Journal of Mathematical Physics, Analysis, Geometry
2009, vol. 5, No. 3, pp. 245–274

Bulk Universality for Unitary Matrix Models

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Received April 25, 2008

A proof of universality in the bulk of spectrum of unitary matrix models, assuming that the potential is globally C^2 and locally C^3 function (see Theorem 1.2), is given. The proof is based on the determinant formulas for correlation functions in terms of polynomials orthogonal on the unit circle. The *sin*-kernel is obtained as a unique solution of a certain nonlinear integro-differential equation without using asymptotics of orthogonal polynomials.

Key words: unitary matrix models, local eigenvalue statistics, universality.

Mathematics Subject Classification 2000: 15A52, 15A57.

1. Introduction

In the paper we study a class of random matrix ensembles known as unitary matrix models. These models are defined by the probability law

$$p_n(U)\,d\mu_n(U) = Z_{n,2}^{-1}\exp\left\{-n\mathrm{Tr}V\left(\frac{U+U^*}{2}\right)\right\}d\mu_n(U), \tag{1.1}$$

where $U = \{U_{jk}\}_{j,k=1}^n$ is an $n\times n$ unitary matrix, $\mu_n(U)$ is the Haar measure on the group $U(n)$, $Z_{n,2}$ is the normalization constant and $V : [-1,1] \to \mathbb{R}^+$ is a continuous function called the potential of the model. Denote $e^{i\lambda_j}$ the eigenvalues of unitary matrix U. The joint probability density of λ_j, corresponding to (1.1), is given by (see [1])

$$p_n(\lambda_1,\dots,\lambda_n) = \frac{1}{Z_n}\prod_{1\le j<k\le n}\left|e^{i\lambda_j}-e^{i\lambda_k}\right|^2\exp\left\{-n\sum_{j=1}^n V(\cos\lambda_j)\right\}. \tag{1.2}$$

To simplify notations, below we will write $V(x)$ instead of $V(\cos x)$. Normalized Counting Measure of eigenvalues (NCM) is given by

$$N_n(\Delta) = n^{-1}\sharp\left\{\lambda_l^{(n)} \in \Delta,\, l = 1, \ldots, n\right\}, \quad \Delta \subset [-\pi, \pi]. \tag{1.3}$$

The random matrix theory deals with several asymptotic regimes of the eigenvalue distribution. The global regime is centered around weak convergence of NCM (1.3). Global regime for unitary matrix models was studied in [2]. We will use the main result of [2]:

Theorem 1.1. *Assume that the potential V of the model* (1.1) *is a $C^2(-\pi,\pi)$ function. Then:*

- *there exists a measure $N \in \mathcal{M}_1([-\pi,\pi])$ with a compact support σ such that NCM N_n converges in probability to N;*
- *N has a bounded density ρ;*
- *denote $\rho_n := p_1^{(n)}$ the first marginal density, then for any $\phi \in H^1(-\pi,\pi)$*

$$\left|\int \phi(\lambda)\rho_n(\lambda)\,d\lambda - \int \phi(\lambda)\rho(\lambda)\,d\lambda\right| \le C\,\|\phi\|_2^{1/2}\,\|\phi'\|_2^{1/2}\,n^{-1/2}\ln^{1/2} n, \tag{1.4}$$

where $\|\cdot\|_2$ denotes L_2 norm on $[-\pi,\pi]$

One of the main topics of local regime is a universality of local eigenvalue statistics. Let

$$p_l^{(n)}(\lambda_1,\ldots,\lambda_l) = \int p_n(\lambda_1,\ldots,\lambda_l,\lambda_{l+1},\ldots,\lambda_n)\,d\lambda_{l+1}\ldots d\lambda_n \tag{1.5}$$

be the l-th marginal density of p_n.

Definition 1.1. *We call by the bulk of the spectrum the set*

$$\{\lambda \in \sigma \,:\, \rho(\lambda) > 0\}, \tag{1.6}$$

where ρ is defined in Theorem 1.1.

The main result of the paper is the proof of universality conjecture in the bulk of spectrum

$$\lim_{n\to\infty}[n\rho_n(\lambda)]^{-l}\,p_l^{(n)}\left(\lambda + \frac{x_1}{n\rho_n(\lambda)}, \ldots, \lambda + \frac{x_l}{n\rho_n(\lambda)}\right) = \det\{S(x_j - x_k)\}_{j,k=1}^{l}, \tag{1.7}$$

where

$$S(x) = \frac{\sin \pi x}{\pi x}. \tag{1.8}$$

By (1.7), the limiting local distributions of eigenvalues do not depend on potential V in (1.1), modulo some weak condition (see Theorem 1.2). The conjecture of universality of all correlation functions was suggested by F.J. Dyson (see [3]) in the early 60s who proved (1.7)–(1.8) for $V(x) = 0$. First rigorous proofs for Hermitian matrix models with nonquadratic V appeared only in the 90s. The case of general V which is locally C^3 function was studied in [4]. The case of real analytic potential V was studied in [5], where the asymptotics of orthogonal polynomials were obtained. For unitary matrix models the bulk universality was proved for $V = 0$ (see [3]) and in the case of a linear V (see [6]).

To prove the main result we need some properties of the polynomials orthogonal with respect to varying weight on the unit circle. Consider a system of functions $\left\{e^{ik\lambda}\right\}_{k=0}^{\infty}$ and use for them the Gram–Schmidt procedure in $L_2\left([-\pi,\pi], e^{-nV(\lambda)}\right)$. For any n we get the system of functions $\left\{P_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$ which are orthogonal and normalized in $L_2\left([-\pi,\pi], e^{-nV(\lambda)}\right)$. Since V is even, it is easy to see that all coefficients of these functions are real. Denote

$$\psi_k^{(n)}(\lambda) = P_k^{(n)}(\lambda)\, e^{-nV(\lambda)/2}. \tag{1.9}$$

Then we obtain the orthogonal in $L_2(-\pi,\pi)$ functions

$$\int_{-\pi}^{\pi} \psi_k^{(n)}(\lambda)\, \overline{\psi_l^{(n)}(\lambda)}\, d\lambda = \delta_{kl}. \tag{1.10}$$

The reproducing kernel of the system (1.9) is given by

$$K_n(\lambda,\mu) = \sum_{j=0}^{n-1} \psi_l^{(n)}(\lambda)\, \overline{\psi_l^{(n)}(\mu)}. \tag{1.11}$$

From (1.10) we obtain that the reproducing kernel satisfies the relation

$$\int_{-\pi}^{\pi} K_n(\lambda,\nu)\, K_n(\nu,\mu)\, d\nu = K_n(\lambda,\mu), \tag{1.12}$$

and from the Cauchy inequality we have

$$|K_n(\lambda,\mu)|^2 \le K_n(\lambda,\lambda)\, K_n(\mu,\mu). \tag{1.13}$$

We also use below the determinant form of the marginal densities (1.5) (see [1])

$$p_l^{(n)}(\lambda_1,\dots,\lambda_l)=\frac{(n-l)!}{n!}\det\|K_n(\lambda_j,\lambda_k)\|_{j,k=1}^{l}. \tag{1.14}$$

In particular,

$$\rho_n(\lambda)=n^{-1}K_n(\lambda,\lambda), \tag{1.15}$$

$$p_2^{(n)}(\lambda,\mu)=\frac{K_n(\lambda,\lambda)K_n(\mu,\mu)-|K_n(\lambda,\mu)|^2}{n(n-1)}. \tag{1.16}$$

The main result of the paper is

Theorem 1.2. *Assume that $V(\lambda)$ is a $C^2(-\pi,\pi)$ function, and there exists an interval $(a,b)\subset\sigma$ such that*

$$\sup_{\lambda\in(a,b)}|V'''(\lambda)|\le C_1,\ \rho(\lambda)\ge C_2,\ \lambda\in(a,b). \tag{1.17}$$

Then for any $d>0$ and $\lambda_0\in[a+d,b-d]$ for K_n defined in (1.11) we have

$$\lim_{n\to\infty}[K_n(\lambda_0,\lambda_0)]^{-1}K_n\left(\lambda_0+\frac{x}{K_n(\lambda_0,\lambda_0)},\lambda_0+\frac{y}{K_n(\lambda_0,\lambda_0)}\right)=e^{i(x-y)/2\rho(\lambda_0)}\frac{\sin\pi(x-y)}{\pi(x-y)} \tag{1.18}$$

uniformly in (x,y), varying on a compact set of $\mathbb{R}^2$.

R e m a r k 1.3. It is easy to see that the universality conjecture (1.7) follows from Theorem 1.2 by (1.14).

The method of the proof is a version of the one used in [4]. An important part of the proof is a uniform convergence of ρ_n to ρ in a neighborhood of λ_0:

Theorem 1.4. *Under the assumptions of Theorem 1.2 for any $d>0$ there exists $C(d)>0$ such that for any $\lambda\in[a+d,b-d]$*

$$|\rho_n(\lambda)-\rho(\lambda)|\le C(d)\,n^{-2/9}. \tag{1.19}$$

2. Proof of Basic Results

P r o o f of Theorem 1.4. We will use some facts from the integral transformations theory (see [7]).

Definition 2.1. *Assume that $g(\lambda)$ is a continuous function on the interval $[-\pi,\pi]$. Then its Germglotz transformation is given by*

$$F[g](z) = \int_{-\pi}^{\pi} \frac{e^{i\lambda}+e^{iz}}{e^{i\lambda}-e^{iz}} g(\lambda)\, d\lambda, \tag{2.1}$$

where $z \in \mathbb{C}\backslash\mathbb{R}$.

