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A q-Analog of the Hua Equations

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A necessary condition is established for a function to be in the image of a quantum Poisson integral operator associated to the Shilov boundary of the quantum matrix ball. A quantum analogue of the Hua equations is introduced.

Key words: quantum matrix ball, Shilov boundary, Poisson integral operator, invariant kernel, Hua equations.

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Dedicated to the memory of L.L. Vaksman

1. Introduction

In late 90s three groups of specialists became advanced in putting the basics of quantum theory of bounded symmetric domains.

T. Tanisaki and his team introduced q-analogs for the prehomogeneous vector spaces of commutative parabolic type and found an explicit form of the associated Sato–Bernstein polynomials [16–18, 21].

On the other hand, H. Jakobsen suggested a less intricate method of producing the above quantum vector spaces. Actually, he was on the way to quantum Hermitian symmetric spaces of noncompact type [13, 12]. A similar approach was used by W. Baldoni and P. Frajria [1] for q-analogs of algebras of invariant differential operators and the Harish-Chandra homomorphism for these quantum symmetric spaces. During the same period H. Jakobsen obtained a description of all the unitarizable highest weight modules over the Drinfeld–Jimbo algebras [11].

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The authors named above made no use of the full symmetry of the quantum prehomogeneous vector spaces in question [35, 31], what became an obstacle in producing the quantum theory of bounded symmetric domains.

The paper [36] laid the foundations of this theory. The subsequent results were obtained in the works by L. Vaksman, D. Proskurin, S. Sinel'shchikov, A. Stolin, D. Shklyarov, L. Turowska, H. Zhang [39, 23, 34, 42, 27, 41, 32]. The compatibility of the approaches described above [38, 11, 36] was proved by D. Shklyarov [28]. The study of quantum analogs of the Harish-Chandra modules related to quantum bounded symmetric domains and their geometric realizations has been started in [29, 37, 30]. The present work proceeds with this research.

Recall that a bounded domain $\mathbb{D}$ in a finite dimensional vector space is said to be symmetric if every point $p \in \mathbb{D}$ is an isolated fixed point of the biholomorphic involutive automorphism $\varphi_p : \mathbb{D} \to \mathbb{D}$, $\varphi_p \circ \varphi_p = \mathrm{id}$.

Equip the vector space of linear maps in $\mathbb{C}^n$ and the canonically isomorphic vector space Mat_n of complex $n \times n$ matrices with the operator norms. It is known [8] that the unit ball $\mathbb{D} = \{\mathbf{z} \in \mathrm{Mat}_n \,|\, \mathbf{z}\mathbf{z}^* < 1\}$ is a bounded symmetric domain.

Denote by $S(\mathbb{D})$ the Shilov boundary of $\mathbb{D}$, $S(\mathbb{D}) = \{\mathbf{z} \in \mathrm{Mat}_n \,|\, I - \mathbf{z}\mathbf{z}^* = 0\} \cong U_n$. It is well known that both $\mathbb{D}$ and $S(\mathbb{D})$ are homogeneous spaces of the group $SU_{n,n}$. Consider a function on $\mathbb{D} \times S(\mathbb{D})$ given by

$$P(\mathbf{z}, \zeta) = \frac{\det(1 - \mathbf{z}\mathbf{z}^*)^n}{|\det(1 - \zeta^* \mathbf{z})|^{2n}}, \qquad \mathbf{z} \in \mathbb{D},\ \zeta \in S(\mathbb{D}). \tag{1.1}$$

It is called the Poisson kernel [10] associated to the Shilov boundary.

General concepts on the boundaries of Hermitian symmetric spaces of non-compact type and the associated Poisson kernels are exposed in [20, 19].

The Poisson kernel, together with the $S(U_n \times U_n)$-invariant integral $\int\limits_{U_n} \cdot d\nu(\zeta)$ on U_n, allows one to define the Poisson integral operator

$$\mathcal{P} : f(\zeta) \mapsto \int\limits_{U_n} P(\mathbf{z}, \zeta) f(\zeta) d\nu(\zeta), \qquad \mathbf{z} \in \mathbb{D},\ \zeta \in S(\mathbb{D}).$$

It intertwines the actions of $SU_{n,n}$ in the spaces of continuous functions on the domain $\mathbb{D}$ and on the Shilov boundary $S(\mathbb{D})$. However, not every continuous function on $\mathbb{D}$ can be produced by applying the Poisson integral operator to a continuous function on the Shilov boundary $S(\mathbb{D})$.

In [10], L.K. Hua obtained the initial results on differential equations whose solutions included the functions of the form $\mathcal{P}(f)$. A later result of K. Johnson and A. Korányi [15] provided a system of differential equations by giving a complete characterization of these functions. Their version of the Hua equations is as follows:

$$\sum_{i,j,k=1}^{n}\left(\delta_{ij}-\sum_{c=1}^{n} z_i^c\overline{z}_j^c\right)\left(\delta_{k\alpha}-\sum_{c=1}^{n}\overline{z}_c^k z_c^\alpha\right)\frac{\partial^2}{\partial\overline{z}_j^k\partial z_i^a}u(\mathbf{z}) = 0,$$
$$\sum_{i,j,k=1}^{n}\left(\delta_{ai}-\sum_{c=1}^{n} z_a^c\overline{z}_i^c\right)\left(\delta_{jk}-\sum_{c=1}^{n}\overline{z}_c^j z_c^k\right)\frac{\partial^2}{\partial\overline{z}_i^j\partial z_\alpha^k}u(\mathbf{z}) = 0$$

for $a,\alpha=1,2,\ldots,n$.

This can be also represented in the form

$$\sum_{c=1}^{n}\left.\frac{\partial^2 u(g\cdot\mathbf{z})}{\partial z_c^\beta\partial\overline{z}_c^\alpha}\right|_{\mathbf{z}=0}=0,\qquad g\in SU_{n,n},\quad \alpha,\beta\in\{1,2,\ldots,n\}. \tag{1.2}$$

It is known that the Poisson kernel (1.1) as a function of $\mathbf{z}$ is a solution of the above equation system. While proving this, one may stick to the special case of $g=1$, because $P(g\mathbf{z},\zeta)=\mathrm{const}(g,\zeta)P(\mathbf{z},g^{-1}\zeta)$, cf. [15, p. 597]. What remains is to note that the relations

$$\sum_{c=1}^{n}\left.\frac{\partial^2 P}{\partial z_c^\beta\partial\overline{z}_c^\alpha}\right|_{\mathbf{z}=0}=0,\qquad \zeta\in U_n,\quad \alpha,\beta\in\{1,2,\ldots,n\} \tag{1.3}$$

follow from (1.1).

Thus, in the classical case, the Poisson integral operator applied to a function on the Shilov boundary is a solution of (1.2). We are going to obtain a quantum analog of this well-known result.

The work suggests a quantum analog for the Hua equations. We thus get a quantization of the necessary condition for a function to be a Poisson integral of a function on the Shilov boundary.

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2. A Background on Function Theory in Quantum Matrix Ball

In what follows we assume $\mathbb{C}$ to be a ground field and all algebras to be associative and unital.

Recall a construction of the quantum universal enveloping algebra for the Lie algebra $\mathfrak{sl}_N$. The quantum universal enveloping algebras were introduced by V. Drinfeld and M. Jimbo in an essentially more general way than it is described below. We follow the notation of [4, 5, 14, 25].

Let $q \in (0,1)$. The Hopf algebra $U_q\mathfrak{sl}_N$ is given by its generators K_i, K_i^{-1}, E_i, F_i, $i = 1, 2, \ldots, N-1$, and the relations:

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_j = q^{a_{ij}}E_jK_i, \qquad K_iF_j = q^{-a_{ij}}F_jK_i,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0, \qquad |i - j| = 1,$$

$$F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 = 0, \qquad |i - j| = 1,$$

$$E_iE_j - E_jE_i = F_iF_j - F_jF_i = 0, \qquad |i - j| \neq 1,$$

with $a_{ii} = 2$, $a_{ij} = -1$ for $|i - j| = 1$, $a_{ij} = 0$ otherwise, and the comultiplication Δ, the antipode S, and the counit ε being defined on the generators by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i,$$

$$S(E_i) = -K_i^{-1}E_i, \qquad S(F_i) = -F_iK_i, \qquad S(K_i) = K_i^{-1},$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \varepsilon(K_i) = 1,$$

(see also [14, Ch. 4]).

Consider the Hopf algebra $U_q\mathfrak{sl}_{2n}$. Equip $U_q\mathfrak{sl}_{2n}$ with a structure of Hopf $*$-algebra determined by the involution

$$K_j^* = K_j, \qquad E_j^* = \begin{cases} K_jF_j, & j \neq n, \\ -K_jF_j, & j = n, \end{cases} \qquad F_j^* = \begin{cases} E_jK_j^{-1}, & j \neq n, \\ -E_jK_j^{-1}, & j = n. \end{cases}$$

This Hopf $*$-algebra $(U_q\mathfrak{sl}_{2n}, *)$ is denoted by $U_q\mathfrak{su}_{n,n}$.

Denote by $U_q\mathfrak{k} = U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$ the Hopf $*$-subalgebra generated by

$$E_i, F_i, \quad i \neq n; \qquad K_j^{\pm 1}, \quad j = 1, 2, \ldots, 2n-1.$$

Now we introduce the notation to be used in the sequel and recall some known results on the quantum matrix ball $\mathbb{D}$.

