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On a Class of Verblunsky Parameters that Corresponds to Guseinov's Class of Jacobi Parameters

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The direct and inverse Geronimus relations between Verblunsky parameters of measures on the unit circle and Jacobi parameters of their Szegő transforms have been used to prove that Guseinov's class of Jacobi parameters $\sum_{n=0}^{\infty} n(|a_n - 1| + |b_n|) < \infty$ is in a canonical correspondence with the following class of Verblunsky parameters $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} n|\alpha_{n+2} - \alpha_n| < \infty$.

Key words: Verblunsky coefficients, Szegő transform, direct and inverse Geronimus relations, Jacobi matrices; spectral measure.

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1. Introduction

In the late 1950s L. Faddeev [1] developed a scattering theory for the one-dimensional Schrödinder equation

$$-y'' + q(x)y = \lambda^2 y \tag{1.1}$$

under the following assumption on the potential q

$$\int_{-\infty}^{\infty} (1+|x|)|q(x)|dx < \infty. \tag{1.2}$$

In 1979 Deift and Trubowitz [2] found a gap in Faddeev's construction and analyzed completely the case of potentials with a finite second moment

$$\int_{-\infty}^{\infty} (1+x^2)|q(x)|dx < \infty. \tag{1.3}$$

As it turned out, they were unaware of the book [3] by V.A. Marchenko where he had given a correct proof of Faddeev's theorem for the class of potentials (1.2). That is why we call (1.2) the Faddeev–Marchenko condition.

In the mid 1970s Guseinov [4, 5] suggested a discrete version of the Faddeev–Marchenko theory for Jacobi matrices

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = (\lambda + 1/\lambda)y_n \tag{1.4}$$

under the condition

$$\sum_{n=0}^{\infty} n(|a_n - 1| + |b_n|) < \infty. \tag{1.5}$$

We say that a Jacobi matrix $J = J(\{a_n\}, \{b_n\})$ belongs to Guseinov's class (G) if its parameters satisfy (1.5).

In his recent papers [6–8] and PhD thesis [9] E. Ryckman came up with a new class of Jacobi matrices for which a complete spectral description is available. Moreover, this class extends Guseinov's class (see Prop. 3.3 below).

Let us write

$$\beta = \{\beta_n\} \in \ell_s^2 \quad \text{if} \quad \|\beta\|_{\ell_s^2}^2 := \sum_n |n|^s |\beta_n|^2 < \infty.$$

Definition 1.1. *A Jacobi matrix is said to be in Ryckman's class (R), or $a, b \in (R)$, if the series $\sum_n (a_n^2 - 1)$ and $\sum_n b_n$ are conditionally summable, and*

$$\lambda_n := -\sum_{k=n+1}^{\infty} b_k \in \ell_1^2, \quad \kappa_n := -\sum_{k=n+1}^{\infty} (a_k^2 - 1) \in \ell_1^2.$$

Ryckman's argument is based on two main ingredients. First, he shows that under an appropriate Szegő transform the spectral measures of the class (R) correspond to measures μ on the unit circle that are from the B. Golinskii–Ibragimov (GI) class:

$$\sum_{n=1}^{\infty} n|\alpha_n|^2 < \infty,$$

where α_n are the Verblunsky parameters of μ. The second step in Ryckman's argument is the Strong Szegő theorem, which provides a complete spectral characteristic of (GI) class.

A problem we address in this note is to find the class of measures on the unit circle that corresponds to Guseinov's class (1.5) under an appropriate Szegő transform. It turns out that this class is described as follows.

Definition 1.2. *We say that $\mu \in (K)$ (or $\alpha = \{\alpha_n\} \in (K)$) if*

$$\alpha_n \to 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} n|\alpha_{n+2} - \alpha_n| < \infty. \tag{1.6}$$

(K) is a proper subclass of (GI) class, and it solves the above problem.

Ryckman also studies the class of Jacobi matrices with $\{\lambda_n\}, \{\kappa_n\} \in \ell_1^2 \cap \ell^1$ and shows that the corresponding class of measures on the unit circle satisfies $\{\alpha_n\} \in \ell_1^2 \cap \ell^1$. It is an open problem to characterize a class of Jacobi matrices which corresponds in this sense to the whole Baxter's class $\{\alpha_n\} \in \ell^1$.