The inverse transformation is given by

$$g(\mu) = \frac{1}{2\pi} \lim_{z\to\mu+i0} \Re F[g](z). \tag{2.2}$$

For $z = \mu + i\eta$, $\eta \neq 0$, we set

$$f_n(z) = \int_{-\pi}^{\pi} \frac{e^{i\lambda}+e^{iz}}{e^{i\lambda}-e^{iz}}\, \rho_n(\lambda)\, d\lambda. \tag{2.3}$$

Bellow we will derive a "square" equation for f_n. Denote

$$\mathcal{I}_n(z) = \int_{-\pi}^{\pi} V'(\lambda) \frac{e^{i\lambda}+e^{iz}}{e^{i\lambda}-e^{iz}}\, \rho_n(\lambda)\, d\lambda. \tag{2.4}$$

Integrating by parts in (2.4), from (1.5) we obtain

$$\mathcal{I}_n(z) = \frac{1}{Z_n}\int V'(\lambda_1) \frac{e^{i\lambda_1}+e^{iz}}{e^{i\lambda_1}-e^{iz}} \prod_{j<k} \left|e^{i\lambda_j}-e^{i\lambda_k}\right|^2 \exp\left\{-n\sum_{j=1}^{n} V(\lambda_j)\right\} \prod_{j=1}^{n} d\lambda_j$$

$$= \frac{1}{nZ_n}\int e^{-nV(\lambda_1)} \frac{d}{d\lambda_1}\left(\frac{e^{i\lambda_1}+e^{iz}}{e^{i\lambda_1}-e^{iz}} \prod_{j<k} \left|e^{i\lambda_j}-e^{i\lambda_k}\right|^2 \exp\left\{-n\sum_{j=2}^{n} V(\lambda_j)\right\}\right) \prod_{j=1}^{n} d\lambda_j.$$

The integrated term equals 0, because all functions here are 2π -periodic. After differentiation we have the sum of n terms under integral sign. Denote

$$I_0(z) = \frac{1}{nZ_n}\int \frac{d}{d\lambda_1}\left(\frac{e^{i\lambda_1}+e^{iz}}{e^{i\lambda_1}-e^{iz}}\right) \prod_{j<k} \left|e^{i\lambda_j}-e^{i\lambda_k}\right|^2 \exp\left\{-n\sum_{j=1}^{n} V(\lambda_j)\right\} \prod_{j=1}^{n} d\lambda_j,$$

$$I_m(z) = \frac{1}{nZ_n} \int \frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \prod_{2 \le j < k \le n} \left| e^{i\lambda_j} - e^{i\lambda_k} \right|^2 \frac{d}{d\lambda_1} \left| e^{i\lambda_1} - e^{i\lambda_m} \right|^2$$
$$\times \prod_{k \neq m} \left| e^{i\lambda_1} - e^{i\lambda_k} \right|^2 \exp\left\{ -n \sum_{j=1}^{n} V(\lambda_j) \right\} \prod_{j=1}^{n} d\lambda_j, \quad m = \overline{2,n}.$$

From symmetry with respect to λ_j we obtain that all $I_m(z)$, except $I_0(z)$, are equal, hence

$$\mathcal{I}_n(z) = I_0(z) + (n-1) I_2(z).$$

$$I_0(z) = \frac{1}{n} \int_{-\pi}^{\pi} \frac{d}{d\lambda_1} \left(\frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \right) \rho_n(\lambda_1)\, d\lambda_1$$
$$= -\frac{2i}{n} \int_{-\pi}^{\pi} \frac{e^{i\lambda_1} e^{iz}}{\left(e^{i\lambda_1} - e^{iz}\right)^2} \rho_n(\lambda_1)\, d\lambda_1 = -\frac{i}{2n} \int_{-\pi}^{\pi} \left(\frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \right)^2 \rho_n(\lambda_1)\, d\lambda_1 + \frac{i}{2n}.$$

To transform I_2, we use the symmetry of $p_2^{(n)}$ ($p_2^{(n)}(\lambda_1, \lambda_2) = p_2^{(n)}(\lambda_2, \lambda_1)$).

$$I_2(z) = \frac{1}{n} \int \frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \frac{\frac{d}{d\lambda_1} \left| e^{i\lambda_1} - e^{i\lambda_2} \right|^2}{\left| e^{i\lambda_1} - e^{i\lambda_2} \right|^2} p_2^{(n)}(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2$$
$$= \frac{i}{n} \int \frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \frac{e^{i\lambda_1} + e^{i\lambda_2}}{e^{i\lambda_1} - e^{i\lambda_2}} p_2^{(n)}(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2$$
$$= \frac{i}{2n} \int \left(\frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} - \frac{e^{i\lambda_2} + e^{iz}}{e^{i\lambda_2} - e^{iz}} \right) \frac{e^{i\lambda_1} + e^{i\lambda_2}}{e^{i\lambda_1} - e^{i\lambda_2}} p_2^{(n)}(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2$$
$$= -\frac{i}{2n} \int \frac{2\left(e^{i\lambda_1} + e^{i\lambda_2}\right) e^{iz}}{\left(e^{i\lambda_1} - e^{iz}\right)\left(e^{i\lambda_2} - e^{iz}\right)} p_2^{(n)}(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2$$
$$= \frac{i}{2n} - \frac{i}{2n} \int \frac{e^{i\lambda_1} + e^{iz}}{e^{i\lambda_1} - e^{iz}} \frac{e^{i\lambda_2} + e^{iz}}{e^{i\lambda_2} - e^{iz}} p_2^{(n)}(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2.$$

Therefore, from (1.5) and (1.14) we obtain

$$\mathcal{I}_n(z) = \frac{i}{2} - \frac{i}{2} f_n^2(z) - \frac{i}{n^2} \int \left| K_n(\lambda_1, \lambda_2) \right|^2 \frac{\left(e^{i\lambda_1} - e^{i\lambda_2}\right)^2 e^{2iz}}{\left(e^{i\lambda_1} - e^{iz}\right)^2 \left(e^{i\lambda_2} - e^{iz}\right)^2} d\lambda_1 d\lambda_2. \tag{2.5}$$

On the other hand, denoting

$$Q_n(z) = \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} \left(V'(\lambda) - V'(\mu)\right) \rho_n(\lambda)\, d\lambda, \tag{2.6}$$

for $z = \mu + i\eta$, from (2.3) we get

$$\mathcal{I}_n(z) = Q_n(z) + V'(\mu) f_n(z). \tag{2.7}$$

Finally, from (2.5) and (2.7) we obtain the "square" equation

$$f_n^2(z) - 2iV'(\mu) f_n(z) - 2iQ_n(z) - 1 = -\frac{2}{n^2} G_n(z), \tag{2.8}$$

with

$$G_n(z) = \int |K_n(\lambda_1, \lambda_2)|^2 \frac{\left(e^{i\lambda_1} - e^{i\lambda_2}\right)^2 e^{2iz}}{\left(e^{i\lambda_1} - e^{iz}\right)^2 \left(e^{i\lambda_2} - e^{iz}\right)^2} d\lambda_1 d\lambda_2.$$

To proceed further we have to prove the following properties of the reproducing kernel K_n.

Lemma 2.1. *Let $K_n(\lambda, \mu)$ be defined by* (1.11)*. Then under the conditions of Theorem 1.2 for any $\delta > 0$*

$$\left| \int \left(e^{i\lambda} - e^{i\mu}\right) |K_n(\lambda, \mu)|^2 \, d\mu \right| \le \frac{1}{2} \left[\left|\psi_{n-1}^{(n)}(\lambda)\right|^2 + \left|\psi_n^{(n)}(\lambda)\right|^2 \right], \tag{2.9}$$

$$\int \left|e^{i\lambda} - e^{i\mu}\right|^2 |K_n(\lambda, \mu)|^2 \, d\mu \le \left[\left|\psi_{n-1}^{(n)}(\lambda)\right|^2 + \left|\psi_n^{(n)}(\lambda)\right|^2 \right], \tag{2.10}$$

$$\int \left|e^{i\lambda} - e^{i\mu}\right|^2 |K_n(\lambda, \mu)|^2 \, d\lambda d\mu \le 2, \tag{2.11}$$

$$\int\limits_{\left|e^{i\lambda} - e^{i\mu}\right| > \delta} |K_n(\lambda, \mu)|^2 \, d\mu \le \delta^{-2} \left[\left|\psi_{n-1}^{(n)}(\lambda)\right|^2 + \left|\psi_n^{(n)}(\lambda)\right|^2 \right], \tag{2.12}$$

$$\int\limits_{\left|e^{i\lambda} - e^{i\mu}\right| > \delta} |K_n(\lambda, \mu)|^2 \, d\lambda d\mu \le 2\delta^{-2}. \tag{2.13}$$

It is easy to see that $\left|e^{i\lambda} - e^{iz}\right| > C|\eta|$ if $|\eta| < 1$ for some $C > 0$. Hence, from (2.11) and (2.8) we derive

$$f_n^2(z) - 2iV'(\mu) f_n(z) - 2iQ_n(z) - 1 = O\left(n^{-2}\eta^{-4}\right). \tag{2.14}$$

Lemma 2.2. *Under the conditions of Theorem 1.2 for any $d > 0$ and $\lambda \in [a + d, b - d]$*

$$\rho_n(\lambda) \le C, \tag{2.15}$$

$$\left| \frac{d\rho_n(\lambda)}{d\lambda} \right| \le C_1 \left(\left|\psi_n^{(n)}(\lambda)\right|^2 + \left|\psi_{n-1}^{(n)}(\lambda)\right|^2 \right) + C_2. \tag{2.16}$$

From the conditions of Theorem 1.2, we obtain that $V''(\lambda)$ is bounded on the interval $[a,b]$. Hence, for $\mu \in [a+d, b-d]$ and sufficiently small η we have

$$|Q_n(\mu+i\eta) - Q_n(\mu)| \le \left|e^{-\eta}-1\right| \int\limits_{-\pi}^{\pi} \frac{|V'(\lambda) - V'(\mu)|\,\rho_n(\mu)}{|e^{i\lambda}-e^{i\mu}|\,|e^{i\lambda}-e^{iz}|}\,d\lambda$$

$$\le C\eta\left(\int\limits_{|\lambda-\mu|<d/2} \frac{d\lambda}{\left|(\lambda-\mu)^2+\eta^2\right|^{1/2}} + \int\limits_{|\lambda-\mu|>d/2} \frac{\rho_n(\lambda)\,d\lambda}{\left|(\lambda-\mu)^2+\eta^2\right|^{1/2}|\lambda-\mu|}\right)$$

$$\le C\eta\ln^{-1}\eta + C\eta d^{-2} \le C\eta\ln^{-1}\eta. \quad (2.17)$$

Besides, applying (1.4), for $\phi(\lambda) = \dfrac{e^{i\lambda}+e^{i\mu}}{e^{i\lambda}-e^{i\mu}}\left(V'(\lambda)-V'(\mu)\right)$ we get

$$Q_n(\mu) = Q(\mu) + O\left(n^{-1/2}\ln^{1/2} n\right), \quad (2.18)$$

where

$$Q(\mu) = \int\limits_{-\pi}^{\pi} \frac{e^{i\lambda}+e^{i\mu}}{e^{i\lambda}-e^{i\mu}}\left(V'(\lambda)-V'(\mu)\right)\rho(\lambda)\,d\lambda. \quad (2.19)$$

Combining (2.17) and (2.18), we find

$$Q_n(\mu+i\eta) = Q(\mu) + O\left(\eta\ln^{-1}\eta\right) + O\left(n^{-1/2}\ln^{1/2} n\right). \quad (2.20)$$

From (2.20) and (2.14) for $z = \mu + in^{-4/9}$ we have

$$f_n^2(z) - 2iV'(\mu)f_n(z) - 2iQ(\mu) - 1 = O(n^{-2/9}). \quad (2.21)$$

Lemma 2.3.

$$\rho(\mu) = \frac{1}{2\pi}\sqrt{2iQ(\mu) + 1 - (V'(\mu))^2}. \quad (2.22)$$

Lemma 2.3 and the equation (2.21) imply that for $z = \mu + in^{-4/9}$

$$\frac{1}{2\pi}\Re f_n(z) = \rho(\mu) + O\left(n^{-2/9}\right)\rho^{-1}(\mu). \quad (2.23)$$

Lemma 2.4. *For $d>0$, $k = n-1, n$ and $\mu \in [a+d, b-d]$*

$$\int\limits_{|\lambda-\mu|<n^{-1/4}} \left|\psi_k^{(n)}(\lambda)\right|^2 d\lambda \le Cn^{-1/4}, \quad (2.24)$$

$$\left|\psi_k^{(n)}(\lambda)\right|^2 \le Cn^{7/8},\ |\mu-\lambda| \le n^{-1/4}. \quad (2.25)$$

