Journal of Mathematical Physics, Analysis, Geometry
2009, vol. 5, No. 4, pp. 347–358

Planar Lebesgue Measure of Exceptional Set in Approximation of Subharmonic Functions

Markiyan Girnyk

Lviv Academy of Commerce
8 Tuhan-Baranovskyi Str., Lviv, 79005, Ukraine
E-mail:hirnyk@lac.lviv.ua

Received November 17, 2008

We consider the pointwise approximation of a subharmonic function having finite order by the logarithm of the modulus of an function up to a bounded quantity. We prove an estimate from below of the planar Lebesgue measure of the exceptional Set in such approximation.

Key words: subharmonic functions, entire functions, approximation.

Mathematics Subject Classification 2000: 31C05, 30E10.

Results on approximation of a subharmonic function by the logarithm of the modulus of an entire function have numerous applications in complex analysis and potential theory (see, for example, [1–6]). The pointwise approximation is possible only outside an exceptional set, and for this reason, the principal question concerning its minimal size arises. In this article we prove that the planar Lebesgue measure of an exceptional set in approximation of the subharmonic function $|z|^\rho$ by the logarithm of the modulus of an entire function of at most order ρ and normal type cannot be arbitrary small in a certain sense.

We use the main results and standard notations of potential theory [7] and theory of distribution of values [8]. Let us recall some of them. We denote by $D(a,r) := \{z : |z-a| < r\}$, $C(a,r) := \{z : |z-a| \le r\}$, $S(a,r) := \{z : |z-a| = r\}$, $A(t,T] := \{z : t < |z| \le T\}$, m_d the Lebesgue measure on $\mathbb{R}^d$, letters C with indices stand for positive constants, in parentheses we indicate dependence on parameters. As usually, $a^+ = \max(a,0)$, $a^- = \max(-a,0)$. Let u be a subharmonic function, then μ_u is its Riesz measure, $B(r,u) := \max\{u(z) : z \in C(0,r)\}$ is the maximum, $n(a,r,u) := \mu(C(a,r))$, $n(r) := n(0,r,u)$ are the counting functions of the Riesz measure, $h(z,u,\mathcal{D})$ is the minimal harmonic majorant of the function u in domain $\mathcal{D}$ (which is sometimes omitted in notations), $T(r,u) := \frac{1}{2\pi}\int_0^{2\pi} u^+(re^{i\varphi})\,d\varphi$ is the Nevanlinna characteristic of u. We notice that the Nevanlinna characteristic of a meromorphic function f is also denoted by $T(r, f)$. It will not make any difficulties for the readers as it is clear from the context which characteristic is used. We denote by Λ the class of nondecreasing slowly changing functions $\lambda : [1, \infty) \mapsto [1, \infty)$ (in particular, $\lambda(2r) \sim \lambda(r)$ if $r \to \infty$).

The notation $a \asymp b$ means that $|a| \le \text{const} \cdot |b|$ and $|b| \le \text{const} \cdot |a|$. Let $\Theta \supset \Lambda$ be the class of nondecreasing functions $\lambda : [1, \infty) \mapsto [1, \infty)$ having the property: $\lambda(2R) \asymp \lambda(R)$. This implies the finiteness of order

$$\tau = \limsup_{R \to \infty} \frac{\log \lambda(R)}{\log R}$$

of the function λ. The content of this paper is closely associated with the theorem by I. Chyzhykov [9], which strengthens and specifies the result by Yu. Lubarskii and Eu. Malinnikova [10], as well as with the theorem by R. Yulmukhametov [11]. For the reader's convenience we quote these theorems in somewhat modificated but equivalent formulations.

