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Almost Periodic Discrete Sets

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We study the almost periodic discrete sets in $\mathbb{R}^k$. Also, we show the connection between these sets and almost periodic measures and calculate their Fourier transform.

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Introduction

Almost periodic (with respect to real shifts) sets were first regarded by M.G. Krein and B.Ya. Levin [1] (see also monograph [2]) when studying the zero distribution of entire almost periodic functions.

Definition. A discrete set $\{a_n\} \subset \{z : |Imz| < M < \infty\}$ is called almost periodic if for each $\varepsilon > 0$ there exists $L < \infty$ such that every real interval of the length L contains at least one number τ such that for some bijection $\rho : \mathbb{Z} \to \mathbb{Z}$

$$|(a_n + \tau) - a_{\rho(n)}| < \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

There were M.G. Krein and B.Ya. Levin who showed that the zero set of an almost periodic function from some special class (the class $[\Delta]$, see [1]) is an almost periodic set.

As shown in [3], any almost periodic set in a strip can be completely described as the zero set of some analytic function with almost periodic modulus.

In the present paper we study the sets in $\mathbb{R}^k$ which are almost periodic with respect to arbitrary shifts. These sets arise in the models describing quasicrystallic structures (see, for example, [4]).

Using a special metric in the space of sequences, we give a geometric description of almost periodic sets. We prove the completeness of the space of almost periodic sets and some analogue of the Bochner criterion of almost periodicity. In addition, we determine the method allowing to construct various examples of these sets. Further, following [5], we introduce a notion of almost periodic measure in $\mathbb{R}^k$. We show that a set is almost periodic if and only if the discrete measure with unit masses at the points of this set is almost periodic. In conclusion we calculate the Fourier transform for a wide class of almost periodic sets.

Throughout the paper we denote i-coordinate of a point $x \in \mathbb{R}^k$ by x^i, an open k-dimensional ball of radius R with center $x \in \mathbb{R}^k$ by $B(x, R)$, a k-dimensional cube $\{y \in \mathbb{R}^k \mid x^i - L/2 \le y^i < x^i + L/2, i = \overline{1,k}\}$ by $Q(x, L)$. For any set $A \subset \mathbb{R}^k$ and for any $\rho > 0$ we write $A_\rho = \bigcup_{x \in A} B(x, \rho)$ and $A_{-\rho} = \{x \in \mathbb{R}^k \mid B(x, \rho) \subset A\}$. We denote the interior of A by $IntA$, the k-dimensional Lebesgue measure of A by $m(A)$, the inner product of the elements $x, y \in \mathbb{R}^k$ by $\langle x, y\rangle$, the usual Euclidean norm by $|x|$. For the mapping $g(x) : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ and $\tau \in \mathbb{R}^k$ we put $g^\tau = g(x - \tau)$, for the measure μ on $\mathbb{R}^k$ we put $\mu^\tau(E) = \mu(E + \tau)$ for any Borel set $E \subset \mathbb{R}^k$.

1. Almost Periodic Mappings

Call to mind some definitions concerning almost periodicity.

Let g be a continuous mapping from $\mathbb{R}^k$ to $\mathbb{R}^n$. A vector $\tau \in \mathbb{R}^k$ is called an *ε-almost period* of g if

$$|g^\tau(x) - g(x)| < \varepsilon \quad \text{for all } x \in \mathbb{R}^k.$$

A set $E \subset \mathbb{R}^k$ is called *relatively dense* if there exists $L < \infty$ such that every k-dimensional ball of the radius L has a nonempty intersection with E. It is obvious that we can replace a ball with a cube in this definition.

A continuous mapping g from $\mathbb{R}^k$ to $\mathbb{R}^n$ is called *almost periodic* if for every $\varepsilon > 0$ the set of ε-almost periods of g is relatively dense in $\mathbb{R}^k$. In the same way one can define the almost periodicity of mappings from $\mathbb{Z}^k$ to $\mathbb{R}^n$, of course, considering the shifts on vectors from $\mathbb{Z}^k$.

Point out some properties of almost periodic mappings. First of all, the almost periodicity of a mapping from $\mathbb{R}^k$ to $\mathbb{R}^n$ is equivalent to the almost periodicity of its coordinates, therefore, it is sufficient to consider almost periodic functions on $\mathbb{R}^k$.

The following assertions are well known for almost periodic functions on the real axis (see, for example, [6, 7]). Their proofs do not change much for those in the case of higher dimension (see, for example, [5]).

Proposition 1. *a) An almost periodic function on $\mathbb{R}^k$ is bounded and uniformly continuous;*

b) if for an almost periodic function g the sequence (g^{τ_n}) converges uniformly on $\mathbb{R}^k$, then its limit $\widetilde{g}$ is an almost periodic function as well. In addition, $g \not\equiv const$ if and only if $\widetilde{g} \not\equiv const$.

