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Global Weak Solutions to the Navier–Stokes–Vlasov–Poisson System

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We consider the Navier–Stokes–Vlasov–Poisson system describing the flow of a viscous incompressible fluid containing small solid charged particles. The existence result for weak global solutions of the corresponding boundary value problem is obtained.

Key words: Navier–Stokes equation, Vlasov–Poisson equation, suspensions, global weak solution, modified Galerkin method, compactness of approximations.

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1. Introduction

An increasing interest in studying the flow of small solid particles in fluids and gases is stimulated by numerous applications of these processes in a wide range of engineering problems as well as by ecological needs. For example, we refer here to the problem of transport of fine–dispersed suspensions by aerial or liquid flows, the work of hydraulic or pneumatic transport devices, dust–collecting units, etc. There is an extensive literature on the motion of such suspensions.

We will not attempt a review of the literature here but merely mention [1, 2], and the references therein. One of the models often used in the simulation of such processes is a two phase flow model. The main feature of this model is that the system of small solid particles is considered as a continuous matter. Then, the motion of a fluid with particles suspended therein is described as a motion of two miscible continuous phases — the carrying fluid and the "fluid of particles". However, this model is applicable only in the case when the size and the specific density of the particles are identical or slightly different.

There is another model, which is known as Navier–Stokes–Vlasov system. This model describes the motion of a fluid with small solid particles in the case when the sizes of the particles are strongly different. In the framework of this model the solid phase of the mixture is assumed to be a system of spherical particles of high specific density described by a distribution function of the particles depending on their location, velocities and radii. This model is based on the homogenized Navier–Stokes system describing the perturbation of the fluid by the solid particles (see [3, 4]). This system involves an unknown distribution function of the particles $f(x, v, r, t)$. The distribution function satisfies the Vlasov equation where the Stokes forces are taking into account. Combining this equation and the perturbated Navier–Stokes system, we obtain a closed system of equations. Namely, the Navier–Stokes–Vlasov system. The existence of a global weak solution of the corresponding boundary value problem as well as the existence and uniqueness of a smooth solution in a small time interval was proved in [5–7].

In the present paper we consider a similar model which describes the motion of small solid charged particles with high dispersion of radii in a viscous incompressible and non-conducting fluid. We assume that the charges of all particles are of the same sign and proportional to their electric capacities. This means that the charge of a particle of radius r' is equal to $q\,r'$. In this case our model is described by the following system of equations:

$$\frac{\partial u}{\partial t} + (u\nabla_x)u - \nu\Delta u + \alpha \int_a^b\int_{\mathbb{R}^3} r(u(x,t) - v)f(x,v,r,t)dvdr - \nabla p = g, \tag{1.1}$$

$$\mathrm{div} u = 0, \tag{1.2}$$

$$-\Delta\varphi = q\int_a^b\int_{\mathbb{R}^3} rf(x,v,r,t)dvdr, \tag{1.3}$$

$$\frac{\partial f}{\partial t} + (v\nabla_x)f + \mathrm{div}_v[G(u,v,\nabla\varphi,r)f] = 0, \tag{1.4}$$

$$G = \beta r^{-2}[u(x,t) - v] - \gamma r^{-2}\nabla\varphi + g, \tag{1.5}$$

where $u = u(x,t)$ and $p = p(x,t)$ are the velocity and the pressure of the fluid, respectively; $\varphi = \varphi(x,t)$ is the potential of the electric field, generated by the charged particles; $f(x,v,r,t)$ is a normalized distribution function related to the initial distribution function $f_\varepsilon(x,v,r,t)$ as

$$f_\varepsilon(x,v,r',t) = \frac{1}{\varepsilon} f(x,v,\frac{r'}{\varepsilon},t).$$

Here $x = (x_1,x_2,x_3)$, $v = (v_1,v_2,v_3)$ denote the velocities of the particles, $r' = \varepsilon r$ $(0 < a \le r \le b < \infty)$ are the radii, ε is the mean radius of particles (small parameter), α, β and γ are positive constants:

$$\alpha = 6\pi\nu; \quad \beta = \frac{9\rho_f\nu}{2\rho_p\varepsilon^2}; \quad \gamma = \frac{3q}{4\pi\rho_p\varepsilon^2};$$

ν is the kinematic viscosity of the fluid; ρ_f and ρ_p are the specific densities of the fluid and the particles, respectively; $g = g(x)$ denotes the gravity forces.

We consider system (1.1)–(1.5) in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with a sufficiently smooth boundary $\partial\Omega$. We assume the following boundary conditions on $\partial\Omega$:

$$u(x,t) = 0 \quad \text{on } S_T \equiv \partial\Omega \times [0,T], \tag{1.6}$$

$$\varphi(x,t) = 0 \quad \text{on } S_T, \tag{1.7}$$

$$f(x,v,r,t)(v,n) \ge 0 \quad \text{on } \partial\Omega \times \mathbb{R}^3 \times [a,b] \times [0,T], \tag{1.8}$$

where $n = n(x)$ is the outward unit normal vector to $\partial\Omega$ at the point x; $(\cdot\,,\cdot)$ in (1.8) denotes the scalar product in $\mathbb{R}^3$. Condition (1.6) corresponds to the adhesion of the fluid to $\partial\Omega$. Condition (1.7) means that the boundary is a perfectly conducting one. Finally, condition (1.8) means that a particle, which reaches the boundary $\partial\Omega$ will rest on the $\partial\Omega$.

The system (1.1)–(1.5) has to be completed by the initial conditions:

$$u(x,0) = u_0(x) \quad \text{in } \Omega; \tag{1.9}$$

$$f(x,v,r,0) = f_0(x,v,r) \quad \text{in } \Omega \times \mathbb{R}^3 \times [a,b]. \tag{1.10}$$

We call system (1.1)–(1.5) the Navier–Stokes–Vlasov–Poisson system. It is a union of the Navier–Stokes and the Vlasov–Poisson systems. The existence and uniqueness results for both of these systems were studied separately by many authors and by various methods (see, e.g., [8–18]).

The goal of the paper is to prove the existence of a global weak solution to problem (1.1)–(1.10). The approach which is used here is a generalization of the methods developed in [9, 10].

The outline of the paper is the following. In Section 2 we introduce the definition of the weak solution of (1.1)–(1.10) and formulate the main result of the paper. We begin Section 3 by regularizing problem (1.1)–(1.10) and defining its weak solution. Then we construct finite–dimensional approximations (u^n, f^n, φ^n) of the solution. To this end, we use the modification of Galerkin's method developed in [9]. Following [10], we use an explicit construction for the solution of the Vlasov equation (1.4). The compactness of the approximations (u^n, f^n, φ^n) is proved in Section 4. Finally, in Section 5 we pass to the limit in the integral identities, which define the weak solution of the regularized problem, and obtain the corresponding identities for the weak solution of the initial problem.

2. Definition of Weak Solution and Formulation of the Main Result

Let Ω be a bounded convex domain in $\mathbb{R}^3$ with a sufficiently smooth boundary. We introduce the following notation: $\Omega_T = \Omega\times[0,T]$, $Q = \Omega\times\mathbb{R}^3$, $Q_T = Q\times[0,T]$, $\mathbb{R}^6 = \mathbb{R}^3\times\mathbb{R}^3$, $\mathbb{R}^6_T = \mathbb{R}^6\times[0,T]$. We also introduce the Hilbert spaces $L_2(\Omega)$ and $L_2(\mathbb{R}^6)$ with the scalar products

$$(f,g)_{2,\Omega} = \int\limits_{\Omega}\sum_{i=1}^{3} f_i(x)g_i(x)dx, \quad (F,G)_{2,\mathbb{R}^6} = \int\limits_{\mathbb{R}^6} F(x,v,r)G(x,v,r)dxdv,$$

and the spaces $J(\Omega)$, $J^1(\Omega)$ that are the closures of divergent-free $C^\infty(\bar\Omega)$ functions with compact support in $L_2(\Omega)$ and $\overset{\circ}{W}{}^1_2(\Omega)$, respectively. We denote by P an extension operator from $L_2(\Omega)$ to $L_2(\mathbb{R}^3)$ such that for any $u\in L_2(\Omega)$, $Pu = u$ in Ω and $Pu = 0$ in $\mathbb{R}^3\setminus\Omega$, and by S a restriction operator from $L_2(\mathbb{R}^3)$ to $L_2(\Omega)$ such that for any $u\in L_2(\mathbb{R}^3)$ $Su = \chi_\Omega u$, where χ_Ω is the characteristic function of Ω.

We assume that the initial functions $u_0(x)$ and $f_0(x,v,r)$ in (1.9), (1.10) satisfy the following conditions:

$$\begin{gathered}\mathrm{div} u_0 = 0, x\in\Omega, \quad u_0(x) = 0, x\in\partial\Omega,\\ 0\le f_0(x,v,r)\le A_1<\infty, (x,v,r)\in Q\times[a,b], \qquad (2.1)\\ \int\limits_a^b\int\limits_Q f_0(x,v,r)dxdvdr = A_2<\infty, \quad \int\limits_a^b\int\limits_Q v^2 f_0(x,v,r)dxdvdr = A_3<\infty.\end{gathered}$$

We consider the triple of functions $\langle u(x,t), \varphi(x,t), f(x,v,r,t)\rangle$, where $u(x,t)$ is a vector function, and $\varphi(x,t)$, $f(x,v,r,t)$ are functions such that

$$u\in L_\infty(0,T;J(\Omega))\cap L_2(0,T;J^1(\Omega)), \qquad (2.2a)$$

and $u(x,t)$ is a continuous function in t in the weak topology of $L_2(\Omega)$

$$\varphi \in L_2(0,T;\overset{\circ}{W}{}_2^1(\Omega)), \tag{2.2b}$$

$$f(x,v,r,t) = S\tilde{f}(x,v,r,t). \tag{2.2c}$$

Here $\tilde{f} \in L_\infty(\mathbb{R}^6_T \times [a,b])$, $\tilde{f} \in L_1(\mathbb{R}^6 \times [a,b])$ uniformly in $t \in [0,T]$, and $\tilde{f}$ is continuous in time in the weak topology of $L_1(\mathbb{R}^6 \times [a,b])$.

Definition 1. *The triple of the functions $\langle u(x,t), \varphi(x,t), f(x,v,r,t)\rangle$ is a weak solution of problem (1.1)–(1.10) if the following integral identities hold:*

$$\int\limits_0^T \Big\{(u, \zeta_t + (u\nabla_x)\zeta)_{2,\Omega} - \nu(u,\zeta)_{J^1(\Omega)}$$

$$-\alpha\left(\int\limits_a^b\int\limits_{\mathbb{R}^3} r(u(x,t)-v)S\tilde{f}dvdr, \zeta\right)_{2,\Omega} + (g,\zeta)_{2,\Omega}\Big\}dt + (u_0,\zeta(0))_{2,\Omega} = 0, \tag{2.3}$$

$$\int\limits_0^T \left\{(\nabla\varphi, \nabla\Phi)_{2,\Omega} - q\left(\int\limits_a^b\int\limits_{\mathbb{R}^3} rS\tilde{f}dvdr, \Phi\right)_{2,\Omega}\right\}dt = 0, \tag{2.4}$$

$$\int\limits_0^T\int\limits_a^b (\tilde{f}, \Psi_t + (v\nabla_x)\Psi + (PG\nabla_v)\Psi)_{2,\mathbb{R}^6}drdt + \int\limits_a^b (Pf_0, \Psi(0))_{2,\mathbb{R}^6}dr = 0 \tag{2.5}$$

for any ζ, Φ and Ψ such that:

$$\zeta \in L_\infty(0,T;J(\Omega)) \cap L_4(0,T;J^1(\Omega)), \quad \zeta_t \in L_2(\Omega_T), \quad \zeta(x,T) = 0; \tag{2.6a}$$

$$\Phi \in L_2(0,T;\overset{\circ}{W}{}_2^1(\Omega)), \tag{2.6b}$$

$\Psi(x,v,r,t)$ is a function with compact support in the space $\mathbb{R}^6_T \times [a,b]$ in x and v,

$$\nabla_x\Psi \in L_1(\mathbb{R}^6_T \times [a,b]), \quad \nabla_v\Psi \in L_\infty(\mathbb{R}^6_T \times [a,b]), \tag{2.6c}$$
$$\Psi_t \in L_1(\mathbb{R}^6_T \times [a,b]), \quad \Psi(x,v,r,T) = 0.$$

Remark 1. *The operators P and S are introduced for the following reason. First, we will construct the solution of (1.4) in $\mathbb{R}^6_T$ and then restrict this solution to Q_T. The convexity of the domain Ω implies condition (1.8).*

The main result of the paper is the following

Theorem 1. *Let $g \in L_\infty(0,T;C^1(\Omega))$, $u_0 \in J(\Omega)$ and $f_0(x,v,r)$ satisfy (2.1). Then there exists a weak solution of problem (1.1)–(1.10), such that*

$$\max_{0\le t\le T}\|u\|_{2,\Omega} + \max_{0\le t\le T}\int_a^b\int_Q v^2 f\,dxdvdr + \int_0^T \|u(t)\|^2_{J^1(\Omega)}dt + \max_{0\le t\le T}\|\nabla\varphi(t)\|^2_{2,\Omega}$$

$$< C\left[\|u_0\|_{2,\Omega} + \int_a^b\int_Q (1+v^2)f_0(x,v,r)dxdvdr + \|g\|^2_{L_\infty(0,T;C^1(\Omega))}\right],$$

where C is a constant that depends on Ω only.

The proof of Theorem 1 is given in Sections 3–5.

3. Regularized Model

In this section we introduce the regularization of problem (1.1)–(1.10). The regularized model allows us to prove the existence and uniqueness of finite-dimensional approximations of its solution. The Section is organized as follows. In Section 3.1 we introduce the regularization of problem (1.1)–1.10) and define its weak solution. In Section 3.2 we construct the finite-dimensional approximations of the solution for the regularized problem. The *a priori* estimates for these approximations are obtained in Section 3.3. Finally, in Section 3.4 we prove the existence of the desired approximations.