Theorem A. *Let u be a subharmonic function with the Riesz measure μ_u. If for a function $\lambda \in \Lambda$ there exists a number R_0, such that for every number $R > R_0$ the condition*

$$\mu_u(A(R, R\lambda(R)]) > 1 \tag{1}$$

holds, then there exists an entire function f and a constant C_1 satisfying

$$\int\limits_{A(R,2R]} |u(z) - \log|f(z)||\, dm_2(z) < C_1 R^2 \log \lambda(R). \tag{2}$$

Moreover, for every real number $\varepsilon > 0$ there exists a constant $C_2(\varepsilon)$ and a set $E(\varepsilon)$ such that

$$|u(z) - \log|f(z)|| < C_2(\varepsilon) \log \lambda(|z|),\ z \notin E(\varepsilon), \tag{3}$$

and

$$m_2(E(\varepsilon) \cap A(R, 2R])/R^2 < \varepsilon. \tag{4}$$

Theorem B. *Let u be a subharmonic function of finite order ρ and a number $\alpha > \rho$. Then there exists an entire function f, a constant $C_3 = a_0 + a_1\alpha$, $a_1 > 1$, depending only on α , and an exceptional set E depending on the functions u, f and the number α such that*

$$|u(z) - \log|f(z)|| \le C_3(\alpha) \log|z|,\ z \notin E, \tag{5}$$

where $E \subset \cup_j D(z_j, r_j)$ and

$$\sum_{R < |z_j| \le 2R} r_j = o(R^{\rho - \alpha}), \quad R \to \infty. \tag{6}$$

Let us formulate the result of our work.

Theorem 1. *For all real numbers $\varepsilon > 0, \rho > 0$, and for every entire function f satisfying the condition*

$$B(r, \log|f|) < C_4 r^{\rho}, \tag{7}$$

for every function $\lambda \in \Theta$ of order τ, and for any measurable set E, the condition

$$||z|^{\rho} - \log|f(z)|| < C_5 \log\lambda(|z|),\ z \notin E, \tag{8}$$

implies the existence of a constant $C_6(\varepsilon)$ such that

$$m_2(E \cap A(R, 2R]) > C_6(\varepsilon) R^{\chi+\rho},\ R > 1, \tag{9}$$

where $\chi = \min(2 - 2\rho - \varepsilon, 2 - \rho - 4C_5\tau - \varepsilon)$.

As the formulation of Theorem 1 is cumbersome, we write its statement in the important case of a bounded function λ (its order $\tau = 0$, then $\chi = 2-2\rho-\varepsilon$):

$$m_2(E \cap A(R, 2R]) > C_6(\varepsilon) R^{2-\rho-\varepsilon},\ R > 1.$$

Let us comment the content of Theorem 1. It states that the number ε on the right-hand side of (4) cannot be replaced by an arbitrary function $\varepsilon(R) \to 0$, $R \to \infty$. Then condition (7) seems to be natural because in Theorem 1 it holds for subharmonic functions of finite order ρ and normal type 1, following from (2). Indeed, let us consider the inequalities

$$\begin{aligned} 2\pi T(r, f) &\le \int_0^{2\pi} |\log|f(re^{i\varphi})||\, d\varphi + O(1) \\ &\le \int_0^{2\pi} (|\log|f(re^{i\varphi})| - u(re^{i\varphi})| + u(re^{i\varphi}))\, d\varphi + O(1), \end{aligned}$$

which imply

$$2\pi R^2 T(R, f) < 2\pi \int_R^{2R} T(r, f) r\, dr \le \int_{A(R,2R]} |\log|f(z)| - u(z)|\, dm_2(z)$$

$$+ \int_{A(R,2R]} u(z)\, dm_2(z) + O(R^2) \le C_1 R^2 \log\lambda(R) + O(R^{\rho+2}) + O(R^2),$$

$$R \to \infty,$$

and growths of the functions $B(r, \log|f|)$ and $T(r, f)$ are equal. We also note the possibility of good approximation of a subharmonic function with finite order by the logarithm of the modulus of some entire function with infinite order. We give a simple example. Let

$$||u(z) - \log|f(z)|| < C_5 \log\lambda(|z|),\ z \notin E,$$

where u is a subharmonic function of finite order, f is an entire function, then

$$||u(z) - \log|f(z) \cdot V(z)|| < C_5 \log\lambda(|z|) + 2,$$

$$z \notin E \bigcup S := \{z = x + iy : x \geq 1, |y| \leq \pi\},$$

where (see [8, p. 256–258]) the function

$$V(z) = \begin{cases} \exp(\exp z) + \Psi_1(z)/z,\ z \in S; \\ \Psi_2(z)/z,\ z \notin S; \end{cases}$$

and $|\Psi_j(z)| < 2$, $|z| > r_0 > 1$, $j = 1, 2$. We also notice that in [11] and [12] the estimates from below of the sum of radii for any disk covering of the exceptional set are obtained, but no estimate from below of the planar measure follows from those estimates.

