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On the Estimation of the Norms of Intermediate Derivatives in Some Abstract Spaces

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The theorems on the exact estimates of norms of intermediate derivatives in some Sobolev type abstract spaces are obtained. The formulas for calculating the norms are given.

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Let H be a separable Hilbert space, A be a positive-definite selfadjoint operator in H. The domain of definition of the operator A^{γ}, $\gamma \geq 0$, becomes a Hilbert space H_{γ} with respect to the scalar product $(x,y)_{\gamma} = (A^{\gamma}x, A^{\gamma}y)$, $x, y \in H_{\gamma}(H_0 = H)$.

By $L_2(R_+; H_{\gamma})$ we denote a Hilbert space of the vector functions $f(t)$ with values in H_{γ}, determined almost everywhere in $R_+ = (0, \infty)$, measurable by Bochner, for which

$$\|f\|_{L_2(R_+;H_\gamma)} = \left(\int_0^\infty \|f(t)\|_\gamma^2 \, dt \right)^{1/2} < \infty.$$

Further, by $L(X,Y)$ denote a space of linear bounded operators acting from the space X to the space $Y, \sigma(\cdot)$ is a spectrum of the operator $(\cdot), \rho(\cdot)$ is a regular set of the operator $(\cdot), E$ is a unique operator in H.

In the sequel, everywhere $\frac{du}{dt} = u', \frac{d^2u}{dt^2} = u''$ are derivatives of the vector function $u(t)$ in the sense of distribution theory [1].

Let us introduce the following spaces:

$$W_2^2(R_+;H) = \{u : u \in L_2(R_+;H_2), u'' \in L_2(R_+;H)\},$$
$$\overset{\circ}{W}{}_2^2(R_+;H;0,1) = \{u : u \in W_2^2(R_+;H), u(0) = u'(0) = 0\},$$
$$W_2^2(R_+;H;T) = \{u : u \in W_2^2(R_+;H), u(0) = Tu'(0), \ T \in L(H_{1/2};H_{3/2})\},$$
$$W_2^2(R_+;H;K) = \{u : u \in W_2^2(R_+;H), u'(0) = Ku(0), \ K \in L(H_{3/2};H_{1/2})\}$$

(in these denotation the spaces $W_2^2(R_+;H;T)$ and $W_2^2(R_+;H;K)$ depend on the choice of the letters T and K, but it does not lead to misunderstandings in the text).

Each of these linear sets becomes a Gilbert space with respect of the norm [1, p. 23–29]

$$\|u\|_{W_2^2(R_+;H)} = \left(\|u\|_{L_2(R_+;H)} + \|u''\|_{L_2(R_+;H)}\right)^{1/2}.$$

For $T = 0$ we get the space

$$\overset{\circ}{W}{}_2^2(R_+;H;0) = \{u : u \in W_2^2(R_+;H), u(0) = 0\},$$

and for $K = 0$ we have

$$\overset{\circ}{W}{}_2^2(R_+;H;1) = \{u : u \in W_2^2(R_+;H), u'(0) = 0\}.$$

Notice that it follows from the theorem on traces [1, Sect. 1, Th. 3.2] that $u(0) \in H_{3/2}$, $u'(0) \in H_{1/2}$.

The space $W_2^2(R;H)$, where $R = (-\infty,\infty)$ [1], is defined in the similar way.

By the theorem on intermediate derivatives [1, Sect. 1, Th. 2.3], the operator

$$A\frac{d}{dt} : W_2^2(R_+;H) \to L_2(R_+;H)$$

is bounded.

In this paper we will find the exact values of the norm of intermediate derivative operators acting from the indicated spaces to the space $L_2(R_+;H)$. Notice that for the scalar functions $(H = R, A = E)$ the exact values of the operator

$$\frac{d}{dt} : W_2^2(R_+) \to L_2(R_+)$$

were found in [2–5]. Similar problems were considered in [6, 7] for some abstract spaces.