Proposition 2. *For a continuous mapping $g(x)$ on $\mathbb{R}^k$ the following conditions are equivalent:*

a) $g(x)$ is almost periodic;

b) for each sequence $(h_n) \subset \mathbb{R}^k$ there exists a subsequence $(h_{n'})$ such that $|g(x + h_{l'}) - g(x + h_{m'})| \underset{l',m'\to\infty}{\longrightarrow} 0$ uniformly on $\mathbb{R}^k$;

c) there is a sequence of finite exponential sums

$$S_n(x) = \sum_{i=1}^{N(n)} c_{i,n} e^{\langle \lambda_{i,n}, x\rangle} \qquad (1)$$

that converges to $g(x)$ uniformly on $\mathbb{R}^k$.

The assertion b) is called the *Bochner criterion* for almost periodic mappings.

A limit

$$M\{g(x)\} = \lim_{T\to\infty} \frac{1}{(2T)^k} \int\limits_{y+[-T,T]^k} g(x)dm(x)$$

is called the *mean value* of the almost periodic function $g(x)$ on $\mathbb{R}^k$. This limit exists uniformly with respect to $y \in \mathbb{R}^k$ and it is independent of y. The Fourier coefficient of $g(x)$ corresponding to $\lambda \in \mathbb{R}^k$ is calculated by the formula

$$a(\lambda, g) = M\{g(x)e^{-i\langle \lambda, x\rangle}\}.$$

A set $\{\lambda \in \mathbb{R}^k : a(\lambda, g) \neq 0\}$ is called the *spectrum* of the function $g(x)$; the spectrum is at most countable. Notice that the spectrum of exponential polynomial (1) is the set of exponents $\{\lambda_{i,n}\}$ and the Fourier coefficient corresponding to the exponent $\lambda_{i,n}$ is just $c_{i,n}$.

The same theorems with evident modifications hold for almost periodic functions on $\mathbb{Z}^k$ as well. A limit

$$M_g = M\{g(n)\} = \lim_{T\to\infty} \frac{1}{(2T)^k} \sum_{\substack{n^i = y^i - T \\ i=\overline{1,k}}}^{y^i+T} g(n),$$

where $y \in \mathbb{Z}^k$, T takes only integers, is called the *mean value* of an almost periodic function $g(x)$ on $\mathbb{Z}^k$. This limit exists uniformly with respect to $y \in \mathbb{Z}^k$ and is it independent of y. The Fourier coefficient of $g(x)$ corresponding to the exponent $\lambda \in [0, 2\pi)$ is calculated by the formula

$$a(\lambda, g) = M\{g(n)e^{-i\langle\lambda,n\rangle}\}.$$

A set $\{\lambda \in [0, 2\pi) : \ a(\lambda, g) \neq 0\}$ is called the *spectrum* of a function $g(n)$; this set is at most countable.

2. Almost Periodic Discrete Multiple Sets

As it was introduced in [8], we call a value set of a sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{R}^k$ a *discrete multiple set* (we write $\{a_n\}_{n\in\mathbb{N}}$) if this sequence does not possess any finite limit points. In other words, a discrete multiple set is a discrete set where each point has a finite multiplicity.

For any two discrete multiple sets $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ we define a *distance* between them by the formula

$$dist(\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}) = \inf_{\sigma} \sup_{n\in\mathbb{N}} | a_n - b_{\sigma(n)}|,$$

where infimum is taken over all bijections $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$. As shown in [8], this function satisfies all the axioms of metric except finiteness.

Definition 1. *A vector $\tau \in \mathbb{R}^k$ is called an ε-almost period of a discrete multiple set $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$ if*

$$dist(\{a_n\}_{n\in\mathbb{N}}, \{a_n + \tau\}_{n\in\mathbb{N}}) < \varepsilon.$$

Definition 2. *A discrete multiple set $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$ is called almost periodic if for each $\varepsilon > 0$ the set of its ε-almost periods is relatively dense in $\mathbb{R}^k$.*

Notice that the sum and the difference of any two ε-almost periods are 2ε-almost periods.

We will indicate the construction method of almost periodic multiple sets. Let $F(x) = (f_1(x), \dots, f_k(x))$ be a mapping from $\mathbb{R}^k$ to $\mathbb{R}^k$ with almost periodic components $f_j(x)$. Then the restriction of $F(x)$ to $\mathbb{Z}^k \subset \mathbb{R}^k$ is the almost periodic mapping from $\mathbb{Z}^k$ to $\mathbb{R}^k$. (This is evident for $f_j(x)$ that is similar to (1), the general case follows from Prop. 2 c).) Thus, for each $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods of mapping $F(m)$. Therefore, for each $\beta > 0$ a discrete multiple set

$$\{\beta m + F(m)\}_{m\in\mathbb{Z}^k} \tag{2}$$

is almost periodic.