We cite two above mentioned results.

Theorem C. *Let a number $\varepsilon > 0$ and an entire function f satisfy the inequality*

$$||z| - \log|f(z)|| = o(\log|z|),\ E \not\ni z \to \infty,$$

where $E \subset \bigcup_j \{z : |z - z_j| < r_j\}$, and radii r_j are uniformly bounded. Then the estimate

$$\sum_{R \leq |z_j| < 2R} r_j > R^{1-\varepsilon},\ R > R(\varepsilon),$$

holds.

Theorem D. *Let numbers $\rho > 0$, $\varepsilon > 0$, and an entire function f satisfy the inequality*

$$||z|^\rho - \log|f(z)|| < C_5 \log|z|,\ z \notin E,$$

where $E \subset \bigcup_j \{z : |z - z_j| < r_j\}$, $r_j \leq |z_j|^{1-\rho/2+\varepsilon}$. Then the estimate

$$\sum_{R \leq |z_j| < 2R} r_j > R^{1+\rho/2-2C_5-\varepsilon},\ R > R(\varepsilon),$$

holds.

Let us consider the accuracy of Theorem 1. From Theorem B it follows that there exists an entire function f and an exceptional set E satisfying (8) with $\lambda(R) = R$, $\tau = 1$, and $m_2(E \bigcap A(R, 2R]) = o(R^{2\rho - 2C_5/a_1 - 2a_0/a_1})$, $R \to \infty$. We draw a conclusion that the planar Lebesgue measure of the exceptional set in the annulus $A(R, 2R]$ is a power function of R with the exponent $\asymp C_5$. The dependence of the exponent on ρ is not clear.

P r o o f o f T h e o r e m 1. We begin with the idea of the proof. At first, we prove that any disk of the form $D(a, |a|^{1-\rho/2})$, where $f(a) = 0$, contains a rather big portion of the exceptional set, namely, the estimate

$$m_2(E \cap D(a, |a|^{1-\rho/2})) > C_7 |a|^{\chi} \tag{10}$$

holds. The proof of estimate (10) is the key point of the proof of Theorem 1. Here we follow the arguments from [12]. A new approach is that we use the theorem by Edrei and Fuchs on the integral over a small set [13, 8]. Next, it is proved that every disk with a somewhat greater radius has the same property without the demand that the center of the disk is zero of the function f. More exactly, we prove that for every b, $|b| > R_1$, and every $\varepsilon > 0$ the inequality

$$m_2(E \cap D(b, |b|^{1-\rho/2+\varepsilon/2})) > C_8 |b|^{\chi} \tag{11}$$

holds. To finish the proof, we put sufficiently many nonoverlapping disks with enlarged radii into the annulus $A(R, 2R)$. By comparing the areas of the annulus and the disks we obtain estimate (9).

We denote $r(a) := |a|^{1-\rho/2}$, $v(z) := |z|^{\rho}$. Let $h(z, v)$ and $h(z, \log|f|)$ be the minimal harmonic majorants, respectively v and $\log|f|$, in the disk $D(a, t)$, $t \in [(1 - \varepsilon/4) r(a),\, r(a)]$. Under the conditions of Theorem 1 we prove the estimate $(0 < \delta < 2\pi)$

$$|h(z, v) - h(z, \log|f|)| \le C_9 \log \lambda(|a|) + C_{10} \delta |a|^{\rho} \log|a| \log\left(\frac{2\pi e}{\delta}\right),$$
$$z \in D(a,\, \varepsilon./4r(a)). \tag{12}$$

