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Small Transversal Vibrations of Elastic Rod with Point Mass at One End Subject to Viscous Friction

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A spectral problem describing small transversal vibrations of an elastic rod with a concentrated mass (bead) at the right end subject to viscous friction is considered. The left end is hinge-jointed. The location of the spectrum of this problem is described and the asymptotic formula of the eigenvalues is obtained.

Key words: eigenvalues, operator pencil, algebraic multiplicity, boundary conditions.

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1. Introduction

Small transversal vibrations of a homogeneous rod of density $\rho = 1$, stretched by a distributed longitudinal force proportional to $g(x) \geqslant 0$, $g \in C^1[0, l]$ subject to homogeneous viscous friction of the coefficient $k > 0$, can be described by the equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(g(x)\frac{\partial u}{\partial x}\right) = 0. \tag{1}$$

Here x is the longitudinal coordinate measured from the left end of the rod, t is the time, $u(x, t)$ is the transversal displacement of a point lying at the distance x from the left end of the rod at the time t. The left end of the rod is hinge-jointed without damping while at the right end there is a massive ring with mass $m > 0$ that is able to move in the direction orthogonal to the equilibrium position of the rod. The hinge connection of the left end is described by the boundary conditions

$$u(0, t) = 0, \tag{2}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0. \tag{3}$$

The boundary conditions at the right end are

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=l} = 0, \tag{4}$$

$$-\left.\frac{\partial^3 u}{\partial x^3}\right|_{x=l} + m\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=l} + g(x)\left.\frac{\partial u}{\partial x}\right|_{x=l} + \beta\left.\frac{\partial u}{\partial t}\right|_{x=l} = 0, \tag{5}$$

where $l > 0$ is the length of the rod, $\beta > 0$ is the coefficient of viscous friction (damping) of the ring. Condition (4) shows that the rod is hinge-connected to the ring while condition (5) means that the ring can move with damping in the direction orthogonal to the equilibrium position of the rod. Various boundary conditions for undamped rods were considered in [1]. Condition (5) is physically motivated for the case of damping (see, e.g., [1–4]. It should be mentioned that the results similar to the ones obtained in this paper can be found in [5] for another type of boundary conditions. Excluding the time by usual transformation $u(x,t) = e^{i\lambda t} y(\lambda, x)$, we arrive at the following spectral problem:

$$y^{(4)} - \lambda^2 y - (gy')' + ik\lambda y = 0, \tag{6}$$

$$y(\lambda, 0) = 0, \tag{7}$$

$$y^{(2)}(\lambda, 0) = 0, \tag{8}$$

$$y^{(2)}(\lambda, l) = 0, \tag{9}$$

$$-y^{(3)}(l) - m\lambda^2 y(l) + g(l)y'(l) + i\lambda\beta y(l) = 0. \tag{10}$$

2. Operator Theory Approach

To consider the operator-theoretical approach to problem (6)–(10) let us introduce the operators acting in the Hilbert space $L_2(0,l) \oplus \mathbb{C}$ according to

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} y(x) \\ c \end{pmatrix} : \begin{array}{l} y(x) \in W_2^4(0,l), \\ c = y(l), \quad y(0) = y^{(2)}(0) = 0, \quad y^{(2)}(l) = 0 \end{array} \right\},$$

$$A\begin{pmatrix} y(x) \\ c \end{pmatrix} = \begin{pmatrix} y^{(4)} \\ -y^{(3)}(l) \end{pmatrix}, \quad G\begin{pmatrix} y \\ c \end{pmatrix} = \begin{pmatrix} -(gy')' \\ g(l)y'(l) \end{pmatrix},$$

$$M = \begin{pmatrix} I & 0 \\ 0 & m \end{pmatrix}, \quad K = \begin{pmatrix} k & 0 \\ 0 & \beta \end{pmatrix}$$

and introduce the quadratic operator pencil

$$L(\lambda) = (A - \lambda^2 M + G + i\lambda K)$$

with the domain $\mathcal{D}(L) = \mathcal{D}(A)$ independent of the spectral parameter λ by definition. The spectrum of $L(\lambda)$ can be identified with the spectrum of problem (6)–(10) because the components of the vectorial equation $L(\lambda)Y = 0$ are nothing but (6) and (10) while conditions (7)–(8) can be found in the description of $D(A)$. The coefficients in the equations of problem (6)–(10) are entire functions of λ. Therefore (see [6, p. 27]), the spectrum of problem (6)–(10), i.e., the spectrum of the pencil L, consists of normal eigenvalues accumulating at infinity. We use the following definitions.