Denote

$$N_{0,0}=\sup_{0\neq u\in \overset{\circ}{W}{}_2^2(R_+;H;0,1)}\left\|Au'\right\|_{L_2(R_+;H)}\|u\|^{-1}_{W_2^2(R_+;H)}, \tag{1}$$

$$N=\sup_{0\neq u\in W_2^2(R_+;H)}\left\|Au'\right\|_{L_2(R_+;H)}\|u\|^{-1}_{W_2^2(R_+;H)}, \tag{2}$$

$$N_T=\sup_{0\neq u\in W_2^2(R_+;H;T)}\left\|Au'\right\|_{L_2(R_+;H)}\|u\|^{-1}_{W_2^2(R_+;H)}, \tag{3}$$

$$N_K=\sup_{0\neq u\in W_2^2(R_+;H;K)}\left\|Au'\right\|_{L_2(R_+;H)}\|u\|^{-1}_{W_2^2(R_+;H)}. \tag{4}$$

In particular, for $T=0$ and $K=0$ we denote the norms by N_0 and N_1, respectively. Find the exact values of these norms.

First, we prove the following statement.

Lemma 1. *For any $u\in W_2^2(R_+;H)$ and $\beta\in(0,2)$ there exists the identity*

$$\|u\|^2_{W_2^2(R_+;H)}-\beta\left\|Au'\right\|^2_{L_2(R_+;H)}=\|\Phi\left(d/dt:\beta:A\right)u\|^2_{L_2(R_+;H)}+\left(\tilde{R}\left(\beta\right)\tilde{\varphi},\tilde{\varphi}\right)_{H^2}, \tag{5}$$

where

$$\Phi\left(d/dt:\beta:A\right)u=\frac{d^2u}{dt^2}+\sqrt{2-\beta}A\frac{du}{dt}+Au^2, \tag{6}$$

$$\tilde{R}\left(\beta\right)=\begin{pmatrix}\sqrt{2-\beta}E & E\\ E & \sqrt{2-\beta}E\end{pmatrix}=R\left(\beta\right)\otimes\tilde{E},$$

$$R\left(\beta\right)=\begin{pmatrix}\sqrt{2-\beta} & 1\\ 1 & \sqrt{2-\beta}\end{pmatrix},\ \tilde{E}=\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix}.$$

P r o o f. By $D\left(R_+;H_2\right)$ we denote a set of all infinitely differentiable in H vector functions with values in H_2 that have compact supports in R_+. Then by the theorem on density [1, Sect. 1, Th. 2.1] this set is everywhere dense in $W_2^2\left(R_+;H\right)$. Since the operators $A^j\frac{d^{2-j}}{dt^{2-j}}$, $j=\overline{0,2}$, are bounded from $W_2^2\left(R_+;H\right)$ to $L_2\left(R_+;H\right)$, then it follows from the theorem on traces that it suffices to prove validity of equality (5) for the functions from the class $D\left(R_+;H_2\right)$. Obviously, for $u\in D\left(R_+;H_2\right)$ there holds the equality

$$\|\Phi(d/dt:\beta:A)\,u\|^2_{L_2(R_+;H)} = \left\|u'' + \sqrt{2-\beta}Au' + Au^2\right\|^2_{L_2(R_+;H)} = \|u''\|^2_{L_2(R_+;H)}$$

$$+ (2-\beta)\|Au'\|^2_{L_2(R_+;H)} + \left\|A^2u\right\|^2_{L_2(R_+;H)} + 2Re\left(u'',A^2u\right)_{L_2(R_+;H)}$$

$$+ 2\sqrt{2-\beta}Re\,(u'',Au')_{L_2(R_+;H)} + 2\sqrt{2-\beta}Re\left(Au',A^2u\right)_{L_2(R_+;H)}. \tag{7}$$

