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On Commutative Systems of Nonselfadjoint Unbounded Linear Operators

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For a commutative system of nonselfadjoint unbounded operators A_1, A_2 the concept of colligation and associated open system is given. For these open systems, the consistency conditions are established and the conservation laws are obtained.

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The study of nonselfadjoint unbounded operators originates from classic works on the extensions of symmetric operators in Hilbert spaces by G. von Neumann and M.G. Krein. However, only beginning with the work by M.S. Livšic [1] the study of this class of operators gained a proper technique — the characteristic function. Further development of these methods was found in the works by A.V. Kuzhel [2, 3] and A.V. Shtraus [4]. Nonselfadjoint unbounded operators in rigged Hilbert spaces were studied by E.R. Tsekanovski and Yu.L. Shmul'yan [5]. A somewhat different approach to the study of unbounded nonselfadjoint operators, based on the analysis of the space of boundary values, was taken by V.A. Derkach and M.M. Malamud [6] and resulted in the analytical formalism for studying the properties of Weyl functions. The Shrödinger dissipative operator in the context of functional model was analyzed by B.S. Pavlov [7]. As for commutative systems of unbounded nonselfadjoint operators, there have not been appropriate approaches for studying. In the paper the methods of studying this class of operators are presented.

For the commutative systems of nonselfadjoint bounded operators, M.S. Livšic suggested an effective method resulted in the construction of functional and triangular models [8, 9]. The method is based on the generalization of the notion of colligation of these systems of operators and on the study of the consistency conditions for open systems. Therefore, it seems natural to give proper constructions for the case of commutative systems of unbounded nonselfadjoint operators as well. The paper is organized as follows. In Section 1 essential facts from the theory of nonselfadjoint unbounded operators are given, in particular, the notion of colligation and associated open system. In Section 2 these results are generalized for the commutative systems of unbounded nonselfadjoint operators.

1. Preliminary Information

I. The pioneer work by M.S. Livšic [1], where nonselfadjoint unbounded operators were studied, marked the beginning of the researches undertaken in this branch of functional analysis which found its further fruitful development in the papers [2–7]. The definition below plays an important role and it is an analogue of those given in [10] for the unbounded case.

Definition 1. *Let A be a linear operator acting in a separable Hilbert space H such that: a) the domain $\mathfrak{D}(A)$ of an operator A is dense in H, $\overline{\mathfrak{D}(A)} = H$; b) an operator A is dense in H; c) there exists a nonempty domain Ω $(\subset (\mathbb{C}\setminus\mathbb{R}))$ such that the resolvent $R_\alpha = (A-\alpha I)^{-1}$ is regular for all $\alpha \in \Omega$. Consider $E_\pm$, the Hilbert spaces, $\psi_- : E_- \to H$, $\psi_+ : H \to E_+$, $K : E_- \to E_+$, $\sigma_\pm : E_\pm \to E_\pm$, linear bounded operators and selfadjoint operators $\sigma_\pm$, $\sigma_\pm = \sigma_\pm^*$, that are boundedly invertible. A collection*

$$\Delta = \Delta(\alpha) = \left(\sigma_-, H\oplus E_-, \begin{bmatrix} A & \psi_- \\ \psi_+ & K \end{bmatrix}, H\oplus E_+, \sigma_+\right) \tag{1.1}$$

is said to be the colligation of an unbounded operator A if there exists $\alpha \in \Omega$ such that

1. $2\operatorname{Im}\alpha\cdot\psi_-^*\psi_- = K^*\sigma_+K - \sigma_-$, $2\operatorname{Im}\alpha\cdot\psi_+\psi_+^* = K\sigma_-^{-1}K^* - \sigma_+^{-1}$;
2. *the operators*

$$\begin{aligned} \varphi_+ &= \psi_+(A-\alpha I) : \mathfrak{D}(A) \to E_+, \\ \varphi_-^* &= \psi_-^*\left(A^* - \bar\alpha I\right) : \mathfrak{D}\left(A^*\right) \to E_- \end{aligned} \tag{1.2}$$

are such that

3. $K^*\sigma_+\varphi_+ + \psi_-^*(A-\bar\alpha I) = 0$, $K\sigma_-^{-1}\varphi_-^* + \psi_+\left(A^*-\alpha I\right) = 0$;
4. $2\operatorname{Im}\langle Ah, h\rangle = \langle \sigma_+\varphi_+h, \varphi_+h\rangle$, $\forall h \in \mathfrak{D}(A)$,
   $-2\operatorname{Im}\left\langle A^*\tilde h, \tilde h\right\rangle = \left\langle \sigma_-^{-1}\varphi_-^*\tilde h, \tilde\varphi_-^*\tilde h\right\rangle$, $\forall h \in \mathfrak{D}\left(A^*\right)$.

First of all, show that an arbitrary operator A satisfying the conditions a)–c) of the given definition may always be included in the colligation Δ (1.1). Really, let B_α and $\tilde B_\alpha$ be selfadjoint bounded operators (see [2, 3])

$$\begin{aligned} B_\alpha &= iR_\alpha - iR_\alpha^* + 2\operatorname{Im}\alpha\cdot R_\alpha^*R_\alpha, \\ \tilde B_\alpha &= iR_\alpha - iR_\alpha^* + 2\operatorname{Im}\alpha\cdot R_\alpha R_\alpha^*. \end{aligned} \tag{1.3}$$

Consider the subspace

$$E_+ = E_+(\alpha) = \operatorname{span}\{B_\alpha h : h \in H\}, \\ E_- = E_-(\alpha) = \operatorname{span}\left\{\tilde{B}_\alpha h : h \in H\right\}. \tag{1.4}$$

It is obvious that the bounded operator

$$T_\alpha = I + i2\operatorname{Im}\alpha \cdot R_\alpha \tag{1.5}$$

maps the subspace E_+ in E_- since [2, 3]

$$T_\alpha B_\alpha = \tilde{B}_\alpha T_\alpha. \tag{1.6}$$

Moreover, it is easy to see that

$$\langle Ah, f\rangle - \langle h, Af\rangle = i\,\langle B_\alpha(A-\alpha I)h, (A-\alpha I)f\rangle\,, \forall h, f \in (A), \tag{1.7}$$

$$\left\langle A^*\tilde{h}, \tilde{f}\right\rangle - \left\langle \tilde{h}, A^*\tilde{f}\right\rangle = -i\left\langle B_\alpha\left(A^* - \bar{\alpha}I\right)\tilde{h}, \left(A^* - \bar{\alpha}I\right)\tilde{f}\right\rangle, forall \tilde{h}, \tilde{f} \in \mathfrak{D}\left(A^*\right).$$

Now specifying the operators

$$\psi_-^* = \sqrt{\left|\tilde{B}_\alpha\right|}, \quad \psi_+ = \sqrt{|B_\alpha|}, \quad \sigma_- = \operatorname{sign} B_\alpha, \quad \sigma_+ = \operatorname{sign} B_\alpha, \tag{1.8}$$

where $\sqrt{|B|}$ and $\operatorname{sign} B$ for a selfadjoint bounded operator are understood in terms of the spectral decomposition B [10], from (1.6) we obtain

$$T_\alpha\sqrt{|B_\alpha|} = \sqrt{\left|\tilde{B}_\alpha\right|}T_\alpha, \quad T_\alpha \cdot \operatorname{sign} B_\alpha = \operatorname{sign}\tilde{B}_\alpha \cdot T_\alpha. \tag{1.9}$$

Setting $K = -\sigma_+ T_\alpha^*$, it is easy to verify that the colligation relations (1.2) follow from (1.3), (1.7), and (1.9).

R e m a r k 1.1. If the operators B_α, $\tilde{B}_\alpha$ (1.1) are boundedly invertible on E_+ and E_- (1.4), then setting

$$\psi_- = \tilde{B}_\alpha, \quad \psi_+ = P_\alpha, \quad \sigma_- = \tilde{B}_\alpha, \quad \sigma_+ = B_\alpha, \quad K = -T_\alpha^*, \tag{1.10}$$

where P_α is an orthoprojector on E_+ (1.4), it is easy to verify that the conditions of colligation (1.2) also take place.