Taking into account (2.23), to prove Theorem 1.4 it is enough to show that $\frac{1}{2\pi}\Re f_n(z) = \rho_n(\mu) + O\left(n^{-2/9}\right)$. We use an evident relation

$$\Re\frac{e^{i\lambda}+e^{iz}}{e^{i\lambda}-e^{iz}} = \frac{\sinh\eta}{\cosh\eta-\cos(\lambda-\mu)} = \frac{d}{d\lambda}2\arctan\left(\tan\left(\frac{\lambda-\mu}{2}\right)\coth\left(\frac{\eta}{2}\right)\right).$$

Combining the relation $\frac{1}{2\pi}\int\Re\frac{e^{i\lambda}+e^{iz}}{e^{i\lambda}-e^{iz}}d\lambda = 1$ with (2.15), we obtain

$$\left|\frac{1}{2\pi}f_n(z)-\rho_n(\mu)\right|$$
$$= (2\pi)^{-1}\left|\left(\int\limits_{|\mu-\lambda|\le\eta^{1/2}} + \int\limits_{\eta^{1/2}\le|\mu-\lambda|\le d/2} + \int\limits_{|\mu-\lambda|\ge d/2}\right)\frac{\sinh\eta}{\cosh\eta-\cos(\lambda-\mu)}\right.$$
$$\left.\times\left(\rho_n(\lambda)-\rho_n(\mu)\right)d\lambda\right|$$
$$\le C\left|\int\limits_{|s|\le\eta^{1/2}}\frac{\sinh\eta}{\cosh\eta-\cos s}\left(\rho_n(s+\mu)-\rho_n(\mu)\right)ds\right| + C\eta^{1/2}+C\eta.$$

Using (2.16) and (2.24), we get finally

$$\left|\frac{1}{2\pi}f_n(z)-\rho_n(\mu)\right| \le C\int\limits_{|s|<\eta^{1/2}}\left|\rho_n'(\mu+s)\right|ds + C\eta^{1/2} \le C\eta^{1/2}.$$

Theorem 1.4 is proved. ■

Now we pass to the proof of Theorem 1.2. We will use the following representation of K_n, which can be derived from the well-known identities of random matrix theory (see [1])

$$\frac{1}{n}K_n(\lambda,\mu) = \frac{1}{n}\sum_{j=0}^{n-1}\psi_l^{(n)}(\lambda)\overline{\psi_l^{(n)}(\mu)} = Q_{n,2}^{-1}e^{-n(V(\lambda)+V(\mu))/2}$$
$$\times\int\prod_{j=2}^{n}\left(e^{i\lambda}-e^{i\lambda_j}\right)\left(e^{-i\mu}-e^{-i\lambda_j}\right)e^{-nV(\lambda_j)}d\lambda_j\prod_{2\le j<k\le n}\left|e^{i\lambda_j}-e^{i\lambda_k}\right|^2, \quad (2.26)$$

where $Q_{n,2} = n!\prod_{j=0}^{n-1}\left|\gamma_l^{(n)}\right|^{-2}$, and $\gamma_l^{(n)}$ is the coefficient in front of $e^{il\lambda}$ in the function $P_l^{(n)}$.

R e m a r k 2.5. Consider the determinant (see (1.2))

$$\det\left\{e^{ik\lambda_j}\right\}_{k,j=0}^{n-1} = e^{i(n-1)\sum\lambda_j/2}\det\left\{e^{i(k-(n-1)/2)\lambda_j}\right\}_{k,j=0}^{n-1}.$$

Taking the complex conjugate, we obtain

$$\begin{aligned}\overline{\det\left\{e^{ik\lambda_j}\right\}_{k,j=0}^{n-1}} &= e^{-i(n-1)\sum\lambda_j/2}\det\left\{e^{-i(k-(n-1)/2)\lambda_j}\right\}_{k,j=0}^{n-1} \\ &= (-1)^{[n/2]}e^{-i(n-1)\sum\lambda_j/2}\det\left\{e^{i(k-(n-1)/2)\lambda_j}\right\}_{k,j=0}^{n-1}.\end{aligned}$$

Thus, from (2.26) we get that the function $e^{-i(n-1)(\lambda-\mu)/2}K_n(\lambda,\mu)$ is real valued.

Now denote

$$\widetilde{\mathcal{K}}_n(x,y) = \frac{1}{n}K_n\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n}\right), \qquad \mathcal{K}_n(x,y) = e^{-i(n-1)(x-y)/2n}\widetilde{\mathcal{K}}_n(x,y). \tag{2.27}$$

From the above we have that $\mathcal{K}_n(x,y)$ is a real-valued and symmetric function. We get from (1.11)–(1.13)

$$\int_{-n\pi}^{n\pi}\mathcal{K}_n(x,z)\,\mathcal{K}_n(z,y)\,dz = \mathcal{K}_n(x,y), \quad |\mathcal{K}_n(x,y)|^2 \le \mathcal{K}_n(x,x)\,\mathcal{K}_n(y,y), \tag{2.28}$$

$$\mathcal{K}_n(x,x) = \rho_n(\lambda_0+x/n) \le C, \quad |\mathcal{K}_n(x,y)| \le C, \quad \text{for } |x|,\,|y| \le nd_0/2 \tag{2.29}$$

Differentiating in (2.26) $\widetilde{\mathcal{K}}_n(x,y)$ with respect to x for $\lambda=\lambda_0+x/n$, $\mu=\mu_0+y/n$, we get

$$\begin{aligned}\frac{\partial}{\partial x}\widetilde{\mathcal{K}}_n(x,y) &= -\frac{1}{2}V'(\lambda)\,\widetilde{\mathcal{K}}_n(x,y) + \frac{n-1}{Q_{n,2}}e^{-n(V(\lambda)+V(\mu))/2} \\ &\times\int\frac{ie^{i\lambda}}{e^{i\lambda}-e^{i\lambda_2}}\prod_{j=2}^{n}\left(e^{i\lambda}-e^{i\lambda_j}\right)\left(e^{-i\mu}-e^{-i\lambda_j}\right)d\lambda_j\prod_{2\le j<k\le n}\left|e^{i\lambda_j}-e^{i\lambda_k}\right|^2 \\ &= -\frac{1}{2}V'(\lambda)\,\widetilde{\mathcal{K}}_n(x,y) \\ &+\frac{i}{n^2}\int_{-\pi}^{\pi}\frac{e^{i\lambda}}{e^{i\lambda}-e^{i\lambda_2}}\left(K_n(\lambda_2,\lambda_2)\,K_n(\lambda,\mu)-K_n(\lambda,\lambda_2)\,K_n(\lambda_2,\mu)\right)d\lambda_2 \\ &= -\frac{1}{2}V'(\lambda)\,\widetilde{\mathcal{K}}_n(x,y)\end{aligned}$$

$$\frac{i}{2n^2}\int\limits_{-\pi}^{\pi}\frac{e^{i\lambda}+e^{i\lambda_2}}{e^{i\lambda}-e^{i\lambda_2}}\left(K_n\left(\lambda_2,\lambda_2\right)K_n\left(\lambda,\mu\right)-K_n\left(\lambda,\lambda_2\right)K_n\left(\lambda_2,\mu\right)\right)d\lambda_2$$
$$+\frac{i(n-1)}{2n^2}K_n\left(\lambda,\mu\right)=-\frac{1}{2}V'\left(\lambda\right)\widetilde{\mathcal{K}}_n\left(x,y\right)$$
$$+\frac{1}{2n}\int\limits_{-n\pi}^{n\pi}\cot\left(\frac{x-z}{2n}\right)\left(\widetilde{\mathcal{K}}_n\left(z,z\right)\widetilde{\mathcal{K}}_n\left(x,y\right)-\widetilde{\mathcal{K}}_n\left(x,z\right)\widetilde{\mathcal{K}}_n\left(z,y\right)\right)dz$$
$$+\frac{i(n-1)}{2n}\widetilde{\mathcal{K}}_n\left(x,y\right). \tag{2.30}$$

Lemma 2.6. *Denote*

$$D\left(\lambda\right)=V'\left(\lambda\right)+v.p.\int\limits_{-\pi}^{\pi}\cot\frac{s}{2}\rho_n\left(\lambda+s\right)ds.$$

Then for any $d>0$ we have uniformly in $[a+d,b-d]$

$$\left|D\left(\lambda\right)\right|\le Cn^{-1/4}\ln n.$$

The definition of $\mathcal{K}_n$ (2.27), the above Lemma, and the bound (2.29) yield

$$\frac{\partial}{\partial x}\mathcal{K}_n\left(x,y\right)=\frac{1}{2n}v.p.\int\limits_{-n\pi}^{n\pi}\cot\left(\frac{z-x}{2n}\right)\mathcal{K}_n\left(x,z\right)\mathcal{K}_n\left(z,y\right)dz+O(n^{-1/4}\ln n). \tag{2.31}$$

Below we take $|x|,|y|\le\mathcal{L}=\ln n$. Then from the inequality $|z|\le n\pi$ we get $\left|\frac{x-z}{2n}\right|\le 3\pi/4$. The function $x\cot x$ is bounded on $[0,3\pi/4]$, thus

$$\left|\frac{1}{2n}\cot\left(\frac{x-z}{2n}\right)\right|\le C\left|\frac{1}{x-z}\right|.$$

For $|x|,|y|\le\mathcal{L}$ we can restrict integration in (2.31) by the domain $|z|\le 2\mathcal{L}$, substituting $O(n^{-1/4}\ln n)$ by $O\left(\mathcal{L}^{-1}\right)$. This follows from the bound

$$\left|\frac{1}{2n}\int\limits_{2\mathcal{L}\le|z|\le n\pi}\cot\left(\frac{x-z}{2n}\right)\mathcal{K}_n\left(x,z\right)\mathcal{K}_n\left(z,y\right)dz\right|$$
$$\le C\mathcal{L}^{-1}\int\left|\mathcal{K}_n\left(x,z\right)\right|\left|\mathcal{K}_n\left(z,y\right)\right|dz\le C\mathcal{L}^{-1}.$$

Note that
$$\frac{1}{2n}\cot\frac{x}{2n}-\frac{1}{x}=O\left(n^{-2}\ln n\right),\quad \text{for } x=O\left(\ln n\right).$$
Hence, from the above estimates and (2.31) we get
$$\frac{\partial}{\partial x}\mathcal{K}_n\left(x,y\right)=v.p.\int\limits_{|z|\le 2\mathcal{L}}\frac{\mathcal{K}_n\left(x,z\right)\mathcal{K}_n\left(z,y\right)}{z-x}\,dz+O\left(\mathcal{L}^{-1}\right). \tag{2.32}$$

The following lemma shows that $\mathcal{K}_n$ behaves almost like a difference kernel.