Integrating by parts, we get the validity of the following equalities:

$$Re\left(u'',A^2u\right)_{L_2(R_+;H)} = \int_0^\infty \left(u'',A^2u\right)_H dt = Re\left[-(\varphi_1,\varphi_0) - \int_0^\infty \left(Au',Au'\right)_H dt\right]$$

$$= -Re\,(\varphi_1,\varphi_0) - \left\|Au'\right\|^2_{L_2(R_+;H)}. \tag{8}$$

In a similar way we obtain

$$\left(u'',Au'\right)_{L_2(R_+;H)} = \int_0^\infty \left(u'',Au'\right)_H dt = -(\varphi_1,\varphi_1) - \int_0^\infty \left(Au',u''\right)_H dt$$

$$= -\|\varphi_1\|^2 - \left(Au',u''\right)_{L_2(R_+;H)}, \qquad \varphi_1 = A^{1/2}u'(0),$$

i.e.,

$$2Re\left(u'',Au'\right)_{L_2(R_+;H)} = -\|\varphi_1\|^2. \tag{9}$$

Similarly, we get

$$2Re\left(Au',A^2u\right)_{L_2(R_+;H)} = -\|\varphi_0\|^2, \quad \varphi_0 = A^{3/2}u(0). \tag{10}$$

Taking into account (8)–(10) in equality (7), we get

$$\|\Phi(d/dt:\beta:A)\,u\|^2_{L_2(R_+;H)} = \|u''\|^2_{L_2(R_+;H)} - \beta\|Au'\|^2_{L_2(R_+;H)} - [2Re\,(\varphi_0,\varphi_1)$$

$$+\sqrt{2-\beta}\,\|\varphi_0\|^2 + \sqrt{2-\beta}\,\|\varphi_1\|^2\Big] + \|A^2u\|^2_{L_2(R_+;H)}. \tag{11}$$

On the other hand, there hold the equalities:

$$2Re\,(\varphi_0,\varphi_1)=\left(\begin{pmatrix} E & 0\\ 0 & E\end{pmatrix}\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix},\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix}\right)_{H^2},$$

$$\|\varphi_0\|^2=\left(\begin{pmatrix} 0 & 0\\ E & 0\end{pmatrix}\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix},\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix}\right)_{H^2},$$

$$\|\varphi_1\|^2=\left(\begin{pmatrix} 0 & E\\ 0 & 0\end{pmatrix}\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix},\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix}\right)_{H^2}.$$

Thus, the equality

$$\|\Phi\,(d/dt:\beta:A)\,u\|^2_{L_2(R_+;H)}=\left\|u''\right\|^2_{W_2^2(R_+;H)}-\beta\left\|Au'\right\|^2_{L_2(R_+;H)}-\left(\tilde{R}\,(\beta)\,\tilde{\varphi},\tilde{\varphi}\right)_{H^2}$$

holds. The lemma is proved.

Hence we get the following corollaries.

Corollary 1. *For $u\in \overset{\circ}{W}{}_2^2\,(R_+;H;0,1)$ and $\beta\in(0,2)$ there holds the equality*

$$\|\Phi\,(d/dt:\beta:A)\,u\|^2_{L_2(R_+;H)}=\|u\|^2_{W_2^2(R_+;H)}-\beta\left\|Au'\right\|^2_{L_2(R_+;H)}. \qquad (12)$$

Corollary 2. *For $u\in W_2^2\,(R_+;H;T)$ and for $\beta\in(0,2)$ there holds the equality*

$$\|u\|^2_{W_2^2(R_+;H)}-\beta\left\|Au'\right\|^2_{L_2(R_+;H)}=\|\Phi\,(d/dt:\beta:A)\,u\|^2_{L_2(R_+;H)}+\left(R_T\,(\beta)\,\varphi,\varphi\right), \qquad (13)$$

where

$$\left(R_T\,(\beta)\,\varphi,\varphi\right)=2Re\,(C\varphi,\varphi)+\sqrt{2-\beta}\left(\|C\varphi\|^2+\|\varphi\|^2\right),$$
$$C=A^{3/2}TA^{-1/2},\ \varphi=A^{1/2}u'\,(0)\in H. \qquad (14)$$

In particular, when $T=0\,(C=0)$, for $u\in \overset{\circ}{W}{}_2^2\,(R_+;H;0)$ and for $\beta\in(0,2)$ we have

$$\|u\|^2_{W_2^2(R_+;H)}-\beta\left\|Au'\right\|^2_{L_2(R_+;H)}=\|\Phi\,(d/dt:\beta:A)\,u\|^2_{L_2(R_+;H)}+\sqrt{2-\beta}\,\|\varphi\|^2. \qquad (15)$$