Definition 1. *The set of values of λ such that the inverse operator $L(\lambda)^{-1}$ exists as a bounded closed operator is said to be the resolvent set of $L(\lambda)$ and the compliment is said to be the spectrum of $L(\lambda)$.*

Definition 2. *A number $\lambda_0 \in C$ is said to be an eigenvalue ([7]) of the operator pencil $L(\lambda)$ if there exists a vector $y_0 \in D(A)$ called eigenvector such that $y_0 \neq 0$ and $L(\lambda_0)y_0 = 0$. The vectors $y_1, \ldots, y_{p-1}$ are said to be the chain of associated vectors for y_0 if*

$$\sum_{s=0}^{k} \frac{1}{s!} \frac{d^s L(\lambda)}{d\lambda^s}\Big|_{\lambda=\lambda_0} y_{k-s} = 0, \quad k = \overline{1, p-1}.$$

The number p is said to be the length of the chain composed by the eigen- and associated vectors. The geometric multiplicity of an eigenvalue is defined to be the number of the corresponding linearly independent eigenvectors. The algebraic multiplicity of an eigenvalue is defined to be the greatest value of the sum of the lengths of chains corresponding to linearly independent eigenvectors. An eigenvalue is said to be isolated if it has some deleted neighborhood contained in the resolvent set. An isolated eigenvalue λ_0 of finite algebraic multiplicity is said to be normal if the image $\mathrm{Im}L(\lambda_0)$ is closed.

Lemma 1. *The operator A is selfadjoint and nonnegative.*

P r o o f. Let $Y = \begin{pmatrix} y(x) \\ y(l) \end{pmatrix} \in \mathcal{D}(A)$ and $Z = \begin{pmatrix} z(x) \\ z(l) \end{pmatrix}$, where $z(x) \in W_2^4(0,l)$, then taking into account $y(0) = 0$, $y^{(2)}(0) = 0$, we integrate by parts twice and obtain

$$(AY, Z) = \int_0^l y^{(4)}(x)\overline{z}(x)\,dx - y^{(3)}(l)\overline{z}(l) \tag{11}$$

$$= -y^{(3)}(0)\overline{z}(0) + y'(l)\overline{z}^{(2)}(l) - y'(0)\overline{z}^{(2)}(0) - y(l)\overline{z}^{(3)}(l) + y(0)\overline{z}^{(3)}(0) + \int_0^l y\overline{z}^{(4)}\,dx.$$

It is clear that if we set

$$z(0) = z^{(2)}(0) = z^{(2)}(l) = 0, \tag{12}$$

then we arrive at $(AY, Z) = (Y, AZ)$ and $D(A^*) = D(A)$. Let us show that A is nonnegative. To do it we consider the inner product (AY, Y). Setting $Z = Y$ into (11) and taking into account (12), we obtain

$$(AY, Y) = \int_0^l |y^{(2)}|^2\,dx \geqslant 0. \qquad \blacksquare$$

Lemma 2. *The operator G is symmetric and nonnegative.*

P r o o f. Let $Y \in \mathcal{D}(A) = \mathcal{D}(G)$, $Z \in \mathcal{D}(A) = \mathcal{D}(G)$. Consider the inner product

$$(GY, Z) = -\int_0^l (g(x)y')'\overline{z}(x)\,dx + g(l)\overline{z}(l)y'(l)$$

$$= \int_0^l gy'\overline{z}'\,dx = g(l)\overline{z}'(l)y(l) - \int_0^l (g\overline{z}')'y\,dx = (Y, GZ). \tag{13}$$

Thus G is symmetric. Let us show that it is nonnegative on the domain $D(G) = D(A)$. To do it we consider the inner product (GY, Y). With account of definition of G equation (13) implies

$$(GY, Y) = \int_0^l g(x)|y'|^2\,dx \geqslant 0. \qquad \blacksquare$$

Lemma 3. *The spectrum of $L(\lambda)$ is located in the closed upper half-plane.*

This result is known (see [8]) for bounded operator pencils, however it remains true for pencils of unbounded operators (see, e.g., [9]).