Consider a $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$ with generators $\{z_a^\alpha\}_{a,\alpha=1,2,\ldots,n}$ and defining relations

$$z_a^\alpha z_b^\beta = \begin{cases} qz_b^\beta z_a^\alpha, & a = b \;\&\; \alpha < \beta \;\text{ or }\; a < b \;\&\; \alpha = \beta, \\ z_b^\beta z_a^\alpha, & a < b \;\&\; \alpha > \beta, \\ z_b^\beta z_a^\alpha + (q - q^{-1})z_a^\beta z_b^\alpha, & a < b \;\&\; \alpha < \beta, \end{cases} \tag{2.1}$$

$$(z_b^\beta)^* z_a^\alpha = q^2 \sum_{a',b'=1}^{n} \sum_{\alpha',\beta'=1}^{m} R(b,a,b',a')R(\beta,\alpha,\beta',\alpha') z_{a'}^{\alpha'} (z_{b'}^{\beta'})^* + (1-q^2)\delta_{ab}\delta^{\alpha\beta}$$

with δ_{ab}, $\delta^{\alpha\beta}$ being the Kronecker symbols, and

$$R(b,a,b',a') = \begin{cases} q^{-1}, & a \neq b \ \& \ b = b' \ \& \ a = a', \\ 1, & a = b = a' = b', \\ -(q^{-2}-1), & a = b \ \& \ a' = b' \ \& \ a' > a, \\ 0, & \text{otherwise}. \end{cases}$$

Denote by $\mathbb{C}[\mathrm{Mat}_n]_q$ the subalgebra generated by z_a^α, $a, \alpha = 1, 2, \ldots, n$. It is a very well-known quantum analog of the algebra of holomorphic polynomials on Mat_n.

Consider an arbitrary Hopf algebra A and an A-module algebra F. Suppose that A is a Hopf $*$-algebra. The $*$-algebra F is said to be an A-module algebra if the involutions are compatible as follows:

$$(af)^* = (S(a))^* f^*, \qquad a \in A, \ f \in F,$$

where S is an antipode of A.

It was shown in [33] (see Props. 8.12 and 10.1) that one has

Proposition 2.1. *$\mathbb{C}[\mathrm{Mat}_n]_q$ carries a structure of $U_q\mathfrak{sl}_{2n}$-module algebra given by*

$$K_n^{\pm 1} z_a^\alpha = \begin{cases} q^{\pm 2} z_a^\alpha, & a = \alpha = n, \\ q^{\pm 1} z_a^\alpha, & a = n \ \& \ \alpha \neq n \ \textit{or} \ a \neq n \ \& \ \alpha = n, \\ z_a^\alpha, & \textit{otherwise}, \end{cases}$$

$$F_n z_a^\alpha = q^{1/2} \cdot \begin{cases} 1, & a = \alpha = n, \\ 0, & \textit{otherwise}, \end{cases}$$

$$E_n z_a^\alpha = -q^{1/2} \cdot \begin{cases} q^{-1} z_a^n z_n^\alpha, & a \neq n \ \& \ \alpha \neq n, \\ (z_n^n)^2, & a = \alpha = n, \\ z_n^n z_a^\alpha, & \textit{otherwise}, \end{cases}$$

and for $k \neq n$

$$K_k^{\pm 1} z_a^\alpha = \begin{cases} q^{\pm 1} z_a^\alpha, & k < n \ \& \ a = k \ \textit{or} \ k > n \ \& \ \alpha = 2n-k, \\ q^{\mp 1} z_a^\alpha, & k < n \ \& \ a = k+1 \ \textit{or} \ k > n \ \& \ \alpha = 2n-k+1, \\ z_a^\alpha, & \textit{otherwise with} \ k \neq n; \end{cases}$$

$$F_k z_a^\alpha = q^{1/2} \cdot \begin{cases} z_{a+1}^\alpha, & k < n \,\&\, a = k, \\ z_a^{\alpha+1}, & k > n \,\&\, \alpha = 2n-k, \\ 0, & \textit{otherwise with } k \neq n; \end{cases}$$

$$E_k z_a^\alpha = q^{-1/2} \cdot \begin{cases} z_{a-1}^\alpha, & k < n \,\&\, a = k+1, \\ z_a^{\alpha-1}, & k > n \,\&\, \alpha = 2n-k+1, \\ 0, & \textit{otherwise with } k \neq n. \end{cases}$$

Also $\mathrm{Pol}(\mathrm{Mat}_n)_q$ *is equipped this way with a structure of* $U_q\mathfrak{su}_{n,n}$*-module algebra.*

It is well known that in the classical case of $q = 1$ the Shilov boundary of the matrix ball $\mathbb{D}$ is just the set $S(\mathbb{D})$ of all unitary matrices. Our intention is to produce a q-analogue of the Shilov boundary for the quantum matrix ball. Introduce notation for the quantum minors of the matrix $\mathbf{z} = (z_a^\alpha)$:

$$(z^{\wedge k})^{\{\alpha_1,\alpha_2,\ldots,\alpha_k\}}_{\{a_1,a_2,\ldots,a_k\}} \stackrel{\mathrm{def}}{=} \sum_{s \in S_k} (-q)^{l(s)} z_{a_1}^{\alpha_{s(1)}} z_{a_2}^{\alpha_{s(2)}} \cdots z_{a_k}^{\alpha_{s(k)}},$$

with $\alpha_1 < \alpha_2 < \ldots < \alpha_k$, $a_1 < a_2 < \ldots < a_k$, and $l(s)$ being the number of inversions in $s \in S_k$.

As known, the quantum determinant

$$\det_q \mathbf{z} = (z^{\wedge n})^{\{1,2,\ldots,n\}}_{\{1,2,\ldots,n\}}$$

is in the center of $\mathbb{C}[\mathrm{Mat}_n]_q$. The localization of $\mathbb{C}[\mathrm{Mat}_n]_q$ with respect to the multiplicative system $(\det_q \mathbf{z})^{\mathbb{N}}$ is called the algebra of regular functions on the quantum GL_n and is denoted by $\mathbb{C}[GL_n]_q$.

Lemma 2.2 (Lemma 2.1 of [41])**.** *There exists a unique involution* $*$ *in* $\mathbb{C}[GL_n]_q$ *such that*

$$(z_a^\alpha)^* = (-q)^{a+\alpha-2n} (\det_q \mathbf{z})^{-1} \det_q \mathbf{z}_a^\alpha,$$

where $\mathbf{z}_a^\alpha$ *is a matrix derived from* $\mathbf{z}$ *by deleting the row* α *and the column* a.

The $*$-algebra $\mathbb{C}[S(\mathbb{D})]_q = (\mathbb{C}[GL_n]_q, *)$ is a q-analogue of the algebra of regular functions on the Shilov boundary of matrix ball $\mathbb{D}$. It can be verified easily that $\mathbb{C}[S(\mathbb{D})]_q$ is a $U_q\mathfrak{su}_{n,n}$-module algebra (see [41, Th. 2.2 and Prop. 2.7] for the proof).

There exists another definition of the algebra $\mathbb{C}[S(\mathbb{D})]_q$.

Consider the two-sided ideal J of the $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$ generated by the relations

$$\sum_{j=1}^n q^{2n-\alpha-\beta} z_j^\alpha (z_j^\beta)^* - \delta^{\alpha\beta} = 0, \qquad \alpha, \beta = 1, 2, \ldots, n. \tag{2.2}$$

One can prove that this ideal is $U_q\mathfrak{su}_{n,n}$-invariant, what allows to introduce the $U_q\mathfrak{su}_{n,n}$-module algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q/J$. It was proved in [41, p. 381, Prop. 6.1] that $\mathbb{C}[S(\mathbb{D})]_q = \mathrm{Pol}(\mathrm{Mat}_n)_q/J$. The last equality also works as a definition for the algebra of regular functions on the Shilov boundary of the quantum matrix ball.

A module V over $U_q\mathfrak{sl}_{2n}$ is said to be a weight module if

$$V = \bigoplus_{\lambda \in P} V_\lambda, \qquad V_\lambda = \left\{ v \in V \;\middle|\; K_i v = q^{\lambda_i} v, \quad i = 1, 2, \ldots, 2n-1 \right\},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n-1})$, and $P \cong \mathbb{Z}^{2n-1}$ is a weight lattice of the Lie algebra $\mathfrak{sl}_{2n}$. A nonzero summand V_λ in this decomposition is called the weight subspace for the weight λ.

To every weight $U_q\mathfrak{sl}_{2n}$-module V we associate the linear maps H_i, $i = 1, 2, \ldots, 2n-1$, in V such that

$$H_i v = \lambda_i v, \qquad \text{iff} \qquad v \in V_\lambda.$$

Fix the element

$$H_0 = \sum_{j=1}^{n-1} j(H_j + H_{2n-j}) + nH_n.$$

Any weight $U_q\mathfrak{sl}_{2n}$-module V can be equipped with a $\mathbb{Z}$-grading $V = \bigoplus_r V_r$ by setting $v \in V_r$ if $H_0 v = 2rv$.

In what follows some more sophisticated spaces will be used. It is known that

$$\mathrm{Pol}(\mathrm{Mat}_n)_q = \bigoplus_{k,j=0}^{\infty} \mathbb{C}[\mathrm{Mat}_n]_{q,k} \cdot \mathbb{C}[\overline{\mathrm{Mat}}_n]_{q,-j}.$$

Here $\mathbb{C}[\overline{\mathrm{Mat}}_n]_q$ is the subalgebra of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ generated by $(z_a^\alpha)^*$, $a, \ \alpha = 1, 2, \ldots, n$, and $\mathbb{C}[\overline{\mathrm{Mat}}_n]_{q,-j}$, $\mathbb{C}[\mathrm{Mat}_n]_{q,k}$ are homogeneous components related to the grading

$$\deg(z_a^\alpha) = 1, \quad \deg(z_a^\alpha)^* = -1, \qquad a, \alpha = 1, 2, \ldots, n.$$

To rephrase this, every $f \in \mathrm{Pol}(\mathrm{Mat}_n)_q$ is uniquely decomposable as a finite sum

$$f = \sum_{k,j \geq 0} f_{k,j}, \qquad f_{k,j} \in \mathbb{C}[\mathrm{Mat}_n]_{q,k} \cdot \mathbb{C}[\overline{\mathrm{Mat}}_n]_{q,-j}. \tag{2.3}$$

Note that $\dim \mathbb{C}[\mathrm{Mat}_n]_{q,k} \cdot \mathbb{C}[\overline{\mathrm{Mat}}_n]_{q,-j} < \infty$.