Lemma 2.7. *For any $d>0$ we have uniformly in $\lambda_0\in[a+d,b-d]$ and $|x|,|y|\le nd/4$*
$$\left|\frac{\partial}{\partial x}\mathcal{K}_n\left(x,y\right)+\frac{\partial}{\partial y}\mathcal{K}_n\left(x,y\right)\right|\le C\left(n^{-1/8}+|x-y|\,n^{-2}\right), \tag{2.33}$$
$$\left|\mathcal{K}_n\left(x,y\right)-\mathcal{K}_n\left(0,y-x\right)\right|\le C\,|x|\left(n^{-1/8}+|x-y|\,n^{-2}\right). \tag{2.34}$$

R e m a r k 2.8. *Note that the last inequality with λ_0+x_1/n instead of λ_0, and x_2-x_1 instead of x and y, leads to the bound that is valid for any $|x_{1,2}|\le nd_0/8$*
$$\left|\mathcal{K}_n\left(x_2,x_2\right)-\mathcal{K}_n\left(x_1,x_1\right)\right|\le Cn^{-1/8}\left|x_2-x_1\right|. \tag{2.35}$$

Lemma 2.9. *For any $|x|,|y|\le\mathcal{L}$*
$$\left|\frac{\partial}{\partial x}\mathcal{K}_n\left(x,y\right)\right|\le C,\qquad \int\limits_{|x|\le\mathcal{L}}\left|\frac{\partial}{\partial x}\mathcal{K}_n\left(x,y\right)\right|^2dx\le C. \tag{2.36}$$

Denote
$$\begin{aligned}\mathcal{K}_n^*(x) &= \mathcal{K}_n(x,0)\mathbf{1}_{|x|\le\mathcal{L}}+\mathcal{K}_n(\mathcal{L},0)(1+\mathcal{L}-x)\mathbf{1}_{\mathcal{L}<x\le\mathcal{L}+1}\\ &+ \mathcal{K}_n(-\mathcal{L},0)(1+\mathcal{L}+x)\mathbf{1}_{-\mathcal{L}-1\le x<-\mathcal{L}},\end{aligned} \tag{2.37}$$

and observe that for $y=0$ and for any $|x|\le\mathcal{L}/3$, similarly to (2.32), we can restrict the integration in (2.32) to $|z|\le 2\mathcal{L}/3$ with a mistake $O(\mathcal{L}^{-1})$. This and Lemma 2.7 give us the equation
$$\frac{\partial}{\partial x}\mathcal{K}_n^*(x)=\int\limits_{|z|\le 2\mathcal{L}/3}\frac{\mathcal{K}_n^*(z)\mathcal{K}_n^*(x-z)}{z}dz+r_n(x)+O(\mathcal{L}^{-1}), \tag{2.38}$$

where

$$r_n(x) = \int\limits_{|z|\le 2\mathcal{L}/3} \frac{\mathcal{K}_n(z,0)(\mathcal{K}_n(x,z)-\mathcal{K}_n(0,x-z))}{z}dz,$$

and by Lemma 2.7, for $|x| \le \mathcal{L}/3$ we have

$$r_n(x) = O(n^{-1/8}\log n).$$

Now, using the estimates similar to (2.32), we can restrict the integration in (2.38) to the real axis. From Lemma 2.9 and the relations (2.28), (2.29) we get

$$\int |\mathcal{K}_n^*(x)|^2 dx \le \int |\mathcal{K}_n(x,0)|^2 dx + C' \le C, \quad \int \left|\frac{d}{dx}\mathcal{K}_n^*(x)\right|^2 dx \le C. \tag{2.39}$$

Consider the Fourier transform

$$\widehat{\mathcal{K}}_n^*(p) = \int \mathcal{K}_n^*(x)e^{ipx}dx,$$

where the integral is defined in the $L^2(\mathbb{R})$ sense, and write $\mathcal{K}_n^*(x)$ as

$$\mathcal{K}_n^*(x) = (2\pi)^{-1}\int \widehat{\mathcal{K}}_n^*(p)e^{-ipx}dp. \tag{2.40}$$

From (1.19) we have

$$\int \widehat{\mathcal{K}}_n^*(p)dp = 2\pi\rho(\lambda_0) + o(1), \tag{2.41}$$

and from (2.39) and the Parseval equation we obtain

$$\int p^2|\widehat{\mathcal{K}}_n^*(p)|^2 dp \le C. \tag{2.42}$$

From the definition of $\mathcal{K}_n(x,y)$ we get that the kernel is positive definite

$$\int\limits_{-\mathcal{L}}^{\mathcal{L}} \mathcal{K}_n(x,y)f(x)\overline{f}(y)dxdy \ge 0, \quad f \in L_2(\mathbb{R}),$$

therefore from (2.34) we have for any function $f \in L_2(\mathbb{R})$

$$\int \widehat{\mathcal{K}}_n^*(p)|\hat{f}(p)|^2 dp \ge -C||f||^2_{L^2(\mathbb{R})}(n^{-1/8}\log^4 n + O(\mathcal{L}^{-1})). \tag{2.43}$$

From the Parseval equation and (2.34) there follows

$$\int |\widehat{\mathcal{K}}_n^*(p) - \widehat{\mathcal{K}}_n^*(-p)|^2 dp \le 2\pi \int |\mathcal{K}_n^*(x) - \mathcal{K}_n^*(-x)|^2 dx \le Cn^{-1/8}\log^3 n. \tag{2.44}$$

By the definition of singular integrals

$$\int \frac{\mathcal{K}_n^*(z)\mathcal{K}_n^*(x-z)}{z}dz = \lim_{\varepsilon\to+0}\int dz\mathcal{K}_n^*(z)\mathcal{K}_n^*(y-z)\Re(z+i\varepsilon)^{-1}. \tag{2.45}$$

In accordance with the relation

$$\int e^{ipz}\Re(z+i\varepsilon)^{-1}dz = \pi i e^{-\varepsilon|p|}\operatorname{sgn} p$$

and the Parseval equation, we can write the r.h.s. of (2.38) as

$$\begin{aligned}\frac{i}{4\pi}\lim_{\varepsilon\to+0}\int dpdp'\widehat{\mathcal{K}}_n^*(p)\widehat{\mathcal{K}}_n^*(p')e^{-ipx}\operatorname{sign}(p-p')e^{-\varepsilon|p-p'|}\\ = \frac{i}{2\pi}\int dpe^{-ipx}\widehat{\mathcal{K}}_n^*(p)\int_0^p \widehat{\mathcal{K}}_n^*(p')dp'\\ - \frac{i}{4\pi}\int dpe^{-ipx}\widehat{\mathcal{K}}_n^*(p)\int_0^\infty(\widehat{\mathcal{K}}_n^*(p')-\widehat{\mathcal{K}}_n^*(-p'))dp'.\end{aligned} \tag{2.46}$$

Note that both integrals are absolutely convergent because $\widehat{\mathcal{K}}_n^* \in L^1(\mathbb{R})$ by (2.42). Now, using the Schwarz inequality and (2.42), we can estimate the second component

$$\begin{aligned}\left|\int_0^\infty(\widehat{\mathcal{K}}_n^*(p')-\widehat{\mathcal{K}}_n^*(-p'))dp'\right| \le \left|\int_0^{\mathcal{L}^2}(\widehat{\mathcal{K}}_n^*(p')-\widehat{\mathcal{K}}_n^*(-p'))dp'\right|\\ + \int_{|p|>\mathcal{L}^2}|\widehat{\mathcal{K}}_n^*(p')|dp' \le \mathcal{L}\left(\int|\widehat{\mathcal{K}}_n^*(p')-\widehat{\mathcal{K}}_n^*(-p')|^2dp'\right)^{1/2} + C\mathcal{L}^{-1}.\end{aligned}$$

Thus, from (2.44)–(2.46) we have uniformly in $|x| < \mathcal{L}/3$

$$\int \frac{\mathcal{K}_n^*(z)\mathcal{K}_n^*(x-z)}{z}dz = \frac{i}{2\pi}\int dp\widehat{\mathcal{K}}_n^*(p)e^{-ipx}\int_0^p \widehat{\mathcal{K}}_n^*(p')dp' + O(\mathcal{L}^{-1}).$$

This allows us to transform (2.38) into the following asymptotic relation that is valid for $|x| \le \mathcal{L}/3$:

$$\int \widehat{\mathcal{K}}_n^*(p)\left(\int_0^p \widehat{\mathcal{K}}_n^*(p')dp' - p\right)e^{-ipx}dp = O(\mathcal{L}^{-1}). \tag{2.47}$$

Consider the functions

$$F_n(p) = \int_0^p \widehat{\mathcal{K}}_n^*(p')dp'. \tag{2.48}$$

Since $p\widehat{\mathcal{K}}_n^*(p) \in L^2(\mathbb{R})$, the sequence $\{F_n(p)\}$ consists of functions that are uniformly bounded and equicontinuous on $\mathbb{R}$. Thus $\{F_n(p)\}$ is a compact family with respect to uniform convergence. Hence, the limit F of any subsequence $\{F_{n_k}\}$ possesses the properties:

(a) F is bounded and continuous;

(b) $F(p) = -F(-p)$ (see (2.44));

(c) $F(p) \le F(p')$, if $p \le p'$ (see (2.43));

(d) $F(+\infty) - F(-\infty) = 2\pi\rho(\lambda_0)$ (see (2.41));

(e) the following equation is valid for any smooth function g with the compact support (see (2.47)):

$$\int (F(p) - p)g(p)dF(p) = 0. \tag{2.49}$$

The last property implies that $F(p) = p$ or $F(p) = \text{const}$, hence it follows from (a)–(c) that

$$F(p) = p\,\mathbf{1}_{|p|\le p_0} + p_0\,\mathrm{sign}(p)\,\mathbf{1}_{|p|\ge p_0},$$

where $p_0 = \pi\rho(\lambda_0)$ from (d). We conclude that (2.49) is uniquely solvable, thus the sequence $\{F_n\}$ converges uniformly on any compact to the above F. This and (2.48) imply the weak convergence of the sequence $\{\mathcal{K}_n^*\}$ to the function $\rho(\lambda_0)\,S(\rho(\lambda_0)\,x)$, where $S(x)$ is defined in (1.8). But weak convergence combined with (2.29) and (2.36) implies the uniform convergence of $\{\mathcal{K}_n^*\}$ to $\mathcal{K}^*$ on any interval. Thus we have uniformly in (x, y), varying on a compact set of $\mathbb{R}^2$,

$$\lim_{n\to\infty} \mathcal{K}_n(x, y) = \rho(\lambda_0)\,S(\rho(\lambda_0)(x - y)).$$

Recalling all definitions, we conclude that Theorem 1.2 is proved.