II. Open systems associated with colligations [10] play an important role in the study of nonselfadjoint operators. Let $u_-(t)$ be a vector-function from E_- defined on $[0, T]$, and h be a vector from H. The open system $F_\Delta = \{R_\Delta, S_\Delta\}$

associated with the colligation Δ (1.1) is the pair of maps [10], $R_\Delta : H \oplus E_- \to H$, $S_\Delta : H \oplus E_- \to H \oplus E_+$,

$$F_\Delta : \begin{cases} R_\Delta\left(h, u_-(t)\right) = h(t), \\ S_\Delta\left(h, u_-(t)\right) = \left(h_T, u_+(t)\right), \end{cases}$$

defined as follows. The operator R_Δ is specified by using the Cauchy problem

$$R_\Delta : \begin{cases} i\dfrac{d}{dt}h(t) + Ay(t) = \alpha\psi_- u_(t), \\ y(t) = h(t) + \psi_- u_-(t) \in \mathfrak{D}(A), \\ h(0) = h, \quad t \in [0, T], \end{cases} \tag{1.11}$$

and the transfer mapping S_Δ has the form:

$$S_\Delta : \begin{cases} u_+(t) = Ku_-(t) - i\varphi_+ y(t), \\ h_T = h(T), \quad t \in [0, T], \end{cases} \tag{1.12}$$

where $h(t)$ is a solution of (1.11), and $y(t) \in \mathfrak{D}(A)$ is defined by $h(t)$ and $u_-(t)$ using formula (1.11).

R e m a r k 1.2. If $u_-(t) \equiv 0$ in (1.11), then $y(t) = h(t) \in \mathfrak{D}(A)$, and we obtain the Cauchy problem

$$\begin{cases} i\dfrac{d}{dt}h(t) + Ah(t) = 0, \\ h(0) = h \in \mathfrak{D}(A), \quad t \in [0, T]. \end{cases}$$

The solvability of this Cauchy problem is equivalent to the existence of the strongly continuous semigroup $Z_t = \exp\{itA\}$, with $h(t) = Z_t h$. So, the solutions of the Cauchy problem (1.11) exist if the operator A is an infinitesimal operator of the strongly continuous semigroup Z_t. The well-known theorem by Miyader–Feller–Fillips [11] gives the necessary and sufficient conditions for the closed densely defined operator A, when the resolvent $R_\lambda = (A - \lambda I)^{-1}$ is regular in the semiplane $\mathbb{C}_-(\omega) = \{\lambda \in \mathbb{C} : \omega + \operatorname{Im}\lambda < 0\}$, $\omega \in \mathbb{R}$ ($|\omega| < \infty$), and, moreover, when $\lambda \in \mathbb{C}_-(\omega)$, the estimations

$$\|R_\lambda^n\| \le M|\omega + \operatorname{Im}\lambda|^{-n}, \quad \forall n \in \mathbb{Z}_+,$$

take place.

R e m a r k 1.3. Let $u_-(t)$ be differentiable. Then (1.11) yields that $y(t)$ also has the derivative and satisfies the nonhomogenous equation

$$i\frac{d}{dt}y(t) + Ay(t) = \psi_-\left(iu_-'(t) + \alpha u_-(t)\right).$$

The solution of this equation exists if:

1) the operator A meets the conditions of the Miyader–Feller–Fillips theorem;
2) $y(0) = h + \psi_- u_-(0) \in \mathfrak{D}(A)$;
3) the function $f(t) = \psi_- \left(i u'_-(t) + \alpha u_-(t) \right)$ is twice continuously differentiable, and $f(0) \in \mathfrak{D}(A)$.

So, if the conditions 1)–3) are met, then there always exists $y(t)$, and thus $h(t)$ (1.11) exists also.

Theorem 1.1. *The conservation law*

$$\|h\|^2 + \int\limits_0^T \langle \sigma_- u_-(t), u_-(t) \rangle \, dt = \|h_T\|^2 + \int\limits_0^T \langle \sigma_+ u_+(t), u_+(t) \rangle \, dt \qquad (1.13)$$

holds for the open system $F_\Delta = \{R_\Delta, S_\Delta\}$ (1.11), (1.12) associated with the colligation Δ (1.1).

P r o o f. Equation (1.11) yields

$$\frac{d}{dt}\|h(t)\|^2 = \langle iAy(t) - i\alpha\psi_- u_-(t), , y(t) - \psi_- u_-(t) \rangle + \langle y(t) - \psi_- u_-(t),$$
$$iAy(t) - i\alpha\psi_- u_-(t) \rangle = -2 \operatorname{Im}\langle Ay(t), y(t) \rangle - 2 \operatorname{Im} \alpha \, \|\psi_- u_-(t)\|^2$$
$$- \langle i(A - \bar{\alpha} I) y(t), \psi_- u_-(t) \rangle - \langle \psi_- u_-(t), i(A - \bar{\alpha} I) y(t) \rangle .$$

Using the relations 1–4 (1.2), we get

$$\frac{d}{dt}\|h(t)\|^2 = - \langle \sigma_+ \varphi_+ y(t), \varphi_+ y(t) \rangle + \langle \sigma_- u_-(t), \sigma_- u_-(t) \rangle - \langle J_+ K u_-(t),$$
$$K u_-(t) \rangle + \langle i\sigma_+ \varphi_+ y(t), K u_-(t) \rangle + \langle J_+ K u_-(t), i\varphi_+ y(t) \rangle$$
$$= \langle \sigma_- u_-(t), u_-(t) \rangle - \langle \sigma_+ \left[K u_-(t) - i\varphi_+ y(t) \right], \left[K u_-(t) - i\varphi_+ y(t) \right] \rangle .$$

As a result, we obtain the following conservation law:

$$\frac{d}{dt}\|h(t)\|^2 = \langle \sigma_- u_-(t), u_-(t) \rangle - \langle \sigma_+ u_+(t), u_+(t) \rangle , \qquad (1.14)$$

which yields (1.13) after integration. ■

Consider an open system dual to $F_\Delta = \{R_\Delta, S_\Delta\}$. Denote by $\tilde{u}_+(t)$ a vector function from E_+ defined on $[0, T]$ $(0 < T < \infty)$, and by $\tilde{h}$ — a vector from H. A pair of mappings $R_\Delta^+ : H + E_+ \to H$, $S_\Delta^+ : H + E_+ \to H + E_-$,

$$F_\Delta^+ : \begin{cases} R_\Delta^+ \left(\tilde{h}, \tilde{u}_+(t) \right) = \tilde{h}(t), \\ S_\Delta \left(\tilde{h}, \tilde{u}_+(t) \right) = \left(\tilde{h}_0, \tilde{u}_-(t) \right) \end{cases}$$

is said to be the dual open system $F_\Delta^+ = \left\{R_\Delta^+, S_\Delta^+\right\}$ associated with the colligation Δ (1.1). Besides, R_Δ^+ is specified by the Cauchy problem

$$R_\Delta^+ : \begin{cases} i\dfrac{d}{dt}\tilde{h}(t) - A^*\tilde{y}(t) = -\bar{\alpha}\psi_+^*\tilde{u}_+(t), \\ \tilde{y}(t) = \psi_+^*\tilde{u}_+(t) - \tilde{h}(t) \in \mathfrak{D}\left(A^*\right), \\ \tilde{h}(T) = \tilde{h}, \quad t \in [0,T]; \end{cases} \tag{1.15}$$

and S_Δ^+ is given by

$$S_\Delta^+ : \begin{cases} \tilde{u}_-(t) = K^*\tilde{u}_+(t) + i\varphi_-^*\tilde{y}(t), \\ \tilde{h}_0 = \tilde{h}(0), \quad t \in [0,T], \end{cases} \tag{1.16}$$

$\tilde{h}(t)$ and $\tilde{y}(t)$ can be found from (1.15). Similarly to Remarks 1.2 and 1.3, it is easy to establish the solvability of (1.15) under natural restrictions on the class of operators A^* and the class of functions $\tilde{u}_+(t)$.

Theorem 1.2. *Let $F_\Delta^+ = \left\{R_\Delta^+, S_\Delta^+\right\}$ be the dual open system (1.15), (1.16) of the colligation Δ (1.1). Then*

$$\|\tilde{h}\|^2 + \int\limits_0^T \left\langle \sigma_+^{-1}\tilde{u}_+(t), \tilde{u}_+(t)\right\rangle dt = \left\|\tilde{h}_0\right\|^2 + \int\limits_0^T \left\langle \sigma_-^{-1}\tilde{u}_-(t), \tilde{u}_-(t)\right\rangle dt. \tag{1.17}$$

P r o o f. Equation (1.15) yields

$$\frac{d}{dt}\|\tilde{h}(t)\|^2 = \left\langle -iA^*\tilde{y}(t) + i\bar{\alpha}\psi_+^*\tilde{u}_+(t), \psi_+^*\tilde{u}_+(t) - \tilde{y}(t)\right\rangle$$
$$+ \left\langle \psi_+^*\tilde{u}_+(t) - \tilde{y}(t), -iA^*\tilde{y}(t) + i\bar{\alpha}\psi_+^*\tilde{u}_+(t)\right\rangle = -2\operatorname{Im}\left\langle A^*\tilde{y}(t), \tilde{y}(t)\right\rangle$$
$$+2\operatorname{Im}\alpha\left\|\psi_-^*\tilde{u}_+(t)\right\|^2 - \left\langle i\left(A^* - \alpha I\right)\tilde{y}(t), \psi_+^*\tilde{u}_+(t)\right\rangle - \left\langle \psi_+^*\tilde{u}_+(t), i\left(A^* - \alpha I\right)\tilde{y}(t)\right\rangle.$$