By the Poisson–Jensen formula for the disk $D(a, t)$ we have

$$|h(z, v) - h(z, \log|f|)|$$
$$= \frac{1}{2\pi} \left| \int_0^{2\pi} \left(|\,a + te^{i\varphi}|^{\rho} - \log|f(a + te^{i\varphi})|\right) \Re \frac{te^{i\varphi} + z - a}{te^{i\varphi} - z + a}\, d\varphi \right|$$
$$\le \frac{1}{2\pi} \int_0^{2\pi} \left|\,|\,a + te^{i\varphi}|^{\rho} - \log|f(a + te^{i\varphi})|\right| \frac{r(a) + r(a)\varepsilon/4}{(1 - \varepsilon/4) r(a) - r(a)\varepsilon/4}\, d\varphi$$

$$\leq \frac{1+\varepsilon}{2\pi}\int\limits_0^{2\pi}\big||\,a+te^{i\varphi}|^\rho - \log|f(a+te^{i\varphi})|\big|\;d\varphi. \tag{13}$$

By $E(t,a)$ we denote the set

$$\big\{\varphi\in[0,2\pi]:||\,a+te^{i\varphi}|^\rho-\log|f(a+te^{i\varphi})||\geq C_5\log\lambda(|\,a+te^{i\varphi}|)$$
$$> C_5(1-\varepsilon)\log\lambda(|a|)\big\} \tag{14}$$

(the last inequality in definitions (14) and the ones below follows from the properties of the function $\lambda\in\Theta$ and restrictions on t for all sufficiently large values $|a|$). Its complement is

$$[0,\,2\pi]\setminus E(t,a):=\big\{\varphi\in[0,2\pi]:||\,a+te^{i\varphi}|^\rho-\log|f(a+te^{i\varphi})||$$
$$< C_5\log\lambda(|\,a+te^{i\varphi}|) < C_5(1+\varepsilon)\lambda(|a|)\big\}\,.$$

Now we continue estimate (13):

$$|h(z,v)-h(z,\log|f|)|\leq\frac{1+\varepsilon}{2\pi}\int\limits_{E(t,a)}\big||\,a+te^{i\varphi}|^\rho-\log|f(a+te^{i\varphi})|\big|\;d\varphi$$
$$+\frac{1+\varepsilon}{2\pi}\int\limits_{[0,2\pi]\setminus E(t,a)}\big||\,a+te^{i\varphi}|^\rho-\log|f(a+te^{i\varphi})|\big|\;d\varphi$$
$$\leq\frac{1+\varepsilon}{2\pi}\int\limits_{E(t,a)}\big(|\,a+te^{i\varphi}|^\rho+\log^+|f(a+te^{i\varphi})|+\log^-|f(a+te^{i\varphi})|\big)\;d\varphi$$
$$+(1+\varepsilon)^2C_5\log\lambda(|a|). \tag{15}$$

We use the theorem by Edrei and Fuchs [13], [8, p. 58] quoted below for the reader's convenience.

Theorem E. *Let f be a meromorphic function , k and δ be real numbers, $k>1$, $0<\delta<2\pi$, $r>1$. For any measurable set $E_r\subset[0,2\pi]$, such that $m_1(E_r)\leq\delta$, the relation*

$$\int\limits_{E_r}\log^+|f(re^{i\varphi})|\,d\varphi\leq\frac{6k}{k-1}\,\delta\log\frac{2\pi e}{\delta}\,T(kr,f) \tag{16}$$

holds.

We notice that the analysis of the proof of Theorem E shows that its δ-subharmonic version is valid. The assumption $r > 1$ is of technical character; without it the term

$$\delta \log \frac{2\sqrt{k}\pi e}{\delta}\, n(0,f)|\log r|/\log\sqrt{k}$$

should be added to the right-hand side of (16) (see the proof of Lemma 7.1 in [8, p. 55]). For more completeness we give a proof of the above mentioned modification of Theorem E. We begin with the inequalities ($R' > R > 0$)