Notice that if we replace $F(m)$ with $\alpha F(m)$, where $\alpha < \dfrac{\beta}{2\sup|F(m)|}$, then a corresponding almost periodic multiple set is uniformly discrete, i.e., for some

$\kappa > 0$ every ball of the radius κ contains at most one element of this multiple set. Thus, in this case we deal with the almost periodic Delone set (see [4]).

Next, by Theorem 6 of [8], any almost periodic multiple set has the form of (2) with a bounded mapping $F(m)$ from $\mathbb{Z}^k$ to $\mathbb{R}^k$.

Theorem 1. *The limit of almost periodic multiple sets is almost periodic as well.*

P r o o f. Let $(\{a_n^{(p)}\}_{n\in\mathbb{N}})$ be a sequence of almost periodic sets. Let $\{b_n\}_{n\in\mathbb{N}}$ be such a set that $dist\left(\{a_n^{(p)}\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}\right) \underset{p\to\infty}{\to} 0$. Let us take any $\varepsilon > 0$. Let p_0 be such a number that

$$dist\left(\{\{a_n^{(p_0)}\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}\right) < \frac{\varepsilon}{3}.$$

Let $E_{\frac{\varepsilon}{3}}$ be a relatively dense set of $\frac{\varepsilon}{3}$-almost periods of $\{a_n^{(p_0)}\}_{n\in\mathbb{N}}$. For $\tau \in E_{\frac{\varepsilon}{3}}$ we have

$$dist\left(\{a_n^{(p_0)}\}_{n\in\mathbb{N}}, \{a_n^{(p_0)} + \tau\}_{n\in\mathbb{N}}\right) < \frac{\varepsilon}{3}.$$

Therefore,

$$dist\left(\{b_n\}_{n\in\mathbb{N}}, \{b_n + \tau\}_{n\in\mathbb{N}}\right) \le dist\left(\{b_n\}_{n\in\mathbb{N}}, \{a_n^{(p_0)}\}_{n\in\mathbb{N}}\right)$$

$$+dist\left(\{a_n^{(p_0)}\}_{n\in\mathbb{N}}, \{a_n^{(p_0)} + \tau\}_{n\in\mathbb{N}}\right) + dist\left(\{a_n^{(p_0)} + \tau\}_{n\in\mathbb{N}}, \{b_n + \tau\}_{n\in\mathbb{N}}\right) < \varepsilon.$$

Hence, τ is an ε-almost period of $\{b_n\}_{n\in\mathbb{N}}$, and the set of almost periods is relatively dense. ∎

R e m a r k. As it was shown in [8, Th. 2], a space $(X, dist)$ of all discrete multiple sets is complete ⋆. Hence, almost periodic multiple sets form a complete closed subspace in $(X, dist)$.

The following theorem is an analogue of the Bochner criterion for almost periodic multiple sets.

Theorem 2. *A discrete multiple set $\{a_n\}_{n\in\mathbb{N}}$ is almost periodic if and only if for every sequence $(h_p)_{p=1}^\infty \subset \mathbb{R}^k$ there is a subsequence $(h'_p)_{p=1}^\infty$ such that $(\{a_n + h'_p\}_{n\in\mathbb{N}})_{p=1}^\infty$ has a limit.*

The proof of this theorem is based on the following lemma:

⋆Precisely, a metric space of all discrete multiple sets lying at a finite distance from some fixed discrete multiple set is complete.

Lemma 1. *Let $D = \{a_n\}_{n\in\mathbb{N}}$ be an almost periodic multiple set. Then every sequence $(h_p)_{p=1}^{\infty} \subset \mathbb{R}^k$ has a subsequence $(h'_p)_{p=1}^{\infty}$ such that for each $\varepsilon > 0$ and arbitrary numbers $l, m > N = N(\varepsilon)$ one can find an ε-almost period τ of D satisfying the inequality*

$$|h'_l - h'_m - \tau| < \varepsilon.$$

P r o o f. First, put $\varepsilon = 1$. Let $E_{\frac{1}{2}}$ be the relatively dense set of $\frac{1}{2}$-almost periods of D. Consider the sets

$$A_p = \bigcup_{\tau\in E_{\frac{1}{2}}} B(h_p + \tau, 1/2), \quad p \in \mathbb{N}.$$

There exists $L < \infty$ such that for all $p = 1, 2, \ldots$ the ball $B(-h_p, L)$ contains a $\frac{1}{2}$-almost period τ_p of D. Since $\tau_p + h_p \in B(\mathbf{0}, L)$, we have $A_p \bigcap B(\mathbf{0}, L) \neq \emptyset$ for all $p = 1, 2, \ldots$.