Consider the vector space $\mathcal{D}(\mathbb{D})'_q$ of formal series of the form (2.3) with the termwise topology. The $U_q\mathfrak{sl}_{2n}$-action and the involution $*$ admit an extension by continuity from the dense linear subspace $\mathrm{Pol}(\mathrm{Mat}_n)_q$ to $\mathcal{D}(\mathbb{D})'_q$

$$* : \sum_{k,j=0}^{\infty} f_{k,j} \mapsto \sum_{k,j=0}^{\infty} f^*_{k,j}.$$

Moreover, $\mathcal{D}(\mathbb{D})'_q$ is a $U_q\mathfrak{sl}_{2n}$-module bimodule over $\mathrm{Pol}(\mathrm{Mat}_n)_q$. We call the elements of $\mathcal{D}(\mathbb{D})'_q$ distributions on a quantum bounded symmetric domain.

3. Statement of Main Result

We intend to determine the Poisson kernel (1.1) by listing some essential properties of the associated integral operator. For this purpose we use the normalized $S(U_n \times U_n)$-invariant measure on $S(\mathbb{D})$ for integration on the Shilov boundary. A principal property of the Poisson kernel is that the integral operator with this kernel is a morphism of the $SU_{n,n}$-module of functions on $S(\mathbb{D})$ into the $SU_{n,n}$-module of functions on $\mathbb{D}$ which takes 1 to 1.

So, in the quantum case the required Poisson integral operator is a morphism of the $U_q\mathfrak{su}_{n,n}$-module $\mathbb{C}[S(\mathbb{D})]_q$ into the $U_q\mathfrak{su}_{n,n}$-module $\mathcal{D}(\mathbb{D})'_q$ which takes 1 to 1. Recall that every $u \in \mathcal{D}(\mathbb{D})'_q$ is of the form

$$u = \sum_{j,k=0}^{\infty} u_{j,k}, \qquad u_{j,k} \in \mathbb{C}[\mathrm{Mat}_n]_{q,j}\mathbb{C}[\overline{\mathrm{Mat}}_n]_{q,-k},$$

and the set $\left\{ z_b^{\beta} (z_a^{\alpha})^* \right\}_{a,b,\alpha,\beta=1,2,\dots,n}$ is a basis of the vector space $\mathbb{C}[\mathrm{Mat}_n]_{q,1}\mathbb{C}[\overline{\mathrm{Mat}}_n]_{q,-1}$. This allows one to introduce the mixed partial derivatives at zero, the linear functionals $\left.\dfrac{\partial^2}{\partial z_b^{\beta} \partial (z_a^{\alpha})^*}\right|_{\mathbf{z}=0}$, such that

$$u_{1,1} = \sum_{a,b,\alpha,\beta=1}^{n} \left(\left.\frac{\partial^2 u}{\partial z_b^{\beta} \partial (z_a^{\alpha})^*}\right|_{\mathbf{z}=0} \right) z_b^{\beta} \, (z_a^{\alpha})^*, \qquad u \in \mathcal{D}(\mathbb{D})'_q.$$

Now we are in position to produce a quantum analog of the Hua equations.

Theorem 3.1. *If $u \in \mathcal{D}(\mathbb{D})'_q$ belongs to the image of the Poisson integral operator on the quantum $n \times n$-matrix ball, then*

$$\sum_{c=1}^{n} q^{2c} \left.\frac{\partial^2 (\xi u)}{\partial z_c^{\beta} \partial (z_c^{\alpha})^*}\right|_{\mathbf{z}=0} = 0 \qquad (3.1)$$

for all $\xi \in U_q\mathfrak{sl}_{2n}$, $\alpha, \beta = 1, 2, \dots, n$.

The equation system (3.1) is a q-analog of (1.2).

4. Invariant Generalized Kernels and Associated Integral Operators

List some plausible but less known definitions and results on the invariant integral and integral kernels (see [40]). Consider a Hopf algebra A and an A-module algebra F. A linear functional ν on F is called the A-invariant integral if ν is a morphism of A-modules:

$$\nu(af) = \varepsilon(a)f, \qquad a \in A,\ f \in F,$$

with ε being the counit of A.

There exists a unique $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant integral

$$\nu : \mathbb{C}[S(\mathbb{D})]_q \to \mathbb{C}, \qquad \nu : \varphi \mapsto \int\limits_{S(\mathbb{D})_q} \varphi d\nu,$$

which is normalized by $\int\limits_{S(\mathbb{D})_q} 1 d\nu = 1$ (see [41, Ch. 3]). As shown in [41], $\mathbb{C}[S(\mathbb{D})]_q$ is isomorphic to the algebra of regular functions on the quantum U_n as a $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module $*$-algebra. This isomorphism can be used to consider the transfer of ν on the latter algebra, where it is known to be positive [45]. Hence ν is itself positive: $\int\limits_{S(\mathbb{D})_q} \varphi^* \varphi d\nu > 0$ for all nonzero $\varphi \in \mathbb{C}[S(\mathbb{D})]_q$. Our subsequent results demonstrate how this $U_q\mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant integral can be used to produce integral operators which are the morphisms of $U_q\mathfrak{sl}_{2n}$-modules.

Consider A-module algebras F_1, F_2. Given a linear functional $\nu : F_2 \to \mathbb{C}$ and $\mathcal{K} \in F_1 \otimes F_2$, we associate a linear integral operator

$$K : F_2 \to F_1, \qquad K : f \mapsto (\mathrm{id} \otimes \nu)(\mathcal{K}(1 \otimes f)).$$

In this context $\mathcal{K}$ is called the kernel of this integral operator. Assume that the integral ν on F_2 is invariant and the bilinear form

$$f' \times f'' \mapsto \nu(f'\, f''), \qquad f', f'' \in F_2,$$

is nondegenerate. It is easy to understand that the integral operator with the kernel $\mathcal{K}$ is a morphism of A-modules if and only if this kernel is invariant [40]. Another statement from [40] that will be used essentially in a subsequent construction of A-invariant kernels is as follows: A-invariant kernels form a subalgebra of $F_1^{\mathrm{op}} \otimes F_2$, with F_1^{op} being the algebra derived from F_1 by replacement of its multiplication by the opposite one.

Additionally to the algebra of kernels $\mathrm{Pol}(\mathrm{Mat}_n)_q^{\mathrm{op}} \otimes \mathbb{C}[S(\mathbb{D})]_q$, we will use the bimodule of generalized kernels $\mathcal{D}(\mathbb{D} \times S(\mathbb{D}))'_q$ whose elements are just the formal series

$$\sum_{i,j} f_{ij} \otimes \varphi_{ij}, \qquad f_{ij} \in \mathbb{C}[\overline{\mathrm{Mat}}_n]_{q,-j}^{\mathrm{op}} \mathbb{C}[\mathrm{Mat}_n]_{q,i}^{\mathrm{op}}, \quad \varphi_{ij} \in \mathbb{C}[S(\mathbb{D})]_q.$$

It is a bimodule over the algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q^{\mathrm{op}} \otimes \mathbb{C}[S(\mathbb{D})]_q$.

5. A Passage from Affine Coordinates to Homogeneous Coordinates

This section, as well as Sections 2 and 4, contains some preliminary material and known results obtained in [33, 41, 43].

Turn to the quantum group SL_N. We follow the general idea by V.G. Drinfeld [6] in considering the Hopf algebra $\mathbb{C}[SL_N]_q$ of matrix elements of finite-dimensional weight $U_q\mathfrak{sl}_N$-modules. It is accustomed to call it the algebra of regular functions on the quantum group SL_N. The linear maps in $(U_q\mathfrak{sl}_N)^*$ adjoint to the operators of left multiplication by the elements of $U_q\mathfrak{sl}_N$ equip $\mathbb{C}[SL_N]_q$ with a structure of $U_q\mathfrak{sl}_N$-module algebra by duality.

Recall that $\mathbb{C}[SL_N]_q$ can be defined by the generators t_{ij}, $i, j = 1, 2, \ldots, N$, (the matrix elements of the vector representation in a weight basis) and by the relations

$$\begin{aligned}
&t_{ij'}t_{ij''} = q t_{ij''}t_{ij'}, && j' < j'',\\
&t_{i'j}t_{i''j} = q t_{i''j}t_{i'j}, && i' < i'',\\
&t_{ij}t_{i'j'} = t_{i'j'}t_{ij}, && i < i' \;\&\; j > j',\\
&t_{ij}t_{i'j'} = t_{i'j'}t_{ij} + (q - q^{-1})t_{ij'}t_{i'j}, && i < i' \;\&\; j < j',
\end{aligned}$$

which are tantamount to (2.1), together with one more relation

$$\det\nolimits_q \mathbf{t} = 1,$$

where $\det_q \mathbf{t}$ is a q-determinant of the matrix $\mathbf{t} = (t_{ij})_{i,j=1,2,\ldots,N}$:

$$\det\nolimits_q \mathbf{t} = \sum_{s \in S_N} (-q)^{l(s)} t_{1s(1)} t_{2s(2)} \ldots t_{Ns(N)},$$

with $l(s) = \mathrm{card}\{(i, j) | i < j \;\&\; s(i) > s(j)\}$.

It is well known that $\det_q \mathbf{t}$ commutes with all t_{ij}. Thus $\mathbb{C}[SL_N]_q$ appears to be a quotient algebra of $\mathbb{C}[\mathrm{Mat}_N]_q$ by the two-sided ideal generated by $\det_q \mathbf{t} - 1$.