Corollary 3. *For $u \in W_2^2(R_+;H;K)$ and for $\beta \in (0,2)$ there holds the equality*

$$\|u\|^2_{W_2^2(R_+;H)} - \beta \left\|Au'\right\|^2_{L_2(R_+;H)} = \|\Phi(d/dt:\beta:A)\,u\|^2_{L_2(R_+;H)} + (R_K(\beta)\,\varphi,\varphi)\,, \tag{16}$$

where

$$(R_K(\beta)\,\varphi,\varphi) = 2Re\,(S\varphi,\varphi) + \sqrt{2-\beta}\left(\|S\varphi\|^2 + \|\varphi\|^2\right), \qquad S = A^{1/2}KA^{-3/2},\ \varphi = A^{3/2}u(0) \in H. \tag{17}$$

In particular, when $K = 0\,(S = 0)$, for $u \in \overset{\circ}{W}{}_2^2(R_+;H;1)$ and for $\beta \in (0,2)$ we have

$$\|u\|^2_{W_2^2(R_+;H)} - \beta \left\|Au'\right\|^2_{L_2(R_+;H)} = \|\Phi(d/dt:\beta:A)\,u\|^2_{L_2(R_+;H)} + \sqrt{2-\beta}\,\|\varphi\|^2\,. \tag{18}$$

Obviously, the lemma below holds

Lemma 2. *$\sigma\left(\tilde{R}(\beta)\right) = \sigma(R(\beta))$ as a geometrical set, where $\tilde{R}(\beta)$ and $R(\beta)$ are determined in Lemma 1.*

Hence it follows that $\tilde{R}(\beta)$ may have only eigenvalues that coincide with $R(\beta)$.

Now we find the exact values of the norms of intermediate derivative operators $N_{0,0}, N_T, N_K, N_0, N_1$ and N, defined by formulae (1)–(4).

Theorem 1. *The norm $N_{0,0} = \frac{1}{\sqrt{2}}$.*

P r o o f. For $u \in \overset{\circ}{W}{}_2^2(R_+;H;0,1)$ and $\beta \in (0,2)$ equality (12) holds. In this equality passing to the limit as $\beta \to 2$ we can find that for any $u \in \overset{\circ}{W}{}_2^2(R_+;H;0,1)$ the inequality

$$\left\|Au'\right\|_{L_2(R_+;H)} \le \frac{1}{\sqrt{2}}\,\|u\|_{W_2^2(R_+;H)}$$

holds, i.e., $N_{0,0} \le \frac{1}{\sqrt{2}}$. Prove that $N_{0,0} = \frac{1}{\sqrt{2}}$. Show that for any $\varepsilon > 0$ there exists such a vector function $u_\varepsilon(t)$ that

$$\mathcal{E}(u_\varepsilon(t)) \equiv \|u_\varepsilon\|^2_{W_2^2(R;H)} - (2+\varepsilon)\left\|Au'_\varepsilon\right\|^2_{L_2(R;H)} < 0. \tag{19}$$

Find $u_\varepsilon(t)$ in the form $u_\varepsilon(t) = g(t)\,\psi_\varepsilon$, where $\psi_\varepsilon \in H_4\,(\|\psi_\varepsilon\|_0 = 1)$, but $g(t)$

is a scalar function from $W_2^2(R)$. Then by the Plancharel theorem

$$\mathcal{E}(g(t)\psi_\varepsilon) = \|g''(t)\psi_\varepsilon\|^2_{L_2(R;H)} + \left\|g(t)A^2\psi_\varepsilon\right\|^2_{L_2(R;H)} - (2+\varepsilon)\|g'(t)A\psi_\varepsilon\|^2_{L_2(R;H)}$$

$$= \int\limits_{-\infty}^{+\infty}\left(\left(\xi^4 E + A^4 - (2+\varepsilon)\xi^2 A^2\right)\psi_\varepsilon,\psi_\varepsilon\right)|\widehat{g}(\xi)|^2 d\xi \equiv \int\limits_{-\infty}^{+\infty} q(\xi,\psi_\varepsilon)|\widehat{g}(\xi)|^2 d\xi,$$

where $q(\xi,\psi_\varepsilon) = \xi^4 + \left\|A^2\psi_\varepsilon\right\|^2 - (2+\varepsilon)\xi^2\|A\psi_\varepsilon\|^2$, and $\widehat{g}(\xi)$ is a Fourier transform of the function $g(t)$.