Using the second relations of 1–4 (1.2), we have

$$\frac{d}{dt}\|h(t)\|^2 = \left\langle \sigma_-^{-1}\varphi_-^*\tilde{y}(t), \varphi_-^*\tilde{y}(t)\right\rangle + \left\langle \sigma_-^{-1}K^*\tilde{u}_+(t), K^*\tilde{u}_+(t)\right\rangle$$
$$- \left\langle \sigma_+^{-1}\tilde{u}_+(t), \tilde{u}_+(t)\right\rangle + \left\langle i\sigma_-^{-1}\varphi_-^*\tilde{y}(t), K^*\tilde{u}_+(t)\right\rangle + \left\langle \sigma_-^{-1}K^*\tilde{u}_+(t), i\varphi_-^*\tilde{y}(t)\right\rangle$$
$$= \left\langle \sigma_-^{-1}\left[K^*\tilde{u}_+(t) + i\varphi_-^*\tilde{y}(t)\right], \left[K^*\tilde{u}_+(t) + i\varphi_-^*\tilde{y}(t)\right]\right\rangle - \left\langle \sigma_+^{-1}\tilde{u}_+(t), \tilde{u}_+(t)\right\rangle.$$

Therefore

$$\frac{d}{dt}\left\|\tilde{h}(t)\right\|^2 = \left\langle \sigma_-^{-1}\tilde{u}_-(t), \tilde{u}_-(t)\right\rangle - \left\langle \sigma_+^{-1}\tilde{u}_+(t), \tilde{u}_+(t)\right\rangle, \tag{1.18}$$

which yields (1.17) after integration. ■

The following theorem establishes an important correlation between the open systems F_Δ (1.11), (1.12) and F_Δ^+ (1.15), (1.16).

Theorem 1.3. *Let $h(t)$ and $u_+(t)$ be defined by $u_-(t)$ using the relations (1.11), (1.12) of the open system F_Δ, and the vector functions $\tilde{h}(t)$ and $\tilde{u}_-(t)$ be constructed by equalities (1.15), (1.16) of the dual open system F_Δ^+. Then the equality*

$$\left\langle h_T, \tilde{h} \right\rangle + \int_0^T \langle u_+(t), \tilde{u}_+(t) \rangle \, dt = \left\langle h, \tilde{h}_0 \right\rangle + \int_0^T \langle u_-(t), \tilde{u}_-(t) \rangle \, dt \tag{1.19}$$

is true.

P r o o f. From (1.11) and (1.15) it follows that

$$\frac{d}{dt} \left\langle h(t), \tilde{h}(t) \right\rangle = \left\langle iAy(t) - i\alpha\psi_- u_-(t), \psi_+^* \tilde{u}_+(t) - \tilde{y}(t) \right\rangle$$

$$+ \left\langle y(t) - \psi_- u_-(t), -iA\tilde{y}(t) + i\bar{\alpha}\psi_+^* \tilde{u}_+(t) \right\rangle = \left\langle i(A - \alpha I) y(t), \psi_+^* \tilde{u}_+(t) \right\rangle$$

$$+ \left\langle \psi_- u_-(t), i \left(A^* - \bar{\alpha} I \right) \tilde{y}(t) \right\rangle .$$

Taking into account 2 (1.2) and (1.12), (1.16), we obtain

$$\frac{d}{dt} \left\langle h(t), \tilde{h}(t) \right\rangle = \langle i\varphi_+ y(t), \tilde{u}_+(t) \rangle + \left\langle u_-(t), i\varphi_-^* \tilde{y}(t) \right\rangle$$

$$= \langle Ku_-(t) - u_+(t), \tilde{u}_+(t) \rangle + \langle u_-(t), \tilde{u}_-(t) - K^* \tilde{u}_+(t) \rangle$$

$$= \langle u_-(t), \tilde{u}_-(t) \rangle - \langle u_+(t), \tilde{u}_+(t) \rangle$$

and, consequently,

$$\frac{d}{dt} \left\langle h(t), \tilde{h}(t) \right\rangle = \langle u_-(t), \tilde{u}_-(t) \rangle - \langle u_+(t), \tilde{u}_+(t) \rangle . \tag{1.20}$$

Equality (1.19) follows from (1.20) after integration. ■

III. Let $u_-(t)$ in the open system F_Δ (1.11) be the plane wave $u_-(t) = e^{i\lambda t} u_-(0)$. And let the vector functions $h(t)$, $y(t)$, and $u_+(t)$ depend on t in a similar way: $h(t) = e^{i\lambda t} h$, $y(t) = e^{i\lambda t} y$, $u_+(t) = e^{i\lambda t} u_+(0)$, where h, $y \in H$, and $u_+(0)$ do not depend on t. Then (1.11), (1.12) yield

$$\begin{cases} -\lambda h + Ay = \alpha\psi_- u_-(0), \\ h - y = -\psi_- u_-(0), \\ u_+(0) = Ku_-(0) - i\varphi_+ y, \end{cases} \tag{1.21}$$

where $y \in \mathfrak{D}(A)$.

Thus, if $\lambda \in \Omega$, then

$$\begin{cases} y = (\alpha - \lambda)(A - \lambda I)^{-1}\psi_- u_-(0), \\ h = -(A - \alpha I)(A - \lambda I)^{-1}\psi_- u_-(0), \\ u_+(0) = S_\Delta(\lambda) u_-(0), \end{cases} \tag{1.22}$$

where $S_\Delta(\lambda)$ is a characteristic function of the colligation Δ (1.1),

$$S_\Delta(\lambda) = K + i(\lambda - \alpha)\psi_+(A - \alpha I)(A - \lambda I)^{-1}\psi_-. \tag{1.23}$$

The function $S_\Delta(\lambda)$ is normalized at the point $\lambda = \alpha$, $S_\Delta(\alpha) = K$. Consider the operator function

$$T_{\lambda,\alpha} = (A - \alpha I)(A - \lambda I)^{-1} = I + (\lambda - \alpha) R_\lambda. \tag{1.24}$$

Then $S_\Delta(\lambda)$ can be written in the form

$$S_\Delta(\lambda) = K + i(\lambda - \alpha)\psi_+ T_{\lambda,\alpha}\psi_-. \tag{1.25}$$

From (1.14) it follows easily (see, for instance, [10]) that

$$\frac{\sigma_- - S_\Delta^*(w)\sigma_+ S_\Delta(\lambda)}{i(\lambda - \bar{w})} = \psi_-^* T_{w,\alpha}^* T_{\lambda,\alpha}\psi_-. \tag{1.26}$$

Analogously, if $\tilde{u}_+(t)$ in the dual open system $F_\Delta^+ = \left\{R_\Delta^+, S_\Delta^+\right\}$ (1.15), (1.16) is given by $\tilde{u}_+(t) = e^{i\bar{\lambda}(t-T)}\tilde{u}_+(T)$, where $\tilde{u}_+(T)$ is an independent of t vector from E_+, and if $\tilde{h}(t)$, $\tilde{y}(t)$, $\tilde{u}_-(t)$ also have the same dependency on t, $\tilde{h}(t) = e^{i\bar{\lambda}(t-T)}\tilde{h}$, $\tilde{y}(t) = e^{i\bar{\lambda}(t-T)}\tilde{y}$, $\tilde{u}_-(t) = e^{i\bar{\lambda}(t-T)}u_-(T)$, then (1.15), (1.16) yield

$$\begin{cases} \bar{\lambda}\tilde{h} + A^*\tilde{y} = \bar{\alpha}\psi_+^*\tilde{u}_+(T), \\ \tilde{h} + \tilde{y} = \psi_+^*\tilde{u}_+(T), \\ \tilde{u}_-(T) = K^*\tilde{u}_+(T) + i\varphi_-^*\tilde{y}, \end{cases} \tag{1.27}$$

where $\tilde{y} \in \mathfrak{D}\left(A^*\right)$. Hence, if $\lambda \in \Omega$, this implies

$$\begin{cases} \tilde{y} = \left(\bar{\alpha} - \bar{\lambda}\right)\left(A^* - \bar{\lambda} I\right)^{-1}\psi_+^*\tilde{u}_+(T), \\ \tilde{h} = \left(A^* - \bar{\alpha} I\right)\left(A^* - \bar{\lambda} I\right)^{-1}\psi_+^*\tilde{u}_+(T), \\ \tilde{u}_-(T) = \overset{+}{S}_\Delta(\lambda)\tilde{u}_+(T), \end{cases} \tag{1.28}$$

where the function $\overset{+}{S}_\Delta(\lambda)$ is given by

$$\overset{+}{S}_\Delta(\lambda) = K^* - i\left(\bar{\lambda} - \bar{\alpha}\right)\psi_-^*\left(A^* - \bar{\alpha} I\right)\left(A^* - \bar{\lambda} I\right)^{-1}\psi_+^*. \tag{1.29}$$