Auxiliary Results for Theorem 1.2

P r o o f of Lemma 2.1. Denote

$$r_{k,j}^{(n)} = \int_{-\pi}^{\pi} e^{i\lambda}\psi_k^{(n)}(\lambda)\,\overline{\psi_j^{(n)}(\lambda)}\,d\lambda. \tag{2.50}$$

Note that from the orthogonality (2.66) we have $r_{k,j}^{(n)} = 0$ for $j > k+1$. Thus,

$$e^{i\lambda}\,\psi_k^{(n)}(\lambda) = \sum_{j=0}^{k+1} r_{k,j}^{(n)} \psi_j^{(n)}(\lambda). \tag{2.51}$$

Multiplication on $e^{i\lambda}$ is isometric in $L_2[-\pi,\pi]$, therefore

$$\sum_{j=0}^{k+1} \left| r_{k,j}^{(n)} \right|^2 = \left\| \psi_k^{(n)}(\lambda) \right\|_2 = 1.$$

Finally we are ready to prove (2.9)

$$\begin{aligned}
&\int_{-\pi}^{\pi} \left(e^{i\lambda} - e^{i\mu}\right) |K_n(\lambda,\mu)|^2\, d\mu \\
&\quad = e^{i\lambda} K_n(\lambda,\lambda) - \int_{-\pi}^{\pi} e^{i\mu} \sum_{m=0}^{n-1} \psi_m^{(n)}(\mu)\, \overline{\psi_m^{(n)}(\lambda)} \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda)\, \overline{\psi_l^{(n)}(\mu)}\, d\mu \\
&\quad = e^{i\lambda} K_n(\lambda,\lambda) - \sum_{l,m=0}^{n-1} r_{m,l}^{(n)} \psi_l^{(n)}(\lambda)\, \overline{\psi_m^{(n)}(\lambda)} \\
&\quad = r_{n-1,n}^{(n)} \psi_{n-1}^{(n)}(\lambda)\, \overline{\psi_n^{(n)}(\lambda)}.
\end{aligned} \tag{2.52}$$

Now, using the Cauchy inequality and the bound $\left| r_{n-1,n}^{(n)} \right| \le 1$, we get (2.9). Similarly, it is easy to obtain the relation

$$\int_{-\pi}^{\pi} \left| e^{i\lambda} - e^{i\mu} \right|^2 |K_n(\lambda,\mu)|^2\, d\mu = 2\Re \left\{ e^{i\lambda} r_{n-1,n}^{(n)} \overline{\psi_{n-1}^{(n)}(\lambda)} \psi_n^{(n)}(\lambda) \right\},$$

which implies (2.10). The bounds (2.11),(2.12),(2.13) are evident consequences of (2.10). The lemma is proved. ■

P r o o f of Lemma 2.2. Observe that

$$\frac{d\rho_n(\lambda)}{d\lambda} = \left. \frac{d\rho_n(\lambda+t)}{dt} \right|_{t=0}.$$

Changing variables in (1.5) $\lambda_j = \mu_j + t$, in view of periodicity of all functions in the consideration, we have the representation for $\rho_n(\lambda+t)$

$$\rho_n(\lambda+t) = \frac{1}{Z_n} \int e^{-nV(\lambda+t)} \prod_{2 \le j < k \le n} \left| e^{i\mu_j} - e^{i\mu_k} \right|^2 \prod_{j=2}^{n} e^{-nV(\mu_j+t)} \left| e^{i\lambda} - e^{i\mu_j} \right|^2 d\mu_j.$$

After differentiating with respect to t, for $t=0$ we get

$$\frac{d\rho_n(\lambda)}{d\lambda} = -nV'(\lambda)\,p_1^{(n)}(\lambda) - n(n-1)\int_{-\pi}^{\pi} V'(\mu)\,p_2^{(n)}(\lambda,\mu)\,d\mu$$

$$= -V'(\lambda)\,K_n(\lambda,\lambda) - \int_{-\pi}^{\pi} V'(\mu)\left[K_n(\lambda,\lambda)\,K_n(\mu,\mu) - |K_n(\lambda,\mu)|^2\right]d\mu. \quad (2.53)$$

Since $V'(\lambda)$ is an odd function, and $K_n(\lambda,\lambda)$ is an even function, we obtain

$$\int_{-\pi}^{\pi} V'(\lambda)\,K_n(\lambda,\lambda)\,d\lambda = 0.$$

Thus, from (2.53) we get

$$\rho_n'(\lambda) = \int_{-\pi}^{\pi}\left(V'(\mu) - V'(\lambda)\right)|K_n(\lambda,\mu)|^2\,d\mu. \quad (2.54)$$

We split this integral in two parts corresponding to the domains $|\mu-\lambda|\le d/2$ and $|\mu-\lambda|\ge d/2$. In the second integral we use (2.12). It follows from (1.17) that in the first integral we can rewrite $V'(\lambda)$ as

$$V'(\mu) - V'(\lambda) = (\mu-\lambda)\,V''(\lambda) + O\left(|\mu-\lambda|^2\right)$$
$$= \left(e^{i\mu} - e^{i\lambda}\right)\frac{V''(\lambda)}{ie^{i\lambda}} + O\left(\left(e^{i\mu} - e^{i\lambda}\right)^2\right),$$

and using (2.9) and (2.10), we get (2.16). To prove (2.15) we use the following well-known inequality.

Proposition 2.10. *For any function $u:[a_1,b_1]\to\mathbb{C}$ with $u'\in L_1(a_1,b_1)$ we have*

$$\|u\|_\infty \le \left\|u'\right\|_1 + (b_1-a_1)^{-1}\|u\|_1, \quad (2.55)$$

where $\|\cdot\|_1, \|\cdot\|_\infty$ are the L_1 and uniform norms on the interval $[a_1,b_1]$.

This inequality can be obtained easily from the relation

$$u(\lambda) = \frac{1}{b_1-a_1}\int_{a_1}^{b_1}(u(\lambda) - u(\mu))\,d\mu + \frac{1}{b_1-a_1}\int_{a_1}^{b_1}u(\mu)\,d\mu.$$

Using (2.55) for $u=\rho_n$ and the interval $[a+d, b-d]$, we get (2.15). ∎

P r o o f of Lemma 2.3. From (1.4) and (2.21) we have for nonreal z

$$f^2(z) - 2iV'(\mu) f(z) - 2iQ(z) - 1 = 0, \tag{2.56}$$

where $f(z)$ is the Germglotz transformation of the limiting density $\rho(\lambda)$. By (2.19) and (2.2), $Q(\mu + i0)$ is an imaginary valued, bounded, continuous function. And from (2.2) we obtain

$$\rho(\mu) = \frac{1}{2\pi} \Re f(\mu + i0).$$

Computing imaginary and real parts in (2.56), we get the relations

$$\Im f(\mu + i0) = V'(\mu), \tag{2.57}$$

$$\Re f(\mu + i0) = \sqrt{2iQ(\mu) + 1 - (V'(\mu))^2}, \tag{2.58}$$

from which we obtain (2.22). ■

P r o o f of Lemma 2.4. To prove (2.24) with $k = n - 1$ we introduce the probability density

$$p_n^-(\lambda_1, \ldots, \lambda_{n-1}) = \frac{1}{Z_n^-} \prod_{1 \le j < k \le n-1} \left| e^{i\lambda_j} - e^{i\lambda_k} \right|^2 \exp\left\{ -n \sum_{j=1}^{n-1} V(\lambda_j) \right\}. \tag{2.59}$$

Denote

$$\rho_n^-(\lambda) = \frac{n-1}{n} \int p_n^-(\lambda, \lambda_2 \ldots, \lambda_{n-1}) \, d\lambda_2 \ldots d\lambda_{n-1} = \frac{1}{n} \sum_{j=0}^{n-2} \left| \psi_j^{(n)}(\lambda) \right|^2. \tag{2.60}$$

Thus we get

$$\left| \psi_{n-1}^{(n)}(\lambda) \right|^2 = n \left(\rho_n(\lambda) - \rho_n^-(\lambda) \right). \tag{2.61}$$

Analogously to the equation (2.8), we can obtain the "square" equation

$$\frac{i}{2} \left[f_n^-(z) \right]^2 + \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} V'(\lambda) \rho_n^-(\lambda) \, d\lambda = \frac{i}{2} + O\left(n^{-2} \eta^{-4} \right), \tag{2.62}$$

for the Germglotz transformation $f_n^-(z)$ of the function $\rho_n^-(\lambda)$. Denote

$$\Delta_n(z) = n \left(f_n(z) - f_n^-(z) \right) = \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} \left| \psi_{n-1}^{(n)}(\lambda) \right|^2 d\lambda. \tag{2.63}$$

Subtracting (2.62) from (2.8), we obtain for $z = \mu + in^{-1/4}$

$$\frac{i}{2}\Delta_n(z)\left(f_n(z) + f_n^-(z)\right) = -\int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} V'(\lambda)\left|\psi_{n-1}^{(n)}(\lambda)\right|^2 d\lambda + O(1),$$

$$\frac{i}{2}\Delta_n(z)\left(f_n(z) + f_n^-(z) - 2iV'(\mu)\right) = \int_{-\pi}^{\pi} \frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} \left(V'(\mu) - V'(\lambda)\right)\left|\psi_{n-1}^{(n)}(\lambda)\right|^2 d\lambda + O(1) = O(1).$$

Note that $\Re f_n^-(z) > 0$ for $\Im z > 0$ therefore

$$\Re\Delta_n\left(\mu + in^{-1/4}\right) \le \frac{C}{\Re f_n\left(\mu + in^{-1/4}\right)}$$

Analogously to (2.23), we can obtain for $z = \mu + in^{-1/4}$

$$\frac{1}{2\pi}\Re f_n(z) = \rho(\mu) + O\left(n^{-1/8}\right)\rho^{-1}(\mu),$$

hence $\Re f_n(z) \ge C_2$ for sufficiently large n, where C_2 is defined in (1.17). Thus,

$$\Re\Delta_n\left(\mu + in^{-1/4}\right) \le C.$$

Note that

$$\Re\frac{e^{i\lambda} + e^{iz}}{e^{i\lambda} - e^{iz}} = \frac{\sinh\eta}{\cosh\eta - \cos(\mu - \lambda)} \ge C\frac{\eta}{\eta^2 + (\mu - \lambda)^2},$$

for $\eta^2 + (\mu - \lambda)^2 < 1$. Thus,

$$\int_{|\lambda-\mu|<n^{-1/4}} \left|\psi_{n-1}^{(n)}(\lambda)\right|^2 d\lambda \le 2n^{-1/2}\int_{|\lambda-\mu|<n^{-1/4}} \frac{\left|\psi_{n-1}^{(n)}(\lambda)\right|^2}{n^{-1/2} + (\mu - \lambda)^2} d\lambda$$
$$\le Cn^{-1/4}\Re\Delta_n\left(\mu + in^{-1/4}\right) \le Cn^{-1/4}.$$

A similar bound can be obtained for $\psi_n^{(n)}(\lambda)$ by using the densities:

$$p_n^+(\lambda_1, \ldots, \lambda_{n+1}) = \frac{1}{Q_{n,2}^+}\prod_{1\le j\le n+1} e^{-nV(\lambda_j)} \prod_{1\le j<k\le n+1}\left|e^{i\lambda_j} - e^{i\lambda_k}\right|^2,$$

$$\rho_n^+(\lambda) = \frac{n+1}{n}\int p_n^+(\lambda, \lambda_2, \ldots, \lambda_{n+1})\, d\lambda_2 \ldots d\lambda_{n+1} = \frac{1}{n}\sum_{j=0}^{n}\left|\psi_j^{(n)}(\lambda)\right|^2.$$

Analogously, we will have $\left|\psi_n^{(n)}(\lambda)\right|^2 = n\left(\rho_n^+(\lambda) - \rho_n(\lambda)\right)$. Thus, the estimate (2.24) is proved. Now we proceed to prove (2.25) for $k = n$. We use the inequality

Proposition 2.11. *For any C^1 function $u : [a_1, b_1] \to \mathbb{C}$*

$$\|u\|_\infty^2 \le 2\|u\|_2 \left\|u'\right\|_2 + (b_1 - a_1)^{-1}\|u\|_2^2, \tag{2.64}$$

where $\|\cdot\|_2, \|\cdot\|_\infty$ are the L_2 and uniform norms on the interval $[a_1, b_1]$.