Note that $\mathbb{C}[SL_N]_q$ is a domain.

In the classical case of $q = 1$ the matrix ball admits a natural embedding into the Grassmannian $\mathrm{Gr}_{n,2n}$

$$(\text{a contraction } A \in \mathrm{End}\,\mathbb{C}^n) \mapsto (\text{the linear span of } (v, Av), \quad v \in \mathbb{C}^n),$$

where the latter pair is an element of $\mathbb{C}^n \oplus \mathbb{C}^n \simeq \mathbb{C}^{2n}$. We are going to describe a q-analog for this embedding.

Let

$$I = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, 2n\}, \qquad i_1 < i_2 < \ldots < i_k;$$
$$J = \{j_1, j_2, \ldots, j_k\} \subset \{1, 2, \ldots, 2n\}, \qquad j_1 < j_2 < \ldots < j_k.$$

The elements

$$t_{IJ}^{\wedge k} = \sum_{s \in S_k} (-q)^{l(s)} t_{i_{s(1)} j_1} t_{i_{s(2)} j_2} \ldots t_{i_{s(k)} j_k}$$

of $\mathbb{C}[SL_{2n}]_q$ are called quantum minors, and it is easy to check that

$$t_{IJ}^{\wedge k} = \sum_{s \in S_k} (-q)^{l(s)} t_{i_1 j_{s(1)}} t_{i_2 j_{s(2)}} \ldots t_{i_k j_{s(k)}}.$$

Consider the smallest unital subalgebra $\mathbb{C}[X]_q \subset \mathbb{C}[SL_{2n}]_q$ that contains the quantum minors

$$t^{\wedge n}_{\{1,2,\ldots,n\}J}, \quad t^{\wedge n}_{\{n+1,n+2,\ldots,2n\}J}, \qquad J = \{j_1, j_2, \ldots, j_n\} \subset \{1, 2, \ldots, 2n\}.$$

It is a $U_q\mathfrak{sl}_{2n}$-module subalgebra which substitutes the classical coordinate ring of the Grassmannian.

The following results are easy modifications of those of [33].

Proposition 5.1. *There exists a unique antilinear involution $*$ in $\mathbb{C}[X]_q$ such that $(\mathbb{C}[X]_q, *)$ is a $U_q\mathfrak{su}_{n,n}$-module algebra, and*

$$\left(t^{\wedge n}_{\{1,2,\ldots,n\}\{n+1,n+2,\ldots,2n\}}\right)^* = (-q)^{n^2} t^{\wedge n}_{\{n+1,n+2,\ldots,2n\}\{1,2,\ldots,n\}}.$$

Lemma 5.2 (Lemma 11.3 of [33]). *Let $J \subset \{1, 2, \ldots, 2n\}$, $\mathrm{card}(J) = n$, $J^c = \{1, 2, \ldots, 2n\} \setminus J$, $l(J, J^c) = \mathrm{card}\{(j', j'') \in J \times J^c | \, j' > j''\}$. Then*

$$\left(t^{\wedge n}_{\{1,2,\ldots,n\}J}\right)^* = (-1)^{\mathrm{card}(\{1,2,\ldots,n\} \cap J)} (-q)^{l(J,J^c)} t^{\wedge n}_{\{n+1,n+2,\ldots,2n\}J^c}. \tag{5.1}$$

Impose the abbreviated notation

$$t = t^{\wedge n}_{\{1,2,\ldots,n\}\{n+1,n+2,\ldots,2n\}}, \qquad x = tt^*.$$

Note that t, t^* and x quasicommute with all generators t_{ij} of $\mathbb{C}[SL_{2n}]_q$. Then the localization $\mathbb{C}[X]_{q,x}$ of the algebra $\mathbb{C}[X]_q$ with respect to the multiplicative set $x^{\mathbb{Z}_+}$ is well defined. The structure of $U_q\mathfrak{su}_{n,n}$-module algebra is uniquely extendable from $\mathbb{C}[X]_q$ to $\mathbb{C}[X]_{q,x}$ [33].

Proposition 5.3. *[cf. Prop. 3.2 of [33]] There exists a unique embedding of $U_q\mathfrak{su}_{n,n}$-module $*$-algebras $\mathcal{I}:\mathrm{Pol}(\mathrm{Mat}_n)_q \hookrightarrow \mathbb{C}[X]_{q,x}$ such that $\mathcal{I}z_a^\alpha = t^{-1}t^{\wedge n}_{\{1,2,\dots,n\}J_{a\alpha}}$, with $J_{a\alpha}=\{a\}\cup\{n+1,n+2,\dots,2n\}\setminus\{2n+1-\alpha\}$.*

Corollary 5.4. *$\mathcal{I}y = x^{-1}$, with*

$$y = 1+\sum_{k=1}^{m}(-1)^k \sum_{\{J'|\ \mathrm{card}(J')=k\}}\ \sum_{\{J''|\ \mathrm{card}(J'')=k\}} z^{\wedge k}{}^{J'}_{J''}\left(z^{\wedge k}{}^{J'}_{J''}\right)^{*}. \tag{5.2}$$

A formal passage to a limit as $q\to 1$ leads to the relation $y=\det(1-\mathbf{z}\mathbf{z}^*)$.

Proposition 5.5 (see Ch. 11 of [33])**.** *Let $1\le \alpha_1<\alpha_2<\ldots<\alpha_k\le n$, $1\le a_1<a_2<\ldots<a_k\le n$, $J=\{n+1,n+2,\dots,2n\}\setminus\{n+\alpha_1,n+\alpha_2,\dots,n+\alpha_k\}\cup\{a_1,a_2,\dots,a_k\}$. Then*

$$\mathcal{I}z^{\wedge k\{n+1-\alpha_k,n+1-\alpha_{k-1},\dots,n+1-\alpha_1\}}_{\{a_1,a_2,\dots,a_k\}} = t^{-1}t^{\wedge n}_{\{1,2,\dots,n\}J}. \tag{5.3}$$

It is accustomed to identify the generators z_a^α, $a,\alpha=1,2,\dots,n$, with their images under $\mathcal{I}$.

Consider the subalgebra of $\mathbb{C}[X]_{q,x}$ generated by $\mathrm{Pol}(\mathrm{Mat}_n)_q \overset{\mathcal{I}}{\hookrightarrow} \mathbb{C}[X]_{q,x}$ together with $t^{\pm1}$, $t^{*\pm1}$.

The elements of this subalgebra admit a unique decomposition of the form

$$\sum_{(i,j)\notin(-\mathbb{N})\times(-\mathbb{N})} t^i t^{*j} f_{ij}, \qquad f_{ij}\in \mathrm{Pol}(\mathrm{Mat}_n)_q \tag{5.4}$$

(the choice of the set of pairs (i,j) is due to the fact that $t^{-1}t^{*-1}\in\mathrm{Pol}(\mathrm{Mat}_n)_q$).

Equip $\mathbb{C}^n\oplus\mathbb{C}^n$ with the sesquilinear form $(\cdot,\cdot)_1-(\cdot,\cdot)_2$. In the classical case of $q=1$, the Shilov boundary of the matrix ball is just the group U_n. The graphs of unitary operators form the isotropic Grassmannian (its points are the subspaces on which the above scalar product on $\mathbb{C}^{2n}$ vanishes). We are going to describe a q-analog of it when the isotropic Grassmannian is replaced by its homogeneous coordinate ring.

Consider an extension $\mathbb{C}[\Xi]_q$ of the algebra $\mathbb{C}[S(\mathbb{D})]_q$ in the class of $U_q\mathfrak{su}_{n,n}$-module $*$-algebras. This extension is produced by adding a generator t and the relations

$$tt^*=t^*t;\quad tz_a^\alpha = q^{-1}z_a^\alpha t;\quad t^*z_a^\alpha = q^{-1}z_a^\alpha t^*,\quad a,\alpha=1,2,\dots,n. \tag{5.5}$$

The $U_q\mathfrak{su}_{n,n}$-action is extended to $\mathbb{C}[\Xi]_q$ as follows:

$$E_jt=F_jt=(K_j^{\pm1}-1)t=0,\qquad j\ne n,$$
$$F_nt=(K_n^{\pm1}-1)t=0,\qquad E_nt=q^{-1/2}tz_n^n.$$

Let $\xi = tt^*$. One can introduce a localization $\mathbb{C}[\Xi]_{q,\xi}$ of the algebra $\mathbb{C}[\Xi]_q$ with respect to the multiplicative set $\xi^{\mathbb{Z}_+}$. The involution $*$ and the structure of $U_q\mathfrak{su}_{n,n}$-module algebra are uniquely extendable from $\mathbb{C}[\Xi]_q$ to $\mathbb{C}[\Xi]_{q,\xi}$.

The algebra $\mathbb{C}[\Xi]_q$ is equipped with a $U_q\mathfrak{sl}_{2n}$-invariant bigrading

$$\deg z_a^\alpha = \deg(z_a^\alpha)^* = (0,0), \qquad \deg t = (1,0), \qquad \deg t^* = (0,1),$$

which extends to the localization $\mathbb{C}[\Xi]_{q,\xi}$. By [41], the homogeneous component $\mathbb{C}[\Xi]_{q,\xi}^{(-n,-n)}$ carries a nonzero $U_q\mathfrak{sl}_{2n}$-invariant integral η such that

$$\eta(t^{*-n} f t^{-n}) = \int\limits_{S(\mathbb{D})_q} f d\nu.$$

Since t, t^* normalize every $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$-isotypic component of $\mathbb{C}[S(\mathbb{D})]_q$, and the $U_q\mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$-action in $\mathbb{C}[S(\mathbb{D})]_q$ is multiplicity-free [3, Prop. 4], it follows from the definition of integral over the Shilov boundary that

$$\eta(t^{*-n} t^{-n} f) = \eta(f t^{*-n} t^{-n}) = \int\limits_{S(\mathbb{D})_q} f d\nu. \tag{5.6}$$

6. The Poisson Kernel

In this section we construct a morphism of the $U_q\mathfrak{su}_{n,n}$-module $\mathbb{C}[S(\mathbb{D})]_q$ into the $U_q\mathfrak{su}_{n,n}$-module $\mathcal{D}(\mathbb{D})_q'$ which takes 1 to 1. The introduced morphism will be a Poisson integral operator, as mentioned in Section 3.