Let us show that there exists a point $x \in B(\mathbf{0}, L)$ belonging to an infinite number of the sets A_p. Cover the ball $B(\mathbf{0}, L)$ by a finite number of mutually disjoint k-dimensional cubes with edges of length $\dfrac{\varepsilon}{2k}$. The Dirichlet principle implies that there is a cube containing infinite number of points $h_p + \tau$. The diagonal of such a cube is less than $\dfrac{1}{2\sqrt{k}}$, hence this cube is contained in the balls $B(h_p + \tau, \frac{1}{2})$ and, consequently, in the infinite number of the sets A_p. Thereby, there exists a subsequence $(A_p^{(1)})_{p=1}^{\infty} \subset (A_p)_{p=1}^{\infty}$ such that

$$A_p^{(1)} = \bigcup_{\tau\in E_{\frac{1}{2}}} B(h_p^{(1)} + \tau, 1/2),\ p \in \mathbb{N},$$

and

$$\bigcap_{p=1}^{\infty} A_p^{(1)} \neq \emptyset.$$

Take a number $h \in A_i^{(1)} \bigcap A_j^{(1)}$ with some i, j. There exists $\tau' \in E_{\frac{1}{2}}$ such that $|h - (h_i^{(1)} + \tau')| < \frac{1}{2}$ and $\tau'' \in E_{\frac{1}{2}}$ such that $|h - (h_j^{(1)} + \tau'')| < \frac{1}{2}$. Therefore we have

$$|h_i^{(1)} - h_j^{(1)} - (\tau'' - \tau')| \leq |h - (h_j^{(1)} + \tau'')| + |h - (h_i^{(1)} + \tau')| < 1.$$

Thus, for each i, j there exists a 1-almost period $\tau = \tau'' - \tau'$ of D such that the inequality

$$|h_i^{(1)} - h_j^{(1)} - \tau| < 1$$

holds.

Now put $\varepsilon = \frac{1}{2}$. Similarly, construct the sets

$$A_p^{(2)} = \bigcup_{\tau \in E_{\frac{1}{4}}} B(h_p^{(2)} + \tau, 1/4),\ p \in \mathbb{N},$$

and $(h_p^{(2)})_{p=1}^{\infty} \subset (h_p^{(1)})_{p=1}^{\infty}$ such that for all i, j the inequality

$$|h_i^{(2)} - h_j^{(2)} - \tau| < \frac{1}{2}$$

holds for some $\frac{1}{2}$-almost period τ of D.

Repeating this construction for $\varepsilon = \frac{1}{3}, \varepsilon = \frac{1}{4}, \dots$ and choosing a diagonal subsequence $(h'_p)_{p=1}^{\infty}$, we can obtain the assertion of the lemma. ■

P r o o f o f T e o r e m 2. Suppose that for every sequence $(h_p)_{p=1}^{\infty} \subset \mathbb{R}^k$ there is a subsequence $(h'_p)_{p=1}^{\infty}$ such that a sequence $(\{a_n + h'_p\}_{n\in\mathbb{N}})_{p=1}^{\infty}$ has a limit. If $\{a_n\}_{n\in\mathbb{N}}$ is not almost periodic, then for some $\varepsilon_0 > 0$ there exists a sequence of k-dimensional balls $(B_p)_{p=1}^{\infty}$ with infinitely increasing diameters l_p, such that no ball contains any ε_0-almost period of $\{a_n\}_{n\in\mathbb{N}}$.

Let us take an arbitrary $h_1 \in \mathbb{R}^k$ and a number ν_1 such that $l_{\nu_1} > 1$. For some $h_2 \in \mathbb{R}^k$ the difference $h_2 - h_1$ belongs to B_{ν_1}. Let ν_2 be the first number such that $l_{\nu_2} > \max\{2, |h_2 - h_1|\}$. Take $h_3 \in \mathbb{R}^k$ such that the differences $h_3 - h_1, h_3 - h_2$ belong to B_{ν_2} (it is possible, since the latter condition is equivalent to $|h_2 - h_1| < l_{\nu_2}$). Then take a number ν_n such that $l_{\nu_n} > \max\{n, |h_2 - h_1|, |h_3 - h_2|, \dots, |h_n - h_2|, |h_n - h_1|\}$ and $h_{n+1} \in \mathbb{R}^k$ such that the differences $h_{n+1} - h_1, h_{n+1} - h_2, \dots, h_{n+1} - h_n$ belong to B_{ν_n}.

