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Ruled Surfaces as Pseudospherical Congruences

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Two-dimensional ruled surfaces in the spaces of constant curvature R^n, S^n, H^n and in the Riemannian products $S^n \times R^1$, $H^n \times R^1$ are considered. A ruled surface is proved to represent a pseudospherical congruence if and only if it is either an intrinsically flat surface in S^n, or an intrinsically flat surface with constant extrinsic curvature in $S^n \times R^1$.

Key words: pseudo-spherical congruence, ruled surface.

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1. Introduction

The aim of the paper is to discuss two-dimensional ruled surfaces from the point of view of the theory of pseudospherical congruences.

Let M^n be one of the n-dimensional spaces of constant curvature (Euclidean space R^n, unite sphere S^n, unite hyperbolic space H^n) or one of the Riemannian products $S^{n-1} \times R^1$, $H^{n-1} \times R^1$.

A geodesic congruence in M^n is a diffeomorphism $\psi : F^2 \to \tilde{F}^2$ of two surfaces in M^n which possesses the following *bitangency property*: for each point $P \in F^2$ there exists a geodesic of M^n through P and $\psi(P) = \tilde{P} \in \tilde{F}^2$ which is tangent to F^2 at P and to $\tilde{F}^2$ at $\tilde{P}$.

The geodesic congruence $\psi : F^2 \to \tilde{F}^2$ is said to be *pseudospherical* if it satisfies two additional conditions:

(C1) the distance between corresponding points $P \in F^2$ and $\tilde{P} \in \tilde{F}^2$ is equal to a nonzero constant independent of P, $|P\tilde{P}| \equiv l_0 \neq 0$;

(C2) the angle between planes tangent to F^2 and $\tilde{F}^2$ at corresponding points is equal to a nonzero constant independent of P, $\angle(T_P F^2, T_{\tilde{P}} \tilde{F}^2) \equiv \omega_0 \neq 0$.

The constants l_0 and ω_0 are called *the parameters* of the pseudospherical congruence ψ.

This definition corresponds to the classical notion of pseudospherical congruences for n-dimensional submanifolds in $(2n-1)$-dimensional spaces of constant curvature [1–3]. Recall that if two n-dimensional submanifolds Φ^n, $\tilde{\Phi}^n$ in $(2n-1)$-dimensional space of constant curvature are connected by a pseudospherical congruence, then Φ^n, $\tilde{\Phi}^n$ are of the same constant negative extrinsic curvature K_{ext} depending on l_0, ω_0. Moreover, an arbitrary n-dimensional submanifold in $(2n-1)$-dimensional space of constant curvature admits a large family of different pseudospherical congruences. These statements, which generalize the classical results by Backlund, Bianchi, Darboux, were proved by K. Teneblat and C.-L. Terng [4], Yu.A. Aminov [5]. Using the classical terminology, $\tilde{\Phi}^n$ is called a *Backlund transformation* of Φ^n. This geometric construction was of great importance for the soliton theory, where the development of some fundamental ideas and principles was initiated. Actually, the n-dimensional submanifolds of constant negative extrinsic curvature (pseudospherical submanifolds) in $(2n-1)$-dimensional spaces of constant curvature represent one of the most illustrative classical examples of integrable systems [2, 3, 6].

On the other hand, it would be interesting to understand what properties of pseudospherical congruences still hold if we consider either submanifolds in spaces of constant curvature without any restrictions on dimension and codimension, or submanifolds in ambient spaces different from the spaces of constant curvature. In this paper we consider two-dimensional surfaces in n-dimensional spaces of constant curvature R^n, S^n, H^n and two-dimensional surfaces in n-dimensional Riemannian products $S^{n-1} \times R^1$, $H^{n-1} \times R^1$.

A particular class of surfaces, which is both natural and exceptional from the point of view of the pseudospherical congruences, consists of ruled surfaces. Namely, let $F^2 \subset M^n$ be an oriented surface ruled by geodesics of M^n. For any smooth function l on F^2, consider a map $\psi_l : F^2 \to F^2$ that moves each point $P \in F^2$ along the corresponding ruling $\gamma_P \subset F^2$, which passes through P, to a point $\tilde{P} \in F^2$ at the distance l from P. Clearly, ψ_l satisfies the bitangency condition, so it represents a geodesic congruence. Besides, it will satisfy (C1) if we set $l \equiv l_0 > 0$, so every point of F^2 moves along a corresponding ruling of F^2 at a fixed distance. Now the problem is to verify the last property (C2). It turns out that generically (C2) does not hold. However some particular ruled surfaces are able to represent pseudospherical congruences. Namely, we will demonstrate some statements which lead to the following conclusions:

1) if F^2 is a ruled surface in one of R^n, H^n, $H^{n-1} \times R^1$, then $\psi_{l_0} : F^2 \to F^2$ does not represent pseudospherical congruences, for any $l_0 > 0$;

2) if F^2 is a ruled surface in S^n, then $\psi_{l_0} : F^2 \to F^2$ represents a pseudospherical congruence if and only if F^2 is an intrinsically flat surface;

3) if F^2 is a ruled surface in $S^{n-1} \times R^1$, then $\psi_{l_0} : F^2 \to F^2$ represents a pseudospherical congruence if and only if F^2 is an intrinsically flat surface with

constant negative extrinsic curvature.

Notice that there exist intrinsically flat ruled surfaces in $S^{n-1} \times R^1$ with nonconstant extrinsic curvature. (It would be interesting to verify that there exist ruled surfaces in $S^{n-1} \times R^1$ with constant extrinsic curvature and nonconstant intrinsic curvature.) Thus a surface in $S^{n-1} \times R^1$ must be referred to as pseudospherical if it has constant intrinsic curvature and constant negative extrinsic curvature. We hope that one can construct a consistent theory of pseudospherical congruences and Backlund transformations for two-dimensional pseudospherical surfaces in $S^{n-1} \times R^1$ and $H^{n-1} \times R^1$.