To obtain more detailed information about the spectrum of the pencil $L(\lambda)$ it is convenient to transform the spectral parameter

$$\tau = \lambda - \frac{ik}{2}. \tag{14}$$

Then equation (6) takes the form

$$\tau^2 y - y^{(4)} + \frac{k^2}{4} y + (gy')' = 0, \tag{15}$$

while boundary condition (10) is

$$m\tau^2 y(l) + y^{(3)}(l) + (\frac{k\beta}{2} - \frac{mk^2}{4})y(l) - i\tau(\beta - mk)y(l) - g(l)y'(l) = 0. \tag{16}$$

Since problem (7)–(9), (15), (16) is deduced from problem (6)–(10) by spectral parameter transformation (14), the spectrum of problem (7)–(9), (15), (16) also consists of normal eigenvalues accumulating at infinity. We introduce new operators:

$$\hat{A} = A + G + \begin{pmatrix} -\frac{k^2}{4} I & 0 \\ 0 & (\frac{-mk^2}{4} + \frac{\beta k}{2}) \end{pmatrix}, \quad \mathcal{D}(\hat{A}) = \mathcal{D}(A),$$

$$\hat{M} = M, \quad \hat{M} \gg 0, \quad \hat{K} = \begin{pmatrix} 0 & 0 \\ 0 & (\beta - mk) \end{pmatrix}$$

and consider the following quadratic operator pencil with the domain $D(\hat{L}(\tau)) = D(L(\lambda)) = D(A)$:

$$\hat{L}(\tau) = \tau^2 \hat{M} - i\tau \hat{K} - \hat{A}.$$

It is obvious that

$$\begin{array}{llll} I) \text{ if} & mk < \beta, & \text{then} & \hat{K} \geqslant 0; \\ II) \text{ if} & mk = \beta, & \text{then} & \hat{K} \equiv 0. \end{array}$$

Theorem. *I. Let $mk < \beta$, then:*

1) *eigenvalues of the pencil $\hat{L}$ lie in the closed upper half-plane and on the interval of the imaginary axis $[-ik/2, 0)$;*

2) *the interval $[-ik/2, 0)$ contains only finite number of eigenvalues which are semisimple, i.e., the corresponding eigenvectors do not possess associated vectors;*

3) *the geometric multiplicity of each eigenvalue of the pencil $\hat{L}$ does not exceed 2;*

4) *the set of eigenvalues $\{\tau_k\}$ can be arranged as the union of two subsequences*

$$\{\tau_k\} = \{\tau_k^{(1)}\} \cup \{\tau_k^{(2)}\}$$

such that

4a) the set $\{\tau_k^{(2)}\}$ *lies on the imaginary and on the real axes symmetrically with respect to these axes, i.e., under proper enumeration* $\tau_{-k}^{(2)} = -\tau_k^{(2)}$*;*

4b) if $0 \in \{\tau_k^{(2)}\}$*, then* 0 *is a double element of the set* $\{\tau_k^{(2)}\}$ *while all other* $\tau_k^{(2)}$ *are simple;*

4c) the elements of the set $\{\tau_k^{(1)}\}$ *lie in the open upper half-plane and on the interval* $[-ik/2, 0]$ *of the imaginary axis; the elements in the open upper half-plane are located symmetrically with respect to the imaginary axis, i.e.,* $\tau_{-k}^{(1)} = -\overline{\tau_k}^{(1)}$ *for each not pure imaginary* $\tau_k^{(1)}$*, and symmetrically located eigenvalues possess equal multiplicities. Denote the elements of* $\{\tau_k^{(1)}\}$*, which are located in the open lower half-plane (if any) by* $-i\theta_1$*,* $-i\theta_2, \ldots,$ $-i\theta_\varkappa$*, where*

$$\theta_1 > \theta_2 \cdots > \theta_\varkappa,$$

then

4d) $i\theta_j \notin \{\tau_k^{(1)}\}$*,* $(j = 1, 2, \ldots, \varkappa)$*;*

4e) the number of elements of the set $\{\tau_k^{(1)}\}$ *with account of multiplicities in each of the intervals* $(i\theta_{j+1}, i\theta_j)$*,* $j = 1, 2 \ldots \varkappa - 1$ *is odd;*