We will need a quantum analog $\mathbb{C}[\mathrm{Mat}_{n,2n}]_q$ for polynomial algebra on the space of rectangular $n \times 2n$-matrices. The algebra $\mathbb{C}[\mathrm{Mat}_{n,2n}]_q$ is determined by the set of generators $\{t_{ij}\}_{i=1,2,\dots,n;\, j=1,2,\dots,2n}$ and the similar relations as in the case of square matrices (see (2.1)). It is known that this algebra is a domain.

The structure of $U_q\mathfrak{sl}_{2n}$-module algebra is given by

$$K_k t_{ij} = \begin{cases} q t_{ij}, & j = k, \\ q^{-1} t_{ij}, & j = k+1, \\ t_{ij}, & j \notin \{k, k+1\}, \end{cases}$$

$$E_k t_{ij} = \begin{cases} q^{-1/2} t_{i,j-1}, & j = k+1, \\ 0, & j \neq k+1, \end{cases} \qquad F_k t_{ij} = \begin{cases} q^{1/2} t_{i,j+1}, & j = k, \\ 0, & j \neq k, \end{cases}$$

with $k = 1, 2, \dots, 2n-1$.

R e m a r k 6.1. Consider the $U_q\mathfrak{sl}_{2n}$-module subalgebras in $\mathbb{C}[\mathrm{Mat}_{n,2n}]_q$ and in $\mathbb{C}[\Xi]_q$ generated by $t^{\wedge n}_{\{1,2,\dots,n\}\{n+1,n+2,\dots,2n\}}$ and t, respectively. The map

$t^{\wedge n}_{\{1,2,\dots,n\}\{n+1,n+2,\dots,2n\}} \mapsto t$ may be extended up to an isomorphism i of these $U_q\mathfrak{sl}_{2n}$-module subalgebras. In a similar way one can introduce $U_q\mathfrak{sl}_{2n}$-module subalgebras in $\mathbb{C}[\mathrm{Mat}_{n,2n}]_q$ and in $\mathbb{C}[\Xi]_q$, generated by $t^{\wedge n}_{\{1,2,\dots,n\}\{1,2,\dots,n\}}$ and t^*, respectively. The map $t^{\wedge n}_{\{1,2,\dots,n\}\{1,2,\dots,n\}} \mapsto (-q)^{n^2}t^*$ extends up to the isomorphism i' of these $U_q\mathfrak{sl}_{2n}$-module subalgebras. The above isomorphisms can be used together to embed the minors $t^{\wedge n}_{\{1,2,\dots,n\}J}$ and $t^{\wedge n}_{\{n+1,n+2,\dots,2n\}J}$ into $\mathbb{C}[\Xi]_q$.

Consider the map $m : \mathbb{C}[\mathrm{Mat}_{n,2n}]_q \otimes \mathbb{C}[\mathrm{Mat}_{n,2n}]_q \to \mathbb{C}[\mathrm{Mat}_{2n}]_q$ defined as follows. The tensor multipliers are embedded into $\mathbb{C}[\mathrm{Mat}_{2n}]_q$ as subalgebras generated by the entries of, respectively, the upper n and the lower n rows of the matrix $\mathbf{t} = (t_{ij})$, $i = 1, 2, \dots, 2n$, $j = 1, 2, \dots, 2n$, and the map m is just a multiplication in the algebra $\mathbb{C}[\mathrm{Mat}_{2n}]_q$.

Lemma 6.2. *The map m is an isomorphism of $U_q\mathfrak{sl}_{2n}$-modules.*

P r o o f. The map m takes the monomial basis

$$t_{11}^{j_{11}} \dots t_{12n}^{j_{1,2n}} t_{21}^{j_{21}} \dots t_{22n}^{j_{2,2n}} \dots t_{n1}^{j_{n1}} \dots t_{n2n}^{j_{n,2n}} \otimes t_{n+11}^{j_{n+1,1}} \dots t_{n+12n}^{j_{n+1,2n}} t_{n+21}^{j_{n+2,1}} \dots t_{n+22n}^{j_{n+2,2n}} \dots t_{2n1}^{j_{2n,1}} \dots t_{2n2n}^{j_{2n,2n}}$$

of the algebra $\mathbb{C}[\mathrm{Mat}_{n,2n}]_q \otimes \mathbb{C}[\mathrm{Mat}_{n,2n}]_q$ to the monomial basis

$$t_{11}^{j_{11}} \dots t_{12n}^{j_{1,2n}} t_{21}^{j_{21}} \dots t_{22n}^{j_{2,2n}} \dots t_{n1}^{j_{n1}} \dots t_{n2n}^{j_{n,2n}} t_{n+11}^{j_{n+1,1}} \dots t_{n+12n}^{j_{n+1,2n}} \cdot t_{n+21}^{j_{n+2,1}} \dots t_{n+22n}^{j_{n+2,2n}} \dots t_{2n1}^{j_{2n,1}} \dots t_{2n2n}^{j_{2n,2n}}$$

of the algebra $\mathbb{C}[\mathrm{Mat}_{2n}]_q$, $j_{ik} \in \mathbb{Z}_+$. Hence it is a bijective map and a morphism of $U_q\mathfrak{sl}_{2n}$-modules since $\mathbb{C}[\mathrm{Mat}_{2n}]_q$ is a $U_q\mathfrak{sl}_{2n}$-module algebra. ■

It is worthwhile to note that the definition of the $U_q\mathfrak{sl}_{2n}$-module algebra $\mathbb{C}[X]_q$ allows a replacement of $\mathbb{C}[SL_{2n}]_q$ by $\mathbb{C}[\mathrm{Mat}_{n,2n}]_q$.

Consider the elements of the $U_q\mathfrak{sl}_{2n}$-module $\mathbb{C}[\mathrm{Mat}_{n,2n}]_q \otimes \mathbb{C}[\mathrm{Mat}_{n,2n}]_q$ given by

$$\mathcal{L} = \sum_{J\subset\{1,2,\dots,2n\}\&\,\mathrm{card}(J)=n} (-q)^{l(J,J^c)} t^{\wedge n}_{\{1,2,\dots,n\}J} \otimes t^{\wedge n}_{\{n+1,n+2,\dots,2n\}J^c},$$

$$\overline{\mathcal{L}} = \sum_{J\subset\{1,2,\dots,2n\}\&\mathrm{card}(J)=n} (-q)^{-l(J,J^c)} t^{\wedge n}_{\{n+1,n+2,\dots,2n\}J^c} \otimes t^{\wedge n}_{\{1,2,\dots,n\}J}.$$

Here J^c is the complement to J and $l(I,J) = \mathrm{card}\{(i,j) \in I \times J |\, i > j\}$.

Proposition 6.3. *$\mathcal{L}$ and $\overline{\mathcal{L}}$ are $U_q\mathfrak{sl}_{2n}$-invariants.*

P r o o f is expounded for $\mathcal{L}$. In the case of $\overline{\mathcal{L}}$, similar arguments are applicable.

Recall a q-analog for the Laplace formula of splitting the quantum determinant of the $2n \times 2n$-matrix $\mathbf{t} = (t_{ij})$ with respect to the upper n lines:

$$\begin{aligned}\det_q \mathbf{t} &= \sum_{J \subset \{1,2,\ldots,2n\} \,\&\, \mathrm{card}(J)=n} (-q)^{l(J,J^c)} t^{\wedge n}_{\{1,2,\ldots,n\}J} t^{\wedge n}_{\{n+1,n+2,\ldots,2n\}J^c} \\ &= \sum_{J \subset \{1,2,\ldots,2n\} \& \mathrm{card}(J)=n} (-q)^{-l(J,J^c)} t^{\wedge n}_{\{n+1,n+2,\ldots,2n\}J^c} t^{\wedge n}_{\{1,2,\ldots,n\}J}.\end{aligned}$$

Our claim follows from Lemma 6.2, the relation $m\mathcal{L} = \det_q \mathbf{t}$ and $U_q\mathfrak{sl}_{2n}$-invariance of the quantum determinant. ∎

Note that, in view of Remark 6.1, one has $\mathcal{L}, \overline{\mathcal{L}} \in \mathbb{C}[X]_q \otimes \mathbb{C}[\Xi]_q$.

Introduce, firstly, a $U_q\mathfrak{sl}_{2n}$-module of kernels $\mathcal{D}(\mathbb{D} \times \Xi)'_q$ whose elements are formal series with coefficients from $\mathbb{C}[\overline{\mathrm{Mat}}_n]_{q,-j}\mathbb{C}[\mathrm{Mat}_n]_{q,i} \otimes \mathbb{C}[\Xi]_{q,\xi}$, and, secondly, a $U_q\mathfrak{sl}_{2n}$-module of kernels $\mathcal{D}(X \otimes \Xi)'_q$ whose elements are finite sums of the form

$$\sum_{(i,j) \notin (-\mathbb{N}) \times (-\mathbb{N})} (t^i t^{*j} \otimes 1) f_{ij}, \qquad f_{ij} \in \mathcal{D}(\mathbb{D} \times \Xi)'_q$$

(cf. (5.4)). Of course, $\mathcal{D}(\mathbb{D} \times \Xi)'_q$ is a $\mathrm{Pol}(\mathrm{Mat}_n)^{\mathrm{op}}_q \otimes \mathbb{C}[\Xi]_{q,\xi}$-bimodule, and $\mathcal{D}(X \times \Xi)'_q$ is a $\mathbb{C}[X]^{\mathrm{op}}_{q,x} \otimes \mathbb{C}[\Xi]_{q,\xi}$-bimodule.