We emphasize that the constant ω_0 in (C2) is subject to the constrain $0 < \omega_0 \leq \frac{\pi}{2}$. On the other hand, if we allow ω_0 to vanish, then ruled surfaces with vanishing extrinsic curvature would appear as pseudospherical congruences too: for any of these ruled surfaces its tangent plane is stationary along any ruling.

The paper is organized as follows. In Section 2 we consider a rather trivial case of ruled surfaces in R^n. Sections 3–5 are devoted to the most interesting case of ruled surfaces in $S^{n-1} \times R^1$ and S^n. In Section 6. we analyze briefly the ruled surfaces in $H^{n-1} \times R^1$ and H^n. All considerations are local, all functions are supposed to be sufficiently smooth.

2. Ruled Surfaces in R^n

Let F^2 be an oriented ruled surface in n-dimensional Euclidean space R^n, $n \geq 3$. This surface may be represented by the position vector

$$r(u, v) = r_0(v) + ua(v),$$

where $r_0(v)$ is a vector function representing a base curve of F^2, whereas $a(v)$ is a unit vector function determining the directions of straight lines (rulings) which sweep out the surface F^2. Without loss of generality, one can assume that the base curve $u = 0$ is orthogonal to the rulings $v = const$ of F^2,

$$\langle r_0', a\rangle = 0.$$

Besides, we will assume that v is the arc length of the base curve $u = 0$, so $|r_0'| \equiv 1$. Due to the described particular choice of local coordinates u, v, the induced metric on F^2 is $ds^2 = du^2 + g_{22}dv^2$, so u, v form a semigeodesic coordinate system in F^2, and u is an arc length for every ruling $v = const$.

Proposition 1. *Let $F^2 \subset R^n$ be a ruled surface represented by a position-vector $r(u, v) = r_0(v) + ua(v)$, which satisfies $|a| = |r_0'| \equiv 1$ and $\langle r_0', a\rangle = 0$. The surface F^2 has a constant Gauss curvature if and only if it is intrinsically flat, $K \equiv 0$. The ruled surface F^2 is intrinsically flat if and only if $[r_0', a] = 0$.*

The proof of this statement is a rather simple task because of the simple form of ds^2 described above.

Now fix a positive constant l_0 and consider a regular map $\Phi_{l_0} : F^2 \to F^2$, which moves every point of F^2 along the corresponding ruling of F^2 at the distance l_0. This map Φ_{l_0} is represented by the following formula:

$$\tilde{r} = r + l_0 a = r_0 + (u + l_0)a. \tag{1}$$

Evidently, Φ_{l_0} satisfies both the bitangency condition and (C1). Let us analyze the condition(C2): for this purpose choose an arbitrary point $P \in F^2$ and calculate the angle ω between the planes tangent to F^2 at points P and $\tilde{P} = \Phi_{l_0}(P)$.

Recall that if two-dimensional planes R^2_1, R^2_2 in R^n have a common straight line $R^1_{12} = R^2_1 \cap R^2_2$, then the angle between R^2_1 and R^2_2 is defined as the angle between straight lines $R^1_1 \subset R^2_1$ and $R^1_2 \subset R^2_2$ orthogonal to R^1_{12}.

The tangent plane $T_P F^2$ at P is spanned by the vectors

$$\begin{aligned} r_u &= (r_0(v) + ua(v))'_u = a, && (2)\\ r_v &= r'_0 + ua'. && (3) \end{aligned}$$

The tangent plane $T_{\tilde{P}} F^2$ at $\tilde{P}$ is spanned by the vectors

$$\begin{aligned} \tilde{r}_u &= r_u + (l_0 a)_u = a, && (4)\\ \tilde{r}_v &= r_v + (l_0 a)_v = r'_0 + ua' + l_0 a' = r'_0 + a'(u + l_0). && (5) \end{aligned}$$

Remark that a is orthogonal to r_v and $\tilde{r}_v$.

The tangent planes $T_P F^2$ and $T_{\tilde{P}} F^2$ have a common straight line, which is just the corresponding ruling $\gamma_P \subset F^2$. Clearly, $\gamma_P \subset F^2$ is spanned by a, therefore the angle ω between $T_P F^2$ and $T_{\tilde{P}} F^2$ is determined by the vectors r_v and $\tilde{r}_v$ orthogonal to a. So ω is equal to a constant ω_0 if and only if

$$\langle r_v, \tilde{r_v}\rangle^2 = |r_v|^2\,|\tilde{r_v}|^2 \cos^2\omega_0. \tag{6}$$

By (3) and (5), equality (6) may be rewritten as follows:

$$\begin{aligned} &(1 + A(2u + l_0) + u(u + l_0)B)^2\\ &= (1 + 2uA + u^2B)(1 + 2Au + 2l_0A + Bu^2 + 2Bul_0 + Bl_0^2)\cos^2\omega_0, \end{aligned} \tag{7}$$

where $A(v) = \langle r'_0, a'\rangle$, $B(v) = \langle a', a'\rangle$. This equality, which is polynomial with respect to u, holds if the coefficients at corresponding degrees of u coincide. Hence we obtain five equalities:

$$\begin{aligned} &1 + 2Al_0 + A^2l_0^2 = (1 + 2Al_0 + Bl_0^2)\cos^2\omega_0, && (8)\\ &4A + 4A^2l_0 + 2Bl_0 + 2ABl_0^2 = (4A + 4A^2l_0 + 2Bl_0 + 2ABl_0^2)\cos^2\omega_0, && (9)\\ &4A^2 + 2B + 6ABl_0 + B^2l_0^2 = (4A^2 + 2B + 6ABl_0 + B^2l_0^2)\cos^2\omega_0, && (10)\\ &4AB + 2B^2l_0 = (4AB + 2B^2l_0)\cos^2\omega_0, && (11)\\ &B^2 = B^2\cos^2\omega_0. && (12) \end{aligned}$$

Since $\omega_0 \in (0, \frac{\pi}{2}]$, equality (12) holds if and only if $B = 0$. In this case equality (10) reads $4A^2 = 4A^2 \cos^2 \omega_0$, therefore $A = 0$. Then (8) becomes $1 = \cos^2 \omega_0$, so it contradicts to the assumption $\omega_0 \in (0, \frac{\pi}{2}]$. Therefore (6) does not hold if $\omega_0 \in (0, \frac{\pi}{2}]$.