This inequality is a simple consequence of the relation

$$u^2(\lambda) = \frac{1}{b_1 - a_1}\int_{a_1}^{b_1}\left(u^2(\lambda) - u^2(\mu)\right) d\mu + \frac{1}{b_1 - a_1}\int_{a_1}^{b_1} u^2(\mu)\, d\mu.$$

Consider the interval $\Delta = \left[\lambda - n^{-1/4}, \lambda + n^{-1/4}\right]$ and the function $\psi(\lambda) = \psi_n^{(n)}(\lambda)$. From the inequality we have

$$|\psi(\lambda)|^2 \le 2\|\psi\|_{2,\Delta}\left\|\psi'\right\|_{2,\Delta} + \frac{1}{2}n^{1/4}\|\psi\|_{2,\Delta}, \tag{2.65}$$

where $\|\cdot\|_{2,\Delta}$ is L_2 norm on the interval Δ. It is easy to see that

$$\|\psi\|_{2,\Delta} \le \|\psi\|_{2,[-\pi,\pi]} = 1.$$

Denote $P(\lambda) = P_n^{(n)}(\lambda)$ and $\omega(\lambda) = e^{-nV(\lambda)/2}$, then $\psi(\lambda) = P(\lambda)\omega(\lambda)$. Now we estimate $\|\psi'\|_{2,[-\pi,\pi]}$:

$$\left\|\psi'\right\|_{2,[-\pi,\pi]} = \left\|P'\omega + P\omega'\right\|_{2,[-\pi,\pi]} \le \left\|P'\omega\right\|_{2,[-\pi,\pi]} + \left\|P\omega'\right\|_{2,[-\pi,\pi]},$$
$$\left\|P\omega'\right\|_{2,[-\pi,\pi]} = \frac{n}{2}\left\|PV'\omega\right\|_{2,[-\pi,\pi]} \le Cn\|P\omega\|_{2,[-\pi,\pi]} = Cn,$$

$$\left\|P'\omega\right\|_{2,[-\pi,\pi]}^2 = \int P'(\lambda)\overline{P'(\lambda)}\omega^2(\lambda)\, d\lambda = -\int P(\lambda)\overline{P''(\lambda)}\omega^2(\lambda)\, d\lambda$$
$$+ n\int P(\lambda)\overline{P'(\lambda)}V'(\lambda)\omega^2(\lambda)\, d\lambda.$$

Using the orthogonality

$$\int e^{-im\lambda}\omega(\lambda)\psi_k^{(n)}\, d\lambda = 0, \quad \text{for} \quad m < k, \tag{2.66}$$

we obtain

$$\int P(\lambda)\overline{P''(\lambda)}\omega^2(\lambda)\,d\lambda = \int P(\lambda)\gamma_n^{(n)}(-in)^2 e^{-in\lambda}\omega^2(\lambda)\,d\lambda$$
$$= -in\int P(\lambda)\overline{P'(\lambda)}\omega^2(\lambda)\,d\lambda,$$

where $\gamma_n^{(n)}$ is defined in (2.26). Thus,

$$\left\|P'\omega\right\|^2_{2,[-\pi,\pi]} = n\int P(\lambda)\overline{P'(\lambda)}\left(V'(\lambda)+i\right)\omega^2(\lambda)\,d\lambda \le Cn\left\|P'\omega\right\|_{2,[-\pi,\pi]},$$

and we obtain that $\|P'\omega\|_{2,[-\pi,\pi]} \le Cn$. Combining all above bounds, we conclude that $\|\psi'\|_{2,[-\pi,\pi]} \le Cn$. Now, using (2.65) and (2.24), we obtain (2.25) for $k=n$. For $k=n-1$ the proof is the same. ∎

P r o o f of Lemma 2.6. Similarly to (2.21) for $\eta = n^{-3/8}$ and $\mu \in [a+d, b-d]$ for f_n, defined in (2.3), we obtain

$$\left|\Im f_n(\mu+i\eta) - V'(\mu)\right| \le Cn^{-3/8}\ln n. \tag{2.67}$$

Moreover, we estimate $M = \Im f_n(\mu+i\eta) + v.p.\int\limits_{-\pi}^{\pi}\cot\frac{s}{2}\rho_n(\mu+s)\,ds$. Note that

$$\Im\frac{e^{i\lambda}+e^{iz}}{e^{i\lambda}-e^{iz}} = -\frac{\sin(\lambda-\mu)}{\cosh\eta-\cos(\lambda-\mu)}.$$

Hence,

$$M = v.p.\int\left(\cot\frac{s}{2} - \frac{\sin s}{\cosh\eta-\cos s}\right)\rho_n(\mu+s)\,ds$$
$$= \int\limits_{|s|\le d/2}\ln\left(\frac{\cosh\eta-\cos s}{1-\cos s}\right)\rho_n'(\mu+s)\,ds + O(\eta) = I_1+I_2+I_3+O(\eta),$$

where I_1 is the integral over $|s|\le n^{-2}$, I_2 is the integral over $n^{-2}\le|s|\le n^{-1/4}$ and I_3 is the integral over $n^{-1/4}\le|s|\le d/2$. We estimate every term:

$$|I_1| \overset{(2.25)}{\le} Cn^{7/8}\int\limits_{|s|\le n^{-2}}\ln\left(\frac{\cosh\eta-\cos s}{1-\cos s}\right)ds \le Cn^{-9/8}\ln n,$$

$$|I_2| \le C\ln n\int\limits_{n^{-2}\le|s|\le n^{1/4}}\left|\rho_n'(\mu+s)\right|ds \overset{(2.24)}{\le} Cn^{-1/4}\ln n,$$

$$|I_3| \overset{(2.16)}{\leq} Cn^{-1/4} \int\limits_{|s|\leq d/2} \left(\left|\psi_n^{(n)}(\mu+s)\right|^2 + \left|\psi_{n-1}^{(n)}(\mu+s)\right|^2 \right) ds \leq Cn^{-1/4}.$$

Combining the above bounds with (2.67), we obtain that the lemma is proved. ■

P r o o f of Lemma 2.7. To simplify notations we denote for $t \in [0,1]$

$$\lambda_x = \lambda_0 + \frac{x - tx}{n}, \quad \lambda_y = \lambda_0 + \frac{y - tx}{n}. \tag{2.68}$$

Then, similarly to (2.30) and (2.54), we obtain

$$\frac{d}{dt} K_n(\lambda_x, \lambda_y) = x \int\limits_{-\pi+\lambda_0}^{\pi+\lambda_0} K_n(\lambda_x, \lambda) K_n(\lambda, \lambda_y) \left(\frac{1}{2} V'(\lambda_x) + \frac{1}{2} V'(\lambda_y) - V'(\lambda) \right) d\lambda. \tag{2.69}$$

To get our estimates, we split this integral in two parts $|\lambda - \lambda_0| \leq d/2$ and $|\lambda - \lambda_0| \geq d/2$. By the assumption of the lemma, λ_x, λ_y are in $[a + d/2, b - d/2]$, thus in the first integral we can write

$$\begin{aligned} & V'(\lambda) - \frac{1}{2} V'(\lambda_x) - \frac{1}{2} V'(\lambda_y) \\ &= \left(e^{i\lambda} - e^{i\lambda_x}\right) \frac{V''(\lambda_x)}{2ie^{i\lambda_x}} + \left(e^{i\lambda} - e^{i\lambda_y}\right) \frac{V''(\lambda_y)}{2ie^{i\lambda_y}} + O\left(\left|e^{i\lambda} - e^{i\lambda_x}\right|^2 + \left|e^{i\lambda} - e^{i\lambda_y}\right|^2 \right) \\ &= \left(e^{i\lambda} - e^{i\lambda_x}\right) \frac{V''(\lambda_x)}{2ie^{i\lambda_x}} + \left(e^{i\lambda} - e^{i\lambda_y}\right) \frac{V''(\lambda_y)}{2ie^{i\lambda_y}} \\ &\quad + O\left(\left|e^{i\lambda} - e^{i\lambda_x}\right| \left|e^{i\lambda} - e^{i\lambda_y}\right| + \frac{|x-y|^2}{n^2} \right). \end{aligned}$$

Similarly to (2.52), we obtain

$$\int\limits_{-\pi}^{\pi} K_n(\lambda_x, \lambda) K_n(\lambda, \lambda_y) \left(e^{i\lambda} - e^{i\lambda_x}\right) d\lambda = -r_{n-1,n}^{(n)} \psi_n^{(n)}(\lambda_x) \overline{\psi_{n-1}^{(n)}(\lambda_y)}.$$

Hence,

$$\int\limits_{|\lambda-\lambda_0|\leq d/2} K_n(\lambda_x, \lambda) K_n(\lambda, \lambda_y) \left(e^{i\lambda} - e^{i\lambda_x}\right) d\lambda = -r_{n-1,n} \psi_n^{(n)}(\lambda_x) \overline{\psi_{n-1}^{(n)}(\lambda_y)} - I_d,$$

where

$$|I_d| = \left| \int\limits_{|\lambda-\lambda_0|\geq d/2} K_n(\lambda_x,\lambda) K_n(\lambda,\lambda_y) \left(e^{i\lambda} - e^{i\lambda_x}\right) d\lambda \right|$$
$$\leq C \left[\int\limits_{|\lambda-\lambda_0|\geq d/2} |K_n(\lambda_x,\lambda)|^2 \, d\lambda \int\limits_{|\lambda-\lambda_0|\geq d/2} |K_n(\lambda,\lambda_y)|^2 \, d\lambda \right]^{1/2}$$
$$\overset{(2.12)}{\leq} C \left[\left|\psi_{n-1}^{(n)}(\lambda_x)\right|^2 + \left|\psi_n^{(n)}(\lambda_x)\right|^2 + \left|\psi_{n-1}^{(n)}(\lambda_y)\right|^2 + \left|\psi_n^{(n)}(\lambda_y)\right|^2 \right].$$