3.1. Regularization of problem (1.1)–(1.10)

The regularized problem has the form:

$$\frac{\partial u}{\partial t} + (u\nabla_x)u - \nu\Delta u + \alpha\int_a^b\int_{\mathbb{R}^3} r\theta_R((u-v)^2)(u(x,t)-v)f\,dvdr - \nabla p = g, \quad (3.1)$$

$$\mathrm{div} u = 0, \quad (3.2)$$

$$\varepsilon\Delta^2\varphi - \Delta\varphi = q\int_a^b\int_{\mathbb{R}^3} rf(x,v,r,t)dvdr, \quad (3.3)$$

$$\frac{\partial f}{\partial t} + (v\nabla_x)f + \mathrm{div}_v[G_{R,\varepsilon}(u,v,\nabla\varphi,g)f] = 0, \quad (3.4)$$

$$G_{R,\varepsilon}(u,v,\nabla\varphi,g) = \frac{\beta}{r^2}\theta_R((u-v)^2)[u-v] - \frac{\gamma}{r^2}\nabla\varphi + g_\varepsilon(x,t). \tag{3.5}$$

Here $\varepsilon > 0$ is a sufficiently small parameter; $\theta_R \in C^\infty(\mathbb{R})$ such that $0 \le \theta_R(z) \le 1$ if $|z| \le R$, $\theta_R(z) = 0$ if $|z| > 2R$ and $\theta'_R \le 0$ if $z \ge 0$; $g_\varepsilon(x,t) = g(x,t)\chi_\varepsilon(x)$, where $\chi_\varepsilon \in C_0^2(\Omega)$, $\chi_\varepsilon(x) = 1$ if $x \in \Omega_\varepsilon \subset \Omega(\mathrm{dist}(\partial\Omega_\varepsilon, \partial\Omega) = \varepsilon)$ and $\chi_\varepsilon = 0$ if $x \in \partial\Omega$.

We complete the problem (3.1)–(3.5) by boundary conditions (1.6)–(1.8) along with the following one:

$$\frac{\partial\varphi}{\partial n} = 0, \quad (x,t) \in S_T, \tag{3.6}$$

and also by initial conditions (1.9), (1.10).

Remark 2. *In contrast to (1.1), (1.3), (1.4) in (3.1), (3.3), (3.4), we observe the cut-off function θ_R and the regularization term $\varepsilon\Delta^2\varphi$. These modifications will allow us to prove the existence and uniqueness of the global solution of the characteristic system to Vlasov's equation and then obtain its solution explicitly.*

Suppose that the triple of functions $\langle u(x,t), \varphi(x,t), f(x,v,r,t)\rangle$ satisfies conditions (2.2a), (2.2c) and

$$\varphi \in L_2(0,T; \overset{\circ}{W}{}_2^2(\Omega)). \tag{3.7}$$

The triple of functions is called a weak solution of (3.1)–(3.5), (1.6)–(1.8), (3.6), (1.9), (1.10) if

$$\int_0^T \Bigg\{ (u, \zeta_t + (u\nabla_x)\zeta)_{2,\Omega} - \nu(u,\zeta)_{J^1(\Omega)} + (g,\zeta)_{2,\Omega}$$

$$-\alpha \left(\int_a^b\int_{\mathbb{R}^3} r\theta_R((u-v)^2)(u-v)S\tilde{f}dvdr, \zeta \right)_{2,\Omega} \Bigg\} dt + (u_0, \zeta(0))_{2,\Omega} = 0, \tag{3.8}$$

$$\int_0^T \left\{ \varepsilon(\Delta\varphi, \Delta\Phi)_{2,\Omega} + (\nabla\varphi, \nabla\Phi)_{2,\Omega} - \left(q\int_a^b \int_{\mathbb{R}^3} rS\tilde{f}dvdr, \Phi \right)_{2,\Omega} \right\} dt = 0, \tag{3.9}$$

$$\int_0^T\int_a^b (\tilde{f}, \Psi_t + (v\nabla_x)\Psi + PG_{R,\varepsilon}\nabla_v)\Psi)_{2,\mathbb{R}^6} drdt + \int_a^b (Pf_0, \Psi(0))_{2,\mathbb{R}^6} dr = 0, \tag{3.10}$$

for any ζ, Ψ satisfying (2.6a), (2.6c) and $\Phi \in L_2(0,T; \overset{\circ}{W}{}_2^2(\Omega))$.

3.2. Construction of approximations

In what follows we make use of the lemma.

Lemma 1. *Suppose that $f_0(x,v,r)$ satisfies condition (2.1). Then there exists a sequence of nonnegative functions $f_0^n(x,v,r)$ defined in $Q\times[a,b]$ such that for any fixed $n\in\mathbb{N}$ and $r\in[a,b]$, $f_0^n(x,v,r)$ is infinitely-differentiable in x, v, $f_0^n(x,v,r)$ has a compact support in $Q\times[a,b]$ and $f_0^n(x,v,r)$ satisfies the inequalities*

$$\sup_{Q\times[a,b]} f_0^n \le A_1, \quad \int_a^b\int_Q f_0^n(x,v,r)dxdvdr \le A_2, \quad \int_a^b\int_Q v^2 f_0^n(x,v,r)dxdvdr \le A_3.$$

Moreover, $f_0^n\to f_0$ in $L_2(Q\times[a,b])$ as $n\to\infty$.

The prove of the lemma make use of the standard averaging technique.

We construct the approximations by the method developed in [9] which is a modification of Galerkin's method. We are looking for the approximations of (3.1), (3.2) in the form

$$u^n(x,t)=\sum_{l=1}^{n} C_{nl}(t)\Psi^l(x), \tag{3.11}$$

where $C_{nl}\in C^1(0,T)$ are unknown coefficients and $\Psi^l(x)$, $l=1,2,\dots,$ is the orthonormal basis in $L_2(\Omega)$ consisting of the eigenfunctions of the problem

$$\Delta\Psi^l(x)-\nabla g^l=\mu_l\Psi^l(x), \quad \mathrm{div}\Psi^l(x)=0, \quad x\in\Omega, \quad \Psi^l(x)=0, \quad x\in\partial\Omega.$$

The corresponding approximations $\varphi^n(x,t)$, $\tilde f^n(x,v,r,t)$ for the solutions of equations (3.3), (3.4) turn out to be the solutions of

$$\varepsilon\Delta^2\varphi^n-\Delta\varphi^n=q\int_a^b\int_{\mathbb{R}^3} rS\tilde f^n(x,v,r,t)dvdr, \tag{3.12}$$

$$\varphi^n(x,t)=\frac{\partial\varphi^n}{\partial n}=0, \quad (x,t)\in S_T, \tag{3.13}$$

$$\frac{\partial\tilde f^n}{\partial t}+(v\nabla_x)\tilde f^n+\mathrm{div}_v\left\{\left[\frac{\beta}{r^2}\theta_R((Pu^n-v)^2)(Pu^n-v)\frac{\gamma}{r^2}P\nabla\varphi^n+Pg_\varepsilon\right]\tilde f^n\right\}=0, \tag{3.14}$$

$$\tilde f^n|_{t=0}=Pf_0^n, \tag{3.15}$$

where the functions f_0^n are given in Lemma 1.

We define the functions $X^n(x,v,r,t,\tau)$ and $V^n(x,v,r,t,\tau)$ as the solutions of the following system of equations:

$$\left.\begin{aligned}\frac{dX^n}{d\tau} &= V^n,\\ \frac{dV^n}{d\tau} &= \frac{\beta}{r^2}\theta_R((Pu^n(X^n,\tau)-V^n)^2)(Pu^n(X^n,\tau)-V^n)\\ &- \frac{\gamma}{r^2}P\nabla\varphi^n(X^n,\tau)+Pg_\varepsilon(X^n,\tau),\\ X^n|_{\tau=t} &= x, \quad V^n|_{\tau=t}=v, \quad 0\le\tau\le t, \quad t\in[0,T].\end{aligned}\right\} \tag{3.16}$$

The properties of the function Ψ^i (see [8]) imply that $\sup\limits_{\Omega_T}|\nabla u^n(x,t)|<\infty$, $u^n|_{S_T}=0$. If, for any $t\in[0,T]$, the function $\varphi^n(x,t)$ belongs to $C^2(\Omega)$ and condition (3.13) is valid, then the right-hand side of system (3.16) satisfies the Lipschitz condition in X^n and V^n. Thus, we obtain the local solvability of (3.16). For any $\tau\in[0,t]$ and $n\in\mathbb{N}$ the functions X^n, V^n are bounded (see Lem. 2) and, therefore, we can extend them at $\tau=0$.

The solution of problem (3.14), (3.15) is given by

$$\tilde f^n(x,v,r,t)=\exp\left\{\frac{\beta}{r^2}\int\limits_0^t\big[3\theta_R((Pu^n(X^n,\tau)-V^n)^2)2\theta_R'((Pu^n(X^n,\tau)-V^n)^2)\right.$$

$$\left.\times(Pu^n(X^n,\tau)-V^n)^2\big]d\tau\right\}Pf_0^n(X^n(x,v,r,t,0),V^n(x,v,r,t,0),r). \tag{3.17}$$

Lemma 2. *If $Pf_0^n(x,v,r)$ has a compact support with respect to x and v in $\mathbb{R}^6$, then the solution of problem (3.14), (3.15) also has a compact support for any $t\in[0,T]$.*

P r o o f. Suppose that $\operatorname{supp}Pf_0^n\subset\Omega\times K_{R_0}\times[a,b]$, where $K_{R_0}=\{v\in\mathbb{R}^3:|v|\le R_0\}$. Let us show that for any $x\in\mathbb{R}^3$, $r\in[a,b]$, $t\in[0,T]$ and any $\tau\in[0,T]$ the inequality

$$|v|>R_0+\frac{\beta}{a^2}T\sqrt{2R}+\frac{\gamma}{a^2}\|\varphi^n\|_{L_2(0,T;C^2(\Omega))}\sqrt{T}+\|g\|_{L_\infty(0,T;C^1(\Omega))}T=R_{\tilde f^n} \tag{3.18}$$

implies that $|V^n|>R_0$.

To this end, we consider the following system of integral equations equivalent to (3.16):

$$X^n(x,v,r,t,\tau)-x=\int_t^\tau V^n(x,v,r,t,s)ds, \tag{3.19}$$

$$V^n(x,v,r,t,\tau)-v=\frac{\beta}{r^2}\int\limits_t^\tau\left[\theta_R((Pu^n(X^n,s)-v)^2)(Pu^n(X^n,s)-V^n)\right]ds$$
$$-\frac{\gamma}{r^2}\int\limits_t^\tau P\nabla\varphi^n(X^n,s)ds+\int\limits_t^\tau Pg_\varepsilon(X^n,s)ds. \qquad (3.20)$$

From (3.18), (3.20) we obtain

$$|V^n(x,v,r,t,\tau)|\geq|v|-\frac{\beta}{r^2}\left|\int\limits_t^\tau\theta_R((Pu^n(X^n,s)-v)^2)(Pu^n(X^n,s)-V^n)ds\right|$$
$$-\frac{\gamma}{r^2}\left|\int\limits_t^\tau P\nabla\varphi^n(X^n,s)ds\right|-\left|\int\limits_t^\tau Pg_\varepsilon(X^n,s)ds\right|>R_0.$$

On the other hand, it follows from (3.20) that

$$\sup_\tau|V^n|\leq|v|+\frac{\beta}{a^2}T\sqrt{2R}+\frac{\gamma}{a^2}\|\varphi^n\|_{L_2(0,T;C^2(\Omega))}\sqrt{T}+\|g\|_{L_\infty(0,T;C^1(\Omega))}T.$$

From this estimate and (3.19) we conclude that

$$|X^n-x|\leq T\left(|v|+\frac{\beta}{a^2}T\sqrt{2R}+\frac{\gamma}{a^2}\|\varphi^n\|_{L_2(0,T;C^2(\Omega))}\sqrt{T}+\|g\|_{L_\infty(0,T;C^1(\Omega))}T\right).$$

Hence, (3.17) implies that supp$\tilde{f}^n\subset\Omega\times K_{R_{\tilde{f}^n}}$ for any $t\in[0,T]$, $r\in[a,b]$. ∎

Let us show that the convexity of Ω implies the following boundary condition for the function $f^n(x,v,r,t)=S\tilde{f}^n(x,v,r,t)$:

$$f^n(x,v,r,t)(v,n(x))\geq 0,\quad x\in\partial\Omega. \qquad (3.21)$$

In fact, since $f^n(x,v,r,t)\geq 0$, condition (3.21) is equivalent to the following statement: if there exists a point $x_0\in\partial\Omega$ such that $(v,n(x_0))<0$, then $f^n(x_0,v,r,t)=0$. From the convexity of the domain Ω it follows that for $\tau<t$ the particle is "out of Ω" and its motion is described by the equations

$$\frac{dX^n}{d\tau}=V^n,\quad \frac{dV^n}{d\tau}=-\frac{\beta}{r^2}\theta_R((V^n)^2)V^n,$$
$$X^n|_{\tau=t}=x,\quad V^n|_{\tau=t}=v,\quad 0\leq\tau\leq t.$$

Thus, the trajectory of the particle is a straight line if $\tau\in[0,t]$. Therefore, $Pf_0(X^n(x_0,v,t,0),V^n(x_0,v,t,0),r)=0$ and, due to (3.17), the desired boundary condition (3.21) holds.