Theorem 1. *Let F^2 be a ruled surface in R^n. Let $\Phi_{l_0} : F^2 \to F^2$ be a regular map that moves every point $P \in F^2$ along a corresponding ruling of F^2 at a fixed constant distance $l_0 > 0$. Then Φ_{l_0} does not satisfy (C2) for any $\omega_0 \in (0, \frac{\pi}{2}]$.*

Thus *the ruled surfaces in R^n do not represent pseudospherical congruences.*

On the other hand, if one allows ω_0 to be equal to 0, then (C2) turns out to be less restrictive. Namely, it is well known that an arbitrary ruled surface with vanishing Gauss curvature in R^n is tangentially degenerate, its tangent plane is stationary along any ruling.

3. Ruled Surfaces in $S^n \times R^1$

Now let us discuss ruled surfaces in the Riemannian product $S^n \times R^1$, where S^n denotes the unit sphere. We will view $S^n \times R^1$ as the hypersurface in $R^{n+1} \oplus R^1 = R^{n+2}$ represented by the equation $(x^1)^2 + \ldots + (x^{n+1})^2 = 1$ in terms of Cartesian coordinates $(x^1, \ldots, x^{n+2})$ in R^{n+2}. The unit sphere S^n will be considered as the section of $S^n \times R^1$ by the horizontal hyperplane $R^{n+1} \subset R^{n+2}$ given by $x^{n+2} = 0$.

Every curve $\{P\} \times R^1 \subset S^n \times R^1$, where P is a point of S^n, is a geodesic curve of $S^n \times R^1$ usually referred to as a vertical one. It may be represented by the position vector $r(u) = \rho + ue$, here $e = (0, \ldots, 0, 1)$, whereas ρ stands for the position vector of P. A ruled surface in $S^n \times R^1$ swept out by vertical geodesics is a cylindrical surface, *a vertical cylinder*, which may be represented in the following form:

$$r(u, v) = \rho(v) + ue, \tag{13}$$

here $\rho(v)$ determines the base curve of the cylinder. Evidently, the cylindrical surfaces in question are intrinsically and extrinsically flat, $K_{int} = K_{ext} \equiv 0$. Moreover, when a point on such a cylindrical surface moves along the corresponding ruling, the tangent plane is stationary.

An arbitrary nonvertical geodesic curve γ in $S^n \times R^1$ may be represented by the position vector $r(u) = a \cos u + b \sin u + e(pu + q)$, where a, b are orthonormal vectors in the horizontal hyperplane R^{n+1}, p and q are some constants. Notice that the position vector $r^*(u) = a \cos u + b \sin u$ represents a great circle γ^* in the unit sphere $S^n \subset R^{n+1}$. Evidently, γ^* is obtained as the intersection of S^n with a two-plane spanned by a and b. It is natural to say that γ^* is the base of γ.

An arbitrary ruled surface F^2 in $S^n \times R^1$ swept out by nonvertical geodesics may be represented in the following way:

$$r(u, v) = a(v) \cos u + b(v) \sin u + e(p(v)u + q(v)), \tag{14}$$

where $a(v)$, $b(v)$ are orthonormal vector functions, $p(v)$ and $q(v)$ are some functions which depend on v only. We will assume that all functions in question are sufficiently smooth. Since $a(v)$, $b(v)$ are orthonormal, we have

$$|a| = 1, \quad |b| = 1, \quad \langle a, b\rangle = 0. \tag{15}$$

Moreover, without loss of generality, one can specify the choice of $a(v)$, $b(v)$ in such a particular way that the following additional conditions will be satisfied:

$$\langle a', b\rangle = 0, \quad \langle a, b'\rangle = 0. \tag{16}$$

Namely, if it is necessary, one can replace a, b by $a^\sharp = a\cos\zeta + b\sin\zeta$, $b^\sharp = -a\sin\zeta + b\cos\zeta$, where $\zeta(v)$ is some suitably chosen function. We will always assume that (15)–(16) hold. Let us emphasize that the rulings of F^2 are given by $v = const$.

R e m a r k. If $a' = b' = 0$, i.e., if a and b are constant, then F^2 is swept out by nonvertical geodesics $v = const$ as well as by vertical geodesics $u = const$. Hence F^2 is a vertical cylinder, its base curve is a great circle in S^n, but its parametrization (14) is different from (13).

Now let us classify the ruled surfaces in $S^n \times R^1$ with constant intrinsic (Gauss) curvature.

Proposition 2. *A regular ruled surface in $S^n \times R^1$ with position vector (14) that satisfies (15)–(16) has a constant Gauss curvature if and only if:*
1) $|a'| = |b'|$, $\langle a', b'\rangle = 0$, and p is constant, $p \equiv p_0$,
or
2) $[a', b'] = 0$, and p, q are constant, $p \equiv p_0$, $q \equiv q_0$,
or
3) $a' = b' = 0$.
Moreover, $K_{int} = 0$ in the cases 1) and 3), and $K_{int} = \frac{1}{1+p_0^2}$ in the case 2).

P r o o f. The metric of F^2 reads

$$ds^2 = (1+p^2)du^2 + 2p(p'u+q')dudv + \left(X\cos 2u + Y\sin 2u + Z + (p'u+q')^2\right)dv^2, \tag{17}$$

where $X = \frac{|a'|^2-|b'|^2}{2}$, $Y = \langle a', b'\rangle$, $Z = \frac{|a'|^2+|b'|^2}{2}$.