It is obvious that the functions $S_\Delta(\lambda)$ (1.25) and $\overset{+}{S_\Delta}(\lambda)$ (1.29) satisfy the relation

$$\overset{+}{S_\Delta}(\lambda) = S_\Delta^*(\lambda). \tag{1.30}$$

Using (1.12), it is easy to show that

$$\frac{S_\Delta(w)\sigma_-^{-1}S_\Delta^*(\lambda) - \sigma_+^{-1}}{i\left(\bar{\lambda} - w\right)} = \psi_+ T_{w,\alpha} T_{\lambda,\alpha}^* \psi_+^*. \tag{1.31}$$

Finally, (1.18), with (1.29) being taken into account, implies

$$\frac{S_\Delta(\lambda) - S_\Delta(w)}{i(\lambda - w)} = \psi_+ T_{w,\alpha} T_{\lambda,\alpha}\psi_-. \tag{1.32}$$

IV. Consider the operator function $K(\lambda, w) : E_- \oplus E_+ \to E_- \oplus E_+$,

$$K(\lambda, w) = \begin{bmatrix} \dfrac{\sigma_1 - S_\Delta^*(w)\sigma_+ S_\Delta(\lambda)}{i\left(\lambda - \bar{w}\right)} & \dfrac{S_\Delta^*(\lambda) - S_\Delta^*(w)}{i\left(\bar{w} - \bar{\lambda}\right)} \\ \dfrac{S_\Delta(\lambda) - S_\Delta(w)}{i(\lambda - w)} & \dfrac{S_\Delta(w)\sigma_-^{-1}S_\Delta^*(\lambda) - \sigma_+^{-1}}{i\left(\bar{\lambda} - w\right)} \end{bmatrix}, \tag{1.33}$$

assuming that λ, w, $\alpha \in \Omega$. The formulae (1.26), (1.31), (1.32) imply that the kernel $K(\lambda, w)$ (1.33) is positively defined [10].

A subspace $H_1 \subseteq H$ is said to be reducing for a densely defined operator A, if at every point of regularity $\lambda \in \Omega$ of the resolvent $R_\lambda = (A - \lambda I)^{-1}$ there takes place $R_\lambda P_1 = P_1 R_\lambda$, where P_1 is an orthoprojector on H_1. For the colligation Δ (1.1), define the subspace

$$H_1 = \operatorname{span}\left\{R_\lambda \psi_- u_- + R_w^* \psi_+^* u_+ : u_\pm \in E_\pm, \lambda, w \in \Omega\right\}. \tag{1.34}$$

Theorem 1.4. *The subspace H_1 (1.34) reduces the operator A, besides, the contraction of A on $H_0 = H \ominus H_1$ is a selfadjoint operator.*

The proof of the statement follows from the colligation relations (1.2) and it is standard [10].

A colligation Δ (1.1) is said to be simple if $H = H_1$ (1.34).

Let two colligations Δ and Δ' be given such that $E_\pm = E'_\pm$, $\sigma_\pm = \sigma'_\pm$, $K = K'$, and, moreover, $\alpha = \alpha' \in \Omega \cap \Omega'$ $(\neq \varnothing)$. These colligations are called unitarily equivalent [10] if there exists a unitary operator U: $H \to H'$ such that

$$UA = A'U, \quad U\mathfrak{D}(A) = \mathfrak{D}\left(A'\right), \quad UA^* = \left(A'\right)^* U, \quad U\mathfrak{D}\left(A'\right)^* = \mathfrak{D}\left(\left(A'\right)\right),$$
$$U\psi_- = \psi'_-, \quad \psi'_+ U = \psi_+.$$

Theorem 1.5. *Let Δ and Δ' be simple colligations, $E_\pm = E'_\pm$, $\sigma_\pm = \sigma'_\pm$ of which are invertible, and $\alpha = \alpha' \in \Omega \cap \Omega'$ ($\neq \varnothing$). Then if in some neighborhood $U_\delta(\alpha) \subset \Omega \cap \Omega'$ of the point α the characteristic functions (1.23) coincide, $S_\Delta(\lambda) = S_{\Delta'}(\lambda)$, then the colligations Δ and Δ' are unitarily equivalent.*

The proof of the theorem follows easily from (1.26), (1.31), and (1.32).

Thus, the characteristic function $S_\Delta(\lambda)$ (1.23) defines the colligation Δ (1.1) up to the unitary equivalency.

V. Let us describe a class of functions generated by the characteristic functions $S_\Delta(\lambda)$ of the colligations Δ (1.1). Consider the following functions from H_1 (1.34):

$$F(\lambda, u_-) = T_{\lambda,\alpha}\psi_- u_-, \quad \tilde{F}(\lambda, u_+) = T^*_{\lambda,\alpha}\psi^*_+ u_+, \tag{1.35}$$

where $u_\pm \in E_\pm$, and λ, $\alpha \in \Omega$.

Theorem 1.6. *The operator $T_{w,\lambda}$ (1.24) (λ, $w \in \Omega$) acts on $F(\lambda, u_-)$ and $\tilde{F}(\lambda, u_+)$ in the following way:*

$$\begin{array}{ll} 1) & T_{w,\lambda}F(\lambda, u_-) = F(w, u_-); \\ 2) & T^*_{w,\lambda}\tilde{F}(\lambda, u_+) = \tilde{F}(w, u_+); \\ 3) & T_{w,\bar{\lambda}}\tilde{F}(\lambda, u_+) = -F\left(w, \sigma_-^{-1}S_\Delta(\lambda)u_+\right); \\ 4) & T^*_{w,\bar{\lambda}}F(\lambda, u_+) = -\tilde{F}(w, \sigma_+ S_\Delta(\lambda)u_-), \end{array} \tag{1.36}$$

where $S_\Delta(\lambda)$ is the characteristic function of (1.25), and λ, w, $\alpha \in \Omega$.

P r o o f. Equations 1), 2) from (1.36) follow from the chain identity $T_{w,\lambda}T_{\lambda,\alpha} = T_{w,\alpha}$. Since

$$T_{w,\bar{\lambda}}\tilde{F}(\lambda, u_+) = T_{w,\bar{\lambda}}T^*_{\lambda,\alpha}\psi^*_+ u_+ = T_{w,\alpha}T_{\alpha,\bar{\lambda}}T^*_{\lambda,\alpha}\psi^*_+ u_+,$$

we have to find the expression

$$T_{\alpha,\bar{\lambda}}T^*_{\lambda,\alpha}\psi^*_+ u_+ = \left(I + \left(\alpha - \bar{\lambda}\right)R_\alpha\right)T^*_{\lambda,\alpha}\psi^*_+ u_+ = T^*_{\lambda,\alpha}\psi^*_+ u_+$$

$$+ (\alpha - \bar{\alpha})R_\alpha T^*_{\lambda,\alpha}\psi^*_+ u_+ + \left(\bar{\alpha} - \bar{\lambda}\right)R_\alpha T^*_{\lambda,\alpha}\psi^*_+ u_+.$$

Taking into account the equality

$$R_\alpha = -i\psi_-\sigma_-^{-1}\psi_- + R^*_\alpha + (\alpha - \bar{\alpha})R_\alpha R^*_\alpha,$$

which follows from 4. (1.2) and (1.7), and using the Hilbert identity $R_\lambda = T_{\lambda,\alpha}R_\alpha$ for the resolvents R_λ, we obtain

$$T_{\alpha,\bar{\lambda}}T^*_{\lambda,\alpha}\psi^*_+ u_+ = \psi^*_+ u_+ + (\alpha - \bar{\alpha})R_\alpha\psi^*_+ u_+ + \psi_-\sigma_-^{-1}\left[K^* - S^*_\Delta(\lambda)\right]u_+.$$

Finally, since $\psi_+^* + (\alpha - \bar{\alpha}) R_\alpha \psi_+^* u_+ + \psi_+ \sigma_-^{-1} K^* = 0$ (see 2, 3 (1.2)), we obtain that

$$T_{\alpha,\bar{\lambda}} T_{\bar{\lambda},\alpha}^* \psi_+^* u_+ = -\psi_- \sigma_-^{-1} S_\Delta^*(\lambda) u_+,$$

which proves 3) from (1.36). The proof of 4) from (1.36) is analogous. ∎

This theorem implies the statement below.