Take arbitrary $p, m\,(p > m)$. By construction $h_p - h_m \in B_{\nu_{p-1}}$, hence $B_{\nu_{p-1}}$ does not contain any ε_0-almost period of $\{a_n\}_{n\in\mathbb{N}}$. We have

$$dist(\{a_n + h_p\}_{n\in\mathbb{N}}, \{a_n + h_m\}_{n\in\mathbb{N}}) = dist(\{a_n + (h_p - h_m)\}_{n\in\mathbb{N}}, \{a_n\}_{n\in\mathbb{N}}) \geq \varepsilon_0.$$

Thus, there are no convergent subsequences in the sequence $(\{a_n + h_p\}_{n\in\mathbb{N}})_{p=1}^{\infty}$.

Conversely, let a discrete multiple set $\{a_n\}_{n\in\mathbb{N}}$ be almost periodic. Consider an arbitrary sequence $(h_p)_{p=1}^{\infty} \subset \mathbb{R}^k$. By Lemma 1 there exists a subsequence $(h'_p)_{p=1}^{\infty}$ such that for every $\varepsilon > 0$ and arbitrary $l, m > N = N(\varepsilon)$ one can find an $\varepsilon/2$-almost period τ of $\{a_n\}_{n\in\mathbb{N}}$ with the property

$$|h'_l - h'_m - \tau| < \frac{\varepsilon}{2}.$$

We get

$$\begin{aligned} dist(\{a_n + h'_l\}_{n\in\mathbb{N}}, \{a_n + h'_m\}_{n\in\mathbb{N}}) &= \inf_{bij\ \sigma:\mathbb{N}\longrightarrow\mathbb{N}} \sup_{n\in\mathbb{N}} |\, a_n + h'_l - a_{\sigma(n)} - h'_m| \\ &\leq \inf_{bij\ \sigma:\mathbb{N}\longrightarrow\mathbb{N}} \sup_{n\in\mathbb{N}} |\, a_n - a_{\sigma(n)} + \tau| + |h'_l - h'_m - \tau| \\ &< dist(\{a_n\}_{n\in\mathbb{N}}, \{a_{\sigma(n)} - \tau\}_{n\in\mathbb{N}}) + \varepsilon/2 < \varepsilon. \end{aligned}$$

Since ε is arbitrary, we see that the sequence $(\{a_n + h'_p\}_{n\in\mathbb{N}})_{p=1}^{\infty}$ is fundamental. Using the remark following Theorem 1, we finish the proof. ■

Definition 3. ([8]) *A discrete multiple set $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$ possesses S-property if there exists $L < \infty$ such that for any $\tau \in \mathbb{R}^k$ there is a bijection $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ with the property*

$$\sup_{n\in\mathbb{N}} |(a_n + \tau) - a_{\sigma(n)}| \leq L.$$

In other words, a discrete multiple set $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$ possesses S-property if there exists $L < \infty$ such that the inequality $dist(\{a_n\}_{n\in\mathbb{N}}, \{a_n + \tau\}_{n\in\mathbb{N}}) \leq L$ holds for any $\tau \in \mathbb{R}^k$.

Next, by [8], any almost periodic discrete multiple set possesses S-property. Therefore Theorem 4 and Proposition 4 of [8] imply the following statements:

Theorem 3. *Let D be an almost periodic multiple set. Then there exists $M < \infty$ such that*

$$\text{card}\,(D \cap B(c,1)) < M \quad \text{for all } c \in \mathbb{R}^k. \tag{3}$$

Theorem 4. *For any almost periodic multiple set D there exists $C < \infty$ such that for any convex bounded set $E \subset \mathbb{R}^k$ and $t \in \mathbb{R}^k$ the inequality*

$$|\text{card}\,(D \cap E) - \text{card}\,(D \cap (E + t))| < C((\text{diam}\,E)^{k-1} + 1)$$

is fulfilled.

Following [8], the density of a discrete multiple set is the value

$$\Delta = \lim_{T\to\infty} \frac{\text{card}\,(D \cap Q(\mathbf{0}, T))}{T^k}.$$

From Theorem 5 of [8] we get

Theorem 5. *Any almost periodic multiple set possesses finite nonzero shift invariant density.*

3. The Connection Between Almost Periodic Measures and Almost Periodic Discrete Multiple Sets

Definition 4. ([5]) *A locally finite complex-valued Radon measure μ on $\mathbb{R}^k$ is called almost periodic if for each compactly supported continuous function φ on $\mathbb{R}^k$ the convolution*

$$\varphi * \mu(z) = \int_{\mathbb{R}^k} \varphi(y - z)\, d\mu(y)$$

is almost periodic on $\mathbb{R}^k$.

In other words, a locally finite complex-valued Radon measure μ is almost periodic if for each compactly supported continuous function φ on $\mathbb{R}^k$ and any $\varepsilon > 0$ there exists a relatively dense set $E_{\varepsilon,\varphi}$ (the set of (ε, φ)-almost periods of the measure μ) with the property

$$|\varphi * \mu(z) - \varphi * \mu(z - \tau)| \leq \varepsilon \quad \forall \tau \in E_{\varepsilon,\varphi},\ \forall z \in \mathbb{R}^k.$$

Notice that the sum and the difference of any two (ε, φ)-almost periods are 2ε-almost periods.