The intrinsic curvature K_{int} of F^2 may be expressed in terms of coordinates u, v and functions X, Y, Z, p, q depending on v. The expression for K_{int} is rather cumbersome, one can suggest to apply some standard computer programs of symbolic calculus to derive the expression in question. Anyway, it is easy to verify that the intrinsic curvature is constant, $K_{int} \equiv K_0$, if and only if an equality of the following form holds:

$$P_0 + P_1\cos 2u + P_2\sin 2u + P_3\cos 4u + P_4\sin 4u = 0, \tag{18}$$

where P_i are some functions, which are *polynomial* with respect to u. Clearly, (18) is equivalent to the system of five equalities $P_0 = 0, \ldots, P_4 = 0$. In particular, we have

$$P_3 = -\frac{1}{2}(X^2 - Y^2)(1+p^2)\big(K_0(1+p^2) - 1\big) = 0, \tag{19}$$

$$P_4 = -2XY(1+p^2)\big(K_0(1+p^2) - 1\big) = 0. \tag{20}$$

It follows from (19) that either $X = 0$, $Y = 0$, or $K_0(1+p^2) - 1 = 0$.

In the first case, when $X = Y = 0$, the equalities $P_1 = 0$, $P_2 = 0$ hold too, whereas $P_0 = 0$ reads

$$(p')^2 Z + K_0\left((p')^2 \cdot u^2 + 2p'q' \cdot u + (q')^2 + (1+p)^2 Z\right)^2 = 0. \tag{21}$$

This is a polynomial equality of the 4th order with respect to u, it holds identically if and only if all its coefficients vanish. It is easy to verify that (21) holds if and only if either $p' \equiv 0$ and $K_0 = 0$, or $Z \equiv 0$ and $K_0 = 0$. Recall the expressions for X, Y to see that $X = 0$, $Y = 0$ are equivalent to $|a'|^2 = |b'|^2$, $\langle a', b'\rangle = 0$, whereas $X = 0$, $Y = 0$, $Z = 0$ are equivalent to $a' = b' \equiv 0$.

In the second case, when $K_0(1+p^2) - 1 = 0$, the function p has to be constant, $p \equiv p_0$, and $K_0 = \frac{1}{1+p_0^2} \in (0, 1]$. Similarly to the first case, the equalities $P_1 = 0$, $P_2 = 0$ hold too, whereas $P_0 = 0$ reads

$$Z^2 - X^2 - Y^2 + \left(\frac{1}{1+p_0^2}\right)^2 (q')^4 + 2\frac{1}{1+p_0^2} Z (q')^2 = 0. \tag{22}$$

Since $Z = \frac{|a'|^2 + |b'|^2}{2} \geq 0$ and $Z^2 - X^2 - Y^2 = |[a', b']|^2 \geq 0$, it is easy to see that (22) holds if and only if $q' \equiv 0$, $[a', b'] \equiv 0$, q.e.d.

Thus there are three different classes of ruled surfaces with constant intrinsic curvature in $S^n \times R^1$:

1) intrinsically flat ruled surfaces represented by (14) with a, b, p, q satisfying (15)–(16) and $|a'| = |b'| \neq 0$, $\langle a', b'\rangle = 0$, $p \equiv p_0$;

2) ruled surfaces represented by (14) with a, b, p, q satisfying (15)–(16) and $[a', b'] = 0$, $p \equiv p_0$, $q \equiv q_0$;

3) intrinsically flat vertical cylinders represented by (13) or by (14) with a, b satisfying $a' = b' = 0$.

Now let us analyze what kind of ruled surfaces with constant intrinsic curvature in $S^n \times R^1$ are of constant extrinsic curvature.

Proposition 3. *Let $F^2 \subset S^n \times R^1$ be a regular intrinsically flat ruled surface with position vector (14) that satisfies (15)–(16) and $|a'| = |b'| \neq 0$, $\langle a', b'\rangle = 0$, $p \equiv p_0$. Then the extrinsic curvature of F^2 is equal to*

$$K_{ext} = -\frac{|a'|^2}{(1+p_0^2)|a'|^2 + (q')^2}.$$

P r o o f. The plane tangent to F^2 is spanned by the vectors

$$\begin{aligned}\partial_u r &= -a\sin u + b\cos u + ep_0,\\ \partial_v r &= a'\cos u + b'\sin u + eq',\end{aligned}$$

since $p \equiv p_0$ by assumption. The vector

$$N_0 = a\cos u + b\sin u$$

is normal to $S^n \times R^1 \subset R^{n+2}$.

By the above assumptions, the metric of F^2 reads

$$ds^2 = (1+p_0^2)du^2 + 2p_0 q' du dv + (|a'|^2 + (q')^2)dv^2.$$

Let $II = L_{11}du^2 + 2L_{12}dudv + L_{22}dv^2$ denote the normal-valued second fundamental form of $F^2 \subset S^N \times R^1$. The second partial derivative

$$\partial_{uu} r = -a\cos u - b\sin u$$

is evidently equal to $-N_0$, so $L_{11} = 0$. On the other hand, the second partial derivative

$$\partial_{uv} r = -a'\sin u + b'\cos u$$

is orthogonal to $\partial_u r$, $\partial_v r$ and N_0. Therefore $L_{12} = \partial_{uv} r$, so the extrinsic curvature of F^2 is the following:

$$K_{ext} = \frac{-|L_{12}|^2}{g_{11}g_{22} - (g_{12})^2} = \frac{-|a'|^2}{(1+p_0^2)|a'|^2 + (q')^2}.$$

Thus we see that generically the extrinsic curvature of the intrinsically flat ruled surfaces in question is not constant, the constancy of K_{ext} results in some additional restrictions.

Corollary. *Let $F^2 \subset S^n \times R^1$ be as in Proposition 3. The extrinsic curvature of F^2 is constant if and only if $q' = c|a'|$. In this case* $K_{ext} = -\dfrac{1}{1+p_0^2+c^2}$.