4f) the number of elements of $\{\tau_k^{(1)}\}$ *in the interval* $(0, i\theta_\varkappa)$ *is even, if* $0 \notin \{\tau_k^{(1)}\}$ *and is odd, if* $0 \in \{\tau_k^{(1)}\}$*.*

II. If $mk = \beta$*, then the spectrum of the pencil* $\hat{L}$ *lies on the real axis and on the interval* $[-\frac{ik}{2}, \frac{ik}{2}]$*. The spectrum of* $\hat{L}$ *is symmetric with respect to the real axis and to the imaginary axis. The number of pure imaginary eigenvalues is finite. If* 0 *belongs to the spectrum, then its algebraic multiplicity is* 2 *or* 4 *and its geometric multiplicity is 1 or 2, respectively. The algebraic multiplicity of any nonzero eigenvalue coincides with its geometric multiplicity and does not exceed 2.*

P r o o f.

I. As it was mentioned above, for $mk < \beta$ the inequality $\hat{K} \geqslant 0$ is valid. Therefore, it is possible to apply the results of [9] to the pencil $\hat{L}$ and to obtain

1) the eigenvalues of $\hat{L}$ lie in the closed upper half-plane and on the imaginary axis (Lemma 2.2 in [9]). Using Lemma 1, we conclude that the eigenvalues in the open lower half-plane are located only on the interval $[-ik/2, 0]$;

2) applying Lemma 2.3 (Part 1) from [9] to the pencil $\hat{L}$ we obtain that the eigenvalues located on the intervals $[-ik/2, 0)$, $(\infty, 0)$ and $(0, \infty)$ are semisimple. The operator $A \geqslant 0$ implies that $\hat{A}$ is bounded from below and

then it follows by Theorem 2.3 from [9] that the number of eigenvalues on the interval $[-ik/2, 0)$ is finite;

3) the canonical fundamental system of solutions $y_k(\lambda, x)$, $(k = 1, 2, 3, 4)$ of equation (6) ([6, p. 25]) is defined by the boundary conditions: $y_k^{(n-1)}(\lambda, 0) = \delta_{kn}$, $k, n = 1, 2, 3, 4$, where δ_{kn} is the Kronecker symbol. Two of these solutions, $y_2(\lambda, x)$ and $y_4(\lambda, x)$, satisfy conditions (7) and (8). Consequently, the geometric multiplicity of each eigenvalue of the pencil $\hat{L}$ does not exceed 2;

4) it is evident that $\hat{M} + \hat{K}$ is a strictly positive operator for $mk < \beta$, therefore the operators involved in $\hat{L}$ satisfy the conditions of Corollary 3.1 and of Theorem 3.1 in [9]. Thus statements $4a$) and $4f$) follow from Corollary 3.1, while statements 4b)–4e) from Theorem 3.1. ■

R e m a r k.
In the case of $mk > \beta$ the problem can be reduced to the case of $mk < \beta$ by transformation $\tau \to -\tau$.

Let us consider again the pencil $L(\lambda)$.

Corollary.

I. If $mk < \beta$, then:

1) the eigenvalues of the pencil L, i.e., the eigenvalues of problem (6)–(10) lie in the closed half-plane $\mathrm{Im}\ \lambda \geq k/2$ and on the interval $[0, ik/2)$;

2) the interval $[0, ik/2)$ contains only finite number of eigenvalues which are semisimple;

3) the geometric multiplicity of each eigenvalue of L does not exceed 2;

4) the set of eigenvalues $\{\lambda_k\}$ can be arranged as a union of two subsets

$$\{\lambda_k\} = \{\lambda_k^{(1)}\} \cup \{\lambda_k^{(2)}\},$$

such that:

4a) the set $\{\lambda_k^{(2)}\}$ lies on the imaginary axis and on the line $Im\lambda = k/2$ and is located symmetrically with respect to the imaginary axis and to the line $Im\lambda = k/2$;