$$\begin{aligned} N(R') &\geq \int\limits_{R}^{R'} \frac{n(t,f)-n(0,f)}{t}\,dt + n(0,f)\log R' \\ &\geq (n(R,f)-n(0,f))\log\frac{R'}{R} + n(0,f)\log R, \end{aligned}$$

from which the estimate

$$n(R,f) \leq \frac{N(R',f)}{\log\frac{R'}{R}} - \frac{n(0,f)\log R}{\log\frac{R'}{R}}$$

follows. In the proof of Lemma 7.1 it is supposed that $R > 1$, and because of this the negative term $-n(0,f)\log R$ is omitted, but here it is taken into account. Then, in the proof of Theorem E in [8] the term

$$n(\sqrt{k}r, f)\delta\log\frac{2\sqrt{k}e\pi}{\delta}$$

is obtained. To estimate $n(\sqrt{k}r, f)$ we apply the previous inequality with $R' = kr$, $R = \sqrt{k}r$ and obtain

$$\begin{aligned} n(\sqrt{k}r, f)\delta\log\frac{2\sqrt{k}e\pi}{\delta} &\leq \delta\log\frac{2\sqrt{k}e\pi}{\delta}\left(\frac{N(kr)}{\log\sqrt{k}} - \frac{n(0,f)\log(\sqrt{k}r)}{\log\sqrt{k}}\right) \\ &\leq \delta\log\frac{2\sqrt{k}e\pi}{\delta}\left(\frac{N(kr)}{\log\sqrt{k}} - \frac{n(0,f)\log r}{\log\sqrt{k}}\right) \\ &\leq \delta\log\frac{2\sqrt{k}e\pi}{\delta}\left(\frac{N(kr)}{\log\sqrt{k}} + \frac{n(0,f)|\log r|}{\log\sqrt{k}}\right), \end{aligned}$$

then we follow the proof in [8].

Now continue to estimate (15). The integral

$$\int\limits_{E(t,a)} |a+te^{i\varphi}|^{\rho}\,d\varphi \leq (1+\varepsilon)|a|^{\rho}\delta \tag{17}$$

if $|a|$ is sufficiently large. To estimate the integral

$$\int\limits_{E(t,a)} \left(\log^{+}|f(a+te^{i\varphi})| + \log^{-}|f(a+te^{i\varphi})|\right) d\varphi$$

we apply more precise Theorem E, putting $k = 2$ and taking into account the relation $T(r,f) = T(r,1/f) + O(1)$ and (7). We obtain the estimate

$$\int\limits_{E(t,a)} \left(\log^{+}|f(a+te^{i\varphi})| + \log^{-}|f(a+te^{i\varphi})|\right) d\varphi$$

$$= \int\limits_{E(t,a)} \left(\log^{+}|f(a+te^{i\varphi})| + \log^{+}\left|\frac{1}{f(a+te^{i\varphi})}\right|\right) d\varphi$$

$$\le 12\delta \log\frac{2\pi e}{\delta}(2T(2t,f(z+a))+O(1)) + \delta\log\frac{2\sqrt{2}\pi e}{\delta}\, n(0,f(z+a))|\log t|/\log\sqrt{2}$$

$$< 24\delta\log\frac{2\pi e}{\delta}3^{\rho}C_4|a|^{\rho} + 6\delta\log\frac{2\sqrt{2}\pi e}{\delta}\, C_4 2^{\rho}|a|^{\rho}|1-\rho/2|\log|a|. \qquad (18)$$

Combining (15), (17), and (18), we have

$$|h(z,v) - h(z,\log|f|)| \le C_{11}\delta\log\frac{2\pi e}{\delta}|a|^{\rho}\log|a| + C_5(\tau+\varepsilon)\log|a| \qquad (19)$$

if $|a|$ is sufficiently large and $z \in D(a, \varepsilon r(a)/4)$.