E x a m p l e. Consider a ruled surface F^2 in $S^3 \times R^1 \subset R^5$ represented by the following position vector:

$$r(u,v) = \begin{pmatrix}\cos v\\ \sin v\\ 0\\ 0\\ 0\end{pmatrix}\cos u + \begin{pmatrix}0\\ 0\\ \cos v\\ \sin v\\ 0\end{pmatrix}\sin u + \begin{pmatrix}0\\ 0\\ 0\\ 0\\ p_0 u + q(v)\end{pmatrix}.$$

Its intrinsic curvature is $K_{int} = 0$, whereas the extrinsic curvature K_{ext} is equal to $-\frac{1}{1+p_0^2+(q')^2}$, so K_{ext} is constant if and only if $q(v)$ is a linear function. Besides, for an arbitrary $\hat{K}_0 \in [-1, 0)$ one can choose appropriate p_0 and $q(v)$ such that $K_{ext} \equiv \hat{K}_0$.

As for other classes of ruled surfaces with constant intrinsic curvature in $S^n \times R^1$, the situation is more trivial.

Proposition 4. *Let $F^2 \subset S^n \times R^1$ be a regular intrinsically flat ruled surface with position vector (14) that satisfies (15)-(16) and $[a', b'] = 0$, $p \equiv p_0$, $q \equiv q_0$. Then the extrinsic curvature of F^2 vanishes, $K_{ext} = 0$.*

P r o o f. The plane tangent to F^2 is now spanned by the vectors

$$\begin{aligned} \partial_u r &= -a \sin u + b \cos u + ep_0, \\ \partial_v r &= a' \cos u + b' \sin u. \end{aligned}$$

The vector $N_0 = a \cos u + b \sin u$ is normal to $S^n \times R^1 \subset R^{n+2}$.

Let $II = L_{11} du^2 + 2L_{12} du dv + L_{22} dv^2$ still denote the normal-valued second fundamental form of $F^2 \subset S^N \times R^1$. The second partial derivative

$$\partial_{uu} r = -a \cos u - b \sin u$$

is equal to $-N_0$, so $L_{11} = 0$. Besides, the second partial derivative

$$\partial_{uv} r = -a' \sin u + b' \cos u$$

is collinear to $\partial_v r$, since $[a', b'] = 0$. Therefore $L_{12} = 0$. Hence $K_{ext} \equiv 0$, q.e.d.

Finally, it is evident that the vertical cylinders in $S^n \times R^1$ are both intrinsically and extrinsically flat, $K_{int} = K_{ext} \equiv 0$.

4. Pseudospherical Congruences in $S^n \times R^1$

Now let us analyze when a ruled surface in $S^n \times R^1$ represents a pseudospherical congruence. So, let $F^2 \subset S^n \times R^1$ be a ruled surface represented as in the previous section by the position vector (14) that satisfies (15)–(16). Fix a constant l_0 and consider the map $\psi_{l_0} : F^2 \to F^2$ which moves each point $P \in F^2$ along the corresponding ruling $\gamma_P \subset F^2$, which passes through P, to a point $\tilde{P} \in F^2$ at the distance l_0 from P. Take into account (14) to see that ψ_{l_0} may be represented as follows:

$$r(u, v) = a(v) \cos(u + \alpha) + b(v) \sin(u + \alpha) + e \left(p(v)(u + \alpha) + q(v) \right), \qquad (23)$$

where $\alpha = \frac{l_0}{\sqrt{1+p^2}}$ depends on v.

Clearly, ψ_{l_0} satisfies the bitangency condition and represents a geodesic congruence. Besides, it satisfies the requirement (C1). The question is to verify the condition (C2): one has to check when the angle between T_PF^2 and $T_{\tilde{P}}F^2$ is constant, $\angle(T_PF^2, T_{\tilde{P}}F^2) \equiv \omega_0 \in (0, \frac{\pi}{2}]$.

To be more precise, let us consider the map $\Pi : T_P\ S^n \times R^1 \to T_{\tilde{P}}\ S^n \times R^1$ generated by the parallel translation in $S^n \times R^1$ from P to $\tilde{P}$ along γ_P. Evidently, the vector $\dot{\gamma}_P(P)$ tangent to the geodesic γ_P at P is mapped by Π to the vector $\dot{\gamma}_P(\tilde{P})$ tangent to γ_P at $\tilde{P}$. The plane T_PF^2 tangent to F^2 at P is mapped by Π to a two-plane $V \subset T_{\tilde{P}}\ S^n \times R^1$. Since $\dot{\gamma}_P(P) \in T_PF^2$ and $\dot{\gamma}_P(\tilde{P}) \in T_{\tilde{P}}F^2$, the intersection of two-planes V and $T_{\tilde{P}}F^2$ is just the straight line in $T_{\tilde{P}}\ S^n \times R^1$ spanned by $\dot{\gamma}_P(\tilde{P})$.

The angle $\angle(T_PF^2, T_{\tilde{P}}F^2)$ is defined as the angle between V and $T_{\tilde{P}}F^2$. In order to calculate the latter, one has to find vectors $\hat{Z} \in V$ and $\tilde{Z} \in T_{\tilde{P}}F^2$ orthogonal to $\dot{\gamma}_P(\tilde{P})$. In particular, $\angle(V, T_{\tilde{P}}F^2) \equiv \omega_0$ if and only if the following equality holds at every point P:

$$\langle \hat{Z}, \tilde{Z} \rangle^2 = \cos^2 \omega_0 |\hat{Z}|^2 |\tilde{Z}|^2. \tag{24}$$

Let us find $\hat{Z}$ and $\tilde{Z}$ and then analyze (24).

The tangent plane T_PF^2 is spanned by the vectors

$$\partial_u r = -a \sin u + b \cos u + ep, \tag{25}$$

$$\partial_v r = a' \cos u + b' \sin u + e\left(p'u + q'\right). \tag{26}$$

Clearly, $\partial_u r$ is equal to $\dot{\gamma}_P(P)$. Consider the vector $Z = -\langle \partial_u r, \partial_v r \rangle \partial_u r + |\partial_u r|^2 \partial_v r$ in T_PF^2 which is orthogonal to $\partial_u r$. By (25), we have

$$\begin{aligned} Z = &-p(p'u + q')\left(-a \sin u + b \cos u\right) \\ &+ (1 + p^2)\left(a' \cos u + b' \sin u\right) + e(p'u + q'). \end{aligned}$$

Since $Z \in T_PF^2$ is orthogonal to $\dot{\gamma}_P(P)$, then the image of Z under Π belongs to V and it is orthogonal to $\dot{\gamma}_P(\tilde{P})$. Hence one can set $\hat{Z} = \Pi(Z)$. The parallel translation Π along γ_P in $S^n \times R^1$ is generated by a vertical translation in R^{n+2} followed by an orthogonal transformation in R^{n+2} which acts as a rotation in the two-plane spanned by the vectors a, b and as the identity map in the orthogonal compliment of this two-plane. Therefore we obtain

$$\begin{aligned} \hat{Z} = &-p(p'u + q')\left(-a \sin(u + \alpha) + b \cos(u + \alpha)\right) \\ &+(1 + p^2)\left(a' \cos u + b' \sin u\right) + e(p'u + q'), \end{aligned} \tag{27}$$

since a', b' and e are orthogonal to a and b.

