsilon$, $G_j^\varepsilon \cup R_j^\varepsilon$, $G_k^\varepsilon \cup R_k^\varepsilon$,
2) $\widehat{v}_{ijk}^\varepsilon = 1$ on $\widehat{C}_i^\varepsilon$, $\widehat{v}_{ijk}^\varepsilon = 0$ on $\widehat{C}_j^\varepsilon \cup \widehat{C}_k^\varepsilon$,
3) $\widehat{v}_{ijk}^\varepsilon|_{S_i^\varepsilon} = \widehat{v}_{ijk}^\varepsilon|_{S_j^\varepsilon} = \widehat{v}_{ijk}^\varepsilon|_{S_k^\varepsilon} = M$, where M is a constant,
4) $\dfrac{\partial \widehat{v}_{ijk}^\varepsilon}{\partial \vec{n}}|_{S_i^\varepsilon} + \dfrac{\partial \widehat{v}_{ijk}^\varepsilon}{\partial \vec{n}}|_{S_j^\varepsilon} + \dfrac{\partial \widehat{v}_{ijk}^\varepsilon}{\partial \vec{n}}|_{S_k^\varepsilon} = 0$, $\vec{n}$ is the outward (or inward) normal*.

As a result, we obtain

$$a_i^\varepsilon = \frac{2|\ln d^\varepsilon|^{-1}}{3(1+q)}(1 + o(1)) = -2a_j^\varepsilon = -2a_k^\varepsilon,$$
$$A_i^\varepsilon = -a_i^\varepsilon \frac{\sqrt{q_i^\varepsilon}}{d^\varepsilon},\ A_j^\varepsilon = -a_j^\varepsilon \frac{\sqrt{q_j^\varepsilon}}{d^\varepsilon},\ A_k^\varepsilon = -a_k^\varepsilon \frac{\sqrt{q_k^\varepsilon}}{d^\varepsilon},$$
$$b_i^\varepsilon = 1 - a_i^\varepsilon \ln(r_i^\varepsilon/2),\ b_j^\varepsilon = -a_j^\varepsilon \ln(r_j^\varepsilon/2),\ b_k^\varepsilon = -a_k^\varepsilon \ln(r_k^\varepsilon/2),$$
$$B_i^\varepsilon = a_i^\varepsilon \ln d^\varepsilon + b_i^\varepsilon,\ B_j^\varepsilon = a_j^\varepsilon \ln d^\varepsilon + b_j^\varepsilon,\ B_k^\varepsilon = a_k^\varepsilon \ln d^\varepsilon + b_k^\varepsilon.$$

*Here the normal derivatives are taken in an arbitrary point of S_i^ε. It is easy to see that the conditions 1)–3) guarantee that $\dfrac{\partial \widehat{v}_{ijk}^\varepsilon}{\partial \vec{n}}$ are constant on S_i^ε, as on S_j^ε and S_k^ε). The condition 4) determines the constant M from the condition 3).

Direct calculations show that

$$\|\nabla^\varepsilon \widehat{v}^\varepsilon_{ijk}\|^2_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} = \frac{4\pi}{3(1+q)}|\ln d^\varepsilon|^{-1}(1+\bar{o}(1)) \to 0, \ \varepsilon \to 0, \tag{2.3.1}$$

$$(\nabla^\varepsilon \widehat{v}^\varepsilon_{ijk}, \nabla^\varepsilon \widehat{v}^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} = -\frac{2\pi}{3(1+q)}|\ln d^\varepsilon|^{-1}(1+\bar{o}(1)), \ \varepsilon \to 0. \tag{2.3.2}$$

Now we prove that w^ε_{ijk} gives vanishingly small contribution to W^ε_{ijk}. Since v^ε_{ijk} minimizes the functional $I^\varepsilon[v] = \|\nabla^\varepsilon v\|^2_{0\varepsilon}$ in the class of functions from $H^1(\widehat{\Gamma}^\varepsilon_{ijk})$ equal to 1 on $\widehat{S}^\varepsilon_i$ and equal to 0 on $\widehat{S}^\varepsilon_j \cup \widehat{S}^\varepsilon_k$, then $\|\nabla^\varepsilon v^\varepsilon_{ijk}\|^2_{L_2(M^\varepsilon)} \le \|\nabla^\varepsilon \widehat{v}^\varepsilon_{ijk}\|^2_{L_2(M^\varepsilon)}$ and therefore

$$\|\nabla^\varepsilon w^\varepsilon_{ijk}\|^2_{L_2(M^\varepsilon)} \le 2\left|(\nabla^\varepsilon w^\varepsilon_{ijk}, \nabla^\varepsilon \widehat{v}^\varepsilon_{ijk})_{L_2(M^\varepsilon)}\right|.$$

Using the properties of the function $\widehat{v}^\varepsilon_{ijk}$, we obtain

$$\begin{aligned}\|\nabla^\varepsilon w^\varepsilon_{ijk}\|^2_{0\varepsilon} \le 4\pi d^\varepsilon \left|\frac{A^\varepsilon_i}{\sqrt{q^\varepsilon_i}} w^\varepsilon_i + \frac{A^\varepsilon_j}{\sqrt{q^\varepsilon_j}} w^\varepsilon_j + \frac{A^\varepsilon_i}{\sqrt{q^\varepsilon_j}} w^\varepsilon_k\right| &= 4\pi|a^\varepsilon_i w^\varepsilon_i + a^\varepsilon_j w^\varepsilon_j + a^\varepsilon_k w^\varepsilon_k| \\ &= 4\pi|a^\varepsilon_j(w^\varepsilon_j - w^\varepsilon_i) + a^\varepsilon_k(w^\varepsilon_k - w^\varepsilon_i)|,\end{aligned} \tag{2.3.3}$$

where $w^\varepsilon_i, w^\varepsilon_j, w^\varepsilon_k$ are the average values of w^ε_{ijk} in $S^\varepsilon_i, S^\varepsilon_j, S^\varepsilon_k$, respectively.