The same bounds are valid for the term with the $e^{i\lambda_y}$ instead of $e^{i\lambda_x}$. To estimate other terms, we use the Schwarz inequality

$$\int\limits_{|\lambda-\lambda_0|\leq d/2} \left| K_n(\lambda_x,\lambda) K_n(\lambda,\lambda_y) \left(e^{i\lambda} - e^{i\lambda_x}\right) \left(e^{i\lambda} - e^{i\lambda_y}\right) \right| d\lambda$$
$$\leq \left[\int_{-\pi}^{\pi} \left| K_n(\lambda_x,\lambda) \left(e^{i\lambda} - e^{i\lambda_x}\right) \right|^2 d\lambda \int_{-\pi}^{\pi} \left| K_n(\lambda,\lambda_y) \left(e^{i\lambda} - e^{i\lambda_y}\right) \right|^2 d\lambda \right]^{1/2}$$
$$\overset{(2.11)}{\leq} C \left[\left|\psi_{n-1}^{(n)}(\lambda_x)\right|^2 + \left|\psi_n^{(n)}(\lambda_x)\right|^2 + \left|\psi_{n-1}^{(n)}(\lambda_y)\right|^2 + \left|\psi_n^{(n)}(\lambda_y)\right|^2 \right],$$

$$\int\limits_{|\lambda-\lambda_0|\leq d/2} |K_n(\lambda_x,\lambda) K_n(\lambda,\lambda_y)| \, d\lambda \leq n \left(\rho_n(\lambda_x) + \rho_n(\lambda_y)\right) \leq Cn.$$

In the second integral we use the boundedness of $V'(\lambda)$, the Cauchy inequality $|K_n(\lambda_x,\lambda) K_n(\lambda,\lambda_y)| \leq |K_n(\lambda_x,\lambda)|^2 + |K_n(\lambda,\lambda_y)|^2$ and (2.12). Thus,

$$\left| \frac{d}{dt} K_n(\lambda_x,\lambda_y) \right|$$
$$\leq C|x| \left[\left|\psi_{n-1}^{(n)}(\lambda_x)\right|^2 + \left|\psi_n^{(n)}(\lambda_x)\right|^2 + \left|\psi_{n-1}^{(n)}(\lambda_y)\right|^2 + \left|\psi_n^{(n)}(\lambda_y)\right|^2 + \frac{|x-y|}{n} \right]. \tag{2.70}$$

Now, using (2.25), we obtain

$$\left| \frac{d}{dt} K_n(\lambda_x,\lambda_y) \right| \leq C|x| \left(n^{7/8} + |x-y| \, n^{-1} \right). \tag{2.71}$$

Finally, observing that

$$\frac{\partial}{\partial x}\mathcal{K}_n(x,y)+\frac{\partial}{\partial y}\mathcal{K}_n(x,y)=-(xn)^{-1}e^{-i(n-1)(x-y)/2n}\frac{d}{dt}K_n(\lambda_x,\lambda_y)\big|_{t=0},$$

$$\mathcal{K}_n(x,y)-\mathcal{K}_n(0,y-x)=e^{-i(n-1)(x-y)/2n}\cdot\frac{1}{n}\left(K_n(\lambda_x,\lambda_y)\big|_{t=0}-K_n(\lambda_x,\lambda_y)\big|_{t=1}\right),$$

and using (2.71), we conclude that the lemma is proved. ■

P r o o f of Lemma 2.9. First, show that for any $|x|\le nd_0/2$ we have the bound

$$\int_{-1}^{1}\frac{\mathcal{K}_n(x,x)\,\mathcal{K}_n(x+t,x+t)-|\mathcal{K}_n(x,x+t)|^2}{t^2}\,dt\le C. \tag{2.72}$$

Denote

$$\Omega_0=[-\pi+\lambda_0,\pi+\lambda_0],\quad \Omega_0^+=\Omega_0/\Omega_0^-, \tag{2.73}$$

$$\Omega_0^-=\left\{\lambda\in\Omega_0:\left|\sin\frac{\lambda-\lambda_0}{2}\right|\le\sin\frac{1}{2n}\right\}=[\lambda_0-1/n,\lambda_0+1/n],$$

and consider the quantity

$$W=\left\langle\prod_{j=2}^{n}\left|1-\frac{\sin^2 1/2n}{\sin^2(\lambda_j-\lambda_0)/2}\right|\right\rangle, \tag{2.74}$$

where the symbol $<\ldots>$ denotes the average with respect to $p_n(\lambda_0,\lambda_2,\ldots,\lambda_n)$. We will estimate W from above. To do this we use the relation

$$1-\frac{\sin^2\dfrac{1}{2n}}{\sin^2\dfrac{\mu-\lambda}{2}}=\frac{\left(e^{i(\lambda+1/n)}-e^{i\mu}\right)\left(e^{i(\lambda-1/n)}-e^{i\mu}\right)}{\left(e^{i\lambda}-e^{i\mu}\right)^2},$$

(1.2) and the Schwarz inequality. We get that W^2 is not larger than the product of two integrals I_+ and I_-, where

$$I_\pm=Z_n^{-1}\int_{\Omega_0^{n-1}}e^{-nV(\lambda_0)}\prod_{2\le j<k\le n}\left|e^{i\lambda_j}-e^{i\lambda_k}\right|^2$$

$$\times\exp\left\{-n\sum_{j=2}^{n}V(\lambda_j)\right\}\prod_{j=2}^{n}\left|e^{i(\lambda_0\pm1/n)}-e^{i\lambda_j}\right|^2\,d\lambda_j.$$

Moreover, the expression $n\left(V\left(\lambda_0\right)-V\left(\lambda_0\pm 1/n\right)\right)$ is bounded in view of (1.17). Hence, from (1.15) we obtain

$$W\le C\rho_n^{1/2}\left(\lambda_0+1/n\right)\rho_n^{1/2}\left(\lambda_0-1/n\right)\le C. \tag{2.75}$$

On the other hand, W can be represented as follows:

$$W=\left\langle\prod_{j=2}^{n}\left(\phi_1\left(\lambda_j\right)+\phi_2\left(\lambda_j\right)\right)\right\rangle, \tag{2.76}$$

where

$$\phi_1\left(\lambda\right)=\frac{\left(\sin^2\dfrac{1}{2n}-\sin^2\dfrac{\lambda-\lambda_0}{2}\right)^2}{\sin^2\dfrac{1}{2n}\sin^2\dfrac{\lambda-\lambda_0}{2}}\mathbf{1}_{\Omega_0^-}, \tag{2.77}$$

$$\phi_2\left(\lambda\right)=\left(1-\frac{\sin^2\dfrac{\lambda-\lambda_0}{2}}{\sin^2\dfrac{1}{2n}}\right)\mathbf{1}_{\Omega_0^-}+\left(1-\frac{\sin^2\dfrac{1}{2n}}{\sin^2\dfrac{\lambda-\lambda_0}{2}}\right)\mathbf{1}_{\Omega_0^+}. \tag{2.78}$$

Since $0\le\phi_2\left(\lambda\right)\le 1$ and $\phi_1\left(\lambda\right)\ge 0$, it follows from (2.76) that W can be estimated bellow as

$$W\ge(n-1)\int_{\Omega_0}\phi_1\left(\lambda\right)\left\langle\delta\left(\lambda_2-\lambda\right)\exp\left\{\sum_{j=3}^{n}\ln\phi_2\left(\lambda_j\right)\right\}\right\rangle d\lambda.$$

Note that $\left\langle\delta\left(\lambda_2-\lambda\right)\right\rangle=p_2^{(n)}\left(\lambda_0,\lambda\right)$. Therefore the Jensen inequality implies

$$\begin{aligned}W\ge(n-1)\int_{\Omega_0^-}&\phi_1\left(\lambda\right)p_2^{(n)}\left(\lambda_0,\lambda\right)\\ &\times\exp\left\{\left\langle\delta\left(\lambda_2-\lambda\right)\sum_{j=3}^{n}\ln\phi_2\left(\lambda_j\right)\right\rangle\left[p_2^{(n)}\left(\lambda_0,\lambda\right)\right]^{-1}\right\}d\lambda\\ &=(n-1)\int_{\Omega_0^-}\phi_1\left(\lambda\right)p_2^{(n)}\left(\lambda_0,\lambda\right)\\ &\times\exp\left\{(n-2)\int_{\Omega_0}\ln\phi_2\left(\lambda'\right)p_3^{(n)}\left(\lambda_0,\lambda,\lambda'\right)\,d\lambda'\left[p_2^{(n)}\left(\lambda_0,\lambda\right)\right]^{-1}\right\}d\lambda,\end{aligned}$$

where $p_k^{(n)}$ is defined in (1.5). Using (1.14) for $l = 2, 3$, we have

$$p_3^{(n)}\left(\lambda_0, \lambda, \lambda'\right) = \frac{n}{n-2}\rho_n\left(\lambda'\right) p_2^{(n)}\left(\lambda_0, \lambda\right) + \left[\frac{2\Re\left(K_n\left(\lambda_0, \lambda\right) K_n\left(\lambda, \lambda'\right) K_n\left(\lambda', \lambda_0\right)\right)}{n\left(n-1\right)\left(n-2\right)} - \frac{K_n\left(\lambda_0, \lambda_0\right)\left|K_n\left(\lambda, \lambda'\right)\right|^2 + K_n\left(\lambda, \lambda\right)\left|K_n\left(\lambda_0, \lambda'\right)\right|^2}{n\left(n-1\right)\left(n-2\right)}\right]. \quad (2.79)$$

By the Cauchy inequality,

$$2\left|K_n\left(\lambda_0, \lambda\right) K_n\left(\lambda, \lambda'\right) K_n\left(\lambda', \lambda_0\right)\right| \le 2K_n^{1/2}\left(\lambda_0, \lambda_0\right) K_n^{1/2}\left(\lambda, \lambda\right)\left|K_n\left(\lambda, \lambda'\right) K_n\left(\lambda', \lambda_0\right)\right| \le K_n\left(\lambda_0, \lambda_0\right)\left|K_n\left(\lambda, \lambda'\right)\right|^2 + K_n\left(\lambda, \lambda\right)\left|K_n\left(\lambda_0, \lambda'\right)\right|^2,$$

we obtain that the second term in (2.79) is nonpositive, hence

$$p_3^{(n)}\left(\lambda_0, \lambda, \lambda'\right) \le \frac{n}{n-2}\rho_n\left(\lambda'\right) p_2^{(n)}\left(\lambda_0, \lambda\right).$$

Taking into account that $\ln\phi_2\left(\lambda'\right) \le 0$, finally we get

$$W \ge (n-1)\int\limits_{\Omega_0^-}\phi_1\left(\lambda\right) p_2^{(n)}\left(\lambda_0, \lambda\right) d\lambda \cdot \exp\left\{n\int\limits_{\Omega_0}\rho_n\left(\lambda'\right)\ln\phi_2\left(\lambda'\right) d\lambda'\right\}. \quad (2.80)$$