On the other hand, the tangent plane $T_{\tilde{P}}F^2$ is spanned by the vectors

$$\begin{aligned}\partial_u\tilde{r} &= -a\sin(u+\alpha) + b\cos(u+\alpha) + ep,\\ \partial_v\tilde{r} &= a'\cos(u+\alpha) + b'\sin(u+\alpha) + e\,(p'(u+\alpha)+q')\\ &\quad +\alpha'\,(-a\sin(u+\alpha)+b\cos(u+\alpha)+ep)\,.\end{aligned} \tag{28}$$

Evidently, $\partial_u\tilde{r}$ is equal to $\dot{\gamma}_P(\tilde{P})$. The vector $\tilde{Z} = -\langle\partial_u\tilde{r},\partial_v\tilde{r}\rangle\partial_u\tilde{r} + |\partial_u\tilde{r}|^2\partial_v\tilde{r}$ in $T_{\tilde{P}}F^2$ is orthogonal to $\partial_u\tilde{r}$. By (28), we have

$$\begin{aligned}\tilde{Z} = &-p(p'(u+\alpha)+q')\,(-a\sin(u+\alpha)+b\cos(u+\alpha))\\ &+(1+p^2)\,(a'\cos(u+\alpha)+b'\sin(u+\alpha)) + e(p'(u+\alpha)+q').\end{aligned} \tag{29}$$

Substituting (27) and (29) into (24), we get a cumbersome equality of the following form:

$$Q_0 + Q_1\cos 2u + Q_2\sin 2u + Q_3\cos 4u + Q_4\sin 4u = 0, \tag{30}$$

where Q_i are polynomial with respect to u with coefficients expressed in terms of $X(v) = \frac{|a'|^2-|b'|^2}{2}$, $Y(v) = \langle a', b'\rangle$, $Z(v) = \frac{|a'|^2+|b'|^2}{2}$, $p(v)$, $q(v)$ and $\alpha(v)$. Since Q_i are polynomial in u, equality (30) is equivalent to the system of five equalities $Q_0 = 0,\ldots,Q_4 = 0$. In particular, we have

$$\begin{aligned}Q_3 &= -\tfrac{1}{2}\sin^2\omega_0(1+p^2)^4\left((Y^2-X^2)\cos(2\alpha) - 2XY\sin(2\alpha)\right) = 0,\\ Q_4 &= \tfrac{1}{2}\sin^2\omega_0(1+p^2)^4\left((Y^2-X^2)\sin(2\alpha) + 2XY\cos(2\alpha)\right) = 0.\end{aligned} \tag{31}$$

Since $\omega_0 \in (0,\frac{\pi}{2}]$ by assumption, (31) is reduced to a linear system with respect to Y^2-X^2 and $2XY$. The only solution is $Y^2-X^2 = 0$, $2XY = 0$, i.e., $X = 0$, $Y = 0$. In this case the equalities $Q_1 = 0$, $Q_2 = 0$ hold too, whereas $Q_0 = 0$ reads as follows:

$$Q_{04}u^4 + Q_{03}u^3 + Q_{02}u^2 + Q_{01}u + Q_{00} = 0, \tag{32}$$

where the coefficients Q_{0i} are expressed in terms of $Z(v)$, $p(v)$, $q(v)$ and $\alpha(v)$. In particular, $Q_{04} = -2\sin^2\omega_0(p')^4$, so if (32) holds, then $p \equiv p_0$. On the other hand, if $p \equiv p_0$, then (32) becomes as follows:

$$\left((1+p_0^2)Z\cos\alpha + (q')^2\right)^2 = \cos^2\omega_0\left((1+p_0^2)Z + (q')^2\right)^2. \tag{33}$$

Thus the angle between T_PF^2 and $T_{\tilde{P}}F^2$ is constant and equal to $\omega_0 \in (0,\frac{\pi}{2}]$ if and only if $X \equiv 0$, $Y \equiv 0$, $p \equiv p_0$ and (33) holds. In other words, it means that $|a'| = |b'|$, $\langle a', b'\rangle \equiv 0$, $p \equiv p_0$ hold together with

$$\left((1+p_0^2)|a'|^2\cos\alpha + (q')^2\right)^2 = \cos^2\omega_0\left((1+p_0^2)|a'|^2 + (q')^2\right)^2, \tag{34}$$

since $Z = \frac{1}{2}(|a'|^2 + |b'|^2) = |a'|^2$. Due to Proposition 2, $|a'| = |b'|$, $\langle a', b'\rangle \equiv 0$, $p \equiv p_0$ mean exactly that F^2 is intrinsically flat. Moreover, substituting $q' = c|a'|$ into (34), we obtain $\left((1+p_0^2)\cos\alpha + c^2\right)^2 = \cos^2\omega_0\left(1+p_0^2+c^2\right)^2$, so c is constant. Therefore the extrinsic curvature of F^2 is constant, $K_{ext} = -\frac{1}{1+p_0^2+c^2}$, by Proposition 3.

Thus we proved the following statement.