We will say that the measures μ_n converge *weakly uniformly* to some measure μ if

$$\varphi * \mu_n(x) \underset{n\to\infty}{\to} \varphi * \mu(x)$$

uniformly on $\mathbb{R}^k$ for every continuous compactly supported function φ.

Note some properties of almost periodic measures.

Theorem 6. *If almost periodic measures converge weakly uniformly to some measure μ, then μ is almost periodic as well.*

The proof follows immediately from the definition of an almost periodic measure and Proposition 1 of the present paper.[

Theorem 7. *Let μ be an almost periodic measure. Then there exists $M < \infty$ such that the condition*

$$|\mu|(B(c, 1)) \leq M \quad \text{for all } c \in \mathbb{R}^k$$

is fulfilled; here $|\mu|$ is the variation of measure μ.

This fact follows from Theorem 2.1. of [5] with $S = \{0\},\ p = 0$.

The following theorem is an analogue of the Bochner criterion for almost periodic measures.

Theorem 8. *A Radon measure μ is almost periodic if and only if for every sequence $(h_n) \subset \mathbb{R}^k$ there is a subsequence $(h_{n'})$ such that measures $\mu^{h_{n'}}$ converge weakly uniformly to some measure μ'.*

This theorem is a corollary of Theorem 2.2. of [5] with $S = \{0\},\ p = 0$.

Definition 5. *For any Radon measure μ the value*

$$\Delta = \lim_{T\to\infty} \frac{\mu(Q(\boldsymbol{0}, T))}{T^k}$$

is called the density of μ.

Theorem 9. *Any almost periodic Radon measure possesses a finite shift invariant density, i.e.,*

$$\Delta = \lim_{T\to\infty} \frac{\mu(Q(\alpha, T))}{T^k}$$

uniformly over $\alpha \in \mathbb{R}^k$.

(See [5, Th. 2.7], with $S = \{0\}$.)

We can associate with a discrete multiple set $D = \{a_n\}_{n\in\mathbb{N}}$ the measure

$$\mu_D = \sum_{a_n\in D} \delta(x - a_n).$$

Notice that for every continuous function φ

$$\varphi * \mu_D(x) = \sum_{n\in\mathbb{N}} \varphi(x - a_n), \ x \in \mathbb{R}^k.$$

Theorem 10. *Let* $(D_p) = (\{a_n^{(p)}\}_{n\in\mathbb{N}})$ *be a sequence of discrete multiple sets. Let* $D = \{a_n\}_{n\in\mathbb{N}}$ *be a discrete multiple set satisfying (3). Then the following conditions are equivalent:*

a) discrete multiple sets D_p *converge to* D*;*

b) measures μ_{D_p} *converge weakly uniformly to* μ_D*.*

We need the following lemma:

Lemma 2. *Let* $(D_p) = (\{a_n^{(p)}\}_{n\in\mathbb{N}})$ *be a sequence of discrete multiple sets, which converges to a discrete multiple set* $D = \{a_n\}_{n\in\mathbb{N}}$ *satisfying (3). Then for sufficiently large* p *any discrete multiple set* D_p *satisfies (3) (with the constant* $4M$ *instead of* M*).*

P r o o f. For sufficiently large p we have $\inf_{\sigma} \sup_{n\in\mathbb{N}} |\, a_n - a_{\sigma(n)}^{(p)}| < 1$. Hence, card $(D_p \cap B(c, 1)) \le$ card $(D \cap B(c, 2)) \le 4M$ for all $c \in \mathbb{R}^k$. ∎

P r o o f o f T h e o r e m 10. Let the measures μ_{D_p} converge weakly uniformly to μ_D. We will show that $dist\left(\{a_n^{(p)}\}_{n\in\mathbb{N}}, \{a_n\}_{n\in\mathbb{N}}\right) \underset{p\to\infty}{\to} 0$, i.e., for any $\varepsilon > 0$ for each sufficiently large p there exists a bijection $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} |\, a_n^{(p)} - a_{\sigma(n)}| < \varepsilon. \tag{4}$$

First, notice that for any $\varepsilon > 0$ there is $\eta > 0$ such that the diameter of every connected component of the union $\bigcup_n B(a_n, \eta)$ is less than ε. Let us check it.