4b) if $ik/2 \in \{\lambda_k^{(2)}\}$, then $ik/2$ is a double element of the set $\{\lambda_k^{(2)}\}$; all other $\lambda_k^{(2)}$ are simple;

4c) the elements of the set $\{\lambda_k^{(1)}\}$ lie in the open half-plane $Im\lambda > k/2$ and on the interval $[0, ik/2]$; the elements which lie in the half-plane $Im\lambda > k/2$ are located symmetrically with respect to the imaginary axis, i.e., $\lambda_{-k}^{(1)} = -\overline{\lambda_k}^{(1)}$ for each not pure imaginary $\lambda_k^{(1)}$ and symmetrically located eigenvalues are of the same multiplicity. Denote by $i\gamma_1, i\gamma_2, \dots, i\gamma_\kappa$ the elements of the set $\{\lambda_k^{(1)}\}$, which lie on the interval $[0, ik/2)$, indexed such that

$$\gamma_1 > \gamma_2 \cdots > \gamma_\varkappa,$$

then

4d) $i(k - \gamma_j) \notin \{\lambda_k^{(1)}\}$, $j = 1, 2 \dots \varkappa$;

4e) the number of elements of the set $\{\lambda_k^{(1)}\}$ located on each of the intervals $(i(k-\gamma_j), i(k-\gamma_{j+1}))$, $j = 1, 2 \dots \varkappa - 1$, counted with multiplicities is odd;

4f) the interval $(ik/2, i(k - \gamma_1))$ contains even number of elements of $\{\lambda_k^{(1)}\}$, if $ik/2 \notin \{\lambda_k^{(1)}\}$, and odd number if $ik/2 \in \{\lambda_k^{(1)}\}$.

II. If $mk = \beta$, then the spectrum of L lies on the line $Im\lambda = k/2$ and on the interval $[0, ik]$. In this case the spectrum of L is symmetric with respect to the line $Im\lambda = k/2$ and to the imaginary axis. The number of pure imaginary eigenvalues is finite. If 0 belongs to the spectrum, then its algebraic multiplicity is even and does not exceed 4.

III. If $mk > \beta$, then

1) The eigenvalues of the pencil L, i.e., the eigenvalues of problem (6)–(10) lie in the strip $0 < Im\lambda \le k/2$ and on the interval on the imaginary axis $[ik/2, ik]$;

2) the interval $(ik/2, ik]$ contains only finite number of eigenvalues which are semisimple;

3) the geometric multiplicity of each of the eigenvalues of the pencil L does not exceed 2;

4) the spectrum $\{\lambda_k\}$ can be given as a union of two subsequences

$$\{\lambda_k\} = \{\lambda_k^{(1)}\} \cup \{\lambda_k^{(2)}\}$$

such that

4a) the set $\{\lambda_k^{(2)}\}$ is located on the imaginary axis and on the horizontal line $Im\lambda = k/2$ symmetrically with respect to the imaginary axis and to the line $Im\lambda = k/2$;

4b) if $ik/2 \in \{\lambda_k^{(2)}\}$, then $ik/2$ is a double element of the set $\{\lambda_k^{(2)}\}$; all other $\lambda_k^{(2)}$ are simple elements of the set;

4c) all elements of $\{\lambda_k^{(1)}\}$ lie in the strip $0 < Im\lambda < k/2$ and on the interval $[ik/2, ik]$; the elements of this subsequence which lie in the strip $0 < Im\lambda < k/2$ are located symmetrically with respect to the imaginary axis, i.e., $\lambda_{-k}^{(1)} = -\overline{\lambda_k}^{(1)}$ for each not pure imaginary $\lambda_k^{(1)}$ and the multiplicities of symmetrically located elements coincide. Let us denote the elements of $\{\lambda_k^{(1)}\}$ which are located in the interval $(ik/2, ik]$ by $i\zeta_1$, $i\zeta_2, \dots,$ $i\zeta_\varkappa$, where

$$\zeta_1 > \zeta_2 \cdots > \zeta_\varkappa,$$

then

4d) $i(k - \zeta_j) \notin \{\lambda_k^{(1)}\}$, $(j = 1, 2 \dots \varkappa)$;

4e) the number of elements of $\{\lambda_k^{(1)}\}$ with account of multiplicities is odd in each of the intervals $(i(k - \zeta_j), i(k - \zeta_{j+1}))$, $(j = 1, 2 \dots \varkappa - 1)$;