Now we find the coefficients $C_{nl}(t)$ in (3.11). To this end, we assume that identity (3.1.) holds for any $\zeta(x,t) = H(t)\Psi^j(x)$, $j = 1,2,\dots,n$. Here $H \in C^1(0,T)$ with $H(T) = 0$. This assumption implies that

$$\left(\frac{\partial u^n}{\partial t} + (u^n\nabla_x)u^n + \alpha \int\limits_a^b\int\limits_{\mathbb{R}^3} r\theta_R((u^n - v)^2)(u^n - v)S\tilde{f}^n dvdr, \Psi^k\right)_{2,\Omega}$$

$$+\nu(u^n, \Psi^k)_{J^1(\Omega)} = (g, \Psi^k)_{2,\Omega}, \quad k = 1,2,\dots,n. \tag{3.22}$$

One can represent (3.22) as a system of differential functional equations

$$\frac{dC_{nk}}{dt} + \sum_{l,m=1}^{n} \beta^k_{lm} C_{nl}(t)C_{nm}(t) + \sum_{l=1}^{n} \varepsilon^k_l C_{nl}(t)$$

$$+\alpha\left(\int\limits_a^b\int\limits_{\mathbb{R}^3} r\theta_R\left(\left(\sum_{l=1}^{n} C_{nl}(t)\Psi^l - v\right)^2\right)\left(\sum_{l=1}^{n} C_{nl}(t)\Psi^l - v\right) S\tilde{f}^n dvdr, \Psi^k\right)_{2,\Omega} = g^k, k = \overline{1,n}, \tag{3.23}$$

where

$$\beta^k_{lm} = ((\Psi^l\nabla)\Psi^m, \Psi^k)_{2,\Omega}, \quad \varepsilon^k_l = \nu(\Psi^l, \Psi^k)_{J^1(\Omega)}, \quad g^k = (g, \Psi^k)_{2,\Omega}.$$

Now (3.23) defines the coefficient $C_{nl}(t)$. This system has to be completed by the initial conditions for $C_{nl}(t)$. Expanding the function $u_0(x)$ into a series in the basis $\Psi^k(x)$, i.e., $u_0(x) = \sum_{k=1}^{\infty} C_k\Psi^k(x)$, we obtain the initial conditions

$$C_{nk}(0) = C_k, \quad k = 1,2,\dots,n. \tag{3.24}$$

3.3. *A priori* estimates of approximations

Lemma 3. *The following estimates hold:*

$$\sup_{\mathbb{R}^6_T\times[a,b]} \tilde{f}^n \le A, \tag{3.25a}$$

$$\int\limits_a^b\int\limits_{\mathbb{R}^6} \tilde{f}^n(x,v,r,t)dxdvdr \le \int\limits_a^b\int\limits_{Q} f_0(x,v,r)dxdvdr, \tag{3.25b}$$

$$\max_{0\le t\le T} \|u^n(t)\|^2_{2,\Omega} + \max_{0\le t\le T} \int\limits_a^b \int\limits_{\mathbb{R}^3} v^2\tilde{f}^n dxdvdr + \int\limits_0^T \|u^n(t)\|^2_{J^1(\Omega)}dt$$

$$+\int_0^T\int_a^b\int_{\mathbb{R}^3}\theta_R((Pu^n-v)^2)(Pu^n-v)^2\tilde{f}^n dxdvdrdt$$

$$+\varepsilon\max_{0\le t\le T}\|\Delta\varphi^n(t)\|^2_{2,\Omega}+\max_{0\le t\le T}\|\nabla\varphi^n(t)\|^2_{2,\Omega}\le A. \tag{3.25c}$$

Here A is a constant that depends on u_0, f_0, g, α, β, ν, and T only.

P r o o f. Using the boundedness of the functions $f_0^n(x,v,r)$ and the definition of $\theta_R(z)$, one obtains inequality (3.25a) from (3.17).

Let us prove (3.25b). To this end, we integrate equation (3.14) over $\mathbb{R}^6\times[a,b]$. Since $\tilde{f}^n$ has a compact support in $(x,v)\in\mathbb{R}^6$, then we get

$$\frac{d}{dt}\int_a^b\int_{\mathbb{R}^6}\tilde{f}^n dxdvdr=0,$$

and inequality (3.25b) is proved.

Now we prove (3.25c). We multiply the k-th equation of system (3.22) by $C_{nk}(t)$ and summarize over $k=1,2,\ldots,n$. Then we extend the vector-functions u^n and g by zero to the whole $\mathbb{R}^3$. This leads to the following equation:

$$\frac{1}{2}\frac{d}{dt}\|Pu^n\|^2_{2,\mathbb{R}^3}+\nu\|Pu^n\|^2_{J^1(\mathbb{R}^3)}$$

$$+\alpha\left(\int_a^b\int_{\mathbb{R}^3}r\theta_R((Pu^n-v)^2)(Pu^n-v)\tilde{f}^n dvdr, Pu^n\right)_{2,\mathbb{R}^3}=(Pg,Pu^n)_{2,\mathbb{R}^3}. \tag{3.26}$$

We multiply (3.14) by $\frac{\alpha v^2 r^3}{2\beta}$ and integrate over $\mathbb{R}^6\times[a,b]$. We add this equation to (3.26) and get

$$\frac{1}{2}\frac{d}{dt}\|Pu^n\|^2_{2,\mathbb{R}^6}+\nu\|Pu^n\|^2_{J^1(\mathbb{R}^3)}+\alpha\int_a^b\int_{\mathbb{R}^6}r\theta_R((Pu^n-v)^2)(Pu^n-v)^2\tilde{f}^n dxdvdr$$

$$+\frac{\alpha}{2\beta}\frac{d}{dt}\int_a^b\int_{\mathbb{R}^6}r^3v^2\tilde{f}^n dxdvdr+\frac{\alpha\gamma}{\beta}\int_a^b\int_{\mathbb{R}^6}r(\nabla P\varphi^n,v)\tilde{f}^n dxdvdr$$

$$-\frac{\alpha}{\beta}\int_a^b\int_{\mathbb{R}^6}r^3\tilde{f}^n(v,Pg_\varepsilon)dxdvdr=(Pg,Pu^n)_{2,\mathbb{R}^3}. \tag{3.27}$$

Now we differentiate (3.12) with respect to the time variable, multiply it by $\varphi^n(x,t)$ and integrate over Ω. Then, taking into account the boundary conditions (3.13), we obtain

$$\frac{\varepsilon}{2}\frac{d}{dt}\int\limits_{\Omega}(\Delta\varphi^n)^2dx+\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}|\nabla\varphi^n|^2dx=q\int\limits_a^b\!\!\int\limits_Q r\varphi^n(x,t)S\frac{\partial\tilde f^n}{\partial t}dxdvdr.$$

We extend the function $\varphi^n(x,t)$ by zero outside of Ω and due to (3.14) we may deduce that

$$\varepsilon\frac{d}{dt}\int\limits_{\mathbb{R}^3}(\Delta P\varphi^n)^2dx+\frac{d}{dt}\int\limits_{\mathbb{R}^3}|\nabla P\varphi^n|^2dx=q\int\limits_a^b\!\!\int\limits_{\mathbb{R}^6} r(\nabla P\varphi^n,v)\tilde f^n dxdvdr.$$

We multiply this equation by $\frac{\alpha\gamma}{\beta q}$ and insert it into (3.27). Then we have

$$\frac{1}{2}\frac{d}{dt}\|Pu^n\|^2_{2,\mathbb{R}^3}+\nu\|Pu^n\|^2_{J^1(\mathbb{R}^3)}+\alpha\int\limits_a^b\!\!\int\limits_{\mathbb{R}^6} r\theta_R((Pu^n-v)^2)(Pu^n-v)^2\tilde f^n dxdvdr$$

$$+\frac{\alpha}{2\beta}\frac{d}{dt}\int\limits_a^b\!\!\int\limits_{\mathbb{R}^6} r^3v^2\tilde f^n dxdvdr+\frac{\varepsilon\alpha\gamma}{2\beta q}\frac{d}{dt}\int\limits_{\mathbb{R}^3}(\Delta P\varphi^n)^2dx$$

$$+\frac{\alpha\gamma}{2\beta q}\frac{d}{dt}\int\limits_{\mathbb{R}^3}|\nabla P\varphi^n|^2dx=(Pg,Pu^n)_{2,\mathbb{R}^3}+\frac{\alpha}{\beta}\int\limits_a^b\!\!\int\limits_{\mathbb{R}^6} r^3(v,Pg_\varepsilon)\tilde f^n dxdvdr. \quad (3.28)$$

Let us estimate the right-hand side of (3.28). From Cauchy's inequality we get

$$(Pg,Pu^n)_{2,\mathbb{R}^3}+\frac{\alpha}{\beta}\int\limits_a^b\!\!\int\limits_{\mathbb{R}^6} r^3(v,Pg_\varepsilon)\tilde f^n dxdvdr$$

$$\le\frac{\delta}{2}\|Pu^n\|^2_{2,\mathbb{R}^3}+\frac{1}{2\delta}\|g\|^2_{2,\Omega}+\frac{\delta\alpha}{2\beta}\int\limits_a^b\!\!\int\limits_{\mathbb{R}^6} r^3v^2\tilde f^n dxdvdr+\frac{\alpha}{2\delta\beta}\int\limits_a^b\!\!\int\limits_{\mathbb{R}^6} r^3\tilde f^n(Pg_\varepsilon)^2dxdvdr,$$

where δ is an arbitrary positive constant.

Let us integrate (3.28) with respect to t. Taking into account the previous bound, we have

$$\frac{1}{2}\|Pu^n(t)\|^2_{2,\mathbb{R}^3}+\nu\int\limits_0^t\|Pu^n(\tau)\|^2_{J^1(\mathbb{R}^3)}d\tau$$

$$+\alpha\int_0^t\iint_{a\mathbb{R}^6}^{b} r\theta_R((Pu^n-v)^2)(Pu^n-v)^2\tilde f^n dxdvdrd\tau+\frac{\alpha}{2\beta}\iint_{a\mathbb{R}^6}^{b} r^3v^2\tilde f^n(x,v,r,t)dxdvdr$$

$$+\frac{\varepsilon\alpha\gamma}{2\beta q}\int_{\mathbb{R}^3}(\Delta P\varphi^n(x,t))^2dx+\frac{\alpha\gamma}{2\beta q}\int_{\mathbb{R}^3}|\nabla P\varphi^n(x,t)|^2dx$$

$$\le\frac{1}{2}\|u_0\|^2_{2,\Omega}+\frac{\delta}{2}\int_0^t\|Pu^n(x,\tau)\|^2_{2,\mathbb{R}^3}d\tau+\frac{1}{2\delta}\int_0^t\|g\|^2_{2,\Omega}d\tau$$

$$+\frac{\delta\alpha}{2\beta}\int_0^t\int_a^b\int_{\mathbb{R}^6} r^3v^2\tilde f^n(x,v,r,\tau)dxdvdrd\tau+\sum_{i=1}^3 I_i, \tag{3.29}$$

where

$$I_1\equiv\frac{\alpha}{2\delta\beta}\int_0^t\int_a^b\int_{\mathbb{R}^6} r^3\tilde f^n(x,v,r,\tau)(Pg_\varepsilon)^2dxdvdrd\tau,$$

$$I_2\equiv\frac{\alpha}{2\beta}\iint_{a\mathbb{R}^6}^{b} r^3v^2Pf_0^n(x,v,r)dxdvdr,$$

$$I_3\equiv\frac{\varepsilon\alpha\gamma}{2\beta q}\int_{\mathbb{R}^3}(\Delta P\varphi^n(x,0))^2dx+\frac{\alpha\gamma}{2\beta q}\int_{\mathbb{R}^3}|\nabla P\varphi^n(x,0)|^2dx.$$

Now we estimate I_i, $i=1,2,3$.

From (3.25b) and the definition of $g_\varepsilon(x,t)$ we get

$$I_1\le C\|g\|^2_{L_\infty(0,T;C^1(\Omega))}.$$

According to Lemma 1, I_2 is uniformly bounded in n by the constant $\tilde C$.

To estimate I_3, we consider equation (3.12) for $t=0$:

$$\varepsilon\Delta^2\varphi^n(x,0)-\Delta\varphi^n(x,0)=q\iint_{a\mathbb{R}^3}^{b} rf_0^n(x,v,r)dvdr. \tag{3.30}$$

It follows from Lemma 1 that the right-hand side of (3.30) belongs to the space $L_p(\Omega)$ with $p\in(1,\frac{5}{3})$. In fact

$$\int_\Omega\left(\iint_{a\mathbb{R}^3}^{b} rf_0^n(x,v,r)dvdr\right)^p dx\le b^p\int_\Omega\left(\iint_{a\mathbb{R}^3}^{b}\frac{1}{(1+v^2)^{1/p}}(1+v^2)^{1/p}f_0^n(x,v,r)dvdr\right)^p dx.$$

From Hölder's inequality we conclude

$$\int\limits_{\Omega}\left(\iint\limits_{a\mathbb{R}^3}^{b} r f_0^n(x,v,r)dvdr\right)^p dx \le b^p \int\limits_{\Omega}\left(\iint\limits_{a\mathbb{R}^3}^{b} \frac{dvdr}{(1+v^2)^{q/p}}\right)^{p/q}$$

$$\times\left(\iint\limits_{a\mathbb{R}^3}^{b}(1+v^2)[f_0^n(x,v,r)]^p dvdr\right)dx$$

$$= b^p\left(\iint\limits_{a\mathbb{R}^3}^{b}\frac{dvdr}{(1+v^2)^{q/p}}\right)^{p/q}\int\limits_a^b\int\limits_Q(1+v^2)[f_0^n(x,v,r)]^p dxdvdr < C_1.$$

The factor $\iint\limits_{a\mathbb{R}^3}^{b}\frac{dvdr}{(1+v^2)^{q/p}}$ is bounded for $p\in(1,\frac{5}{3})$. Hence, C_1 is a constant that does not depend on n.

We multiply (3.30) by $\varphi^n(x,0)$ and integrate the resulting equation over Ω. We have

$$\varepsilon\int\limits_{\Omega}(\Delta\varphi^n(x,0))^2dx + \int\limits_{\Omega}|\nabla\varphi^n(x,0)|^2dx = q\int\limits_{\Omega}\int\limits_a^b\int\limits_{\mathbb{R}^3} r f_0^n(x,v,r)\varphi^n(x,0)dvdrdx.$$

From Hölder's inequality we get

$$q\int\limits_{\Omega}\iint\limits_{a\mathbb{R}^3}^{b} r f_0^n(x,v,r)\varphi^n(x,0)dvdrdx \le \left\|\iint\limits_{a\mathbb{R}^3}^{b} r f_0^n(x,v,r)dvdr\right\|_{L_p(\Omega)}\|\varphi^n(x,0)\|_{L_q(\Omega)}.$$

As it was shown above, the first term on the right-hand side of this inequality is uniformly bounded in n. To estimate the second term we make use of the embedding theorem and Friedrich's inequality

$$\varepsilon\int\limits_{\Omega}(\Delta\varphi^n(x,0))^2dx + \int\limits_{\Omega}|\nabla\varphi^n(x,0)|^2dx \le C_1^{1/p}\|\varphi^n(x,0)\|_{L_q(\Omega)}$$

$$\le C_2\|\varphi^n(x,0)\|_{W_2^1(\Omega)} \le C_3\|\nabla\varphi^n(x,0)\|_{L_2(\Omega)}.$$

Thus, $\|\nabla\varphi^n(x,0)\|^2_{L_2(\Omega)} \le C_3\|\nabla\varphi^n(x,0)\|_{L_2(\Omega)}$ or $\|\nabla\varphi^n(x,0)\|_{L_2(\Omega)} \le C_3$, and we obtain

$$I_3 \le \frac{\alpha\gamma}{2\beta}C_3^2 \equiv \hat{C},$$

where $\hat{C}$ is a constant that does not depend on n.