It is obvious that the function $q(\xi,\psi_\varepsilon)$ takes its minimal value at the points $\xi = \pm(2+\varepsilon)$ equal to $h(\varepsilon,\psi_\varepsilon) = \left\|A^2\psi_\varepsilon\right\|^2 - \frac{1}{4}(2+\varepsilon)^2\|A\psi_\varepsilon\|^4$.

If the operator A has at least one eigenvector responding to eigenvalue μ, we can take this normed eigenvector as ψ_ε.

Thus in this case $h(\varepsilon,\psi_\varepsilon) = \mu^4 - \frac{1}{4}(2+\varepsilon)^2\mu^4 < 0$. If μ is a point of a continuous spectrum, we can find such a vector ψ_ε ($\|\psi_\varepsilon\| = 1$) that $A^l\psi_\varepsilon = \lambda^l\psi_\varepsilon + o(\delta), l = 1,2,\ldots,$ for $\delta \to 0$. Obviously, for small δ the function $h(\varepsilon,\psi_\varepsilon) < 0$. Now let us fix the vector ψ_ε, for which $h(\varepsilon,\psi_\varepsilon) < 0$, and find the function $g(t)$.

Since the function $q(\xi,\psi_\varepsilon)$ is continuous with respect to the argument ξ, there can be found $(\eta_0(\varepsilon),\eta_1(\varepsilon))$, where $q(\xi,\psi_\varepsilon) < 0$, i.e.,

$$\varepsilon(g(t)\psi_\varepsilon) = \int\limits_{\eta_0(\varepsilon)}^{\eta_1(\varepsilon)} q(\xi,\psi_\varepsilon)\left|\hat{g}(\xi)\right|^2 d\xi < 0.$$

Further, from the continuity of the functional $\mathcal{E}(\cdot)$ in the space $W_2^2(R;H)$ by the theorem on density of finite infinitely differentiable vector function [1, p. 29] there exists a vector function $u_{N,\varepsilon}(t) \in W_2^2(R;H)$ with the support $(-N,N) \subset R$, for which $\mathcal{E}(u_{N,\varepsilon}(t)) < 0$. Assuming $u_\varepsilon(t) = u_{N,\varepsilon}(t+2N)$, we get $u_\varepsilon(t) \in \overset{\circ}{W}{}_2^2(R_+;H;0,1)$ and $\mathcal{E}(u_\varepsilon(t)) = \mathcal{E}(u_{N,\varepsilon}(t+2N)) < 0$. Thus, $N_{0,0} = \frac{1}{\sqrt{2}}$. The theorem is proved.

Since $\overset{\circ}{W}{}_2^2(R_+;H;0,1) \subset W_2^2(R_+;H;T)$, then $N_T \geq \frac{1}{\sqrt{2}}$. Analogously, $N \geq N_K \geq N_{0,0} = \frac{1}{\sqrt{2}}$. Explain when $N_T = \frac{1}{\sqrt{2}}$ or $N_K = \frac{1}{\sqrt{2}}$. The following holds.

Theorem 2. *The norm* $N_T = \frac{1}{\sqrt{2}}\left(N_K = \frac{1}{\sqrt{2}}\right)$ *iff for all* $\beta \in (0,2)$ *and* $\varphi \in H \quad (R_T(\beta)\varphi,\varphi) > 0 \quad ((R_K(\beta)\varphi,\varphi) > 0)$.