Theorem 1.7. *The family of operators $T_{w,z}$ (1.24) $(w,\ z \in \Omega)$ acts on the functions $F(\lambda, u_-)$ and $F(\lambda, u_+)$ (1.35) in the following way:*

$$\begin{aligned}
&1)\quad T_{w,z} F(\lambda, u_-) = \frac{w-z}{w-\lambda} F(w, u_-) + \frac{\lambda - z}{\lambda - w} F(\lambda, u_-);\\
&2)\quad T_{w,z}^* \tilde{F}(\lambda, u_+) = \frac{\bar{w}-\bar{z}}{\bar{w}-\bar{\lambda}} \tilde{F}(w, u_+) + \frac{\bar{\lambda}-\bar{z}}{\bar{\lambda}-\bar{w}} \tilde{F}(\lambda, u_+);\\
&3)\quad T_{w,z} \tilde{F}(\lambda, u_+) = \frac{\bar{\lambda}-z}{\bar{\lambda}-w} \tilde{F}(\lambda, u_+) - \frac{w-z}{w-\bar{\lambda}} F\left(w, \sigma_-^{-1} S_\Delta^*(\lambda) u_+\right);\\
&4)\quad T_{w,z}^* F(\lambda, u_-) = \frac{\lambda-\bar{z}}{\lambda-\bar{w}} F(\lambda, u_-) - \frac{\bar{w}-\bar{z}}{\bar{w}-\lambda} \tilde{F}(w, \sigma_+ S_\Delta(\lambda) u_-);
\end{aligned} \tag{1.37}$$

for all $u_\pm \in E_\pm$ and all $\lambda,\ w,\ z,\ \alpha \in \Omega$.

The class $\Omega_\alpha(\sigma_-, \sigma_+)$. *Let $E_\pm$ be Hilbert spaces, $\sigma_\pm$ be selfadjoint invertible operators in E and $\alpha \in \mathbb{C} \setminus \mathbb{R}$. An operator function $S(\lambda)$: $E_- \to E_+$ belongs to the class $\Omega_\alpha(\sigma_-, \sigma_+)$ if:*

1) the function $S(\lambda)$ is holomorphic in some neighborhood $U_\delta(\alpha) = \{\lambda \in \mathbb{C} : |\lambda - \alpha| < \delta\}$ of the point α and $S(\alpha) \neq 0$;

2) the kernel $K(\lambda, w)$ (1.33) is Hermitian positive for all $\lambda,\ w \in U_\delta(\alpha)$.

It is obvious that the characteristic function $S_\Delta(\lambda)$ (1.25) belongs to the class $\Omega_\alpha(\sigma_-, \sigma_+)$.

Theorem 1.8. *Let an operator function $S(\lambda)$: $E_- \to E_+$ belong to the class $\Omega_\alpha(\sigma_-, \sigma_+)$. Then there exists such a colligation Δ (1.1) that its characteristic function $S_\Delta(\lambda)$ (1.25) coincides with $S(\lambda)$, $S_\Delta(\lambda) = S(\lambda)$, for all $\lambda \in U_\delta(\alpha)$.*

P r o o f. Following [10], denote by $e_\lambda f$ the "δ-function", the support of which is concentrated at the point $\lambda \in U_\delta(\alpha)$ and $e_\lambda f$ in this point takes the value $f = (u_-, u_+) \in E_- \oplus E_+$. Consider the manifold L generated by the finite linear combinations $\sum_1^N e_{\lambda_k} f_k$ $(N \in \mathbb{N})$. Using $K(\lambda, w)$ (1.33), on L specify the nonnegative bilinear form

$$\langle e_\lambda f, e_w g \rangle_K \stackrel{\text{def}}{=} \langle K(\lambda, w) f, g \rangle_{E_- \oplus E_+}. \tag{1.38}$$

Closing L by the norm generated by form (1.38) and factorizing by the kernel of metric (1.38), we obtain the Hilbert space H_K [10].

Proceeding from (1.37), in H_K specify the family of operators $T_{w,z}$:

$$T_{w,z}e_\lambda f = e_w\left(\frac{w-z}{w-\lambda}u_- - \frac{w-z}{w-\bar\lambda}\sigma_-^{-1}S^*(\lambda)u_+, 0\right) + e_\lambda\left(\frac{\lambda-z}{\lambda-w}u_-, \frac{\bar\lambda-z}{\bar\lambda-w}u_+\right), \tag{1.39}$$

where λ, $w \in U_\delta(\alpha)$, and $f = (u_-, u_+)$. It is easy to show that

$$T_{w,z}^*e_\lambda f = e_w\left(0, \frac{\bar w-\bar z}{\bar w-\bar\lambda}u_+ - \frac{\bar w-\bar z}{\bar w-\lambda}\sigma_+S(\lambda)u_-\right) + e_\lambda\left(\frac{\lambda-\bar z}{\lambda-\bar w}u_-, \frac{\bar\lambda-\bar z}{\bar\lambda-\bar w}u_+\right). \tag{1.40}$$

Let

$$K = S(\lambda), \quad \psi_-u_- = e_\alpha u_-, \quad \psi_+^*u_+ = e_\alpha u_+. \tag{1.41}$$

It is obvious that the colligation relations 1 (1.2) are true. Really,

$$\left\langle\psi_-u_-, \psi_-u_-'\right\rangle = \left\langle K(\alpha,\alpha)u_-, u_-'\right\rangle = \left\langle\frac{\sigma_- - K^*\sigma_+K}{i\left(\alpha-\bar\alpha\right)}u_-, u_-'\right\rangle,$$

which proves the first condition in 1. (1.2). Simple calculations show that

$$\begin{aligned}\psi_-^*e_\lambda f &= \frac{\sigma_- - K^*\sigma_+S(\lambda)}{i(\lambda-\bar\alpha)}u_- + \frac{S^*(\lambda)-K^*}{i\left(\bar\alpha-\bar\lambda\right)}u_+,\\ \psi_+e_\lambda f &= \frac{S(\lambda)-K}{i(\lambda-\alpha)}u_- + \frac{K\sigma_-^{-1}S^*(\lambda)-\sigma_+^{-1}}{i\left(\bar\lambda-\alpha\right)}u_+.\end{aligned} \tag{1.42}$$

Since the first relation in 3 (1.2) can be written as $K^*\sigma_+\psi_+ + \psi_-^*T_{\alpha,\bar\alpha} = 0$, its proof follows easily from (1.39) and (1.42).

Relations 4 (1.2) are proven in a similar way after being rewritten in terms of $T_{\alpha,\bar\alpha}$.

From (1.39), it is easy to calculate how the resolvent $R_w = (w-z)^{-1}\left(T_{w,z}-I\right)$ acts on the elements $e_\lambda f$, which, in virtue of the relation $AR_w = I + wR_w$, results in the conclusion that the operator A in H_K has the form

$$Ae_\lambda f = e_\lambda\left(\lambda u_-, \bar\lambda u_+\right), \tag{1.43}$$

where the domain $\mathfrak{D}(A)$ is

$$\mathfrak{D}(A) = \left\{\sum_{p=1}^N e_{\lambda_p}f_p \in H_K : \lambda_p \in U_\delta(\alpha), f_p = \left(u_-^p, u_+^p\right) \in E_-\oplus E_+, u_-^p = \sigma_-^{-1}S^*\left(\lambda_p\right)u_+^p, 1\le p\le N, N\le\infty\right\}. \tag{1.44}$$

Construct the colligation

$$\Delta_K = \left(\sigma_-, H_K \oplus E_-, \begin{bmatrix} A & \psi_- \\ \psi_+ & K \end{bmatrix}, H_K \oplus E_+, \sigma_+\right), \tag{1.45}$$

where K, ψ_-, ψ_+, and A are given by formulae (1.41) and (1.43), (1.44) correspondingly. Finally, show that the characteristic function $S_{\Delta_K}(\lambda)$ (1.25) of the colligation Δ_K (1.45) coincides with $S(\lambda)$. Equations (1.39), (1.41) imply

$$T_{\lambda,\alpha}\psi_- u_- = T_{\lambda,\alpha} e_\alpha (u_-, 0) = e_\lambda (u_-, 0).$$

Using the structure of the operator ψ_+ (1.42), we obtain that

$$\psi_+ T_{\lambda,\alpha}\psi_- u_- = \frac{S(\lambda) - K}{i(\lambda - \alpha)} u_-,$$

which concludes the proof. ■

Conservative and not only (passive and others) systems from one variable were studied in [12].

2. Commutative Colligations and Open Systems. Systems of Unbounded Operators

I. In a Hilbert space, consider a commutative system of the linear unbounded operators $\{A_1, A_2\}$, where the domain $\mathfrak{D}(A_p)$ of each operator A_p is dense in H, $\overline{\mathfrak{D}(A_p)} = H$, $p = 1, 2$, and the commutativity of the operators A_1, A_2 is understood in terms of interchangeability of resolvents, $[R_1, R_2] = 0$, where $R_p = R_p(\alpha) = (A_p - \alpha I)^{-1}$, $p = 1, 2$, assuming that α is a point of regularity of resolvents $R_1(\lambda)$, $R_2(\lambda)$. Obviously, $[R_1, R_2] = 0$ yields $[R_1(\lambda), R_2(w)] = 0$ for all λ and w belonging to the joint domain of regularity of $R_1(\lambda)$ and $R_2(\lambda)$. The following definition plays an important role hereinafter and it is a generalization of Definition 1 (see Sect. 1) for the commutative case.