The kernel $\mathcal{L} \in \mathbb{C}[X]^{\mathrm{op}}_q \otimes \mathbb{C}[\Xi]_q$ can be written in the form

$$\mathcal{L} = \left(1 + \sum_{J \neq \{n+1,n+2,\ldots,2n\}} (-q)^{l(J,J^c)} t^{\wedge n}_{\{1,2,\ldots,n\}J} t^{-1} \otimes t^{\wedge n}_{\{n+1,n+2,\ldots,2n\}J^c} t^{*-1}\right)(t \otimes t^*).$$

Note that $t^{\wedge n}_{\{1,2,\ldots,n\}J} t^{-1}, t^{\wedge n}_{\{n+1,n+2,\ldots,2n\}J^c} t^{*-1} \in \mathbb{C}[\mathrm{Mat}_n]_q$ (see (5.3)). This allows one to write down explicitly such an element $\mathcal{L}^{-n}$ of the space of generalized kernels $\mathcal{D}(X \times \Xi)'_q$ that $\mathcal{L}^n \cdot \mathcal{L}^{-n} = \mathcal{L}^{-n} \cdot \mathcal{L}^n = 1$, where $\cdot$ stands for the (left and right) actions of $\mathcal{L}^n$ on the element $\mathcal{L}^{-n}$ of the bimodule $\mathcal{D}(X \times \Xi)'_q$.

A similar construction produces also a $U_q\mathfrak{sl}_{2n}$-invariant generalized kernel $\overline{\mathcal{L}}^{-n}$.

Note that $\mathcal{L}^{-n} = \sum_i x_i$ is a formal series with $x_i \in \mathbb{C}[\mathrm{Mat}_n]_{q,i} \otimes \mathbb{C}[\Xi]_{q,\xi}$, $i \in \mathbb{Z}_+$, and the terms of the formal series $\overline{\mathcal{L}}^{-n} = \sum_j y_j$ are such that $y_j \in \mathbb{C}[\overline{\mathrm{Mat}}_n]_{q,-j} \otimes \mathbb{C}[\Xi]_{q,\xi}$. This allows one to define the 'product' $\overline{\mathcal{L}}^{-n}\mathcal{L}^{-n}$ as a double series $\sum_{i,j} y_j x_i$ which is thus an element of the module of generalized kernels $\mathcal{D}(X \otimes \Xi)'_q$. Clearly, one has $\overline{\mathcal{L}}^n \cdot \left(\overline{\mathcal{L}}^{-n}\mathcal{L}^{-n}\right) \cdot \mathcal{L}^n = 1$ in $\mathcal{D}(X \times \Xi)'_q$, and this property determines uniquely the generalized kernel $\overline{\mathcal{L}}^{-n}\mathcal{L}^{-n}$. Furthermore, the above uniqueness allows one to verify invariance of the generalized kernel $\overline{\mathcal{L}}^{-n}\mathcal{L}^{-n}$. Of course, the argument should apply the invariance of $\mathcal{L}$, $\overline{\mathcal{L}}$.

Consider the Poisson kernel $P \in \mathcal{D}(\mathbb{D} \times S(\mathbb{D}))'_q$ for the matrix ball having the following properties:

i) up to a constant multiplier, the Poisson kernel is just $(1 \otimes tt^*)^n \overline{\mathcal{L}}^{-n}\mathcal{L}^{-n}$, that is

$$P = \mathrm{const}(q, n)(1 \otimes tt^*)^n \overline{\mathcal{L}}^{-n}\mathcal{L}^{-n};$$

ii) the integral operator with kernel P takes $1 \in \mathbb{C}[S(\mathbb{D})]_q$ to $1 \in \mathcal{D}(\mathbb{D})'_q$.

Lemma 6.4. *The integral operator* $\mathbb{C}[S(\mathbb{D})]_q \to \mathcal{D}(\mathbb{D})'_q$ *with kernel* $(1 \otimes tt^*)^n \overline{\mathcal{L}}^{-n}\mathcal{L}^{-n}$ *is a morphism of* $U_q\mathfrak{sl}_{2n}$*-modules.*

P r o o f. It follows from the existence of an invariant integral $\eta : \mathbb{C}[\Xi]_q^{(-n,-n)} \to \mathbb{C}$ and the invariance of $\overline{\mathcal{L}}^{-n}\mathcal{L}^{-n}$ that the integral operator $\mathbb{C}[S(\mathbb{D})]_q \to \mathcal{D}(\mathbb{D})'_q$ with kernel $\overline{\mathcal{L}}^{-n}\mathcal{L}^{-n}$ is a morphism of $U_q\mathfrak{sl}_{2n}$-modules. Now it remains to move the multiplier $(1 \otimes t^{*-n}t^{-n})$ from $\overline{\mathcal{L}}^{-n}\mathcal{L}^{-n}$ to the left and to apply (5.6). ■

R e m a r k 6.5. Since the integral operator $\mathbb{C}[S(\mathbb{D})]_q \to \mathcal{D}(\mathbb{D})'_q$ with kernel $(1 \otimes tt^*)^n \overline{\mathcal{L}}^{-n}\mathcal{L}^{-n}$ is a morphism of $U_q\mathfrak{sl}_{2n}$-modules, the image of $1 \in \mathbb{C}[S(\mathbb{D})]_q$ is a $U_q\mathfrak{sl}_{2n}$-invariant element of $\mathcal{D}(\mathbb{D})'_q$, that is just a constant. This proves the existence of the Poisson kernel P.

E x a m p l e 6.6. Let us illustrate the Poisson kernel P determined above in the simplest case when $n = 1$. The considered invariant kernels were obtained in [26]. The elements

$$\mathcal{L} = t_{11} \otimes t_{22} - qt_{12} \otimes t_{21}, \qquad \overline{\mathcal{L}} = -q^{-1}t_{21} \otimes t_{12} + t_{22} \otimes t_{11}$$

of the algebra $\mathbb{C}[X]^{\mathrm{op}}_{q,x} \otimes \mathbb{C}[\Xi]_{q,\xi}$ are $U_q\mathfrak{sl}_2$-invariant kernels. We present below an easy computation, which uses, for the sake of brevity, the notation

$$z = qt_{12}^{-1}t_{11}, \qquad z^* = t_{22}t_{21}^{-1},$$

instead of mentioning explicitly the embeddings $\mathcal{I}$, $\mathcal{J}$. One has

$$\mathcal{L} = (1 - z \otimes z^*)(-qt_{12} \otimes t_{21}), \qquad \overline{\mathcal{L}} = (-q^{-1}t_{21} \otimes t_{12})(1 - q^2 z^* \otimes z),$$

and, as $t = t_{12}$, $t^* = -qt_{21}$,

$$\mathfrak{L} = (1 - z \otimes z^*)(t \otimes t^*), \qquad \overline{\mathfrak{L}} = (q^{-2}t^* \otimes t)(1 - q^2 z^* \otimes z).$$

Hence,

$$\overline{\mathfrak{L}}^{-1}\mathfrak{L}^{-1} = q^2(1 \otimes t^{-1}t^{*-1})(1 - z^* \otimes z)^{-1}((1 - z^*z) \otimes 1)(1 - z \otimes z^*)^{-1}.$$

Omit $\otimes$ and in the second tensor multiplier z replace by ζ and z^* by ζ^* (which is standard in function theory) to obtain

$$P = \mathrm{const}(q)(1 - z^*\zeta)^{-1}(1 - z^*z)(1 - z\zeta^*)^{-1}.$$

What remains now is to find $\mathrm{const}(q)$, or, to be more precise, to prove that $\mathrm{const}(q)=1$. In fact, the integral operator with kernel $(1-z^*\zeta)^{-1}(1-z^*z)(1-z\zeta^*)^{-1}$ takes 1 to 1. This is because $\zeta^* = \zeta^{-1}$, and integration of the product of the series in ζ produces the constant term: $\sum\limits_{k=0}^{\infty} z^k(1 - zz^*)z^{*k} = 1$.

Note that the Poisson kernel is a formal series $P = \sum\limits_{j,k=0}^{\infty} p_{jk}$, with $p_{jk} \in \mathbb{C}[\overline{\mathrm{Mat}}_n]_{q,-k}\mathbb{C}[\mathrm{Mat}_n]_{q,j} \otimes \mathbb{C}[S(\mathbb{D})]_q$. In the sequel we will omit $\otimes$ and in the second tensor multiplier replace z by ζ and z^* by ζ^* (that is standard in function theory).

Lemma 6.7. *The following relation is valid:*

$$p_{11} = \mathrm{const}(q,n) \sum_{a,b,\alpha,\beta=1}^{n} \left(\frac{1 - q^{-2n}}{1 - q^{-2}} q^{2(2n-a-\alpha)} \zeta_a^\alpha (\zeta_b^\beta)^* - \delta_{ab}\delta^{\alpha\beta} \right) (z_a^\alpha)^* z_b^\beta, \quad (6.1)$$

with $\mathrm{const}(q,n) \neq 0$.