P r o o f. Let $N_T = \frac{1}{\sqrt{2}}$. Then for any $u \in W_2^2(R_+;H;T)$ and $\beta \in (0,2)$ we have

$$\|u\|^2_{W_2^2(R_+;H)} - \beta \|Au'\|^2_{L_2(R_+;H)}$$

$$= \|u\|^2_{W_2^2(R_+;H)} \left(1 - \beta \|Au'\|^2_{L_2(R_+;H)} \|u\|^{-2}_{W_2^2(R_+;H)}\right)$$

$$\geq \|u\|^2_{W_2^2(R_+;H)} \left(1 - \beta \sup_{u \in W_2^2(R_+;H;T)} \|Au'\|^2_{L_2(R_+;H)} \|u\|^{-2}_{W_2^2(R_+;H)}\right)$$

$$= \|u\|^2_{W_2^2(R_+;H)} \left(1 - \beta \tfrac{1}{2}\right) > 0.$$

Then it follows from equality (13) that for any $u \in W_2^2(R_+;H;T)$ and $\beta \in (0,2)$

$$\|\Phi(d/dt : \beta : A) u\|^2_{L_2(R_+;H)} + (R_T(\beta)\varphi, \varphi) > 0, \forall \varphi \in H \left(\varphi = A^{1/2}u'(0) \in H\right). \tag{20}$$

Since the characteristically polynomial $\Phi(\lambda : \beta : A) = \lambda^2 E + \sqrt{2-\beta}\lambda A + A^2$ is represented in the form

$$\Phi(\lambda : \beta : A) = (\lambda E - \omega_1(\beta) A)(\lambda E - \omega_2(\beta) A),$$

where $\omega_1(\beta) = \overline{\omega_2}(\beta) = \left(-\sqrt{2-\beta} - i\sqrt{2+\beta}\right)/2, (Re\omega_1(\beta) < 0, Re\omega_2(\beta) < 0)$, we get that the Cauchy problem

$$\Phi(d/dt : \beta : A) u = 0, u(0) = Tu'(0), u'(0) = A^{-1/2}\varphi, \forall \varphi \in H, \tag{21}$$

has a unique solution from the space $W_2^2(R_+;H)$

$$u(t,\beta) = \frac{1}{\omega_2 - \omega_1} \left\{ e^{\omega_1(\beta)tA} \left(\omega_2(\beta) TA^{-1/2}\varphi - A^{-3/2}\varphi\right) \right.$$
$$\left. + e^{\omega_2(\beta)tA} \left(A^{-3/2}\varphi - \omega_1(\beta) TA^{-1/2}\varphi\right)\right\}.$$

Obviously, $\|u(t,\beta;\varphi)\| \leq d_1(\beta)\|\varphi\|, d_1(\beta) > 0$. Using the uniqueness of the solution of the Cauchy problem and also using Banach's theorem on invertible operator, we get $\|u(t,\beta;\varphi)\| \geq d_2(\beta)\|\varphi\|$. Thus, it follows from equality (20) that $(R_T(\beta)\varphi, \varphi) > 0$ for $\beta \in (0,2)$ and $\forall \varphi \in H$.

Inversely, if $(R_T(\beta)\varphi, \varphi) > 0$, then from equality (13) it follows that

$$\|u\|^2_{W_2^2(R_+;H)} - \beta \left\|Au'\right\|^2_{L_2(R_+;H)} > 0 \left(\forall \beta \in (0,2), \forall u \in W_2^2(R_+;H;T)\right).$$

By passing to the limit as $\beta \to 2$, we get $N_T \leq \frac{1}{2}$. Consequently, $N_T = \frac{1}{\sqrt{2}}$. We prove in a similar way that $N_K = \frac{1}{\sqrt{2}}$ iff $(R_K(\beta)\varphi, \varphi) > 0$ for $\beta \in (0,2)$ and $\forall \varphi \in H$.

Using this theorem we get the following statement.

Theorem 3. *The norm* $N_T = \frac{1}{\sqrt{2}}$ *iff* $ReC \geq 0$ *(see (14)).*

In fact, if $N_T = \frac{1}{\sqrt{2}}$, then $(R_T(\beta)\varphi, \varphi) > 0, \beta \in (0,2)$, $\varphi \in H$. By passing to the limit as $\beta \to 2$, we get $ReC \geq 0$. Inversely, if $ReC \geq 0$, then $(R_T(\beta)\varphi, \varphi) > 0$, for $\beta \in (0,2)$, i.e., $N_T = \frac{1}{\sqrt{2}}$.