Theorem 2. *The map $\psi_{l_0} : F^2 \to F^2$ represents a pseudospherical congruence if and only if F^2 is an intrinsically flat surface with constant extrinsic curvature.*

We should emphasize that if an intrinsically flat surface with constant extrinsic curvature in $S^n \times R^1$ is given, then for any $l_0 \neq 2\pi k\sqrt{1+p_0^2}$ one can construct a well-defined pseudospherical congruence $\psi_{l_0} : F^2 \to F^2$, whose parameters l_0, ω_0 necessarily satisfy the equality

$$\left((1+p_0^2)\cos\frac{l_0}{\sqrt{1+p_0^2}} + c^2\right)^2 = \cos^2\omega_0\left(1+p_0^2+c^2\right)^2.$$

E x a m p l e. Consider a ruled surface F^2 in $S^3 \times R^1 \subset R^5$ represented by the following position vector:

$$r(u,v) = \begin{pmatrix} \cos v \\ \sin v \\ 0 \\ 0 \\ 0 \end{pmatrix}\cos u + \begin{pmatrix} 0 \\ 0 \\ \cos v \\ \sin v \\ 0 \end{pmatrix}\sin u + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ p_0 u + q_0 v + q_1 \end{pmatrix}.$$

Its intrinsic curvature is $K_{int} \equiv 0$, whereas the extrinsic curvature K_{ext} is equal to $-\frac{1}{1+p_0^2+q_0^2}$. An arbitrary $l_0 > 0$ given, the map $\psi_{l_0} : F^2 \to F^2$ represents a pseudospherical congruence with

$$\cos^2\omega_0 = \left(1 - \frac{1+p_0^2}{1+p_0^2+q_0^2}\left(1-\cos\frac{l_0}{\sqrt{1+p_0^2}}\right)\right)^2.$$

5. Ruled Surfaces and Pseudospherical Congruences in S^n

Since the unit sphere S^n may be viewed as a horizontal section of $S^n \times R^1$, the ruled surfaces in S^n are just the ruled surfaces in $S^n \times R^1$ with $p \equiv 0$ and $q \equiv q_0$. This gives us a simple way to adopt the ideas, methods and results obtained above in Sections 3–4.

An arbitrary ruled surface F^2 in S^n swept out by great circles of S^n may be represented in the following way:

$$r(u,v) = a(v)\cos u + b(v)\sin u, \tag{35}$$

where $a(v)$, $b(v)$ are orthonormal vector functions

$$|a| = 1, \quad |b| = 1, \quad \langle a, b\rangle = 0. \tag{36}$$

Moreover, without loss of generality, one can specify the choice of $a(v)$, $b(v)$ in such a particular way that the following additional conditions will be satisfied:

$$\langle a', b\rangle = 0, \quad \langle a, b'\rangle = 0. \tag{37}$$

The metric of F^2 reads $ds^2 = du^2 + (a'\cos u + b'\sin u)^2 dv^2$, so (u, v) is a semi-geodesic local coordinate system in F^2. The coordinate curves $v = const$ are rulings of F^2, and u is the arc length for any of these rulings.

The following statement describes the ruled surfaces with constant intrinsic curvature in S^n. This is Proposition 2 rewritten for the particular case of $p \equiv 0$, $q \equiv q_0$.

Proposition 2$^\flat$. *A regular ruled surface F^2 in S^n with position vector (35) that satisfies (36)–(37) has a constant Gauss curvature if and only if either:*

1) $|a'| = |b'|$, $\langle a', b'\rangle = 0$,

or

2) $[a', b'] = 0$.

Moreover $K_{int} \equiv 0$ in the Case 1), and $K_{int} \equiv 1$ in the Case 2).

Clearly, for an arbitrary surface in the unit sphere S^n one has $K_{int} = K_{ext} + 1$, so the constancy of K_{int} is equivalent to the constancy of K_{ext}. Therefore an intrinsically flat surface in S^n is pseudospherical, its extrinsic curvature is constant negative, $K_{ext} \equiv -1$. On the other hand, if $K_{int} \equiv 1$, then $K_{ext} \equiv 0$. Let us mention two trivial examples: the Clifford tori are intrinsically flat ($K_{int} \equiv 0$), whereas the totally geodesic two-dimensional spheres have the vanishing extrinsic curvature, so $K_{int} \equiv 1$. A rather substantial discussion of surfaces with constant curvature in S^n one can find, for example, in [7].

Now, fix a constant l_0 and consider the map $\psi_{l_0} : F^2 \to F^2$, which moves each point $P \in F^2$ along the corresponding ruling of F^2 at the distance l_0. This map may be represented as follows:

$$\tilde{r}(u,v) = a(v)\cos(u + l_0) + b(v)\sin(u + l_0).$$

Evidently, ψ_{l_0} satisfies the bitangency condition as well as the requirement (C1). The question is when ψ_{l_0} satisfies the requirement (C2) and represents a pseudo-spherical congruence. The answer is given in Theorem 2, where we have to set $p \equiv 0$, $q \equiv q_0$.

Theorem 2$^\flat$. *The map $\psi_{l_0} : F^2 \to F^2$ represents a pseudospherical congruence if and only if F^2 is intrinsically flat, $K_{int} \equiv 0$. Besides, the parameters ω_0 and l_0 of ψ_{l_0} are related by the equality* $\cos^2 \omega_0 = \cos^2 l_0$.

6. Ruled Surfaces and Pseudospherical Congruences in H^n and $H^n \times R^1$

Similarly to the spherical case one can consider ruled surfaces and pseudospherical congruences in the hyperbolic space H^n and in the Riemannian product $H^n \times R^1$. The hyperbolic space H^n is viewed as the hypersurface in Minkowski space M^{n+1} represented by the relations $|x|^2 = -(x^1)^2 + \ldots + (x^{n+1})^2 = -1$, $x_1 > 0$ in terms of Cartesian coordinates $(x^1, \ldots, x^{n+1})$ in M^{n+1}. The manifold $H^n \times R^1$ is viewed as the hypersurface in $M^{n+2} = M^{n+1} \oplus R^1$ represented by the same relations in terms of Cartesian coordinates $(x^1, \ldots, x^{n+1}, x^{n+2})$ in M^{n+2}. Clearly, H^n may be interpreted as the section of $H^n \times R^1$ by an arbitrary horizontal hyperplane $M^{n+1} \subset M^{n+2}$ given by $x^{n+2} = const$.