4f) the number of elements of $\{\lambda_k^{(1)}\}$ in the interval $(i(k - \zeta_\varkappa), ik/2)$ is even if $ik/2 \notin \{\lambda_k^{(1)}\}$ and is odd if $ik/2 \in \{\lambda_k^{(1)}\}$. ■

3. Eigenvalue Asymptotics

In this section we assume $g = const > 0$. Direct calculation gives:

$$y_2(\lambda, x) = -\frac{z_2^2 \mathrm{sh}(z_1 x)}{z_1(z_1^2 - z_2^2)} + \frac{z_1^2 \mathrm{sh}(z_2 x)}{z_2(z_1^2 - z_2^2)}, \tag{17}$$

$$y_4(\lambda, x) = \frac{\mathrm{sh}(z_1 x)}{z_1(z_1^2 - z_2^2)} - \frac{\mathrm{sh}(z_2 x)}{z_2(z_1^2 - z_2^2)}, \tag{18}$$

where

$$z_1 = \sqrt{\frac{g + \sqrt{g^2 - 4(-\lambda^2 + ik\lambda)}}{2}},$$

$$z_2 = \sqrt{\frac{g - \sqrt{g^2 - 4(-\lambda^2 + ik\lambda)}}{2}}.$$

The solution of (6), which satisfies conditions (7)–(9), is

$$y = y_4^{(2)}(\lambda, l) \cdot y_2(\lambda, x) - y_2^{(2)}(\lambda, l) \cdot y_4(\lambda, x). \tag{19}$$

Substituting (19) into boundary condition (10), we obtain

$$y_4^{(2)}(\lambda,l)\cdot(-y_2^{(3)}(\lambda,l)-m\lambda^2 y_2(\lambda,l)+gy_2^{(1)}(\lambda,l)+i\lambda\beta y_2(\lambda,l))$$
$$-y_2^{(2)}(\lambda,l)\cdot(-y_4^{(3)}(\lambda,l)-m\lambda^2 y_4(\lambda,l)+gy_4^{(1)}(\lambda,l)+i\lambda\beta y_4(\lambda,l))=0,$$

what with account of (17) and (18) gives

$$-z_2^3 z_1^2 \mathrm{sh}(z_1 l)\mathrm{ch}(z_2 l)-z_1^2(m\lambda^2-i\lambda\beta)\cdot\mathrm{sh}(z_1 l)\mathrm{sh}(z_2 l)$$
$$+g(l)z_1^2 z_2\mathrm{sh}(z_1 l)\mathrm{ch}(z_2 l)+z_1^3 z_2^2\mathrm{sh}(z_2 l)\mathrm{ch}(z_1 l)$$
$$+z_2^2(m\lambda^2-i\lambda\beta)\mathrm{sh}(z_1 l)\mathrm{sh}(z_2 l)-g(l)z_1 z_2^2\mathrm{sh}(z_2 l)\mathrm{ch}(z_1 l)=0. \tag{20}$$

To find asymptotics of the roots of equation (20) we substitute

$$\sqrt{\lambda}=\frac{\pi n}{l}+\frac{A}{n}+\frac{B}{n^2}+\frac{C}{n^3}+O(\frac{1}{n^4}). \tag{21}$$

Then the coefficients before the powers of $1/n$ must vanish, what implies

$$\begin{aligned}\lambda_n \quad = \quad & \frac{\pi^2 n^2}{l^2}+\frac{ik}{2}+\frac{g}{2}+\frac{1}{lm}-\frac{1}{2\pi n m^2}-\frac{1}{4\pi^2 m^2 n^2}+\frac{lg}{2\pi^2 m n^2}\\ & -\frac{l^2 g^2}{8\pi^2 n^2}-\frac{l^2 k^2}{8\pi^2 n^2}+\frac{l}{6\pi^2 m^3 n^2}+\frac{il(\beta-mk)}{\pi^2 m^2 n^2}+O\left(\frac{1}{n^3}\right).\end{aligned}$$

This shows how to find the parameters l,k,g,m,β using the spectrum of problem (6)–(10), i.e to solve the inverse problem for the case of $g=const$.

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