Thus, from (3.29), for any $t \in [0,T]$, we conclude

$$\frac{1}{2}\|Pu^n(t)\|^2_{2,\mathbb{R}^3} + \nu \int\limits_0^t \|Pu^n(\tau)\|^2_{J^1(\mathbb{R}^3} d\tau$$

$$+\alpha \int\limits_0^t \int\limits_a^b\int\limits_{\mathbb{R}^6} r\theta_R((Pu^n - v)^2)(Pu^n - v)^2 \tilde{f}^n dxdvdrd\tau$$

$$+\frac{\alpha}{2\beta}\int\limits_a^b\int\limits_{\mathbb{R}^6} r^3 v^2 \tilde{f}^n dxdvdr + \frac{\varepsilon\alpha\gamma}{2\beta q}\int\limits_{\mathbb{R}^3}(\Delta P\varphi^n(x,t))^2 dx + \frac{\alpha\gamma}{2\beta q}\int\limits_{\mathbb{R}^3} |\nabla P\varphi^n(x,t)|^2 dx$$

$$\leq \frac{1}{2}\|u_0\|^2_{2,\Omega} + \frac{\delta T}{2}\max_{0\leq t\leq T}\|Pu^n(t)\|^2_{2,\mathbb{R}^3} + \frac{T}{2\delta}\|g\|^2_{L_\infty(0,T;C^1(\Omega))}$$

$$+\frac{\alpha b^3\delta T}{2\beta}\max_{0\leq t\leq T}\int\limits_a^b\int\limits_{\mathbb{R}^6} v^2\tilde{f}^n(x,v,r,t)dxdvdr + C\|g\|^2_{L_\infty(0,T;C^1(\Omega))} + \tilde{C} + \hat{C}.$$

Therefore,

$$\frac{1}{2}\max_{0\leq t\leq T}\|Pu^n(t)\|^2_{2,\mathbb{R}^3} + \alpha a \int\limits_0^T\int\limits_a^b\int\limits_{\mathbb{R}^6} \theta_R((Pu^n - v)^2)(Pu^n - v)^2\tilde{f}^n dxdvdrdt$$

$$+\nu\int\limits_0^T \|Pu^n\|^2_{J^1(\mathbb{R}^3)}dt + \frac{a^3\alpha}{2\beta}\max_{0\leq t\leq T}\int\limits_a^b\int\limits_{\mathbb{R}^6} v^2\tilde{f}^n(x,v,r,t)dxdvdr$$

$$+\frac{\varepsilon\alpha\gamma}{2\beta q}\max_{0\leq t\leq T}\int\limits_{\mathbb{R}^3}(\Delta P\varphi^n(x,t))^2dx + \frac{\alpha\gamma}{2\beta q}\max_{0\leq t\leq T}\int\limits_{\mathbb{R}^3}|\nabla P\varphi^n(x,t)|^2dx$$

$$\leq \frac{1}{2}\|u_0\|^2_{2,\Omega} + \frac{\delta T}{2}\max_{0\leq t\leq T}\|Pu^n(t)\|^2_{2,\mathbb{R}^3} + \frac{T}{2\delta}\|g\|^2_{L_\infty(0,T;C^1(\Omega))}$$

$$+\frac{\alpha b^3\delta T}{2\beta}\max_{0\leq t\leq T}\int\limits_a^b\int\limits_{\mathbb{R}^6} v^2\tilde{f}^n(x,v,r,t)dxdvdr + C\|g\|^2_{L_\infty(0,T;C^1(\Omega))} + \tilde{C} + \hat{C}.$$

We take δ such that $\delta < \frac{1}{2T}\left(\frac{a}{b}\right)^3$. Now the desired inequality (3.25c) immediately follows from the last bound and Lemma 3 is proved. ■

3.4. Existence result for the approximations $(u^n, \varphi^n, \tilde{f}^n)$

Lemma 4. *For any $n = 1, 2, \ldots$ and any $R > 0$, $\varepsilon > 0$ there exists a unique solution $(u^n, \varphi^n, \tilde{f}^n)$ of problem (3.11)–(3.15), (3.23), (3.24).*

P r o o f. Let $C(0,T)$ be the space of vector functions $e(t) = (e_1(t), \ldots, e_n(t))$ continuous on $[0,T]$. This space is equipped with a norm $|e| = \max\limits_{0\le t\le T}\left[\sum\limits_{i=1}^{n} e_i^2(t)\right]^{1/2}$.
We take $\varphi \in L_2(0,T;C^2(\Omega))$ and denote by $w = (e_1(t), \ldots, e_n(t), \varphi(x,t))$. Then $w \in C(0,T) \oplus L_2(0,T;C^2(\Omega))$. The norm in this space is given by $|w| = |e| + \left(\int\limits_0^T \|\varphi(t)\|^2_{C^2(\Omega)}dt\right)^{1/2}$.

Let K be a bounded closed convex set in $C(0,T) \oplus L_2(0,T;C^2(\Omega))$:

$$K = \{w : |w| \le C_{R,\varepsilon}, e_i(0) = C_i, i = 1, 2, \ldots, n; \varphi(x,t) = \frac{\partial\varphi(x,t)}{\partial n} = 0, (x,t) \in S_T\}.$$

Here $C_{R,\varepsilon}$ is a constant which will be specified later and C_i are the coefficients defined in (3.24).

Let $w^0 = (e_1^0(t), e_2^0(t), \ldots, e_n^0(t), \varphi^0(x,t))$ be an arbitrary element of K.

We set

$$q^0(x,t) = \sum_{i=1}^{n} e_i^0 \Psi^i$$

and consider the problem

$$\frac{\partial\tilde{f}}{\partial t} + (v\nabla_x)\tilde{f} + \mathrm{div}_v\left\{\left[\frac{\beta}{r^2}\theta_R((Pq^0 - v)^2)(Pq^0 - v)\right.\right.$$
$$\left.\left. - \frac{\gamma}{r^2}\nabla(P\varphi^0) + Pg_\varepsilon(x,t)\right]\tilde{f}\right\} = 0, \tag{3.31}$$
$$\tilde{f}|_{t=0} = Pf_0^n(x,v,r).$$

The existence and uniqueness of the solution $\tilde{f}$ to this problem follows from the regular properties of the functions q^0, φ^0 and g_ε. More precisely, $q^0 \in C(0,T;C^1(\Omega))$, $\varphi^0 \in C^2(\Omega) \cap C_0^1(\Omega)$, $g_\varepsilon \in L_\infty(0,T;C^1(\Omega))$.

Define the vector $q^1 = \sum\limits_{i=1}^{n} e_i^1 \Psi^i$ as a solution of the system of ordinary differential equations

$$\left(\frac{\partial q^1}{\partial t} + (q^0\nabla)q^1 + \alpha \int\limits_a^b\int\limits_{\mathbb{R}^3} r\theta_R((q^0 - v)^2)(q^0 - v)S\tilde{f}dvdr, \Psi^k\right)_{2,\Omega}$$

$$+\nu(q^1,\Psi^k)_{J^1(\Omega)}=(g,\Psi^k)_{2,\Omega},\quad k=1,2,\dots,n. \tag{3.32}$$

This system is a linearization of system (3.23) and can be rewritten as

$$\frac{de_k^1}{dt}+\sum_{j,l=1}^n\beta_{jl}^k e_j^0 e_l^1+\sum_{l=1}^n\varepsilon_l^k e_l^1=g^k-\alpha\left(\int_a^b\!\!\int_{\mathbb{R}^3} r\theta_R\left(\left(\sum_{l=1}^n e_l^0\psi^l-v\right)^2\right)\right.$$

$$\left.\times\left(\sum_{l=1}^n e_l^0\Psi^l-v\right)S\tilde f dvdr,\Psi^k\right)_{2,\Omega},\quad k=1,2,\dots,n. \tag{3.33}$$

We complete (3.33) by the following initial data:

$$e_k^1(0)=C_k=(u_0,\Psi^k)_{2,\Omega},\quad k=1,2,\dots,n. \tag{3.34}$$

The linear problem (3.33), (3.34) has a unique solution $\{e_k^1(t),k=1,\dots,n\}$.

Define $\varphi^1(x,t)$ as a solution of the problem:

$$\varepsilon\Delta^2\varphi^1-\Delta\varphi^1=q\int_a^b\!\!\int_{\mathbb{R}^3} rS\tilde f(x,v,r,t)dvdr, \tag{3.35}$$

$$\varphi^1(x,t)=\frac{\partial\varphi^1(x,t)}{\partial n}=0,\quad (x,t)\in S_T. \tag{3.36}$$

As in the case of equation (3.30), one can conclude that the right-hand side of equation (3.35) belongs to the space $L_p(\Omega)$ with $p\in(1;\frac{5}{3})$, uniformly on $t\in[0,T]$. Therefore, there exists a unique generalized solution of problem (3.35), (3.36) (see [19]) satisfying the inequality

$$\|\varphi^1\|_{W_p^4(\Omega)}\le C_\varepsilon\max_{0\le t\le T}\left\|q\int_a^b\int_{\mathbb{R}^3} rS\tilde f(x,v,r,t)dvdr\right\|_{L_p(\Omega)}\le\tilde C_{R,\varepsilon}. \tag{3.37}$$

Taking p from the interval $\left(\frac{3}{2};\frac{5}{3}\right)$ and using the embedding theorem, we conclude that $\varphi^1(x,t)\in C^2(\Omega)$ and

$$\|\varphi^1(t)\|_{C^2(\Omega)}\le C\|\varphi^1(t)\|_{W_p^4(\Omega)}\le C_{R,\varepsilon}^{(1)}. \tag{3.38}$$

Thus, the vector $w^1=(q^1,\varphi^1)$ may be defined as $w^1=\Lambda w^0$, where $w^0\in K$ and Λ is an operator from K to $C(0,T)\oplus L_2(0,T;C^2(\Omega))$. The fixed points of this operator together with the corresponding functions $\tilde f$ give the solution of problem (3.11)–(3.15), (3.23), (3.24).

Now we prove that the operator Λ maps the set K into itself. To this end, we have to prove that $|w^1| \le C_{R,\varepsilon}$ or

$$\max_{0\le t\le T} \|q^1\|_{J(\Omega)} + \left(\int_0^T \|\varphi^1(t)\|^2_{C^2(\Omega)} dt\right)^{1/2} \le C_{R,\varepsilon}. \tag{3.39}$$

Let us prove (3.39). We multiply the k-th equation in (3.32) by $e^1_k(t)$ and summarize over $k = \overline{1,n}$. We get

$$\frac{1}{2}\frac{d}{dt}\|q^1\|^2_{2,\Omega} + \nu\|q^1\|^2_{J^1(\Omega)} = (g, q^1)_{2,\Omega} - \alpha\left(\int_a^b\int_{\mathbb{R}^3} r\theta_R((q^0-v)^2)(q^0-v)S\tilde{f}dvdr, q^1\right)_{2,\Omega}.$$

To estimate the second term in the right-hand side of this equation we make use of (3.37), the definition of $\theta_R(z)$ and the embedding of the space $L_s(\Omega)$ with $s \in [2,6]$ into $J^1(\Omega)$. We have

$$\left|\alpha\left(\int_a^b\int_{\mathbb{R}^3} r\theta_R((q^0-v)^2)(q^0-v)S\tilde{f}dvdr, q^1\right)_{2,\Omega}\right|$$

$$\le \alpha\sqrt{2R}b\int_\Omega |q^1(x,t)| \int_a^b\int_{\mathbb{R}^3} S\tilde{f}(x,v,r,t)dvdrdx$$

$$\le \alpha b\sqrt{2R}\left\{\int_\Omega\left[\int_a^b\int_{\mathbb{R}^3} S\tilde{f}dvdr\right]^p dx\right\}^{1/p}\left\{\int_\Omega |q^1(x,t)|^s dx\right\}^{1/s} \le \hat{C}_{R,\varepsilon}\|q^1\|_{J^1(\Omega)},$$

where $p \in (\frac{3}{2}, \frac{5}{3})$, $s \in (\frac{5}{2}, 3)$, and $\frac{1}{p} + \frac{1}{s} = 1$.

As in the case of the proof of (3.25c), one can obtain

$$\frac{1}{4}\max_{0\le t\le T}\|q^1(t)\|^2_{2,\Omega} + \frac{\nu}{2}\|q^1\|^2_{L_2(0,T;J^1(\Omega))} \le \frac{1}{2}\|u_0\|^2_{2,\Omega}$$

$$+C\left[T^3\|g\|^2_{L_\infty(0,T;C^1(\Omega))} + \hat{C}^2_{R,\varepsilon}T^2\right] \equiv \frac{1}{4}[C^{(2)}_{R,\varepsilon}]^2.$$

Hence, we get

$$\max_{0\le t\le T}\|q^1(t)\|_{2,\Omega} \le C^{(2)}_{R,\varepsilon}.$$

An estimate for the second term in (3.39) follows from (3.38). Namely,

$$\left(\int\limits_0^T \|\varphi^1(t)\|^2_{C^2(\Omega)}dt\right)^{1/2} \le C^{(1)}_{R,\varepsilon}\sqrt{T},$$

and, hence, $|w^1| \le C^{(1)}_{R,\varepsilon}\sqrt{T} + C^{(2)}_{R,\varepsilon} \equiv C_{R,\varepsilon}$.