A vertical cylinder in $H^n \times R^1$, which is swept out by vertical geodesics of $H^n \times R^1$, is represented in the following way:

$$r(u, v) = \rho(v) + ue; \tag{38}$$

here $\rho(v)$ determines the base curve of cylinder, it belongs to $H^n = H^n \times \{0\}$, and $e = (0, \ldots, 0, 1)$. It is evident that the vertical cylinders are intrinsically and extrinsically flat, $K_{int} = K_{ext} \equiv 0$. Moreover, when a point on such a cylindrical surface moves along the corresponding ruling, the tangent plane is stationary.

An arbitrary ruled surface F^2 in $H^n \times R^1$ swept out by nonvertical geodesics may be represented in the following way:

$$r(u, v) = a(v) \cosh u + b(v) \sinh u + e(p(v)u + q(v)), \tag{39}$$

where $a(v)$, $b(v)$ are orthonormal vector functions with values in $M^{n+1} = M^{n+1} \times \{0\}$ which satisfy

$$|a|^2 = -1, \quad |b|^2 = 1, \quad \langle a, b\rangle = 0, \tag{40}$$

$p(v)$ and $q(v)$ are some arbitrary functions depending on v. Without loss of generality, one can always specify the choice of $a(v)$, $b(v)$ in such a way that the following additional conditions will be satisfied:

$$\langle a', b\rangle = 0, \quad \langle a, b'\rangle = 0. \tag{41}$$

Notice that a is timelike, whereas b is spacelike. Since a' and b' are orthogonal to a, they are spacelike or vanishing.

Proposition 5. *A regular ruled surface in $H^n \times R^1$ with position vector (39), which satisfies (40)–(41), is intrinsically flat if and only if $a' = b' = 0$.*

P r o o f. The metric form of F^2 reads

$$ds^2 = (1 + p^2)du^2 + 2p(p'u + q')dudv+$$

$$+ \left(X \cosh 2u + Y \sinh 2u + Z + (p'u + q')^2\right) dv^2,$$

where $X = \frac{|a'|^2+|b'|^2}{2}$, $Y = \langle a', b' \rangle$, $Z = \frac{|a'|^2-|b'|^2}{2}$.

It is easy to verify that F^2 is intrinsically flat, $K_{int} \equiv 0$, if and only if an equality of the following form holds:

$$P_0 + P_1 \cosh 2u + P_2 \sinh 2u + P_3 \cosh 4u + P_4 \sinh 4u = 0, \tag{42}$$

here P_i are some functions which are polynomial with respect to u. The last equality is equivalent to the system of five equalities $P_0 = 0, \ldots, P_4 = 0$. In particular, we have

$$P_3 = -\frac{1}{2}(X^2 + Y^2)(1 + p^2) = 0, \quad P_4 = -2XY(1 + p^2) = 0,$$

so $X = 0$ and $Y = 0$. Recall that $X = \frac{|a'|^2+|b'|^2}{2}$, $Y = \langle a', b' \rangle$. Since a' and b' are either spacelike or vanishing, then $X = Y = 0$ if and only if $a' = b' = 0$.

On the other hand, if $a' = 0$ and $b' = 0$, then (42) holds without any additional requirements, so $K_{int} \equiv 0$, q.e.d.

Notice that if $a' = b' = 0$, i.e., if a and b are constant, then F^2 is swept out by nonvertical geodesics $v = const$ as well as by vertical geodesics $u = const$. Hence F^2 is a vertical cylinder, its base curve is a geodesic of H^n, but its parametrization (38) is different from (39).

Thus *only intrinsically flat ruled surfaces in $H^n \times R^1$ are vertical cylinders*. Since H^n may be viewed as a horizontal section of $H^n \times R^1$, we obtain that *there are no intrinsically flat ruled surfaces in H^n*. It seems to be true that only ruled surfaces of constant intrinsic curvature in H^n are totally geodesic ones. An interesting problem is to classify the ruled surfaces with constant intrinsic (or extrinsic) curvature in $H^n \times R^1$.

Let us briefly discuss when a ruled surface in $H^n \times R^1$ (or in H^n) represents a pseudospherical congruence. A ruled surface F^2 in $H^n \times R^1$ (or in H^n) given, let $\psi_{l_0} : F^2 \to F^2$ be a regular map which moves each point of F^2 along the corresponding ruling of F^2 at a constant distance l_0. Clearly, it satisfies the bitangency condition and the requirement (C1). The problem is to analyze when ψ_{l_0} possesses the property (C2). For this purpose one can directly apply the method used above in the proof of Theorem 2 with some minor changes, for instance, the standard trigonometric functions have to be replaced by their hyperbolic counterparts. It turns out that the key difference between the spherical and hyperbolic cases is that in $S^n \times R^1$ (as well as in S^n) there is an exceptional class of intrinsically flat ruled surfaces different from vertical cylinders (see items 1 of Props. 2–3), whereas $H^n \times R^1$ and H^n do not admit the ruled surfaces of this kind. So the following statements hold.

Theorem 3. *Let F^2 be a ruled surface in $H^n \times R^1$. The map $\psi_{l_0} : F^2 \to F^2$ does not represent pseudospherical congruences for any l_0.*

Corollary. *Let F^2 be a ruled surface in H^n. The map $\psi_{l_0} : F^2 \to F^2$ does not represent pseudospherical congruences for any l_0.*

7. Conclusion

Thus, if we consider ruled surfaces in R^n, S^n, H^n, $S^n \times R^1$, $H^n \times R^1$, then pseudospherical congruences may be represented only by the intrinsically flat surfaces with nonzero constant extrinsic curvature in $S^n \times R^1$ and S^n. This class of pseudospherical congruences is rather exceptional. So, it would be very interesting to construct a more general theory of pseudospherical congruences in $S^n \times R^1$ and $H^n \times R^1$ dealing with nonruled surfaces.

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