The next step is to find the upper bound of the difference $\log|f(z)| - h(z,\log|f|)$ for $z \in D(a,\varepsilon r(a)/4)\backslash E$. This estimate is obtained only in indirect way. Using the standard tools of calculus, we can prove (see [12]) that for $z \in A := A(R-r(R), R+r(R)]$ the inequality

$$|v(z) - h(z,v,A)| \le C_{12} \qquad (20)$$

holds, where

$$h(z,v,A) := (R+r(R))^{\rho}\frac{\log|z| - \log(R-r(R))}{\log(R+r(R)) - \log(R-r(R))}$$

$$+(R-r(R))^{\rho}\frac{-\log|z| + \log(R+r(R))}{\log(R+r(R)) - \log(R-r(R))}$$

is the minimal harmonic majorant of the function $v(z) := |z|^{\rho}$ in the annulus A. From (20) and the definition of the minimal harmonic majorant it follows that

$$|v(z) - h(z,v,D(a,t))| \le C_{12}. \qquad (21)$$

Applying (19), (21), and (8), we obtain the estimate

$$|\log|f(z)|-h(z,\log|f|)|\le|h(z,v)-h(z,\log|f|)|+|v(z)-h(z,v)|$$

$$+|v(z)-\log|f(z)||\le C_{11}\delta\log\frac{2\pi e}{\delta}|a|^{\rho}\log|a|+C_5(\tau+\varepsilon)\log|a|+C_{12}$$

$$+C_5(\tau+\varepsilon)\log|a| \qquad (22)$$

if $z\in D(a,\varepsilon r(a)/4)\setminus E$.

Now we prove the estimate from below for the difference $|\log|f(z)|-h(z,\log|f|)|$. By the Poisson–Jensen formula

$$\log|f(z)|=h(z,\log|f|,D(a,\,t))-\sum_{a_n\in D(a,\,t)}g(z,a_n), \qquad (23)$$

where $g(z,a_n)$ is the Green function of the disk $D(a,\,t)$ with the pole in zero a_n of the function f. Using the known properties of the Green function, from (23) we obtain

$$|\log|f(z)|-h(z,\log|f|,D(a,t))|\ge g(z,a)=\log\frac{t}{|z-a|}. \qquad (24)$$

We face the alternative: either for every $t\in[(1-\varepsilon)r(a),r(a)]$ the measure $m_1(E(t,a))\ge\delta$, or there exists $t\in[(1-\varepsilon)r(a),r(a)]$ for which the measure $m_1(E(t,a))<\delta$, where $\delta=\varepsilon(|a|^{\rho}\log|a|)^{-1}$. In the first case the planar Lebesgue measure

$$m_2(E\cap D(a,r(a)))\ge\varepsilon r(a)\cdot\varepsilon r(a)(|a|^{\rho}\log|a|)^{-1}=\varepsilon^2\frac{|a|^{2-2\rho}}{\log|a|}. \qquad (25)$$

In the second case, as it follows from (24), for $z\in D(a,(1-\varepsilon)r(a)/|a|^{\kappa})$, $\kappa=2C_5(\tau+\varepsilon)+2C_{11}\varepsilon$, the estimate

$$|\log|f(z)|-h(z,\log|f|)|\ge\kappa\log|a| \qquad (26)$$

takes place, and from (22) we obtain that

$$|\log|f(z)|-h(z,\log|f|)|\le C_{11}\varepsilon\log|a|+2C_5(\tau+\varepsilon)\log|a|, \qquad (27)$$

if $z\in D(a,\varepsilon r(a)/4)\setminus E$. Comparing (26) and (27), we conclude that the disk $D(a,(1-\varepsilon)r(a)/|a|^{\kappa})\subset E$. Since its area equals $\pi(1-\varepsilon)^2|a|^{2-\rho-2\kappa}$, and the planar Lebesgue measure of the portion of the exceptional set $E\cap D(a,r(a))$ does not exceed $\varepsilon^2|a|^{2-2\rho}/\log^2|a|$ in the first case, then in any case (it is clear that const $\cdot\,\varepsilon$ can be replaced by ε)

$$m_2(D(a,r(a))\cap E)\ge C_{13}|a|^{\chi},\ \chi=\min(2-2\rho-\varepsilon,2-\rho-4C_5\tau-\varepsilon). \qquad (28)$$