P r o o f. In the algebra of kernels one has

$$\mathfrak{L} = \left(1 - \sum_{a,\alpha=1}^{n} z_a^\alpha (\zeta_a^\alpha)^* + \ldots \right) t\tau^*, \qquad (6.2)$$

$$\overline{\mathfrak{L}} = q^{-2n^2} t^*\tau \left(1 - q^2 \sum_{a,\alpha=1}^{n} q^{2(2n-a-\alpha)} (z_a^\alpha)^* \zeta_a^\alpha + \ldots \right), \qquad (6.3)$$

which is easily deducible from (5.3), (5.1), and the fact that $\mathcal{I}$ is a homomorphism of $*$-algebras. Also, it is seen from (5.2) that

$$y = (tt^*)^{-1} = 1 - \sum_{a,\alpha=1}^{n} (z_a^\alpha)^* z_a^\alpha + \ldots . \qquad (6.4)$$

Here three dots replace the terms whose degree is above two, and the following abbreviated notation is implicit:

$$t = t \otimes 1, \qquad \tau = 1 \otimes \tau, \qquad z_a^\alpha = z_a^\alpha \otimes 1, \qquad \zeta_a^\alpha = 1 \otimes \zeta_a^\alpha,$$

$$t^* = t^* \otimes 1, \qquad \tau^* = 1 \otimes \tau^*, \qquad (z_a^\alpha)^* = (z_a^\alpha)^* \otimes 1, \qquad (\zeta_a^\alpha)^* = 1 \otimes (\zeta_a^\alpha)^*.$$

Apply (6.2)–(6.4), together with the commutation relations (see also (5.5))

$$t^*\tau \left(\sum_{a,\alpha=1}^{n} q^{2(2n-a-\alpha)} (z_a^\alpha)^* \zeta_a^\alpha \right) = q^{-2} \left(\sum_{a,\alpha=1}^{n} q^{2(2n-a-\alpha)} (z_a^\alpha)^* \zeta_a^\alpha \right) t^*\tau;$$

$$t\tau^* \left(\sum_{a,\alpha=1}^{n} z_a^\alpha (\zeta_a^\alpha)^* \right) = q^2 \left(\sum_{a,\alpha=1}^{n} z_a^\alpha (\zeta_a^\alpha)^* \right) t\tau^*,$$

to obtain

$$\begin{aligned} \overline{\mathcal{L}}^{-n} \mathcal{L}^{-n} &= \prod_{j=1}^{n} \left(1 - q^{2j} \sum_{a,\alpha=1}^{n} q^{2(2n-a-\alpha)} (z_a^\alpha)^* \zeta_a^\alpha + \ldots \right)^{-1} \\ &\times \quad q^{2n^3} (\tau^*\tau)^{-n} (tt^*)^{-n} \prod_{j=0}^{n-1} \left(1 - q^{2j} \sum_{a,\alpha=1}^{n} z_a^\alpha (\zeta_a^\alpha)^* + \ldots \right)^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} P = \mathrm{const}(q,n) \prod_{j=0}^{n-1} &\left(1 - q^{-2j} \sum_{a,\alpha=1}^{n} q^{2(2n-a-\alpha)} (z_a^\alpha)^* \zeta_a^\alpha + \ldots \right)^{-1} \\ \times &\left(1 - \sum_{a,\alpha=1}^{n} (z_a^\alpha)^* z_a^\alpha + \ldots \right)^{n} \cdot \prod_{j=0}^{n-1} \left(1 - q^{2j} \sum_{a,\alpha=1}^{n} z_a^\alpha (\zeta_a^\alpha)^* + \ldots \right)^{-1}, \end{aligned} \tag{6.5}$$

where three dots inside every parentheses denote the contribution of the terms whose degree is above two, and the multipliers in the products are written in order of decreasing of index j from left to right.

Now (6.1) follows from (6.5). ■

R e m a r k 6.8. A formal passage to a limit as $q \to 1$ in (6.1) leads to

$$p_{11} = \mathrm{const}(n) \sum_{a,b,\alpha,\beta=1}^{n} \left(n\, \zeta_a^\alpha \overline{\zeta_b^\beta} - \delta_{ab} \delta^{\alpha\beta} \right) \overline{z_a^\alpha} z_b^\beta.$$

This relation is well known (with $\mathrm{const}(n) = n$); see, for example, [15, p. 597] and is a consequence of (1.1).

R e m a r k 6.9. Using the definition of P, (6.5) and the definition of integral over the Shilov boundary, one can compute $\mathrm{const}(q,n)$ explicitly. But we do not need this value on the way to Hua equations.

7. Deducing the Hua Equations

Now we are about to produce a quantum analog of (1.3).

It follows from Lemma 6.7 and the definition of multiplication in the algebra of kernels that

$$\left.\frac{\partial^2 P}{\partial z_b^\beta \partial (z_a^\alpha)^*}\right|_{\mathbf{z}=0} = \mathrm{const}(q,n)\cdot\left(\frac{1-q^{-2n}}{1-q^{-2}} q^{2(2n-a-\alpha)}\zeta_a^\alpha(\zeta_b^\beta)^* - \delta_{ab}\delta^{\alpha\beta}\right).$$

Set here $a=b=c$ to get

$$\left.\frac{\partial^2 P}{\partial z_c^\beta \partial (z_c^\alpha)^*}\right|_{\mathbf{z}=0} = \mathrm{const}(q,n)\cdot\left(\frac{1-q^{-2n}}{1-q^{-2}} q^{2(2n-c-\alpha)}\ \zeta_c^\alpha(\zeta_c^\beta)^* - \delta^{\alpha\beta}\right).$$

On the other hand, the generators of the function algebra on the Shilov boundary are a subject to the relation

$$\sum_{c=1}^{n}\zeta_c^\alpha(\zeta_c^\beta)^* = q^{-2n+\alpha+\beta}\delta^{\alpha\beta}, \qquad \alpha,\beta=1,2,\dots,n,$$

see (2.2). Hence

$$\begin{aligned}
&\frac{1}{\mathrm{const}(q,n)}\sum_{c=1}^{n} q^{2c}\left.\frac{\partial^2 P}{\partial z_c^\beta \partial (z_c^\alpha)^*}\right|_{\mathbf{z}=0}\\
&= \frac{1-q^{-2n}}{1-q^{-2}} q^{2(2n-\alpha)}\sum_{c=1}^{n}\zeta_c^\alpha(\zeta_c^\beta)^* - q^2\frac{1-q^{2n}}{1-q^2}\delta^{\alpha\beta}\\
&= \frac{1-q^{-2n}}{1-q^{-2}} q^{2(2n-\alpha)} q^{-2n+\alpha+\beta}\delta^{\alpha\beta} - q^2\frac{1-q^{2n}}{1-q^2}\delta^{\alpha\beta} = 0,
\end{aligned}$$

so that the following statement is valid.

Lemma 7.1. *If $u\in\mathcal{D}(\mathbb{D})'_q$ is a Poisson integral on the quantum $n\times n$-matrix ball, then*

$$\sum_{c=1}^{n} q^{2c}\left.\frac{\partial^2 u}{\partial z_c^\beta \partial (z_c^\alpha)^*}\right|_{\mathbf{z}=0} = 0$$

for all $\alpha,\beta=1,2,\dots,n$.

Since the subspace of Poisson integrals

$$u = \int\limits_{S(\mathbb{D})_q} P(\mathbf{z}, \zeta) f(\zeta) d\nu(\zeta), \qquad f \in \mathbb{C}[S(\mathbb{D})]_q,$$

is a $U_q\mathfrak{sl}_{2n}$-submodule, the above lemma implies Theorem 3.1.

It is known [9] that in the classical case of $q = 1$ the Poisson kernel P is a solution of one more equation system

$$\sum_{\gamma=1}^{n} \frac{\partial^2 u(g \cdot \mathbf{z})}{\partial z_b^\gamma \partial \overline{z}_a^\gamma}\Bigg|_{\mathbf{z}=0} = 0, \qquad g \in SU_{n,n}, \quad a, b \in \{1, 2, \ldots, n\}.$$

An argument similar to the above allows one to obtain a q-analog of this result.

Proposition 7.2. *If $u \in \mathcal{D}(\mathbb{D})'_q$ is a Poisson integral on the quantum $n \times n$-matrix ball, then*

$$\sum_{\gamma=1}^{n} q^{2\gamma} \frac{\partial^2 (\xi u)}{\partial z_a^\gamma \partial (z_b^\gamma)^*}\Bigg|_{\mathbf{z}=0} = 0$$

for all $\xi \in U_q\mathfrak{sl}_{2n}$, $a, b = 1, 2, \ldots, n$.

8. Addendum. Hint to a More General Case

Turn from the special case of $n \times n$-matrix ball to a more general case of bounded symmetric domain of tube type. We intend to introduce the Hua operator, which can be used in order to rewrite the Hua equations in a more habitual form (see [15, p. 593]).

Let $\mathfrak{g}$ be a simple complex Lie algebra, and $(a_{ij})_{i,j=1,2,\ldots,l}$ be an associated Cartan matrix. We refer to a well known (see [14]) description of universal enveloping algebra $U\mathfrak{g}$ in terms of its generators e_i, f_i, h_i, $i = 1, 2, \ldots, l$, and standard relations. Consider also the linear span $\mathfrak{h}$ of the set $\{h_i | i = 1, 2, \ldots, l\}$ (a Cartan subalgebra), and the simple roots $\{\alpha_i \in \mathfrak{h}^* | i = 1, 2, \ldots, l\}$ given by $\alpha_i(h_j) = a_{ji}$. Let δ be the maximal root, $\delta = \sum\limits_{i=1}^{l} c_i \alpha_i$. Assume that it is possible to choose $l_0 \in \{1, 2, \ldots, l\}$ so that $c_{l_0} = 1$. Fix an element $h_0 \in \mathfrak{h}$ with the following properties:

$$\alpha_i(h_0) = 0, \quad i \neq l_0; \qquad \alpha_{l_0}(h_0) = 2.$$

In this case the Lie algebra $\mathfrak{g}$ is equipped with the $\mathbb{Z}$-grading as follows:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}, \quad \mathfrak{g}_j = \{\xi \in \mathfrak{g} | \, [h_0, \xi] = 2j\xi\} \tag{8.1}$$

(that is, $\mathfrak{g}_i = \{0\}$ for all i with $|i| > 1$).