Now we show that Λ, Λ: $K \to K$, is a compact operator and estimate the derivative $\dfrac{dw^1}{dt}$. Multiplying the k-th equation in (3.32) by $\dfrac{de^1_k}{dt}$ and summarizing over $k = \overline{1,n}$, we get

$$\|q^1_t\|^2_{2,\Omega} + ((q^0\nabla)q^1, q^1_t)_{2,\Omega} + \frac{\nu}{2}\frac{d}{dt}\|q^1\|^2_{J^1(\Omega)}$$

$$+\alpha\left(\int\limits_a^b\int\limits_{\mathbb{R}^3} r\theta_R((q^0-v)^2)(q^0-v)S\tilde{f}dvdr, q^1_t\right)_{2,\Omega} = (g, q^1_t)_{2,\Omega}.$$

Then we obtain

$$\|q^1_t\|^2_{2,\Omega} + \frac{\nu}{2}\frac{d}{dt}\|q^1\|^2_{J^1(\Omega)}$$

$$\le \|q^1_t\|_{2,\Omega}\left[\|q^0\|_{C(\Omega)}\|q^1\|_{J^1(\Omega)} + \|g\|_{L_\infty(0,T;C^1(\Omega))}(\mathrm{mes}\Omega)^{1/2} + C\right],$$

where $C \equiv \alpha b\sqrt{2R}A(b-a)\frac{4}{3}\pi R^3_{\tilde{f}}(\mathrm{mes}\Omega)^{1/2}$, A is the constant defined in (3.25a), and $R_{\tilde{f}}$ is defined in Lemma 2. We observe that the functions Ψ^k are smooth and $\|q^0\|_{C(\Omega)} \le C_n$. Then, applying Young's inequality and integrating with respect to t, we get

$$\int\limits_0^T \|q^1_t\|^2_{2,\Omega}dt \le C_n.$$

This gives $\|e^1\|^2_{W^1_2(0,T)} \le C_n$. Therefore, the function $e^1 \in W^1_2(0,T)$. Moreover, $W^1_2(0,T)$ is compactly embedded in $C(0,T)$ [20].

To complete the proof of the compactness of the operator Λ we make use of the following Lemma (see [21]):

Lemma. *Let B_0, B and B_1 be Banach spaces such that $B_0 \subset B \subset B_1$. B_0 and B_1 are reflexive, and the embedding of B_0 in B_1 is compact. Consider the Banach space*

$$W = \left\{v : v \in L_{p_0}(0,T;B_0), v' = \frac{dv}{dt} \in L_{p_1}(0,T;B_1)\right\},$$

where $0 < T < +\infty$ *is fixed and* $1 < p_i < \infty$, $i = 0, 1$. *The norm in the space* W *is defined by* $\|v\|_{L_{p_0}(0,T;B_0)} + \|v'\|_{L_{p_1}(0,T;B_1)}$.

Then the embedding of W *in* $L_{p_0}(0, T; B)$ *is compact.*

This lemma implies that the Banach space

$$W = \{\varphi(x,t) : \varphi(x,t) \in L_2(0,T; W_p^4(\Omega)), \varphi_t' \in L_2(0,T; L_2(\Omega))\}$$

with the norm $\|\varphi\|_{L_2(0,T;W_p^4(\Omega))} + \|\varphi_t'\|_{L_2(0,T;L_2(\Omega))}$ is compactly embedded in $L_2(0, T; C^2(\Omega))$.

Therefore, it remains to prove that $\dfrac{\partial \varphi^1}{\partial t} \in L_2(0, T; L_2(\Omega))$.

By differentiating equations (3.35) and (3.36) with respect to t, we obtain the following problem for φ_t^1:

$$\varepsilon\Delta^2\varphi_t^1 - \Delta\varphi_t^1 = q \int_a^b\!\!\int_{\mathbb{R}^3} rS\frac{\partial \tilde{f}(x,v,r,t)}{\partial t} dvdr, \quad (x,t) \in \Omega_T, \tag{3.40}$$

$$\varphi_t^1 = \frac{\partial \varphi_t^1}{\partial n} = 0, \quad (x,t) \in S_T. \tag{3.41}$$

Using equation (3.31) for the function $\tilde{f}$, we can rewrite (3.40) in the form

$$\varepsilon\Delta^2\varphi_t^1 - \Delta\varphi_t^1 = -q \int_a^b\!\!\int_{\mathbb{R}^3} r(v\nabla_x)S\tilde{f}dvdr.$$

Multiplying this equation by φ_t^1, integrating it over Ω and taking into account (3.41), we obtain

$$\varepsilon \int_\Omega (\Delta\varphi_t^1)^2 dx + \int_\Omega |\nabla\varphi_t^1|^2 dx = q \int_\Omega \int_a^b \int_{\mathbb{R}^3} r(v, \nabla\varphi_t^1)S\tilde{f}dvdrdx.$$

Let us estimate the right-hand side of this equation. Using Lemmas 2, 3, we get

$$q \int_\Omega \int_a^b\!\!\int_{\mathbb{R}^3} r(v, \nabla\varphi_t^1)S\tilde{f}dvdrdx$$

$$\leq qb \left\{ \int_\Omega \int_a^b\!\!\int_{\mathbb{R}^3} v^2 S\tilde{f}dvdrdx \right\}^{1/2} \left\{ \int_\Omega \int_a^b\!\!\int_{\mathbb{R}^3} S\tilde{f}|\nabla\varphi_t^1|^2 dvdrdx \right\}^{1/2}$$

$$\leq qb\sqrt{A}\left(A(b-a)\frac{4}{3}\pi R_{\tilde f}^3\right)^{1/2}\left\{\int\limits_{\Omega}|\nabla\varphi_t^1|^2dx\right\}^{1/2}$$

$$=\bar C_{R,\varepsilon}\left\{\int\limits_{\Omega}|\nabla\varphi_t^1|^2dx\right\}^{1/2}\leq\frac{1}{2}\int\limits_{\Omega}|\nabla\varphi_t^1|^2dx+\frac{1}{2}\bar C_{R,\varepsilon}^2.$$

Then,

$$\varepsilon\int\limits_{\Omega}(\Delta\varphi_t^1)^2dx+\int\limits_{\Omega}|\nabla\varphi_t^1|^2dx\leq\frac{1}{2}\int\limits_{\Omega}|\nabla\varphi_t^1|^2dx+\frac{1}{2}\bar C_{R,\varepsilon}^2,$$

and, therefore

$$\max_{0\leq t\leq T}\int\limits_{\Omega}|\nabla\varphi_t^1|^2dx\leq\bar C_{R,\varepsilon}^2.$$

Taking into account (3.41), we conclude that $\varphi_t^1\in L_2(0,T;W_2^1(\Omega))$. Thus, it is proved that the image $\Lambda[K]$ of the set K is a compact set in $C(0,T)\oplus L_2(0,T;C^2(\Omega))$. The continuity of the operator Λ follows from the continuous dependence of the solutions of (3.33) on the initial data, the coefficients and the right-hand sides; the continuous dependence of the solution of (3.14) on the coefficients that follows from (3.16) and (3.17); the *a priori* estimate of the right-hand side of (3.12) and the embedding theorem of $W_p^4(\Omega))$ $(p\in(\frac{3}{2};\frac{5}{3}))$ in $C^2(\Omega)$.

Schauder's theorem implies that the operator Λ has a fixed point in K. We denote it by $w=(e_1(t),\ldots,e_n(t),\varphi(x,t))$.

The proof of the uniqueness of the solution of (3.11)–(3.14), (3.23), (3.24) is carried out in a standard way. Lemma 4 is proved. ■

4. Convergence Properties of the Approximations $(u^n,\varphi^n,\tilde f^n)$

Due to the *a priori* estimates (3.25a), (3.25c) one can extract subsequences $\{u^n\}$, $\{\varphi^n\}$, and $\{\tilde f^n\}$ (still denoted by n) such that:

$u^n\to u$ $*$-weakly in $L_\infty(0,T;J(\Omega))$ and weakly in $L_2(0,T;J^1(\Omega))$;

$\tilde f^n\to\tilde f$ $*$-weakly in $L_\infty(\mathbb{R}_T^6\times[a,b])$;

$\varphi^n\to\varphi$ $*$-weakly in $L_\infty(0,T;W_2^1(\Omega))$.

These types of convergence are not sufficient to pass to the limit as $n\to\infty$. Therefore, in Lemmas 5, 7, and 9 the additional properties of sequences $\{\tilde f^n\}$ and $\{u^n\}$ will be obtained. In Lemmas 6 and 8 we show that the limit functions $\tilde f(x,v,r,t)$ and $u(x,t)$ satisfy the conditions from the definition of weak solution and Theorem 1. The additional properties of the solutions of problems (1.3), (1.7) and (3.3), (1.7), (3.6) are established in Lemma 10 proved in [22].

Lemma 5. *There exists a subsequence $\{\tilde f^n\}$ that converges uniformly with respect to $t\in[0,T]$ in the weak topology of $L_2(\mathbb{R}^6\times[a,b])$.*

P r o o f. We denote by $\{g_i(x,v,r)\}$ an orthonormal total sequence of smooth functions with compact support in $L_2(\mathbb{R}^6 \times [a,b])$. Let us consider the sequence

$$\alpha_{ni}(t) = \int_a^b\int_{\mathbb{R}^6} \tilde{f}^n(x,v,r,t) g_i(x,v,r)\,dxdvdr, \quad i = 1,2,\ldots .$$

Due to estimates (3.25a), (3.25b) this sequence is bounded for any fixed i uniformly in n. Moreover, from (3.14), (3.25a) and (3.25b) after simple rearrangements we get

$$\left|\frac{d\alpha_{ni}(t)}{dt}\right| = \left|\int_a^b\int_{\mathbb{R}^6} g_i(x,v,r)\frac{\partial \tilde{f}^n}{\partial t}dxdvdr\right| \leq \tilde{C}_i\left(1 + \|u^n\|_{2,\Omega} + \|\nabla\varphi^n\|_{2,\Omega}\right).$$

This estimate, along with (3.25c), implies that the sequence $\{\alpha_{ni}(t)\}$ is equicontinuous for any i, $i = 1,2,\ldots$. One can extract a subsequence that converges uniformly in t from any fixed interval $(0,T]$ and for any i. We keep the same notation for this subsequence.

Let $b(x,v,r)$ be an arbitrary function from $L_2(\mathbb{R}^6 \times [a,b])$ and β_i be its Fourier coefficients with respect to $\{g_i(x,v,r)\}$. Then, we have

$$\sup_{0\leq t\leq T}\left|\int_a^b\int_{\mathbb{R}^6} b(x,v,r)[\tilde{f}^n(x,v,r,t) - \tilde{f}^m(x,v,r,t)]dxdvdr\right|$$

$$= \sup_{0\leq t\leq T}\left|\int_a^b\int_{\mathbb{R}^6}\left\{\left(b(x,v,r) - \sum_{i=1}^N \beta_i g_i(x,v,r)\right)[\tilde{f}^n - \tilde{f}^m]\right.\right.$$

$$\left.\left. + \sum_{i=1}^N \beta_i g_i(x,v,r)[\tilde{f}^n - \tilde{f}^m]\right\}dxdvdr\right| \leq \sup_{0\leq t\leq T}\left|\sum_{i=1}^N \beta_i(\alpha_{in}(t) - \alpha_{im}(t))\right|$$

$$+2A\left\{\int_a^b\int_{\mathbb{R}^6}\left|b(x,v,r) - \sum_{i=1}^N \beta_i g_i(x,v,r)\right|^2 dxdvdr\right\}^{\frac{1}{2}},$$

where A is the constant defined in Lemma 3.

For sufficiently large N, n and m, the right-hand side of this inequality is arbitrarily small. This proves the lemma. ■

Lemma 6. *The limit function $\tilde{f}(x,v,r,t)$ is such that:*

$$\tilde{f}(x,v,r,t) \geq 0 \text{ almost everywhere in } \mathbb{R}^6_T \times [a,b]; \tag{4.1}$$

$$\int_a^b \int_{\mathbb{R}^6} \tilde{f}(x,v,r,t)dxdvdr = \int_a^b \int_Q f_0(x,v,r)dxdvdr; \tag{4.2}$$

$$\sup_{0<t<T} \int_a^b \int_{\mathbb{R}^6} v^2 \tilde{f}(x,v,r,t)dxdvdr < \infty. \tag{4.3}$$

P r o o f. We denote by B an arbitrary measurable set in $\mathbb{R}^6 \times [a,b]$ with $\text{mes}B < \infty$. According to Lemma 5, we have

$$\int_B \tilde{f}(x,v,r,t)dxdvdr = \lim_{n\to\infty} \int_B \tilde{f}^n(x,v,r,t)dxdvdr.$$

Due to (3.17), $\tilde{f}^n(x,v,r,t) \geq 0$, and (4.1) is proved.

One can easily see that

$$\int_a^b \int_{\mathbb{R}^6} (1+x^2)^{\delta/2} \tilde{f}^n(x,v,r,t)dxdvdr \leq \delta At + \hat{A}, \tag{4.4}$$

where $\hat{A}$ is a constant such that $\hat{A} = \hat{A}(f_0)$, A is the constant defined in Lemma 3, and $\delta \in (0,1)$. In fact, from equation (3.14), we have

$$\left| \frac{d}{dt} \int_a^b \int_{\mathbb{R}^6} (1+x^2)^{\delta/2} \tilde{f}^n(x,v,r,t)dxdvdr \right| = \left| \int_a^b \int_{\mathbb{R}^6} (1+x^2)^{\delta/2} (v\nabla_x) \tilde{f}^n dxdvdr \right|$$

$$= \left| \delta \int_a^b \int_{\mathbb{R}^6} (v,x)(1+x^2)^{\frac{\delta}{2}-1} \tilde{f}^n dxdvdr \right| \leq \delta \int_a^b \int_{\mathbb{R}^6} (1+v^2)\tilde{f}^n dxdvdr \leq \delta A.$$

Here we used the boundedness of the support of $\tilde{f}^n(x,v,r,t)$, and (3.25c). Inequality (4.4) immediately follows from the last bound.