We put $r_1(b) := r(b)|b|^{\varepsilon/2}$. For arbitrary disk of the form $D(b, r_1(b))$ with sufficiently large $|b|$ we prove the estimate

$$m_2(D(b, r_1(b)) \cap E) \geq C_{14}|b|^{\chi}. \tag{29}$$

Without lost of generality, we may suppose $b \notin E$. In the opposite case either $D(b, r(b)) \subset E$ and then (29) holds, or there exists $c \in D(b, r(b)) \setminus E$. It is important that the disk $D(c, \frac{3}{4}r_1(c)) \subset D(b, r_1(b))$, this is used under. Suppose the disk $D(c, \frac{3}{4}r_1(c))$ does not contain zeros of the entire function f, then $n(c, t, \log|f|) = 0$ for $t \in [0, \frac{3}{4}r_1(c)]$.

The Poisson–Jensen formula for the difference $|z|^\rho - \log|f(z)|$ in the disk $D(c, t/2)$ has the form

$$-|c|^\rho + \log|f(c)| + \frac{1}{2\pi}\int\limits_0^{2\pi}\left(\left|c + \frac{1}{2}te^{i\varphi}\right|^\rho - \log\left|f\left(c + \frac{1}{2}te^{i\varphi}\right)\right|\right) d\varphi$$
$$= \int\limits_0^{t/2} \frac{n(c, s, |z|^\rho)}{s}\, ds. \tag{30}$$

The estimating of the integral on the left-hand side of (30) similarly to the one in (13) results

$$\left|\frac{1}{2\pi}\int\limits_0^{2\pi}\left(\left|c + \frac{1}{2}te^{i\varphi}\right|^\rho - \log\left|f(c + \frac{1}{2}te^{i\varphi})\right|\right) d\varphi\right|$$
$$\leq C_{11}\delta \log\frac{2\pi e}{\delta} T(t, f(w + c)) + C_5(\tau + \varepsilon)\log|c| + C_{15}\delta|c|^\rho, \tag{31}$$

where $m_1(E(t/2, c)) \leq \delta$. Next, for $t \in [0, \frac{3}{4}r_1(c)]$ the estimate

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \log^+\left|f\left(c + \frac{1}{2}te^{i\varphi}\right)\right| d\varphi \leq C_4(|c| + t/2)^\rho \leq C_4 2^\rho|c|^\rho \tag{32}$$

takes place if $|c|$ is sufficiently large. Here we make use of (7). From (32) and the definition of the Nevanlinna characteristic of a meromorphic function we obtain the estimate

$$T(t, f(w + c)) \leq 2^\rho C_4|c|^\rho. \tag{33}$$

We put $\delta := |c|^{-\rho}$. Again we face the alternative: either for every $t \in [\frac{1}{2}r_1(c), \frac{3}{4}r_1(c)]$ the measure $m_1(E(t/2, c)) \geq \delta$, or there exists $t \in [\frac{1}{2}r_1(c), \frac{3}{4}r_1(c)]$ for which $m_1(E(t/2, c)) < \delta$. In the first case

$$m_2(E \cap D(b, r_1(b))) \geq m_2\left(E \cap D\left(c, \frac{3}{4}r_1(c)\right)\right) \geq |c|^{-\rho}\frac{1}{4}r_1(c)\frac{1}{4}r_1(c)$$

$$\asymp |c|^{2-2\rho+\varepsilon} \asymp |b|^{2-2\rho+\varepsilon}.$$

In the second case (31) and (33) imply the inequality

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \left| c + \frac{1}{2} t e^{i\varphi} \right|^\rho - \log^+ \left| f\left(c + \frac{1}{2} t e^{i\varphi} \right) \right| d\varphi \right| \le C_{16} \log |c| + C_5(\tau + \varepsilon) \log |c|. \tag{34}$$

Since $c \notin E$, therefore

$$||c|^\rho - \log |f(c)|| \le C_5(\tau + \varepsilon) \log |c|. \tag{35}$$

Combining (34) and (35), we conclude that the right-hand side of (30) does not exceed $C_{17} \log |c|$.