Similarly is proved

Theorem 4. *The norm* $N_K = \frac{1}{\sqrt{2}}$ *iff* $ReS \geq 0$ *(see (17)).*

Notice that if ReC is not a non negative operator, then the following theorem holds.

Theorem 5. *Let* $\inf\limits_{\varphi \in H} Re(C\varphi, \varphi) < 0$, $\left(\inf\limits_{\varphi \in H} Re(S\varphi, \varphi) < 0\right)$. *Then the norm*

$$N_T = \frac{1}{\sqrt{2}}\left(1 - 2\left|\inf_{\|\varphi\|=1} \frac{Re(C\varphi, \varphi)}{1 + \|C\varphi\|^2}\right|^2\right)^{-1/2} \tag{22}$$

$$\left(N_K = \frac{1}{\sqrt{2}}\left(1 - 2\left|\inf_{\|\varphi\|=1} \frac{Re(S\varphi, \varphi)}{1 + \|S\varphi\|^2}\right|^2\right)^{-1/2}\right) \tag{23}$$

(see (14), (17)).

P r o o f. Let $\inf\limits_{\varphi \in H} ReC < 0$. Then by Theorem 3 $N_T > \frac{1}{\sqrt{2}}$. Therefore $N_T^{-2} \in (0,2)$. Then if in equality (13) as $u(t)$ we take the solution of the Cauchy problem (see (21)), for $\beta \in \left(0, N_T^{-2}\right)$ and $\|\varphi\| = 1$ we get

$$(R_T(\beta)\varphi, \varphi) = \|u(t,\beta;\varphi)\|^2_{W_2^2(R_+;H)} - \left\|Au'(t,\beta;\varphi)\right\|^2_{L_2(R_+;H)}$$

$$\geq \|u(t,\beta;\varphi)\|^2_{W_2^2(R_+;H)}\left(1 - \beta N_T^{-2}\right) > 0.$$

Thus, for $\beta \in \left(0, N_T^{-2}\right)$ the function

$$m(\beta) = \inf_{\|\varphi\|=1} (R(\beta)\varphi, \varphi) > 0.$$

And for $\beta \in \left(N_T^{-2}, 2\right)$, by definition of N_T, there can be found a vector function $\upsilon(t,\beta) \in W_2^2(R_+; H; T)$ such that

$$\|\upsilon(t,\beta)\|^2_{W_2^2(R_+;H)} - \left\|A\upsilon'(t,\beta)\right\|^2_{L_2(R_+;H)} < 0.$$

Consequently, for $\beta \in \left(N_T^{-2}, 2\right)$ it follows from equality (13) that

$$(R_T(\beta)\varphi_\beta, \varphi_\beta) + \|\Phi\,(d/dt : \beta : A)\, v(t,\beta)\|^2_{L_2(R_+;H)} < 0$$

$\left(\varphi_\beta = A^{-1/2} v(0,\beta)\right)$, i.e., $m(\beta) < 0$ for $\beta \in \left(N_T^{-2}, 2\right)$. Thus, the continuous function $m\,(\beta)$, determined for $\beta \in (0,2)$, changes its sign at the point N_T^{-2}, i.e., $m\left(N_T^{-2}\right) = 0$. Hence, it follows easily that

$$\sqrt{2 - N_T^{-2}} = -2 \inf_{\|\varphi\|=1} Re\,(C\varphi,\varphi) \Big/ \left[1 + \|C\varphi\|^2\right],$$

i.e.,

$$N_T = \frac{1}{\sqrt{2}} \left(1 - 2\left|\inf_{\|\varphi\|=1} \frac{Re\,(C\varphi,\varphi)}{1+\|C\varphi\|^2}\right|^2\right)^{-1/2}.$$

Formula (23) is proved in a similar way. The theorem is proved.

It follows from Theorems 3–5 that $N_0 = N_1 = \frac{1}{\sqrt{2}}\,(C = S = 0)$.

Now we find the norm N.

There holds the following.