Now we prove the following statement. For any $\varepsilon_1 > 0$, there exists $R_1(\varepsilon_1) < \infty$ such that for any n and $t \in [0,T]$

$$\int_a^b \int_{\mathbb{R}^3} \int_{|x|>R_1(\varepsilon_1)} \tilde{f}^n(x,v,r,t)dxdvdr + \int_a^b \int_{|v|>R_1(\varepsilon_1)} \int_{\mathbb{R}^3} \tilde{f}^n(x,v,r,t)dxdvdr < \varepsilon_1. \tag{4.5}$$

In fact,

$$\int_a^b \int_{\mathbb{R}^3} \int_{|x|>R} \tilde{f}^n dxdvdr + \int_a^b \int_{|v|>R} \int_{\mathbb{R}^3} \tilde{f}^n dxdvdr$$

$$\leq \frac{1}{(1+R^2)^{\delta/2}} \int_a^b \int_{\mathbb{R}^6} (1+x^2)^{\delta/2} \tilde{f}^n dxdvdr + \frac{1}{1+R^2} \int_a^b \int_{\mathbb{R}^6} (1+v^2) \tilde{f}^n dxdvdr$$

$$\leq \frac{\delta AT + A_1}{(1+R^2)^{\delta/2}} + \frac{A}{1+R^2}.$$

As it follows from (4.1), (4.5) and Lemma 1

$$\int_a^b \int_{\mathbb{R}^6} \tilde{f}(x,v,r,t) dxdvdr = \lim_{R\to\infty} \lim_{n\to\infty} \int_a^b \int_{|v|<R} \int_{|x|<R} \tilde{f}^n dxdvdr$$

$$= \lim_{n\to\infty} \int_a^b \int_{\mathbb{R}^6} \tilde{f}^n(x,v,r,t) dxdvdr = \lim_{n\to\infty} \int_a^b \int_{\mathbb{R}^6} \tilde{f}^n(x,v,r,0) dxdvdr$$

$$= \lim_{n\to\infty} \int_Q f_0^n(x,v,r) dxdvdr = \int_a^b \int_Q f_0(x,v,r) dxdvdr.$$

Equation (4.2) is proved.

It remains to prove (4.3). Let $b_R(x,v)$ be a function such that $|b_R(x,v)| \leq 1$ and $b_R(x,v) = 0$ if $|x| \geq R$ and $|v| \geq R$, where R is a positive parameter. We have

$$\int_a^b \int_{\mathbb{R}^6} v^2 b_R(x,v) \tilde{f}(x,v,r,t) dxdvdr =$$

$$= \int_a^b \int_{\mathbb{R}^6} v^2 b_R(x,v)(\tilde{f} - \tilde{f}^n) dxdvdr + \int_a^b \int_{\mathbb{R}^6} v^2 b_R(x,v) \tilde{f}^n dxdvdr.$$

According to Lemma 5, the first term on the right-hand side tends to zero as $n \to \infty$ for any fixed R. It also follows from the definition of the function $b_R(x,v)$ and (3.25c) that the second term is bounded uniformly in n and R by the constant A defined in Lemma 3. Lemma 6 is proved. ■

Lemma 7. *The sequence $\{\tilde{f}^n\}$ converges (up to a subsequence) to $\tilde{f}$ in the weak topology of $L_1(\mathbb{R}^6 \times [a,b])$ uniformly in $t \in [0,T]$.*

P r o o f. Let $g(x,v,r) \in L_\infty(\mathbb{R}^6 \times [a,b])$. Then, we have

$$\left| \int\limits_a^b \int\limits_{\mathbb{R}^6} g(x,v,r)[\tilde{f}^n(x,v,r,t) - \tilde{f}^m(x,v,r,t)]dxdvdr \right| \tag{4.6}$$

$$\leq \left| \int\limits_a^b \int\limits_{|v|\leq R_1} \int\limits_{|x|\leq R_1} g(x,v,r)[\tilde{f}^n(x,v,r,t) - \tilde{f}^m(x,v,r,t)]dxdvdr \right|$$

$$+ \left[\int\limits_a^b \int\limits_{\mathbb{R}^3} \int\limits_{|x|>R_1} (\tilde{f}^n + \tilde{f}^m)dxdvdr + \int\limits_a^b \int\limits_{|v|>R_1} \int\limits_{\mathbb{R}^3} (\tilde{f}^n + \tilde{f}^m)dxdvdr \right] \|g\|_{L_\infty(\mathbb{R}^6\times[a,b])}.$$

Due to inequality (4.5) and the choice of R_1, the second term on the right-hand side of (4.6) is sufficiently small, uniformly in n, m and t. According to Lemma 5, the first term on the right-hand side of (4.6) is smaller than any $\varepsilon_1 > 0$ for any fixed R_1 and $n, m > N(\varepsilon_1)$. Thus, the sequence $\{\tilde{f}^n\}$ is weakly fundamental in $L_1(\mathbb{R}^6 \times [a,b])$ uniformly in t. Therefore, it is weakly convergent in $L_1(\mathbb{R}^6 \times [a,b])$ uniformly in t. Lemma 7 is proved. ■

Corollary 1. *The limit function $\tilde{f}(x,v,r,t)$ is continuous in $t \in [0,T]$ in the weak topology of $L_1(\mathbb{R}^6 \times [a,b])$.*

Lemma 8. *The vector function $u(x,t)$ is weakly continuous in t in the norm of $L_2(\Omega)$.*

P r o o f. First, we show that for any fixed k and $n \geq k$, the functions $C_{nk}(t)$ in (3.11) represent a uniformly bounded and equicontinuous set of functions on $[0,T]$. The uniform boundedness of $C_{nk}(t)$ is a consequence of the *a priori* estimate (3.25c). One can obtain the corresponding equicontinuity from (3.22). Indeed, integrating (3.22) with respect to $\tau \in (t, t+\Delta t)$, estimating the right-hand side and using Cauchy's inequality, we have

$$|C_{nk}(t+\Delta t) - C_{nk}(t)| \leq \nu \|\Psi^k\|_{J^1(\Omega)} \|u^n\|_{L_2(0,T;J^1(\Omega))} \sqrt{\Delta t}$$

$$+ \max_{x\in\Omega} |\Psi^k(x)| \|u^n\|_{L_\infty(0,T;J(\Omega))} \sqrt{\Delta t} \|u^n\|_{L_2(0,T;J^1(\Omega))}$$

$$+ \int\limits_t^{t+\Delta t} \|g(\tau)\|_{2,\Omega} d\tau + \alpha b \max_{x\in\Omega} |\Psi^k(x)| \left\{ \int\limits_t^{t+\Delta t} \int\limits_\Omega \int\limits_a^b \int\limits_{\mathbb{R}^3} S\tilde{f}^n(x,v,r,\tau)dvdrdxd\tau \right\}^{1/2}$$

$$\times\left\{\int_0^T\int_\Omega\int_a^b\int_{\mathbb{R}^3}\theta_R((u^n(x,\tau)-v)^2)|u^n(x,\tau)-v|^2 S\tilde{f}^n(x,v,r,\tau)dvdrdxd\tau\right\}^{1/2}.$$

From (3.25b), (3.25c) and the properties of the functions $\Psi^k(x)$, we get

$$|C_{nk}(t+\Delta t)-C_{nk}(t)|\le A(k)\sqrt{\Delta t}+\int_t^{t+\Delta t}\|g(\tau)\|_{2,\Omega}d\tau.$$

It is clear that for any fixed k and $n\ge k$ the right-hand side of this inequality tends to zero uniformly in t as $\Delta t\to 0$. By the usual diagonal process we extract a subsequence n_l. For any fixed k, the functions $C_{n_lk}(t)$ converge uniformly to a continuous function $C_k(t)$ as $l\to\infty$. For this subsequence we keep the same notation $C_{nk}(t)$.

Now we prove that the sequence of functions $u^n(x,t)$ converges to the function $u(x,t)$ in the weak topology of $L_2(\Omega)$ uniformly with respect to $t\in[0,T]$.

We denote by $g(x)$ an arbitrary vector function from $L_2(\Omega)$ and by g_k the Fourier coefficients of this function with respect to the system $\{\Psi^k(x)\}$. Then,

$$\sup_{[0,T]}\left|\int_\Omega(u^n(x,t)-u^m(x,t),g(x))dx\right|$$

$$=\sup_{[0,T]}\left|\int_\Omega\left[\left(g(x)-\sum_{k=1}^N g_k\Psi^k(x),u^n(x,t)-u^m(x,t)\right)\right.\right.$$

$$\left.\left.+\sum_{k=1}^N g_k(\Psi^k(x),u^n(x,t)-u^m(x,t))\right]dx\right|\le\sup_{[0,T]}\left\{\int_\Omega\left|g(x)-\sum_{k=1}^N g_k\Psi^k(x)\right|^2dx\right\}^{1/2}$$

$$\times\left\{\left[\int_\Omega|u^n(x,t)|^2dx\right]^{1/2}+\left[\int_\Omega|u^m(x,t)|^2dx\right]^{1/2}\right\}+\sup_{[0,T]}\left|\sum_{k=1}^N g_k(C_{nk}(t)-C_{mk}(t))\right|$$

$$\le\left(\|u^n\|_{L_\infty(0,T;J(\Omega))}+\|u^m\|_{L_\infty(0,T;J(\Omega))}\right)\left\{\int_\Omega\left|g(x)-\sum_{k=1}^N g_k\Psi^k(x)\right|^2dx\right\}^{1/2}$$

$$+\sup_{[0,T]}\left|\sum_{k=1}^N g_k(C_{nk}(t)-C_{mk}(t))\right|.$$

It is clear that the right-hand side of this inequality, for N, n, m sufficiently large, is arbitrary small.

It follows from the convergence of the sequence $\{u^n(x,t)\}$ to the function $u(x,t)$ that this limit is continuous in t in the weak topology of $L_2(\Omega)$. Lemma 8 is proved. ■

Lemma 9. *Galerkin's approximations $\{u^n\}$ satisfy the following inequality:*

$$\int_0^{T-\rho} \|u^n(t+\rho)-u^n(t)\|^2_{2,\Omega}dt < C\rho^{1/2},$$

where ρ is an arbitrary constant from $(0,T)$, and C is a constant that does not depend on n.

P r o o f. We will obtain this inequality by the arguments similar to those used in [8, 9]. For any fixed $\rho \in (0,T)$, $t \in [0,T-\rho]$ and $\tau \in [t,t+\rho]$ from (3.22) we have

$$\left(\frac{\partial u^n}{\partial \tau} + (u^n\nabla_x)u^n + \alpha \int_a^b\int_{\mathbb{R}^3} r\theta_R((u^n-v)^2)(u^n-v)S\tilde{f}^n dvdr, \Phi\right)_{2,\Omega}$$

$$+\nu(u^n,\Phi)_{J^1(\Omega)} = (g,\Phi)_{2,\Omega}, \tag{4.7}$$

where Φ is an arbitrary function from $J^1(\Omega)$ such that $\Phi = \sum_{k=1}^n d_k\Psi^k$. We set $\Phi = u^n(x,t+\rho) - u^n(x,t)$. Integrating (4.7) with respect to τ in the interval $[t,t+\rho]$, we get

$$\|u^n(t+\rho)-u^n(t)\|^2_{2,\Omega} = \int_t^{t+\rho}\Big\{(u^n(\tau),(u^n(\tau)\nabla)[u^n(t+\rho)-u^n(t)])_{2,\Omega}$$

$$-\nu(u^n(\tau),u^n(t+\rho)-u^n(t))_{J^1(\Omega)} + (g(\tau),u^n(t+\rho)-u^n(t))_{2,\Omega}$$

$$-\alpha\left(\int_a^b\int_{\mathbb{R}^3} r\theta_R((u^n(\tau)-v)^2)(u^n(\tau)-v)S\tilde{f}^n(v,a,\tau)dvdr, u^n(t+\rho)-u^n(t)\right)_{2,\Omega}\Big\}d\tau.$$

Therefore,

$$\|u^n(t+\rho)-u^n(t)\|^2_{2,\Omega} \le \sum_{k=1}^8 I_k(t), \tag{4.8}$$

where

$$I_1(t)+I_2(t)\equiv\int\limits_t^{t+\rho}\int\limits_\Omega |u^n(x,\tau)|^2\big(|Du^n(x,t+\rho)|+|Du^n(x,t)|\big)dxd\tau,$$

$$I_3(t)+I_4(t)\equiv\nu\int\limits_t^{t+\rho}\int\limits_\Omega |Du^n(x,\tau)|(|Du^n(x,t+\rho)|+|Du^n(x,t)|)dxd\tau,$$

$$I_5(t)+I_6(t)\equiv\int\limits_t^{t+\rho}\int\limits_\Omega |g(x,\tau)|(|u^n(x,t+\rho)|+|u^n(x,t)|)dxd\tau,$$

$$I_7(t)+I_8(t)\equiv\alpha\int\limits_t^{t+\rho}\int\limits_\Omega\int\limits_a^b\int\limits_{\mathbb{R}^3} S\tilde{f}^n(x,v,r,\tau)\theta_R((u^n(x,\tau)-v)^2)|u^n(x,\tau)-v|$$
$$\times\big(|u^n(x,t+\rho)|+|u^n(x,t)|\big)dvdrdxd\tau,$$

and

$$|Du^n|=\left(\sum_{i,j=1}^{3}\left(\frac{\partial u_i^n}{\partial x_j}\right)^2\right)^{1/2}.$$

Integrating inequality (4.8) with respect to t in the interval $[0,T-\rho]$ and estimating the terms $I_k(t)$, $(k=1,2,\dots,8)$ on the right-hand side, one can show that

$$\int\limits_0^{T-\rho} I_k(t)dt\le\eta_k\rho^{1/2},\quad k=1,2,\dots,8,\tag{4.9}$$

where η_k are constants that do not depend on n.

Using Cauchy's inequality and the embedding theorem of $J^1(\Omega)$ in $L_4(\Omega)$, we get

$$\int\limits_0^{T-\rho} I_1(t)dt\le C\int\limits_0^{T-\rho}\int\limits_t^{t+\rho}\|u^n(\tau)\|^2_{J^1(\Omega)}\|u^n(t+\rho)\|_{J^1(\Omega)}d\tau dt.$$

We change the order of integration supposing that $u^n(x,t)=0$ for $t>T$ and $t<0$. Then,

$$\int\limits_0^{T-\rho} I_1(t)dt\le C\int\limits_0^{T}\|u^n(\tau)\|^2_{J^1(\Omega)}\int\limits_{\tau-\rho}^{\tau}\|u^n(t+\rho)\|_{J^1(\Omega)}dtd\tau$$

$$\leq C\sqrt{\rho}\left(\int_0^T \|u^n(t)\|^2_{J^1(\Omega)}dt\right)^{3/2} \equiv \eta_1\sqrt{\rho}.$$

In the same way one can obtain the estimate for $I_2(t)$.