Definition 2. *Let a system of the linear unbounded operators $\{A_1, A_2\}$ be given in a Hilbert space H such that: a) the domain $\mathfrak{D}(A_p)$ of the operator A_p is dense in H, $\overline{\mathfrak{D}(A_p)} = H$, $p = 1, 2$; b) every operator A_p is closed in H, $p = 1, 2$; c) there exists a nonempty domain $\Omega \subset \mathbb{C}\backslash\mathbb{R}$ such that the resolvents $R_p(\lambda) = (A_p - \lambda I)^{-1}$ are regular for all $\lambda \in \Omega$, $p = 1, 2$; d) the resolvents R_1 $(= R_1(\alpha))$, R_2 $(= R_2(\alpha))$ commute at least at one point $\alpha \in \Omega$.*

And let the Hilbert spaces $E_\pm$, the linear bounded operators $\psi_- : E_- \to H$, $\psi_+ : H \to E_+$ and $\{\sigma_p^-\}_1^2$, $\{\tau_p^-\}_1^2$, $\{N_p\}_1^2$, $\Gamma : E_- \to E_-$, $\{\sigma_p^+\}_1^2$, $\{\tau_p^+\}_1^2$, $\left\{\tilde{N}_p\right\}_1^2$,

$\tilde{\Gamma}: E_+ \to E_+$ be given, where $\left\{\sigma_p^{\pm}\right\}_1^2$ and $\left\{\tau_p^{\pm}\right\}_1^2$ are selfadjoint. The totality

$$\Delta = \Delta(\alpha) = \Bigg(\left\{\sigma_p^-\right\}_1^2, \left\{\tau_p^-\right\}_1^2, \left\{N_p\right\}_1^2, \Gamma, H \oplus E_-, \left\{ \begin{bmatrix} A_p & \psi_- \\ \psi_+ & K \end{bmatrix} \right\}_1^2, \\ H \oplus E_+, \tilde{\Gamma}, \left\{\tilde{N}_p\right\}_1^2, \left\{\tau_p^+\right\}_1^2, \left\{\sigma_p^+\right\}_1^2 \Bigg) \tag{2.1}$$

is said to be the commutative colligation if there exists $\alpha \in \Omega$ such that:

1) $2\,\mathrm{Im}\,\alpha \cdot N_p^* \psi_-^* \psi_- N_p = K^* \sigma_p^+ K - \sigma_p^-$, $2\,\mathrm{Im}\,\alpha \cdot \tilde{N}_p \psi_+ \psi_+^* \tilde{N}_p^* = K \tau_p^- K^* - \tau_p^+$;
2) *the operators*

$$\varphi_+^p = \psi_+ \left(A_p - \alpha I\right) : \mathfrak{D}\left(A_p\right) \to E_+,$$

$$\left(\varphi_-^p\right)^* = \psi_-^* \left(A_p^* - \bar{\alpha} I\right) : \mathfrak{D}\left(A_p^*\right) \to E_-$$

are such that:

3) $K^* \sigma_p^+ \varphi_+^p + N_p^* \psi_-^* \left(A_p - \bar{\alpha} I\right) = 0$, $K \tau_p^- \left(\varphi_-^p\right)^* + \tilde{N}_p \psi_1 \left(A_p^* - \alpha I\right) = 0$;
4) $2\,\mathrm{Im}\left\langle A_p h_p, h_p\right\rangle = \left\langle \sigma_p^+ \varphi_+^p h_p, \varphi_+^p h_p \right\rangle$, $\forall h_p \in \mathfrak{D}\left(A_p\right)$,

$$-2\,\mathrm{Im}\left\langle A_p^* \tilde{h}_p, \tilde{h}_p \right\rangle = \left\langle \tau_p^- \left(\varphi_-^p\right)^* \tilde{h}_p, \left(\varphi_-^p\right)^* \tilde{h}_p \right\rangle, \quad \forall \tilde{h}_p \in \mathfrak{D}\left(A_p\right), \tag{2.2}$$

where $p = 1, 2$. And, moreover, the relations:

5) $R_2 \psi_- N_1 - R_1 \psi_- N_2 = \psi_- \Gamma$, $\tilde{N}_1 \psi_+ R_2 - \tilde{N}_2 \psi_+ R_1 = \tilde{\Gamma} \tilde{\psi}_+$;
6) $\tilde{\Gamma} K - K \Gamma = i\left(\tilde{N}_1 \psi_+ \psi_- N_2 - \tilde{N}_2 \psi_+ \psi_- N_1\right)$;
7) $K N_p = \tilde{N}_p K$, *are true, where $R_p = R_p(\alpha)$, $p = 1, 2$.*

Show that for every operator system $\{A_1, A_2\}$ satisfying the suppositions a)–d) there always exist such Hilbert spaces $E_\pm$ and corresponding operators $\psi_\pm$, K, $\left\{\sigma_p^{\pm}\right\}_1^2$, $\left\{\tau_p^{\pm}\right\}_1^2$, $\left\{N_p\right\}_1^2$, $\left\{\tilde{N}_p\right\}_1^2$, Γ, $\tilde{\Gamma}$, that the relations 1–7 (2.1) hold. To do this, similarly to (1.5), consider two commuting bounded operators

$$T_p = I + i2\,\mathrm{Im}\,\alpha \cdot R_p, \quad p = 1, 2, \tag{2.3}$$

and let (see (1.3))

$$\begin{aligned} B_p &= iR_p - iR_p^* + 2\,\mathrm{Im}\,\alpha \cdot R_p^* R_p, \quad p = 1, 2, \\ \tilde{B}_p &= iR_p - iR_p^* + 2\,\mathrm{Im}\,\alpha \cdot R_p R_p^* \quad , p = 1, 2. \end{aligned} \tag{2.4}$$

It is easy to see that

$$T_p B_p = \tilde{B}_p T_p, \quad p = 1, 2. \tag{2.5}$$

Analogously as in (1.7),

$$2\,\mathrm{Im}\left\langle A_p h_p, h_p\right\rangle = \left\langle B_p \left(A_p - \alpha I\right) h_p, \left(A_p - \alpha I\right) h_p\right\rangle, \forall h_p \in \mathfrak{D}\left(A_p\right), \tag{2.6}$$

$$-2\operatorname{Im}\left\langle A_p^*\tilde{h}_p, \tilde{h}_p\right\rangle = \left\langle \tilde{B}_p\left(A_p^* - \bar{\alpha}I\right)\tilde{h}_p, \left(A_p^* - \bar{\alpha}I\right)\tilde{h}_p\right\rangle, \forall \tilde{h}_p \in \mathfrak{D}\left(A_p^*\right)$$

take place, where $p = 1, 2$. Define the bounded operators in H

$$\begin{gathered} \sigma_p^+ = B_p, \quad p = 1,2, \quad \sigma_1^- = T_2\tilde{B}_1T_2^*, \quad \sigma_2^- = T_1B_2T_1^*, \\ N_1 = \tilde{B}_1T_2^*, \quad N_2 = \tilde{B}_2T_1^*, \quad \Gamma = \tilde{B}_1R_2^* - \tilde{B}_2R_1^*, \\ \tau_p^- = \tilde{B}_p, \quad p = 1,2, \quad \tau_1^+ = T_2^*B_1T_2, \quad \tau_2^+ = T_1^*B_2T_1, \\ \tilde{N}_1 = T_2^*B_1, \quad \tilde{N}_2 = T_1^*B_2, \quad \tilde{\Gamma} = R_2^*B_1 - R_1^*B_2. \end{gathered} \tag{2.7}$$

Consider the Hilbert spaces

$$\begin{aligned} E_- &= \operatorname{span}\left\{\tilde{B}_1H + \tilde{B}_2H + N_1^*H + N_2^*H\right\}, \\ E_+ &= \operatorname{span}\left\{B_1H + B_2H + \tilde{N}_1H + \tilde{N}_2H\right\}, \end{aligned} \tag{2.8}$$

and let

$$K = -T_1^*T_2^*, \quad \psi_- = P_-, \quad \psi_+ = P_+, \tag{2.9}$$

where $P_\pm$ are the orthoprojectors on $E_\pm$ (2.8). It is easy to see that the relations 1–4 (2.2) follow from equalities (2.3)–(2.7). Consequently, 7 (2.2) follows from (2.5). By simple calculations we may check the conditions 5, 6. Finally, it is easy to show that the operator K (2.9) maps E_- in (2.8).