Theorem 6. *The norm $N = 1$, where N is determined by formula (2).*

P r o o f. It is obvious that $N \geq \frac{1}{\sqrt{2}}$. Show that $N \neq \frac{1}{\sqrt{2}}$. In fact, if $N = \frac{1}{\sqrt{2}}$, then it follows from equality (5) that

$$\|\Phi\,(d/dt : \beta : A)\, u\|^2_{L_2(R_+;H)} + \left(\tilde{R}\,(\beta)\,\tilde{\varphi}, \tilde{\varphi}\right)_{H^2} \geq \|u\|^2_{W_2^2(R_+;H)}$$

$$\times\left(1 - \beta \sup_{u \in W_2^2(R_+;H)} \|Au'\|^2\, \|u\|^{-2}_{W_2^2(R_+;H)}\right) \geq \|u\|^2_{W_2^2(R_+;H)}\left(1 - \beta\tfrac{1}{2}\right) > 0. \tag{24}$$

Then for $\beta \in (0,2)$ the Cauchy problem

$$\Phi\,(d/dt : \beta : A)\, u = 0, u\,(0) = A^{-3/2}\varphi_0, u'\,(0) = A^{-1/2}\varphi_1,\ \forall \varphi_0, \varphi_1 \in H, \tag{25}$$

has a unique solution from $W_2^2\,(R_+;H)$, therefore for $\beta \in (0,2)$ $(\tilde{R}\,(\beta)\,\tilde{\varphi}, \tilde{\varphi})_{H^2} > 0$. By Lemma 2 all eigenvalues of the matrix $R\,(\beta)$ are positive. But it is seen from the form $R\,(\beta)$ (see Lem. 1) that for $\beta \in (1,2)$, $R\,(\beta)$ has also the negative eigenvalue $\lambda_1\,(R\,(\beta)) = 1 - \beta < 0$. Thus, $N > \frac{1}{2}$, i.e., $N^{-2} \in (0,2)$. Then for $\beta \in \left(0, N^{-2}\right)$ we have

$$\|\Phi\,(d/dt : \beta : A)\, u\|^2_{L_2(R_+;H)} + \left(\tilde{R}\,(\beta)\,\tilde{\varphi}, \tilde{\varphi}\right) \geq \|u\|^2_{W_2^2}\left(1 - \beta N^{-2}\right) > 0.$$

Hence it follows that if in this inequality we replace u by the solution of the Cauchy problem (25) for $\beta \in \left(0, N_T^{-2}\right)$, then we obtain $\left(\tilde{R}\,(\beta)\,\tilde{\varphi}, \tilde{\varphi}\right) > 0$.

Consequently, all eigenvalues of the matrix $R(\beta)$ are positive for $\beta \in \left(0, N^{-2}\right)$. In particular, $\lambda_1(\beta) > 0$ ($\lambda_1(\beta)$ is the first eigenvalue of the matrix $R(\beta)$). And for $\beta \in \left(N_T^{-2}, 2\right)$ it follows from the definition of N that there exists such $\upsilon(t,\beta) \in W_2^2(R_+;H)$ that

$$\|\upsilon\|^2_{W_2^2(R_+;H)} - \left\|A\upsilon'\right\|^2_{L_2(R_+;H)} < 0.$$

Consequently, $\left(\tilde{R}(\beta)\tilde{\varphi_\beta}, \tilde{\varphi_\beta}\right) < 0 \quad \left(\varphi_{\beta,0} = A^{-3/2}\upsilon_\beta(0), \varphi_{\beta,1} = A^{-1/2}\upsilon'_\beta(0)\right)$.

We again obtain that $\lambda_1(\beta)$ is the first eigenvalue of the matrix $R(\beta)$, is negative for $\beta \in \left(N^{-2}, 2\right)$. Consequently, $\lambda_1\left(N^{-2}\right) = 0$.

I.e.,

$$\begin{vmatrix} \sqrt{2-N^{-2}} & 1 \\ 1 & \sqrt{2-N^{-2}} \end{vmatrix} = 0.$$

Hence we find that $N^{-2} = 1$, i.e., $N = 1$. The theorem is proved.

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