Now we consider $I_3(t)$. Cauchy's inequality and estimate (3.25c) imply that

$$\int_0^{T-\rho} I_3(t)dt \leq \nu \int_0^{T-\rho}\int_t^{t+\rho} \|u^n(\tau)\|_{J^1(\Omega)}\|u^n(t+\rho)\|_{J^1(\Omega)}d\tau dt$$

$$\leq \nu \int_0^{T-\rho} \|u^n(t+\rho)\|_{J^1(\Omega)} \left\{\rho\int_t^{t+\rho} \|u^n(\tau)\|^2_{J^1(\Omega)}d\tau\right\}^{1/2} dt$$

$$\leq \nu \left\{\rho\int_0^T \|u^n(\tau)\|^2_{J^1(\Omega)}d\tau\right\}^{1/2} \int_0^T \|u^n(t)\|_{J^1(\Omega)}dt$$

$$\leq \nu(A\rho)^{1/2}\left(T\int_0^T \|u^n(t)\|^2_{J^1(\Omega)}dt\right)^{1/2} \leq \nu AT\rho^{1/2} \leq \eta_3\sqrt{\rho}.$$

A similar bound can be easily proved for $I_4(t)$.

Consider now the terms $I_5(t)$ and $I_6(t)$. It is easy to see that

$$I_k(t) \leq \max_{0\leq t\leq T}\|u^n(t)\|_{J(\Omega)}\int_t^{t+\rho}\|g(\tau\|_{2,\Omega}d\tau, \quad k=5,6.$$

Changing the order of integration, we get

$$\int_0^{T-\rho}\int_t^{t+\rho}\|g(\tau)\|_{2,\Omega}d\tau dt \leq \rho\int_0^T \|g(\tau)\|_{2,\Omega}dt.$$

This inequality implies bound (4.9) for $I_5(t)$, $I_6(t)$.

Finally, we consider the terms $I_7(t)$, $I_8(t)$. Using Cauchy's inequality, we get

$$I_7(t) \leq \alpha\int_t^{t+\rho}\int_\Omega |u^n(x,t+\rho)|\left\{\int_a^b\int_{\mathbb{R}^3} S\tilde{f}^n(x,v,r,\tau)dvdr\right\}^{1/2}$$

$$\times \left\{ \int_a^b \int_{\mathbb{R}^3} S\tilde{f}^n(x,v,r,\tau)\theta_R((u^n(x,\tau)-v)^2)|u^n(x,\tau)-v|^2 dvdr \right\}^{1/2} dxd\tau.$$

To estimate the integral over Ω, we make use of Hölder's inequality. Namely, we have

$$I_7(t) \le \alpha \int_t^{t+\rho} \left\{ \int_\Omega |u^n(x,t+\rho)|^6 dx \right\}^{1/6} \left\{ \int_\Omega \left(\int_a^b \int_{\mathbb{R}^3} S\tilde{f}^n(x,v,r,\tau) dvdr \right)^{3/2} dx \right\}^{1/3}$$

$$\times \left\{ \int_\Omega \int_a^b \int_{\mathbb{R}^3} S\tilde{f}^n(x,v,r,\tau)\theta_R((u^n(x,\tau)-v)^2)|u^n(x,\tau)-v|^2 dvdrdx \right\}^{1/2} d\tau. \quad (4.10)$$

Let us consider the second factor on the right-hand side of (4.10). From the *a priory* estimates (3.25a), (3.25c) we have

$$\left\{ \int_\Omega \left(\int_a^b \int_{\mathbb{R}^3} S\tilde{f}^n(x,v,r,\tau) dvdr \right)^{3/2} dx \right\}^{1/3}$$

$$\le \int_\Omega \left(\int_a^b \int_{\mathbb{R}^3} (1+v^2)[S\tilde{f}^n(x,v,r,\tau)]^{3/2} dvdr \right)$$

$$\times \left(\int_a^b \int_{\mathbb{R}^3} \frac{dvdr}{(1+v^2)^2} \right)^{1/2} dx \le C_1 \int_\Omega \int_a^b \int_{\mathbb{R}^3} (1+v^2) S\tilde{f}^n(x,v,r,\tau) dvdrdx \le C_2.$$

Therefore, it follows from (4.10) that

$$I_7(t) \le C_3 \rho \left[\int_0^T \left\{ \int_\Omega |u^n(x,t)|^6 dx \right\}^{1/3} dt \right]^{1/2}$$

$$\times \left\{ \int_0^T \int_a^b \int_Q S\tilde{f}^n(x,v,r,t)\theta_R((u^n(x,t)-v)^2)(u^n(x,t)-v)^2 dvdxdrdt \right\}^{1/2}.$$

Due to (3.25c) and the embedding of $J^1(\Omega)$ in $L_6(\Omega)$, we have

$$\int_0^{T-\rho} I_7(t)dt \leq \eta_7\rho.$$

The term $I_8(t)$ can be estimated in a similar way.

Thus, inequality (4.9) is obtained. The proof of Lemma 9 is completed. ■

Lemma 9 and estimates (3.25a)–(3.25c) imply that $\{u^n\}$ is compact set in $L_2(\Omega_T)$. Then there exists a subsequence (still denoted by $\{u^n\}$) which strongly converges to $u(x,t)$ in $L_2(\Omega_T)$.

Finally, the additional convergence properties of the sequence $\{\varphi^n\}$ are given by the following Lemma (see [22]).

Lemma 10. *Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary and $\varphi_\varepsilon(x)$ be a solution of*

$$\varepsilon\Delta^2\varphi_\varepsilon - \Delta\varphi_\varepsilon = F, \quad \text{in}\,\Omega,$$

$$\varphi_\varepsilon = 0, \quad \varepsilon\frac{\partial\varphi_\varepsilon}{\partial n} = 0, \quad \text{on}\,\partial\Omega,$$

where $\varepsilon \geq 0$, $F \in L_p(\Omega)$ $(p > \frac{6}{5})$.

Then,

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla\varphi_\varepsilon - \nabla\varphi_0|dx = 0$$

uniformly with respect to F such that $\|F\|_{L_p(\Omega)} \leq C$.

5. Passage to the Limit in (3.1.)–(3.10)

In this section we pass to the limit in (3.1.)–(3.10) and obtain (2.3)–(2.5). To this end, we set $R = n$, $\varepsilon = \frac{1}{n}$.

5.1. Derivation of identity (2.3)

We multiply (3.22) by $H_i(t)$ and summarize over j. Then, integrating by parts, we obtain (3.1.) for u^n and $\tilde{f}^n$, where the test functions ζ are defined by

$$\zeta(x,t) = \sum_{j=1}^{n} H_j(t)\Psi^j(x), \quad H_j(t) \in C^1(0,T), \quad H_i(T) = 0.$$

Notice that the set of functions ζ is dense in the set of functions satisfying (2.6a).

Now, we show that the limits of the subsequences $\{u^n\}$ and $\{\tilde{f}^n\}$ satisfy (2.3). Due to the strong convergence of $\{u^n\}$ to u in $L_2(\Omega_T)$, the uniform boundedness of $\|u^n\|_{L_2(\Omega_T)}$ in n, and (2.6a) we get

$$\lim_{n\to\infty}(u^n,(u^n\nabla_x)\zeta)_{2,\Omega_T} = (u,(u\nabla_x)\zeta)_{2,\Omega_T}.$$

Pass to the limit as $n\to\infty$ in the term of (3.1.) containing $\tilde{f}^n$. Notice that for any $\varepsilon > 0$ there exists $R_1(\varepsilon) > 0$ such that

$$I \equiv \int\limits_a^b \int\limits_{Q_T\cap\{v:|v|\geq R_1\}} r\theta_n((u^n-v)^2)S\tilde{f}^n(x,v,r,t)|u^n-v||\zeta(x,t)|dxdvdrdt < \varepsilon \tag{5.1}$$

uniformly in n. Following the lines of the derivation of the bound of I_7 (see Lem. 9), taking into account Lemma 3, and the fact that $\zeta \in L_4(0,T;J^1(\Omega))$, we get

$$I \leq \sqrt{A}\left\{\int\limits_a^b \int\limits_{|v|\geq R_1} \frac{dvda}{(1+v^2)^2}\right\}^{1/6}.$$

Inequality (5.1) immediately follows from this estimate.

Next we prove that for any $R_1 > 0$

$$\lim_{n\to\infty}\left(\int\limits_a^b \int\limits_{Q_T\cap\{v:|v|\leq R_1\}} r\theta_n((u^n-v)^2)S\tilde{f}^n(x,v,r,t)(u^n-v,\zeta)dxdvdrdt\right.$$

$$\left. - \int\limits_a^b \int\limits_{Q_T\cap\{v:|v|\leq R_1\}} rS\tilde{f}(x,v,r,t)(u(x,t)-v,\zeta(x,t))dxdvdrdt\right) = 0. \tag{5.2}$$

To this end, we represent the left hand side of this equation as a sum of the following integrals:

$$I_1 \equiv \int\limits_a^b \int\limits_{Q_T\cap\{v:|v|\leq R_1\}} rS(\tilde{f}^n-\tilde{f})(u-v,\zeta)dxdvdtdr,$$

$$I_2 \equiv \int\limits_a^b \int\limits_{Q_T\cap\{v:|v|\leq R_1\}} rS\tilde{f}^n[\theta_n((u^n-v)^2)-\theta_n((u-v)^2)](u^n-v,\zeta)dxdvdtdr,$$

$$I_3 \equiv \int_a^b \int_{Q_T \cap \{v:|v|\le R_1\}} rS\tilde{f}^n[\theta_n((u-v)^2)-1](u^n-v,\zeta)dxdvdtdr,$$

$$I_4 \equiv \int_a^b \int_{Q_T \cap \{v:|v|\le R_1\}} rS\tilde{f}^n(u^n-u,\zeta)dxdvdtdr. \tag{5.3}$$

We recall that $\tilde{f}^n \to \tilde{f}$ $*$-weakly in $L_\infty(\mathbb{R}^6_T \times [a,b])$. Moreover, it is easy to see that $(u-v,\zeta) \in L_1(Q_T \cap \{v : |v| \le R_1|\})$. Then $I_1 \to 0$ as $n \to \infty$.

Let us estimate I_2. First, we show that

$$\lim_{n\to\infty} \int_0^T \int_{|v|\le R_1} \int_\Omega |\theta_n((u^n-v)^2) - \theta_n((u-v)^2)|^3 dxdvdt = 0. \tag{5.4}$$

In fact, since $|\theta_n((u^n-v)^2) - \theta_n((u-v)^2)| \le C|(u^n-v)^2 - (u-v)^2| \le C(|u^n - u||u^n| + |u||u^n-u| + 2|v||u^n-u|)$, then

$$\int_0^T \int_{|v|\le R_1} \int_\Omega |\theta_n((u^n-v)^2) - \theta_n((u-v)^2)| dxdvdt$$

$$\le \hat{C}\|u^n-u\|_{2,Q_T}\left(\|u^n\|_{L_\infty(0,T;J(\Omega))} + \|u\|_{L_\infty(0,T;J(\Omega))} + 1\right).$$

Thus, the convergence of u^n to u in $L_2(Q_T)$ (as $n \to \infty$) and the bound $|\theta_n((u^n - v)^2) - \theta_n((u-v)^2)| \le 2$ imply (5.4).

From (3.25a) we have

$$|I_2| \le A(b-a)\int_0^T \int_{|v|\le R_1} \int_\Omega |u^n-v||\theta_n((u-v)^2) - \theta_n((u^n-v)^2)||\zeta| dxdvdt.$$

Using Hölder's inequality, we obtain

$$|I_2| \le A(b-a)\int_0^T \left[\left\{\int_{|v|\le R_1} \int_\Omega |u^n-v|^2 dxdv\right\}^{1/2} \times \left\{\int_{|v|\le R_1} \int_\Omega |\zeta(x,t)|^6 dxdv\right\}^{1/6}\right.$$

$$\left. \times \left\{\int_{|v|\le R_1} \int_\Omega |\theta_n((u^n-v)^2) - \theta_n((u-v)^2)|^3 dxdv\right\}^{1/3}\right] dt$$

$$\leq \tilde{A}(1 + \|u^n\|_{L_\infty(0,T;J(\Omega))}) \times \left\{ \int_{|v|\leq R_1} \int_{\Omega} |\zeta(x,t)|^6 dxdv \right\}^{1/6}$$

$$\times \int_0^T \left\{ \int_{|v|\leq R_1} \int_{\Omega} |\theta_n((u^n - v)^2) - \theta_n((u-v)^2)|^3 dxdv \right\}^{1/3} dt.$$

Applying Hölder's inequality to the integral with respect to the time variable, we get

$$|I_2| \leq \hat{A} \left\{ \int_0^T \int_{|v|\leq R_1} \int_{\Omega} |\theta_n((u^n - v)^2) - \theta_n((u-v)^2)|^3 dxdvdt \right\}^{1/3}$$

$$\times \left\{ \int_0^T \left[\int_{|v|\leq R_1} \int_{\Omega} |\zeta(x,t)|^6 dxdv \right]^{1/4} dt \right\}^{2/3}.$$

Using (5.4) and the embedding of $J^1(\Omega)$ in $L_6(\Omega)$, we finally obtain

$$\lim_{n\to\infty} I_2 = 0.$$

To estimate I_3 we decompose the domain Ω_T into two parts. Namely, we set

$$\Omega_T^1 = \{(x,t) \in \Omega_T : |u(x,t)| \leq B\}, \quad \Omega_T^2 = \Omega_T \setminus \Omega_T^1,$$

where B is a positive constant. Since $u \in L_2(\Omega_T)$, then for any $\delta > 0$ there exists B such that $\operatorname{mes} \Omega_T^2 < \delta$. Hence,

$$|I_3| \leq A\Big(\int_a^b \int_{|v|\leq R_1} \int_{\Omega_T^1} |\theta_n((u-v)^2) - 1||(u^n - v, \zeta)| dxdvdrdt$$

$$+ \int_a^b \int_{|v|\leq R_1} \int_{\Omega_T^2} |\theta_n((u-v)^2) - 1||(u^n - v, \zeta)| dxdvdrdt \Big).$$

The argument of the function θ_n is bounded in $\Omega_T^1 \times \{v : |v| \leq R_1\}$. The sequence $\{\theta_n\}$ converges uniformly to 1 as $n \to \infty$ on any compact set. Therefore, since $(u^n - v, \zeta) \in L_1(Q_T \cap \{v : |v| \leq R_1\})$, then for sufficiently large n the integral over

$\Omega_T^1 \times \{v : |v| \le R_1 \times [a,b]\}$ is arbitrary small. The second integral is arbitrary small due to the choice of δ. This means that $I_3 \to 0$ as $n \to \infty$.