The following estimate is valid

$$\begin{aligned}|w^\varepsilon_i - w^\varepsilon_j| + |w^\varepsilon_i - w^\varepsilon_k| &\le C\sqrt{|\ln\tan\frac{\theta^\varepsilon}{2}|}\cdot\|\nabla^\varepsilon v^\varepsilon_{ijk}\|_{0\varepsilon} \\ \le C\sqrt{|\ln\tan\frac{\theta^\varepsilon}{2}|}\cdot\|\nabla^\varepsilon \widehat{v}^\varepsilon_{ijk}\|_{0\varepsilon} &\le C\sqrt{|\ln\tan\frac{\theta^\varepsilon}{2}|/|\ln d^\varepsilon|} \underset{\varepsilon\to 0}{\to} 0.\end{aligned} \tag{2.3.4}$$

The proof is similar to that of (1.2.20).

It follows from (2.3.3), (2.3.4) and from the form of the coefficients $a^\varepsilon_i, a^\varepsilon_j, a^\varepsilon_k$ that

$$\|\nabla^\varepsilon w^\varepsilon\|^2_{0\varepsilon} = \bar{o}(|\ln d^\varepsilon|^{-1}). \tag{2.3.5}$$

We have

$$\begin{aligned}W^\varepsilon_{ijk} = -\big[&(\widehat{v}^\varepsilon_{ijk}, \widehat{v}^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} + (\widehat{v}^\varepsilon_{ijk}, w^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} \\ &+ (w^\varepsilon_{ijk}, \widehat{v}^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} + (w^\varepsilon_{ijk}, w^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})}\big].\end{aligned} \tag{2.3.6}$$

It follows from (2.3.1), (2.3.2), (2.3.5), (2.3.6) that

$$W^\varepsilon_{ijk} \sim -(\widehat{v}^\varepsilon_{ijk}, \widehat{v}^\varepsilon_{jik})_{L_2(\widehat{\Gamma}^\varepsilon_{ijk})} \sim \frac{2\pi}{3(1+q)}|\ln d^\varepsilon|^{-1}.$$

Let $w(x,y) \in C^\infty(\Omega)$. Then

$$\langle W^\varepsilon, w\rangle = \sum_{i,j,k:A_{ijk}=1} \frac{2\pi}{3(1+q)} w(x_i^\varepsilon, x_j^\varepsilon) |\ln d^\varepsilon|^{-1}.$$

By the construction of the manifold M^ε, for any three squares $K_\alpha^\varepsilon, K_\beta^\varepsilon, K_\gamma^\varepsilon$ there are only three holes $D^\varepsilon_{i_{\alpha\beta\gamma}}, D^\varepsilon_{j_{\alpha\beta\gamma}}, D^\varepsilon_{k_{\alpha\beta\gamma}}$ such that

$$D^\varepsilon_{i_{\alpha\beta\gamma}} \subset K_\alpha^\varepsilon,\ D^\varepsilon_{j_{\alpha\beta\gamma}} \subset K_\beta^\varepsilon,\ D^\varepsilon_{k_{\alpha\beta\gamma}} \subset K_\gamma^\varepsilon,\ A_{i_{\alpha\beta\gamma} j_{\alpha\beta\gamma} k_{\alpha\beta\gamma}} = 1.$$

Therefore the sum above can be easily rewritten as follows:

$$\langle W^\varepsilon, w\rangle = \frac{2\pi}{3(1+q)} \sum_{\alpha,\beta,\gamma=1}^{n^2(\varepsilon)} w(x_{i_{\alpha\beta\gamma}}, x_{j_{\alpha\beta\gamma}}) |\ln d^\varepsilon|^{-1}$$

$$= \frac{2\pi}{3(1+q)} \sum_{\alpha,\beta,\gamma=1}^{n^2(\varepsilon)} w(x_{i_{\alpha\beta\gamma}}, x_{j_{\alpha\beta\gamma}}) |K_\alpha^\varepsilon| \cdot |K_\beta^\varepsilon| \cdot |K_\gamma^\varepsilon| \underset{\varepsilon\to 0}{\to} \int\limits_K\int\limits_K\int\limits_K \frac{2\pi}{3(1+q)} w(x,y)\,dxdydz.$$

Thus

$$W(x,y) = \chi_K(x)\chi_K(y) \int\limits_K \frac{2\pi}{3(1+q)} dz = \frac{2\pi l^2}{3(1+q)} \chi_K(x)\chi_K(y),$$

where χ_K is the characteristic function of K.

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