R e m a r k 2.1. Equations 2, 4 (2.2) imply

$$B_p = \psi_+^*\sigma_p^+\psi_+, \quad \tilde{B}_p = \psi_-\tau_p^-\psi_-^*, \quad p = 1, 2, \tag{2.10}$$

by virtue of the density of the domains $\mathfrak{D}(A_p)$, $\mathfrak{D}\left(A_p^*\right)$, $p = 1$, 2.

II. Before turning to the open system associated with the commutative colligation Δ (2.1), which is a two-variable analogue of the system $F_\Delta = \{R_\Delta, S_\Delta\}$ (1.11), (1.12), write the main equations (1.11), (1.12) in other form. Since (1.11), (1.12) are given by

$$\begin{cases} i\partial_t h(t) + Ay(t) = \alpha\psi_- u_-(t), \\ y(t) = h(t) + \psi_- u_-(t) \in \mathfrak{D}(A), \\ h(0) = h, \quad t \in [0, T], \\ u_+(t) = Ku_-(t) - i\varphi_+ y(t), \end{cases} \tag{2.11}$$

where $\partial_t = \dfrac{\partial}{\partial t}$, by multiplying the second equality by α and subtracting it from the first one, we obtain

$$Lh(t) + \hat{y}(t) = 0, \tag{2.12}$$

where the operator L and the function $\hat{y}(t)$ are such that

$$L = i\partial_t + \alpha, \quad y(t) = R_\alpha \hat{y}(t) \in \mathfrak{D}(A). \tag{2.13}$$

Therefore equations (2.11) can be written in the following form:

$$\begin{cases} Lh(t) + \hat{y}(t) = 0, \\ R_\alpha \hat{y}(t) = h(t) + \psi_- u_-(t) \in \mathfrak{D}(A), \\ h(0) = h, \quad t \in [0, T], \\ u_+(t) = Ku_-(t) - i\psi_+ \hat{y}(t). \end{cases} \tag{2.14}$$

The first two equalities yield that $\hat{y}(t)$ is a solution of the equation

$$LR_\alpha \hat{y}(t) + \hat{y}(t) = \psi_- Lu_-(t). \tag{2.15}$$

Applying the operator L to equalities (2.11), we obtain

$$\begin{cases} -i\partial_t \hat{y}(t) + ALy(t) = \alpha\psi_- Lu_-(t), \\ Ly(t) = -\hat{y}(t) + \psi_- Lu_-(t) \in \mathfrak{D}(A), \\ Lu_+(t) = KLu_-(t) - i\varphi_+ Ly(t). \end{cases} \tag{2.16}$$

Since these equalities coincide with the relations (2.11) after substitutions $h(t) \to -\hat{y}(t)$, $y(t) \to Ly(t)$, $u_\pm(t) \to Lu_\pm(t)$, by using the conservation law (1.14), we obtain

$$\partial_t \left\| \hat{y}(t) \right\|^2 = \left\langle \sigma_- Lu_-(t), Lu_-(t) \right\rangle - \left\langle \sigma_+ Lu_+(t), Lu_+(t) \right\rangle. \tag{2.17}$$

III. Denote by $D = [0, T_1] \times [0, T_2]$ the rectangle in $\mathbb{R}_+^2$, $0 < T_p < \infty$, $p = 1, 2$, and let $u_-(t)$ be a vector function in E_- specified when $t = (t_1, t_2) \in D$. The system of the relations

$$R_\Delta : \begin{cases} i\partial_1 h_1(t) + A_1 y_1(t) = \alpha\psi_- N_1 u_-(t), \\ y_1(t) = h_1(t) + \psi_- N_1 u_-(t) \in \mathfrak{D}\left(A_1\right), \\ i\partial_2 h_2(t) + A_2 y_2(t) = \alpha\psi_- N_2 u_-(t), \\ y_2(t) = h_2(t) + \psi_- N_2 u_-(t) \in \mathfrak{D}\left(A_2\right), \\ h_1(0) = h_1, \quad h_2(0) = h_2, \quad t = (t_1, t_2) \in D, \end{cases} \tag{2.18}$$

where $\partial_p = \partial/\partial t_p$, $p = 1,\ 2$, is said to be the open system $F_\Delta = \{R_\Delta, S_\Delta\}$ associated with the colligation Δ (2.1). Let the vector functions $y_1(t)$ and $y_2(t)$ be such that

$$y_1(t) = R_1 y(t), \quad y_2(t) = R_2 y(t), \tag{2.19}$$

and $y(t)$ be a vector function from H. Thus, the functions $\{y_p(t)\}_1^2$ have a joint generatrix $y(t)$, moreover, (2.19) implies that

$$R_1 y_2(t) = R_2 y_1(t). \tag{2.20}$$

As for the initial data h_1 and h_2, we assume that

$$h_p = R_p y(0) - \psi_- N_p u_-(0), \quad p = 1, 2. \tag{2.21}$$

The mapping S_Δ is given by

$$S_\Delta : u_+(t) = K u_-(t) - i\psi_+ y(t). \tag{2.22}$$

Similarly to (2.13), consider the differential operators

$$L_p = i\partial_p + \alpha, \quad p = 1, 2. \tag{2.23}$$

Then the main equations (2.18) can be written in the following form:

$$\begin{cases} L_1 h_1(t) + y(t) = 0, \\ R_1 y(t) = h_1(t) + \psi_- N_1 u_-(t) \in \mathfrak{D}\left(A_1\right), \\ L_2 h_2(t) + y(t) = 0, \\ R_2 y(t) = h_2(t) + \psi_- N_2 u_-(t) \in \mathfrak{D}\left(A_2\right), \end{cases} \tag{2.24}$$

which is similar to (2.14). Consequently, $L_1 h_1(t) = -y(t) = L_2 h_2(t)$. Therefore, taking into account (2.23) and (2.18), we obtain that (cf. (2.15))

$$\begin{cases} R_1 L_1 y(t) + y(t) = \psi_- N_1 L_1 u_-(t), \\ R_2 L_2 y(t) + y(t) = \psi_- N_2 L_2 u_-(t), \\ y(0) = y_0, \quad t = (t_1, t_2) \in D, \\ u_+(t) = K u_-(t) - i\psi_+ y(t). \end{cases} \tag{2.25}$$

So, if the vector function $y(t)$ satisfies the relations (2.25), then the functions $h_1(t)$, $h_2(t)$ (2.24) as well as $y_1(t)$, $y_2(t)$ (2.19) can be defined by it.

Theorem 2.1. *The system of equations (2.18) is consistent if the vector function $u_-(t)$ is a solution of the equation*

$$\left\{N_1 L_1 - N_2 L_2 + \Gamma L_1 L_2\right\} u_-(t) = 0, \tag{2.26}$$

given that (2.19), (2.21) hold and L_p has the form of (2.23), $p = 1, 2$.

P r o o f. The consistency condition follows from (2.25) if one takes into account the commutativity $[R_1 L_1, R_2 L_2] = 0$. Since

$$L_1 R_1 L_2 R_2 y(t) = y(t) - \psi_- N_1 L_1 u_-(t) + R_- \psi_- N_2 L_1 L_2 u_-(t)$$

and similarly

$$L_2 R_2 L_1 R_1 y(t) = y(t) - \psi_- N_2 L_2 u_-(t) + R_2 \psi_- N_1 L_1 L_2 u_-(t),$$

then, subtracting these equalities, we obtain

$$\psi_- \left(N_1 L_1 - N_2 L_2\right) u_-(t) + \left(R_2 \psi_- N_1 - R_1 \psi_- N_2\right) L_1 L_2 u_-(t) = 0.$$

Taking into account 5 (2.2), we obtain

$$\psi_- \left\{N_1 L_1 - N_2 L_2 + \Gamma L_1 L_2\right\} u_-(t) = 0,$$

which proves (2.26). ■

Theorem 2.2. *If for the vector function $y(t)$ equation (2.25) takes place, and $u_-(t)$ is a solution of (2.26), then $u_+(t)$ (2.22) satisfies the equation*

$$\left\{\tilde{N}_1 L_1 - \tilde{N}_2 L_1 + \tilde{\Gamma} L_1 L_2\right\} u_+(t) = 0. \tag{2.27}$$