On the other hand, for the function $v(z) = |z|^\rho$ its Riesz measure $d\mu_v(z) = \frac{1}{2\pi}\Delta v \asymp |z|^{\rho-2}\, dm_2(z)$, and because of this $n(c, s, v) \asymp |c|^{\rho-2} s^2$ if $s \le \frac{3}{4} r_1(c)$, and the right-hand side of (30), i.e., $\int_0^{t/2} \frac{n(c,s,v)}{s}\, ds \asymp |c|^{\rho-2} r_1(c)^2 \asymp |c|^\varepsilon (t \ge \frac{1}{2} r_1(c))$, what contradicts the previous estimate. We draw a conclusion that there exists a zero a of the entire function f such that $a \in D(c, \frac{3}{4} r_1(c))$. If $|b|$ is a sufficiently large number, then $D(a, r(a)) \subset D(b, r_1(b))$. In any case, the measure

$$m_2(E \cap D(b, r_1(b))) \ge C_8 |b|^\chi.$$

To finish the proof of Theorem 1, into the annulus $A[R, 2R)$ we put nonoverlapping disks $D(b, r_1(b)))$ at a rate of $\asymp \frac{R^2}{R^{2-\rho+\varepsilon}} = R^{\rho-\varepsilon}$. The union of these disks contains such a portion of the exceptional set E that

$$m_2(E \cap A[R, 2R)) \ge C_6(\varepsilon) R^{\chi+\rho-\varepsilon}.$$

Theorem 1 is proved.

Acknowledgement. I would like to thank all the referees of the paper.

References

[1] *S.N. Mergelyan*, Uniform Approximations of Functions of Complex Variable. — *Usp. Mat. Nauk.* **7** (1952), No. 2, 31–122. (Russian)

[2] *A. Beurling and P. Malliavin*, On Fourier Transforms of Measures with Compact Support. — *Acta Math.* **107** (1962), 291–309.

[3] *N.U. Arakelyan*, Entire Functions of Finite Order with Infinite Set of Deficiencies. — *Dokl. AN SSSR* **170** (1966), No. 5, 999–1002. (Russian)

[4] *Yu.I. Lyubarskii and M.L. Sodin*, Analogs of Sine-Type Functions for Convex Domains. Preprint 17-86, FTINT AN USSR, Kharkov, 1986. (Russian)

[5] *P.Z. Agranovich and V.N. Logvinenko*, Many Term Asymptotic Representation of Function Subharmonic in the Plane. — *Sib. Mat. J.* **32** (1991), No. 1, 3–21. (Russian)

[6] *D. Drasin*, Approximation of Subharmonic Functions with Applications. Approximation, Complex Analysis, and Potential Theory. — Proc. NATO ASI. Montreal, Canada, 3 to 14 July 2000. Kluwer Acad. Publ., Dordrecht, Boston, London, 2001, 163–189.

[7] *W.K. Hayman and P.B. Kennedy*, Subharmonic Functions. V. 1. Acad. Press, London, New York, San Francisco, 1976.

[8] *A.A. Goldberg and I.V. Ostrovskii*, Distribution of Values of Meromorphic Functions. Nauka, Moscow, 1970. (Russian)

[9] *I.E. Chyzhykov*, Approximation of Subharmonic Functions. — *St. Petersburg Math. J.* **16** (2004), No. 3, 211–237. (Russian)

[10] *Yu. Lyubarskii and Eu. Malinnikova*, On Approximation of Subharmonic Functions. — *J. d'Analyse Math.* **83** (2001), 121–149.

[11] *R.S. Yulmukhametov*, Approximation of Subharmonic Functions. — *Anal. Math.* **11** (1985), No. 3, 257–282. (Russian)

[12] *M. Girnyk*, Accuracy of Approximation of Subharmonic Function by Logarithm of Modulus of Analytic One in Chebyshev Metric. — *Zap. Nauch. Sem. POMI* **327** (2005), 55–73. (Russian)

[13] *A. Edrei and W.H.J. Fuchs*, Bounds for the Number of Deficient Values of Certain Classes of Meromorphic Functions. — *Proc. London Math. Soc.* **12** (1962), 113–145.