Put $\varepsilon \in (0, 1)$ and put $\eta < \varepsilon/(2M+1)$, where M satisfies (3). Let A be an arbitrary connected component of the union $\bigcup_n B(a_n, \eta)$ and c be an arbitrary point of A. By Theorem 3, the ball $B(c, 1)$ contains at most M points of D. Choose from them the points $a_{n_1}, \dots, a_{n_N}$ $(N \le M)$ belonging to A (we suppose that $B(a_{n_i}, \eta) \cap B(a_{n_{i+1}}, \eta) \ne \emptyset,\ i = \overline{1, N-1}$). If A is not contained in $B(c, 1)$, then there exists a point $a' \in A \setminus B(c, 1)$ with the property $|a' - a_{n_i}| < 2\eta$ for some n_i, $i = \overline{1, N}$. Since $c \in B(a_{n_p}, \eta)$ for some n_p, $p = \overline{1, N}$, we have

$$|a' - c| \le |a' - a_{n_i}| + (N-1) \max_{j=\overline{1,N-1}} |a_{n_j} - a_{n_{j+1}}| + |c - a_{n_p}|$$

$$< 2\eta + (N-1)2\eta + \eta = \eta(2N+1) < 1,$$

The latter assertion contradicts the connectivity of A. Thus $A = \bigcup_{i=\overline{1,N}} B(a_{n_i}, \eta)$ and

$$\operatorname{diam} A < 2\eta N < \varepsilon.$$

Take a nonnegative continuous function φ with the support in $B(\mathbf{0}, \eta/2)$ such that $0 \le \varphi(x) \le \varphi(0) = 1$. Put $\nu = \int \varphi dm$. We may assume that $\eta < 2/\sqrt{\pi}$, then $\nu < 1$. Since the measures μ_{D_p} converge weakly uniformly to μ_D, for sufficiently large p we have

$$|\varphi * \mu_{D_p}(x) - \varphi * \mu_D(x)| < \nu/2 \quad \text{for all } x \in \mathbb{R}^k. \tag{5}$$

The distance between an arbitrary pair of terms $a_n \in A$ and $a_m \notin A$ is at least 2η, therefore

$$\int_A \left(\int_{\mathbb{R}^k} \varphi(x-y) d\mu_D(y) \right) dm(x) = \int_A \left(\int_A \varphi(x-y) d\mu_D(y) \right) dm(x)$$

$$= \int_A \left(\int_{\mathbb{R}^k} \varphi(x-y) dm(x) \right) d\mu_D(y) = \nu \operatorname{card}(D \cap A). \tag{6}$$

Furthermore, let $a_n^{(p)}$ be a term of D_p such that $a_n^{(p)} \in A$. We have $\varphi * \mu_{D_p}(a_n^{(p)}) \ge \varphi(0) = 1$. In view of (5), we get

$$\varphi * \mu_D(a_n^{(p)}) \ge 1 - \nu/2 > 1/2.$$

Hence, there exists a term $a_{n'}$ of D such that $|a_{n'} - a_n^{(p)}| < \eta/2$. Then the distance between $a_{n'}$ and A is less than $\eta/2$. Therefore, $a_{n'} \in A$. Similarly, for any $a_j^{(p)} \notin A$

there exists a term $a_{j'}$ of D such that $|a_{j'} - a_j^{(p)}| < \eta/2$. Consequently, the distance between $a_{j'}$ and $\mathbb{R}^k \backslash A$ is less than $\eta/2$, so $a_{j'} \notin A$. Thus

$$\int_A \left(\int_{\mathbb{R}^k} \varphi(x-y) d\mu_{D_p}(y) \right) dm(x) = \int_A \left(\int_A \varphi(x-y) d\mu_{D_p}(y) \right) dm(x)$$

$$= \int_A \left(\int_{\mathbb{R}^k} \varphi(x-y) dm(x) \right) d\mu_{D_p}(y) = \nu \,\text{card}\, (D_p \cap A). \tag{7}$$

Since

$$\left| \int_A (\varphi * \mu_{D_p}(x) - \varphi * \mu_D(x)) dm(x) \right| \leq m(A)\, \nu/2 < \nu/2,$$

the values from equalities (6) and (7) coincide. Therefore, card $(D \cap A) =$ card $(D_p \cap A)$. The same is valid for all connected components of the union $\bigcup_n B(a_n, \eta)$. On the other hand, we have just proved that each term $a_n^{(p)}$ of D_p belongs to $B(a_{n'}, \eta)$ for some term $a_{n'}$ of D. Hence, it belongs to a connected component of $\bigcup_n B(a_n, \eta)$. Consequently, there is a bijection σ such that (4) is fulfilled.

Conversely, suppose that the discrete multiple sets D_p converge to a discrete multiple set D. Let φ be a nonnegative continuous function with compact support G in $\mathbb{R}^k$. We will show that the convolutions $\varphi * \mu_{D_p}(x)$ converge to $\varphi * \mu_D(x)$ as $p \to \infty$ uniformly over $x \in \mathbb{R}^k$.