P r o o f. Calculate

$$\left[\tilde{N}_1 L_1 - \tilde{N}_2 L_2\right] u_+(t) = K\left[N_1 L_1 - N_2 L_2\right] u_-(t)$$

$$-i\left[\tilde{N}_1 \psi_+ L_1 - \tilde{N}_2 \psi_+ L_2\right] y(t) = -K\Gamma L_1 L_2 u_-(t)$$

$$-i\tilde{N}\psi_+ L_1 \left(\psi_- N_2 L_2 u_-(t) - L_2 R_2 y(t)\right) + iN_2 \psi_+ L_2 \left(\psi_- N_1 L_1 u_-(t) - L_1 R_1 y(t)\right)$$

$$= \left\{-K\Gamma - i\tilde{N}_1 \psi_+ \psi_- N_2 + i\tilde{N}_2 \psi_+ \psi_- N_1\right\} L_1 L_2 u_-(t)$$

$$+i\left(\tilde{N}_1 \psi_+ R_2 - \tilde{N}_2 \psi_+ R_1\right) L_1 L_2 y(t) = -\tilde{\Gamma} K L_1 L_2 u_-(t) + i\tilde{\Gamma}\psi_+ L_1 L_2 y(t)$$

$$= -\tilde{\Gamma} L_1 L_2 u_+(t),$$

which proves (2.27) in virtue of (2.26), (2.25) and 5, 6 (2.2). ■

Theorem 2.3. *For the open system $F_\Delta = \{R_\Delta, S_\Delta\}$ (2.18), (2.22) associated with the colligation Δ (2.1), the following conservation laws are true:*

$$\begin{array}{ll} 1) & \partial_p \left\|h_p(t)\right\|^2 = \left\langle \sigma_p^- u_-(t), u_-(t)\right\rangle - \left\langle \sigma_p^+ u_+(t), u_+(t)\right\rangle, \ p = 1, 2; \\ 2) & \partial_2 \left\{\left\langle \sigma_1^- L_1 u_-(t), L_1 u_-(t)\right\rangle - \left\langle \sigma_1^+ L_1 u_+(t), L_1 u_+(t)\right\rangle\right\} \\ & = \partial_1 \left\{\left\langle \sigma_2^- L_2 u_-(t), L_2 u_-(t)\right\rangle - \left\langle \sigma_2^+ L_2 u_+(t), L_2 u_+(t)\right\rangle\right\}. \end{array} \tag{2.28}$$

P r o o f. The relations 1) (2.27) are proved in the same way as equality (1.14). Since the conservation laws 1) (2.28) can be written as (see (2.17))

$$\partial_p \|y(t)\|^2 = \left\langle \sigma_p^- L_p u_-(t), L_p u_-(t)\right\rangle - \left\langle \sigma_p^+ L_p u_+(t), L_p u_+(t), \right\rangle, \quad p = 1, 2,$$

then, taking into account the equality of mixed derivatives $\partial_2 \partial_1 \|y(t)\|^2 = \partial_1 \partial_2 \|y(t)\|^2$, we obtain 2) (2.28). ■

IV. Along with the open system $F_\Delta = \{R_\Delta, S_\Delta\}$ (2.18), (2.22) characterizing the evolution generated by $\{A_1, A_2\}$, consider also the dual situation responding to the dynamics specified by the adjoint operator system $\{A_1^*, A_2^*\}$.

Let a vector function $\tilde{u}_+(t)$ in E_+, $t = (t_1, t_2)$, be specified in the rectangle $D = [0, T_1] \times [0, T_2]$ from $\mathbb{R}_+^2$, $0 < T_p < \infty$, $p = 1$, 2. The equation system

$$R_\Delta^+ : \begin{cases} i\partial_1 \tilde{h}_1(t) - A_1^* \tilde{y}_1(t) = -\bar{\alpha}\psi_+^* \tilde{N}_1^* \tilde{u}_+(t), \\ \tilde{y}_1(t) = \psi_+^* \tilde{N}_1^* \tilde{u}_+(t) - \tilde{h}_1(t) \in \mathfrak{D}\left(A_1^*\right), \\ i\partial_2 \tilde{h}_2(t) - A_2^* \tilde{y}_2(t) = -\bar{\alpha}\psi_+^* \tilde{N}_2^* \tilde{u}_+(t), \\ \tilde{y}_2(t) = \psi_+^* \tilde{N}_2^* u_+(t) - \tilde{h}_2(t) \in \mathfrak{D}\left(A_2^*\right), \\ \tilde{h}_1(T) = \tilde{h}_1, \quad \tilde{h}_2(T) = \tilde{h}_2, \quad t = (t_1, t_2) \in D, \end{cases} \tag{2.29}$$

where, as usually, $\partial_p = \partial/\partial t_p$, $p = 1, 2$ and $\tilde{y}_1(t)$, $\tilde{y}_2(t)$ are such that

$$\tilde{y}_1(t) = R_1^* \tilde{y}(t), \quad \tilde{y}_2(t) = R_2^* \tilde{y}(t) \tag{2.30}$$

is said to be the dual open system $F_\Delta^+ = \left\{R_\Delta^+, S_\Delta^+\right\}$ associated with the colligation Δ (2.1). Thus the vector functions $\{\tilde{y}_p(t)\}_1^2$ have the joint generatrix $\tilde{y}(t) \in H$, besides,

$$R_1^* \tilde{y}_2(t) = R_2^* \tilde{y}_1(t). \tag{2.31}$$

The initial data $\tilde{h}_1$, $\tilde{h}_2$ of problem (2.28) can be found from the equalities

$$\tilde{h}_p = \psi_+^* \tilde{N}_p^* u_+(T) - R_p^* \tilde{y}(T), \quad p = 1, 2. \tag{2.32}$$

The mapping S_Δ^+ is given by

$$S_\Delta^+ : \tilde{u}_-(t) = K^* \tilde{u}_+(t) + i\psi_-^* \tilde{y}(t). \tag{2.33}$$

Consider (see (2.23)) the differential operators

$$L_p^+ = i\partial_p + \bar{\alpha}, \quad p = 1, 2. \tag{2.34}$$

Similarly to the considerations in Section 3, we obtain that the vector function $\tilde{y}(t)$ satisfies the relations

$$\begin{cases} R_1^* L_1^+ \tilde{y}(t) + \tilde{y}(t) = \psi_+^* \tilde{N}_1^* L_1^+ \tilde{u}_+(t), \\ R_2^* L_2^* \tilde{y}(t) + \tilde{y}(t) = \psi_+^* \tilde{N}_2^* L_2^+ \tilde{u}_+(t), \\ \tilde{y}(T) = \tilde{y}_T, \quad t = (t_1, t_2) \in D, \\ \tilde{u}_-(t) = K^* \tilde{u}_+(t) + i\psi_-^* \tilde{y}(t). \end{cases} \tag{2.35}$$

It is not difficult to obtain the analogues of Theorems 2.1–2.3 using the equalities above.

Theorem 2.4. *The system of the equations (2.29) of the dual open system $F_\Delta^+ = \left\{R_\Delta^+, S_\Delta^+\right\}$ (2.29)–(2.33) corresponding to the commutative colligation Δ (2.1) is consistent if $\tilde{u}_+(t)$ satisfies the equation*

$$\left\{\tilde{N}_1^* L_1^+ - \tilde{N}_2^* L_2^+ + \tilde{\Gamma}^* L_1^+ L_2^+\right\} \tilde{u}_+(t) = 0 \qquad (2.36)$$

under the condition that (2.30) and (2.32) take place.

The proof of this theorem is similar to that of Theorem 2.1.

Theorem 2.5. *Let $\tilde{y}(t)$ be the solution of (2.35), and $\tilde{u}_+(t)$ satisfy equation (2.36). Then for the vector functions $\tilde{u}_-(t)$ (2.33), we have*

$$\left\{N_1^* L_1^+ - N_2^* L_2^+ + \Gamma^* L_1^+ L_2^+\right\} \tilde{u}_-(t) = 0. \qquad (2.37)$$

Theorem 2.6. *For the dual open system $F_\Delta^+ = \left\{R_\Delta^+, S_\Delta^+\right\}$ (2.29)–(2.33), the conservation laws*

$$\begin{array}{ll} 1) & \partial_p \left\|\tilde{h}_p(t)\right\|^2 = \left\langle \tau_p^- \tilde{u}_-(t), \tilde{u}_-(t)\right\rangle - \left\langle \tau_p^+ \tilde{u}_+(t), \tilde{u}_+(t)\right\rangle, \, p = 1, 2; \\ 2) & \partial_2 \left\{\left\langle \tau_1^- L_1^+ \tilde{u}_-(t), L_1^+ \tilde{u}_-(t)\right\rangle - \left\langle \tau_1^+ L_1^+ \tilde{u}_+(t), \tilde{u}_+(t)\right\rangle\right\} \\ & = \partial_1 \left\{\left\langle \tau_2^- L_2^+ \tilde{u}_-(t), L_1^+ \tilde{u}_-(t)\right\rangle - \left\langle \tau_2^+ L_2^+ \tilde{u}_+(t), L_2^+ \tilde{u}_+(t)\right\rangle\right\} \end{array} \qquad (2.38)$$

hold.

As it will be shown later, relations 2) (2.28) and 2) (2.38) play an important role in the study of the properties of characteristic functions for commutative systems of the linear unbounded operators $\{A_k\}_1^n$ for any $n \in \mathbb{N}$.

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