Consider now the last integral in (5.1.). We have

$$|I_4| \le A\frac{4}{3}\pi R_1^3 \int\limits_0^T \int\limits_\Omega |u^n - u||\zeta| dxdt \le A\frac{4}{3}\pi R_1^3 \left\{ \int\limits_{\Omega_T} |u^n - u|^2 dxdt \right\}^{1/2}$$

$$\times \left\{ \int\limits_0^T \int\limits_\Omega |\zeta|^2 dxdt \right\}^{1/2} \le \tilde{A}\|u^n - u\|_{2,\Omega_T} \|\zeta\|_{L_\infty(0,T;J(\Omega))}.$$

Thus, $\lim\limits_{n\to\infty} I_4 = 0$, and (5.2) is proved.

It follows from (5.1), (5.2) that

$$\int\limits_a^b \int\limits_{Q_T} rS\tilde{f}|u(x,t) - v|\zeta(x,t) dxdvdrdt < \infty.$$

This estimate, along with (5.1), (5.2), allows us to pass to the limit as $n \to \infty$ in the fourth term of (3.1.). The remaining terms in (3.1.) are linear with respect to u^n, and the passage to the limit as $n \to \infty$ is evident. Identity (2.3) is proved.

5.2. Derivation of identity (2.4)

Multiplying (3.12) by $\Phi(x,t)$ and integrating by parts, we obtain (3.9) for φ^n and $\tilde{f}^n$ with $\varepsilon = \frac{1}{n}$.

The first term in (3.9) tends to zero as $n \to \infty$. Indeed,

$$\left| \frac{1}{n} \int\limits_0^T (\Delta\varphi^n, \Delta\Phi)_{2,\Omega} dt \right| \le \frac{1}{\sqrt{n}} \int\limits_0^T \frac{1}{\sqrt{n}} \|\Delta\varphi^n\|_{2,\Omega} \|\Delta\Phi\|_{2,\Omega} dt \le \frac{1}{\sqrt{n}}\sqrt{A} \int\limits_0^T \|\Delta\Phi\|_{2,\Omega} dt, \tag{5.5}$$

where A is the constant defined in (3.25c).

We recall (see the beginning of Section 4) that the sequence $\{\varphi^n\} \to \varphi$ $*$-weakly in $L_\infty(0,T; W_2^1(\Omega))$. Taking into account the properties of the functions $\Phi(x,t)$, we can pass to the limit in the second term of (3.9).

Lemma 7 and Lebesgue's theorem allow us to pass to the limit in the third term of (3.9) for $\Phi \in L_\infty(0,T; \overset{\circ}{W}{}_2^2(\Omega))$. The functional space $L_\infty(0,T; \overset{\circ}{W}{}_2^2(\Omega))$ is dense in $L_2(0,T; \overset{\circ}{W}{}_2^1(\Omega))$. Identity (2.4) is proved.

5.3. Derivation of identity (2.5)

Taking into account (3.5) from identity (3.10), we get

$$\int_0^T\int_a^b \Big(\tilde f^n, \Psi_t + (v\nabla_x)\Psi + (g_n(x,t)\nabla_v)\Psi\Big)_{2,\mathbb{R}^6} drdt + \int_a^b (Pf_0^n, \Psi(0))_{2,\mathbb{R}^6} dr$$

$$+\beta \int_0^T\int_a^b \frac{1}{r^2}\Big(\tilde f^n, (P\theta_n((u^n-v)^2)[u^n(x,t)-v]\nabla_v)\Psi\Big)_{2,\mathbb{R}^6} drdt$$

$$-\gamma \int_0^T\int_a^b \frac{1}{r^2}\Big(\tilde f^n, (P\nabla_x\varphi^n(x,t)\nabla_v)\Psi\Big)_{2,\mathbb{R}^6} drdt = 0, \qquad (5.6)$$

where $\Psi(x,v,r,t)$ is an arbitrary function satisfying conditions (2.6c).

Due to the $*$-weak convergence of the sequence $f^n(x,v,r,t)$ in $L_\infty(\mathbb{R}^6_T\times[a,b])$ (see Lem. 3) and the properties of the functions $g_n(x,t)$, $\Psi(x,v,r,t)$ we can pass to the limit in the first integral of (5.6). According to Lemma 1 and the properties of the functions $f_0^n(x,v,r)$, we can pass to the limit in the second integral of (5.6). Applying the arguments similar to the ones used in Section 5.1 for the term containing $\tilde f^n$ in (3.1.), we pass to the limit in the third integral of (5.6).

Consider the fourth integral in (5.6). Let $\mathcal{G}_n(x,y)$ be Green's function of problem (3.12), (3.13) with $\varepsilon = \frac{1}{n}$ and $\mathcal{G}(x,y)$ be Green's function of problem (1.3), (1.7). We introduce the following notation:

$$F_n(x,t) = q\int_a^b\int_{\mathbb{R}^3} r\tilde f^n(x,v,r,t)dvdr, \quad F(x,t) = q\int_a^b\int_{\mathbb{R}^3} r\tilde f(x,v,r,t)dvdr,$$

$$\varphi^n(x,t) = \int_\Omega \mathcal{G}_n(x,y)F_n(y,t)dy, \quad \varphi(x,t) = \int_\Omega \mathcal{G}(x,y)F(y,t)dy,$$

$$\tilde\varphi^n(x,t) = \int_\Omega \mathcal{G}(x,y)F_n(y,t)dy,$$

where $\tilde f(x,v,r,t)$ is the $*$-weak limit of the sequence $\{\tilde f^n(x,v,r,t)\}$ in $L_\infty(\mathbb{R}^6_T\times[a,b])$.

The functions F_n belong to $L_p(\Omega)$ ($p\in(\frac{3}{2};\frac{5}{3})$) uniformly in n and $t\in[0,T]$ (see Lem. 4). According to Lemma 7, the sequence $\{F_n(x,t)\}$ converges to $F(x,t)$ in the weak topology of $L_1(\Omega)$ uniformly in $t\in[0,T]$. Thus, $F(x,t)\in L_p(\Omega)$ ($p\in(\frac{3}{2};\frac{5}{3})$).

The operator with the integral kernel $\nabla\mathcal{G}(x,t)$ acting from $L_1(\Omega)$ to $L_1(\Omega)$ is completely continuous (see [23]). Therefore, the sequence $\{\nabla\tilde{\varphi}^n(x,t)\}$ converges to $\nabla\varphi(x,t)$ in $L_1(\Omega)$ for any $t\in[0,T]$. Moreover, the strong convergence in $L_1(\Omega_T)$ follows from Lebesgue's theorem. It is evident that the functions $\varphi^n(x,t)$ and $\varphi(x,t)$ are the solutions of the problems (3.12), (3.13) and (1.3), (1.7), respectively.

Thus, we can rewrite the fourth integral in (5.6) as follows:

$$\gamma\int_0^T\int_a^b\frac{1}{r^2}\left(\tilde{f}^n,(P\nabla_x\varphi^n(x,t)\nabla_v)\Psi\right)_{2,\mathbb{R}^6}drdt=I_1^{(n)}+I_2^{(n)}+I_3^{(n)},$$

where

$$I_1^{(n)}=\gamma\int_0^T\int_a^b\int_\Omega\int_{\mathbb{R}^3}\frac{1}{r^2}S\tilde{f}^n(x,v,r,t)(\nabla_x\varphi(x,t)\nabla_v)\Psi(x,v,r,t)dvdxdrdt,$$

$$I_2^{(n)}=\gamma\int_0^T\int_a^b\int_\Omega\int_{\mathbb{R}^3}\frac{1}{r^2}S\tilde{f}^n(x,v,r,t)\left([\nabla_x\tilde{\varphi}^n(x,t)-\nabla_x\varphi(x,t)]\nabla_v\right)\Psi dvdxdrdt,$$

$$I_3^{(n)}=\gamma\int_0^T\int_a^b\int_\Omega\int_{\mathbb{R}^3}\frac{1}{r^2}S\tilde{f}^n(x,v,r,t)\left([\nabla_x\varphi^n(x,t)-\nabla_x\tilde{\varphi}^n(x,t)]\nabla_v\right)\Psi dvdxdrdt.$$

The function $\Psi(x,v,r,t)$ satisfies conditions (2.6c) and $\nabla_x\varphi\in L_1(\Omega_T)$. Then the $*$-weak convergence of the sequence $\{\tilde{f}^n(x,v,r,t)\}$ to $\tilde{f}(x,v,r,t)$ in $L_\infty(\mathbb{R}^6_T\times[a,b])$ implies that

$$\lim_{n\to\infty}I_1^{(n)}=\int_0^T\int_a^b\frac{\gamma}{r^2}\left(\tilde{f},(P\nabla_x\varphi(x,t)\nabla_v)\Psi\right)_{2,\mathbb{R}^6}drdt.$$

Now taking into account the convergence of $\{\nabla_x\tilde{\varphi}^n(x,t)\}$ to $\nabla_x\varphi(x,t)$ in $L_1(\Omega_T)$ and the uniform boundedness of the approximations $\tilde{f}^n(x,v,r,t)$ (see (3.25a)), we get

$$\lim_{n\to\infty}I_2^{(n)}=0.$$

Finally, using Lemma 10, we obtain

$$\lim_{n\to\infty}I_3^{(n)}=0.$$

Thus, equality (2.5) is proved and the proof of the Theorem 1 is completed.

References

[1] *S.L. Soo*, Multiphase Fluid Dynamics. Hemisphere Publish. Corp., New York, 1989.

[2] *A. Fortier*, Mechanique of Suspensions. Mir, Moscow, 1971.

[3] *V.A. L'vov*, Convergence of Solutions of an Initial-Boundary Value Problem for a System of Navier–Stokes Equations in Domains with a Moving Fine-Grained Boundary. — *Dokl. Akad. Nauk UkrSSR, Ser. A* **85** (1987), No. 7, 21–24. (Russian)

[4] *V.A. L'vov*, An Initial-Boundary Value Problem for the Navier–Stokes Equations in Domains with a Random Boundary. — *Dokl. Akad. Nauk UkrSSR, Ser. A* **85** (1988), No. 3, 20–23.

[5] *O.A. Anoshchenko*, Global Existence of a Generalized Solution of a System of Equations of Motion of a Suspension. Dynamical Systems and Complex Analysis. Naukova Dumka, Kyiv, 1992. (Russian)

[6] *O.A. Anoshchenko*, Existence and Uniqueness of the Solution of a System of Equations of Motion of a Suspension in Holder Classes. — Mat. Fiz., Anal., Geom. **1** (1994), 31–40. (Russian)

[7] *O.A. Anoshchenko and A. Boutet de Monvel-Berthier*, The Existence of a Global Generalized Solution of the System of Equations Describing Suspension Motion. — Math. Meth., Appl. Sci. **20** (1997), 495–519.

[8] *O.A. Ladyzhenskaya*, The mathematical theory of viscous incompressible flow. New York, London, Paris: Gordon and Breach Sci. Publ., New York, **XVIII**, (1969).

[9] *S.N. Antontsev, A.V. Kazhikhov, and V.N. Monakhov*, Boundary Value Problems in the Mechanics of Inhomogeneous Fluids. Izd-vo "Nauka" (Sibirsk. Otd.), Novosibirsk, 1983. (Russian)

[10] *A.A. Arsen'ev*, Existence in the Large of a Weak Solution of Vlasov's System of Equations. — *Z. Vychisl. Mat. i Mat. Fiz.* **15** (1975), 136–147, 276. (Russian)

[11] *C. Bardos and P. Degond*, Global Existence for the Vlasov-Poisson Equation in 3 Space Variables with Small Initial Data. — *Ann. Inst. H. Poincare Anal. Non Lineaire* **2** (1985), No. 2, 101–118.

[12] *J. Schaeffer*, Global Existence of Smooth Solutions to the Vlasov–Poisson System in Three Dimensions. — *Comm. Part. Diff. Eq.* **16** (1991), No. 8-9, 1313–1335.

[13] *K. Pfaffelmoser*, Global classical Solutions to the Vlasov–Poisson System in Three Dimensions for General Initial Data. — *J. Diff. Eq.* **95** (1992), No. 2, 281–303.

[14] *J. Batt and G. Rein*, Global Classical Solutions of the Periodic Vlasov–Poisson System in Three Dimensions. — *C.R.Acad. Sci. Paris* **313** (1991), 411–416.

[15] *R. Alexandre*, Weak Solutions of the Vlasov–Poisson Initial Boundary Value Problem. — *Math. Meth. Appl. Sci.* **16** (1993), 587–607.

[16] *K. Hamdache*, Global Existence and Large Time Behaviour of Solutions for the Vlasov–Stokes Equations. — *Jap. J. Industr. Appl. Math.* **15** (1998), 51–74.

[17] *L. Boudin, L. Desvillettes, C. Grandmont, and A. Moussa*, Global Existence of Solutions for the Coupled Vlasov and Navier–Stokes Equations. (To appear in Diff. Integr. Eqs, 2009.)

[18] *A. Mellet and A. Vasseur*, Global Weak Solutions for a Vlasov–Fokker–Planck/Navier–Stokes Equations. — *Math. Mod. Meth. Appl. Sci.* **17** (2007), No. 7, 1039–1063.

[19] *A.I. Koshelev*, A Priori Estimates in L_p. — *Usp. Mat. Nauk* **13** (1958), No. 4, 29–88.

[20] *S.L. Sobolev*, Applications of Functional Analysis in Mathematical Physics. Amer. Math. Soc. **VII**. Providence, RI, (1963).

[21] *J-L. Lions*, Quelques Methodes de Resolution des Problemes aux Limites non Lineaires. Dunod, Paris, 1969.

[22] *O. Anoshchenko, E. Khruslov, and O. Lysenko*, On Convergence of Solutions of Singularily Perturbated Boundary-value Problems. — *J. Math. Phys., Anal., Geom.* **5** (2009), 115–122.

[23] *M. Krasnoselskii, P. Zabreiko, E. Pustilnik, and P. Sobolevskii*, The Integral Operators in the Spaces of Summarized Functions. Nauka, Moscow, 1966.