Take $z \in \mathbb{R}^k$ and $\varepsilon > 0$. There is $\delta > 0$ such that the inequality $|x_1 - x_2| < \delta$ implies $|\varphi(x_1) - \varphi(x_2)| < \varepsilon$. By assumption, for each sufficiently large p there exists a bijection σ such that $|a_n - a_{\sigma(n)}^{(p)}| < \delta$ for all $n \in \mathbb{N}$. By Lemma 2, the set G contains at most $M' = M'(G) < \infty$ terms of D_p. Hence,

$$|\varphi * \mu_{D_p}(x) - \varphi * \mu_D(x)| \leq M' \sup_{|x_1 - x_2| < \delta} |\varphi(x_1) - \varphi(x_2)| < M'\varepsilon.$$

■

Now we prove the main result of this section.

Theorem 11. *A discrete multiple set D is almost periodic if and only if the corresponding measure μ_D is almost periodic.*

P r o o f. By Theorem 2, a discrete multiple set D is almost periodic if and only if for every sequence $(h_p)_{p=1}^{\infty} \subset \mathbb{R}^k$ there is a subsequence $(h'_p)_{p=1}^{\infty}$ such that the sequence of almost periodic multiple sets $(\{a_n + h'_p\}_{n \in \mathbb{N}})_{p=1}^{\infty}$ has a limit.

By Theorem 10, it is true if and only if for every sequence $(h_p)_{p=1}^{\infty} \subset \mathbb{R}^k$ there is a subsequence $(h'_p)_{p=1}^{\infty}$ such that the sequence of almost periodic measures $(\mu_{D+h'_p})_{p=1}^{\infty}$ converges weakly uniformly to some measure. By Theorem 8, it is true if and only if the measure μ is almost periodic. ■

To complete the paper we will show the formula for calculation the Fourier coefficients of an almost periodic multiple set D. Since the Fourier coefficients of D are defined as the Fourier coefficients of the corresponding measure μ_D, we have (see [5, Sect. 4])

$$a(\lambda, D) = \lim_{T\to\infty} \frac{1}{(2T)^k} \int\limits_{n\in[-T,T]^k} e^{-i\langle\lambda,\, t\rangle} d\mu_D(t).$$

Furthermore, the set D has the form of (2) with a bounded function $F : \mathbb{Z}^k \to \mathbb{R}^k$. Notice that for $T \to \infty$

$$\text{card}\,\{n:\ \beta n + F(n) \in [-T,T]^k\} = \text{card}\,\{n:\ n \in [-T/\beta, T/\beta]^k\} + O(T^{k-1}).$$

Thus, we have

$$a(\lambda, D) = \lim_{T\to\infty} \frac{1}{(2T)^k} \sum_{\beta n+F(n)\in[-T,T]^k} e^{-i\langle\lambda,\, \beta n+F(n)\rangle}$$

$$= \lim_{T\to\infty} \frac{1}{(2\beta T)^k} \sum_{n\in[-T,T]^k} e^{-i\langle\beta\lambda,\, n\rangle} e^{-i\langle\lambda,\, F(n)\rangle}.$$

References

[1] *M.G. Krein and B.Ja. Levin*, On Entire Almost Periodic Functions of Exponential Type. — *Dokl. AN SSSR* **64** (1949), 285–287. (Russian)

[2] *B.Ja. Levin*, Distributions of Zeros of Entire Functions. Transl. Math. Monogr. **5**. AMS, Providence, RI, 1980.

[3] *S.Yu. Favorov, A.Yu. Rashkovskii, and A.L. Ronkin*, On Almost Periodic Sets in the Complex Plane. — *Dop. NAN Ukr.* (1998), No. 12, 37–39. (Russian)

[4] *J.C. Lagarias*, Mathematical Quasicristals and the Problem of Diffraction. Directions in Mathematical Quasicrustals (M. Baake and R. Moody, Eds.). CRM Monogr. Ser. **13**. AMS, Providence, RI, 2000, 61–93.

[5] L.I. Ronkin, Almost Periodic Distributions and Divisors in Tube Domains. — *Zap. Nauchn. Sem. POMI* **247**, (1997), 210–236. (Russian)

[6] *C. Corduneanu*, Almost Periodic Functions. Intersci. Publ., New York, London, Sydney, 1968.

[7] *S.Yu. Favorov*, Sunyer-i-Balaguer's Almost Elliptic Functions and Yosida's Normal Functions. — *J. Anal. Math.* **104** (2008), 307–340.

[8] *Ye.Yu. Kolbasina*, On the Property of Discrete Set in $\mathbb{R}^k$. — *Visn. Kharkiv. Univ.*, Ser. Mat. Prykl. Mat. Mekh. (2008), No. 826, 52–66. (Russian)