Denote by $\mathfrak{k} \subset \mathfrak{g}$ the Lie subalgebra generated by

$$e_i, f_i, \quad i \neq l_0; \qquad h_i, \quad i = 1, 2, \ldots, l.$$

If (8.1) is true, then $\mathfrak{g}_0 = \mathfrak{k}$, and the pair $(\mathfrak{g}, \mathfrak{k})$ is called the Hermitian symmetric pair. In what follows, we obey the conventions of the theory of Hermitian symmetric spaces, where the notation $\mathfrak{p}^{\pm}$ is used instead of $\mathfrak{g}_{\pm 1}$ as in (8.1).

Harish-Chandra introduced a standard realization of an irreducible bounded symmetric domain $\mathbb{D}$, considered up to biholomorphic isomorphisms, as a unit ball in the normed space $\mathfrak{p}^-$ [7, 44]. Let G be a simply connected complex linear algebraic group with $\mathrm{Lie}(G) = \mathfrak{g}$, and $K \subset G$ such connected linear algebraic subgroup that $\mathrm{Lie}(K) = \mathfrak{k}$. In this context one has the well-known Harish-Chandra embedding

$$i : K \backslash G \hookrightarrow \mathfrak{p}^-.$$

Let W be the Weyl group of the root system R of $\mathfrak{g}$, and $w_0 \in W$ be the longest element. The irreducible bounded symmetric domain $\mathbb{D}$ associated to the pair $(\mathfrak{g}, \mathfrak{k})$ is a tube type domain if and only if $\varpi_{l_0} = -w_0 \varpi_{l_0}$.

Let $U_q\mathfrak{g}$ be the quantum universal enveloping algebra of $\mathfrak{g}$. Recall that it is a Hopf algebra and it can be described in terms of its generators E_i, F_i, $K_i^{\pm 1}$, $i = 1, 2, \ldots, l$, and the standard Drinfeld–Jimbo relations.

Introduce quantum analogs of invariant differential operators to use them for producing the Hua operator.

Let V be a finite-dimensional weight $U_q\mathfrak{g}$-module. In an obvious way, $V \otimes_{U_q\mathfrak{k}} U_q\mathfrak{g}$ is equipped with a structure of right $U_q\mathfrak{g}$-module. It is easy to see that in the category of left $U_q\mathfrak{g}$-modules

$$\begin{aligned} &\mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g}, V) \cong (V^* \otimes_{U_q\mathfrak{k}} U_q\mathfrak{g})^*, \\ &f \mapsto \widetilde{f}, \qquad \widetilde{f}(l \otimes \xi) = l(f(\xi)), \qquad l \in V^*, \xi \in U_q\mathfrak{g}. \end{aligned}$$

The vector space $\mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g}, V)$ is a quantum analog of the space of sections of homogeneous vector bundle on the homogeneous space $K \backslash G$. Suppose we are given two finite-dimensional weight $U_q\mathfrak{g}$-modules V_1, V_2 and a morphism of right $U_q\mathfrak{g}$-modules

$$A : V_2^* \otimes_{U_q\mathfrak{k}} U_q\mathfrak{g} \to V_1^* \otimes_{U_q\mathfrak{k}} U_q\mathfrak{g}.$$

To the latter morphism associate the adjoint linear map

$$A^* : \mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g}, V_1) \to \mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g}, V_2),$$

which is also a morphism of $U_q\mathfrak{g}$-modules. These dual operators are treated as quantum analogs of invariant differential operators.

Thus the invariant differential operators are in one-to-one correspondence with the elements of the space

$$\begin{aligned}&\mathrm{Hom}_{U_q\mathfrak{g}}(V_2^*\otimes_{U_q\mathfrak{k}}U_q\mathfrak{g},V_1^*\otimes_{U_q\mathfrak{k}}U_q\mathfrak{g})\cong\mathrm{Hom}_{U_q\mathfrak{k}}(V_2^*,V_1^*\otimes_{U_q\mathfrak{k}}U_q\mathfrak{g}),\\ &f\mapsto\widetilde{f},\qquad \widetilde{f}(l)=f(l\otimes 1),\qquad l\in V_2^*.\end{aligned}$$

Turn to a construction of the Hua operator. Set $\mathfrak{p}^+=U_q\mathfrak{k}E_{l_0}$, $\mathfrak{p}^-=U_q\mathfrak{k}(K_{l_0}F_{l_0})$, both are finite-dimensional weight $U_q\mathfrak{k}$-modules [13]. The morphisms of right $U_q\mathfrak{k}$-modules

$$\begin{aligned}&\mathfrak{p}^+\to\mathbb{C}\otimes_{U_q\mathfrak{k}}U_q\mathfrak{g}, &&E_{l_0}\mapsto 1\otimes E_{l_0},\\ &\mathfrak{p}^-\to\mathbb{C}\otimes_{U_q\mathfrak{k}}U_q\mathfrak{g}, &&K_{l_0}F_{l_0}\mapsto 1\otimes K_{l_0}F_{l_0}\end{aligned}$$

determine the invariant linear differential operators

$$\mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g},\mathbb{C})\to\mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g},\mathfrak{p}^\pm).$$

Recall that $U_q\mathfrak{k}$-modules form a tensor category, and that the comultiplication $\triangle:U_q\mathfrak{g}\to U_q\mathfrak{g}\otimes U_q\mathfrak{g}$ is a morphism of this category.

Consider the morphisms of $U_q\mathfrak{g}$-modules

$$\mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g},\mathbb{C})\to\mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g}\otimes U_q\mathfrak{g},\mathfrak{p}^+\otimes\mathfrak{p}^-),\tag{8.2}$$

$$\mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g}\otimes U_q\mathfrak{g},\mathfrak{p}^+\otimes\mathfrak{p}^-)\to\mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g},\mathfrak{p}^+\otimes\mathfrak{p}^-).\tag{8.3}$$

Let $\mathfrak{k}_q$ be the finite-dimensional weight $U_q\mathfrak{k}$-module with the same weights and weight multiplicities as the $U\mathfrak{k}$-module $\mathfrak{k}$. There exists a unique $U_q\mathfrak{k}$-submodule $\mathcal{H}_q\subset\mathfrak{p}^+\otimes\mathfrak{p}^-$ such that $(\mathfrak{p}^+\otimes\mathfrak{p}^-)/\mathcal{H}_q\approx\mathfrak{k}_q$ (because a similar fact is well known in the classical case of $q=1$ (see [2, Prop. 4.2])). Fix a surjective morphism $\mathfrak{p}^+\otimes\mathfrak{p}^-\to\mathfrak{k}_q$ and consider the associated invariant 'formal' differential operator

$$\mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g},\mathfrak{p}^+\otimes\mathfrak{p}^-)\to\mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g},\mathfrak{k}_q).\tag{8.4}$$

Denote by $\mathcal{D}_q$ the composition of the maps (8.2), (8.3), and (8.4). By definition, $\mathcal{D}_q$ is an invariant differential operator.

Recall the standard definitions of quantum analogs for the algebras of regular functions on the group G and on the homogeneous space $K\backslash G$. Denote by $\mathbb{C}[G]_q\subset(U_q\mathfrak{g})^*$ the Hopf algebra of all matrix elements of weight finite dimensional $U_q\mathfrak{g}$-modules. $\mathbb{C}[G]_q$ is equipped with the structure of $U_q^{\mathrm{op}}\mathfrak{g}\otimes U_q\mathfrak{g}$-module algebra via quantum analogs of the standard right and left regular actions $(\xi'\otimes\xi'')f=\mathcal{L}_{\mathrm{reg}}(\xi')\mathcal{R}_{\mathrm{reg}}(\xi'')f$, where

$$\mathcal{L}_{\mathrm{reg}}(\xi'f)(\eta)=f(\xi'\eta),\quad \mathcal{R}_{\mathrm{reg}}(\xi''f)(\eta)=f(\eta\xi''),\ \xi',\xi'',\eta\in U_q\mathfrak{g},\ f\in\mathbb{C}[G]_q.$$

($U_q^{\mathrm{op}}\mathfrak{g}$ is the Hopf algebra with the opposite multiplication.) $\mathbb{C}[G]_q$ is called the algebra of regular functions on the quantum group G.

Introduce the notation

$$\mathbb{C}[K\backslash G]_q = \{\xi \in \mathbb{C}[G]_q |\, \mathcal{L}_{\mathrm{reg}}(\eta)\xi = 0, \quad \eta \in U_q\mathfrak{k}\}.$$

This Hopf subalgebra is a quantum analog for the algebra of regular functions on the homogeneous space $K\backslash G$. It is easy to prove that $\mathbb{C}[K\backslash G]_q \subset \mathrm{Hom}_{U_q\mathfrak{k}}(U_q\mathfrak{g}, \mathbb{C})$, so one can consider the restriction of $\mathcal{D}_q$ to $\mathbb{C}[K\backslash G]_q$.

Now introduce a localization $\mathbb{C}[K\backslash G]_{q,x}$ of the algebra $\mathbb{C}[K\backslash G]_q$ with respect to the Ore set $x^{\mathbb{Z}_+}$. It can be proved that the extension of $\mathcal{D}_q$ up to $\mathbb{C}[K\backslash G]_{q,x}$ is well-defined. Pass from $\mathbb{C}[K\backslash G]_{q,x}$ to $\mathrm{Pol}(\mathfrak{p}^-)_q$ (via the Harish-Chandra embedding $\mathcal{I}$, see Section 5 for the special case) and $\mathcal{D}(\mathbb{D})'_q$ to get a q-analog for the Hua operator.

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