Now we will show that the second multiplier in (2.80) is bounded from below

$$n\int\limits_{\Omega_0}\rho_n\left(\lambda'\right)\ln\phi_2\left(\lambda'\right) d\lambda' = \left(\int\limits_{|s|\le 1} + \int\limits_{1\le|s|\le nd_0/2} + \int\limits_{nd_0/2\le|s|\le n\pi}\right)\rho_n\left(\lambda_0 + s/n\right)\ln\phi_2\left(\lambda_0 + s/n\right) ds$$
$$\ge C\left(\int\limits_{|s|\le 1}\ln\left(1 - \frac{\sin^2 s/\left(2n\right)}{\sin^2 1/\left(2n\right)}\right) ds + \int\limits_{1\le|s|\le nd_0/2}\ln\left(1 - \frac{\sin^2 1/\left(2n\right)}{\sin^2 s/\left(2n\right)}\right) ds\right) + \ln\left(1 - \frac{\sin^2 1/\left(2n\right)}{\sin^2 d_0/4}\right)\int\limits_{|s|\le n\pi}\rho_n\left(\lambda_0 + s/n\right) ds \ge C\left(I_1 + I_2\right) + O\left(n^{-1}\right).$$

$$I_1 = \int_0^1 \ln\left(\frac{\cos(s/n) - \cos(1/n)}{1 - \cos(1/n)}\right) ds = -n \int_0^{1/n} \frac{\sin t}{\sin(t+1/n)} \frac{t - 1/n}{2\sin\dfrac{t-1/n}{2}} dt \geq -C$$

$$I_2 = n \int_{1/n}^{d_0/2} \ln\left(\frac{\cos(1/n) - \cos t}{1 - \cos t}\right) dt = (nd_0/2 - 1)\ln\left(1 - \frac{\sin^2 1/2n}{\sin^2 d_0/2}\right)$$

$$- n\left(1 - \cos 1/n\right) \int_{1/n}^{d_0/2} \cot t/2 \frac{t-1/n}{2\sin\dfrac{t-1/n}{2}} \frac{1}{\sin\dfrac{t+1/n}{2}} dt$$

$$\geq -C - Cn^{-1} \int_{1/n}^{d_0/2} \frac{dt}{t\,(t+1/n)} \geq -C.$$

Thus, from (2.75) and (2.80) we obtain

$$n \int_{\Omega_0^-} \phi_1(\lambda)\, p_2^{(n)}(\lambda_0, \lambda)\, d\lambda \geq -C. \tag{2.81}$$

Then, using (1.14), (2.27), (2.15), (2.77), and the inequality $\dfrac{1}{t^2} \leq C\dfrac{\sin^2 1/2n}{\sin^2 t/2n}$, we obtain (2.72) for $x = 0$ from (2.81). Substituting λ_0 by $\lambda_0 + x/n$, we get (2.72) for any $|x| \leq nd_0/2$.

Now we are ready to prove (2.36). Denote $C_n = \sup\left|\dfrac{\partial}{\partial x}\mathcal{K}_n(x,y)\right|$. In view of (2.32)

$$C_n \leq \left|\left(v.p. \int_{|z-x|\leq 1} + \int_{|z-x|\geq 1}\right) \frac{\mathcal{K}_n(x,z)\,\mathcal{K}_n(z,y)}{z-x}\, dz\right| + o(1)$$
$$\leq |I_1(x,y)| + |I_2(x,y)| + o(1).$$

Using the Schwarz inequality and (2.28) with (2.29), we can estimate I_2 as follows:

$$|I_2(x,y)| \leq \mathcal{K}_n^{1/2}(x,x)\,\mathcal{K}_n^{1/2}(y,y) \leq C.$$

To estimate I_1 denote

$$\hat{t}_n^* = \sup\left\{t > 0 : |x-y| \leq t \Rightarrow \mathcal{K}_n(x,y) \geq \rho_n(\lambda_0)/2\right\},$$
$$t_n^* = \min\left\{\hat{t}_n^*, 1\right\}. \tag{2.82}$$

We will prove that the sequence t_n^* is bounded from below by some nonzero constant. Represent I_1 in the form

$$I_1(x,y) = v.p. \int\limits_{|t|\le t_n^*} \frac{\mathcal{K}_n(x,x+t)\,\mathcal{K}_n(x+t,y) - \mathcal{K}_n(x,x)\,\mathcal{K}_n(x,y)}{t}\,dt$$
$$+ \int\limits_{t_n^*\le|t|\le 1} \frac{\mathcal{K}_n(x,x+t)\,\mathcal{K}_n(x+t,y)}{t}\,dt = I_1' + I_1''.$$

Using (2.29), we have $|I_1''| \le C\,|\ln t_n^*|$. On the other hand, from (1.11) and the Cauchy inequality we obtain for any x, y, z

$$|\mathcal{K}_n(x,z) - \mathcal{K}_n(y,z)|^2 \le (\mathcal{K}_n(x,x) + \mathcal{K}_n(y,y) - 2\mathcal{K}_n(x,y))\,\mathcal{K}_n(z,z)$$
$$= \left(\left(\mathcal{K}_n^{1/2}(x,x) - \mathcal{K}_n^{1/2}(y,y)\right)^2 + 2\left(\mathcal{K}_n^{1/2}(x,x)\,\mathcal{K}_n^{1/2}(y,y) - \mathcal{K}_n(x,y)\right)\right)\mathcal{K}_n(z,z). \tag{2.83}$$

From (2.35) we get that the first term of (2.83) is bounded by $Cn^{-1/4}|x-y|^2$. The second term we rewrite as

$$\mathcal{K}_n^{1/2}(x,x)\,\mathcal{K}_n^{1/2}(y,y) - \mathcal{K}_n(x,y) = \frac{\mathcal{K}_n(x,x)\,\mathcal{K}_n(y,y) - \mathcal{K}_n^2(x,y)}{\mathcal{K}_n^{1/2}(x,x)\,\mathcal{K}_n^{1/2}(y,y) + \mathcal{K}_n(x,y)}.$$

Thus, for $|x-y| \le t_n^*$ we get

$$|\mathcal{K}_n(x,z) - \mathcal{K}_n(y,z)|^2 \le C\left(n^{-1/4}|x-y|^{3/2} + \mathcal{K}_n(x,x)\,\mathcal{K}_n(y,y) - |\mathcal{K}_n(x,y)|^2\right). \tag{2.84}$$

Hence, using (2.84), (2.72) and the Schwarz inequality, we obtain

$$|I_1'| \le C \int\limits_{|t|\le t_n^*} \frac{|\mathcal{K}_n(x,x+t) - \mathcal{K}_n(x,x)| + |\mathcal{K}_n(x+t,y) - \mathcal{K}_n(x,y)|}{|t|}\,dt$$
$$\le C\,(t_n^*)^{1/2}.$$

Finally, from the above estimates we have

$$C_n \le C\left(|\ln t_n^*| + (t_n^*)^{1/2}\right). \tag{2.85}$$

Note that if the sequence t_n^* is not bounded from below, then we have

$$C \le \rho_n(\lambda_0)/2 \le |\mathcal{K}_n(x+t_n^*,x) - \mathcal{K}_n(x,x)| \le C_n t_n^* \le C t_n^* \ln t_n^* + C t_n^*,$$

and we get a contradiction. Thus $t_n^* \geq d^*$ for some n-independent $d^* > 0$. Therefore, from (2.85) we obtain the first inequality of (2.36).

To prove the second inequality of (2.36), we observe that by (2.33) we have

$$\int\limits_{|x|\leq\mathcal{L}} \left|\frac{\partial}{\partial x}\mathcal{K}_n(x,y)\right|^2 dx = \int\limits_{|x|\leq\mathcal{L}} \left|\frac{\partial}{\partial y}\mathcal{K}_n(x,y)\right|^2 dx + o(1).$$

Then we rewrite the analog of (2.32) for $\frac{\partial}{\partial y}\mathcal{K}_n(x,y)$ as

$$\begin{aligned}\frac{\partial}{\partial y}\mathcal{K}_n(x,y) &= \left(v.p.\int\limits_{|z-y|\leq d*} + \int\limits_{|z|\leq 2\mathcal{L}} \mathbf{1}_{|z-y|\geq d*}\right)\frac{\mathcal{K}_n(x,z)\,\mathcal{K}_n(z,y)}{y-z}\,dz + O\left(\mathcal{L}^{-1}\right)\\ &= I_1(x,y) + I_2(x,y) + O\left(\mathcal{L}^{-1}\right).\end{aligned}$$

To complete the proof, it is enough to estimate $I_{1,2}^2$. Since in I_1 the domain of integration is symmetric with respect to y, we can write

$$\begin{aligned}I_1(x,y) = &\int\limits_{|z-y|\leq d^*} \frac{(\mathcal{K}_n(x,z) - \mathcal{K}_n(x,y))\,\mathcal{K}_n(z,y)}{y-z}\,dz\\ &+ \int\limits_{|z-y|\leq d^*} \frac{(\mathcal{K}_n(z,y) - \mathcal{K}_n(y,y))\,\mathcal{K}_n(x,y)}{y-z}\,dz.\end{aligned}$$

Now, using the Schwarz inequality and (2.28), we obtain

$$\begin{aligned}\left|I_1^2(x,y)\right| \leq 2d^*C &\int\limits_{|z-y|\leq d^*} \frac{|\mathcal{K}_n(x,z) - \mathcal{K}_n(x,y)|^2}{(z-y)^2}\,dz\\ &+ 2d^*\mathcal{K}_n^2(x,y) \int\limits_{|z-y|\leq d^*} \frac{|\mathcal{K}_n(z,y) - \mathcal{K}_n(y,y)|^2}{(z-y)^2}\,dz.\end{aligned}$$

Integrating the above inequality with respect to x and using (2.28) with (2.29), we get

$$\begin{aligned}\int \left|I_1^2(x,y)\right| dx \leq C &\int\limits_{|z-y|\leq d^*} \frac{|\mathcal{K}_n(z,y) - \mathcal{K}_n(y,y)|^2}{(z-y)^2}\,dz\\ &+ C \int\limits_{|z-y|\leq d^*} \frac{\mathcal{K}_n(z,z) + \mathcal{K}_n(y,y) - 2\mathcal{K}_n(z,y)}{(z-y)^2}\,dz.\end{aligned}$$

Using the bounds (2.83) in the second integral and (2.84) in the first one, in view of (2.72) we obtain the bound for I_1^2. To estimate I_2, we write

$$\int \left| I_2^2\left(x,y\right)\right| dx \le \int\limits_{|z|,|z'|\le 2\mathcal{L}} \mathbf{1}_{|z-y|>d^*}\mathbf{1}_{|z'-y|>d^*}\left|\frac{\mathcal{K}_n\left(y,z\right)\mathcal{K}_n\left(z,z'\right)\mathcal{K}_n\left(z',y\right)}{\left(z-y\right)\left(z'-y\right)}\right| dzdz'$$

$$\le C\int\limits_{|z|,|z'|\le 2\mathcal{L}} \mathbf{1}_{|z-y|>d^*}\mathbf{1}_{|z'-y|>d^*}\left(\left|\frac{\mathcal{K}_n\left(y,z\right)}{z-y}\right|^2+\left|\frac{\mathcal{K}_n\left(y,z'\right)}{z'-y}\right|^2\right) dzdz' \le C.$$

Above bounds for I_1 and I_2 prove the second inequality of (2.36). Thus, Lemma 2.9 is proved. ∎

Acknowledgement. The author is grateful to Dr. M.V. Shcherbina for the problem statement and fruitful discussions.

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