Note that the scattering theory for orthogonal polynomials on the unit circle (CMV matrices in the modern terminology) and for Jacobi matrices was suggested by Geronimo and Case in [10] for Baxter's class, and in [11], for Guseinov's class.

2. Classes of Verblunsky Parameters

Theorem 2.1 (Szegő)**.** *Let μ be a nontrivial probability measure on $\mathbb{T}$ with Verblunsky parameters $\{\alpha_n\}$, $|\alpha_n| < 1$. Then*

$$\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$$

if and only if $\mu(dt) = w(t)m(dt)$, $m(dt)$ the normalized Lebesgue measure on $\mathbb{T}$, with $\log w \in L^1$.

Theorem 2.2 (B. Golinskii–Ibragimov's version of the Strong Szegő theorem)**.** *Let μ be a nontrivial probability measure on $\mathbb{T}$ with the Verblunsky parameters $\{\alpha_n\}$, $|\alpha_n| < 1$. The following assertions are equivalent:*

1. $\alpha \in \ell^2_1$, *i.e.,*

$$\sum_{n=0}^{\infty} n|\alpha_n|^2 < \infty.$$

2. $\mu(dt) = w(t)m(dt)$ *with* $\widehat{\log w} \in \ell^2_1$. *Here* $\hat{f}$ *is a sequence of Fourier coefficients of a function* f.

Proposition 2.3. $\alpha \in (K) \Longrightarrow \alpha \in \ell^2_1$.

P r o o f. The claim will be proved separately for even and odd n's. So we want to show that

$$c_n \to 0 \quad \text{and} \quad \sum_{n=1}^{\infty} n|c_{n+1} - c_n| < \infty \Longrightarrow \sum_{n=1}^{\infty} n|c_n|^2 < \infty.$$

We have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n|c_n|^2 = \sum_{n=1}^{\infty} n|\sum_{k=n}^{\infty}(c_{k+1} - c_k)|^2 \le \sum_{n=1}^{\infty} n(\sum_{k=n}^{\infty} |c_{k+1} - c_k|)^2 \\
=&\sum_{n=1}^{\infty} n(\sum_{k=n}^{\infty} |c_{k+1} - c_k|)(\sum_{l=n}^{\infty} |c_{l+1} - c_l|) = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\sum_{l=n}^{\infty} n|c_{k+1} - c_k||c_{l+1} - c_l| \\
\le&\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\sum_{l=n}^{\infty} l|c_{k+1} - c_k||c_{l+1} - c_l| \le \sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\sum_{l=1}^{\infty} l|c_{k+1} - c_k||c_{l+1} - c_l| \\
=&\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{l=1}^{\infty} l|c_{k+1} - c_k||c_{l+1} - c_l| = \sum_{k=1}^{\infty} k \sum_{l=1}^{\infty} l|c_{k+1} - c_k||c_{l+1} - c_l| \\
=&(\sum_{k=1}^{\infty} k|c_{k+1} - c_k|)^2.
\end{aligned}$$

■

It is easy to see that also $\alpha \in (K) \Longrightarrow \alpha \in \ell^1$. Indeed,

$$|\alpha_n| = |\sum_{k=n}^{\infty}(\alpha_k - \alpha_{k+2})|,$$

and so

$$\sum_{n=0}^{\infty} |\alpha_n| \le \sum_{n=0}^{\infty} n|\alpha_n - \alpha_{n+2}|.$$

Hence $(K) \subset \ell^2_1 \cap \ell^1$. Obviously, the inclusion is proper.

3. Classes of Jacobi Parameters

Definition 3.1. *We say that* $a, b \in (S)$ *if*

$$\sum_{n=0}^{\infty}(|a_n - 1|^2 + |b_n|^2) < \infty.$$

We say that $a, b \in (R)$ *if*

$$\sum_{n=0}^{\infty} n \left|\sum_{k=n}^{\infty}(a_k - 1)\right|^2 < \infty, \qquad \sum_{n=0}^{\infty} n \left|\sum_{k=n}^{\infty} b_k\right|^2 < \infty.$$

We say that $a, b \in (G)$ *if*

$$\sum_{n=0}^{\infty} n(|a_n - 1| + |b_n|) < \infty.$$

Proposition 3.2. *In Definition 3.1 one can everywhere equivalently replace* $a_n - 1$ *with* $a_n^2 - 1$.

P r o o f. It is obvious for (S) and (G), we prove it for (R). Assume that

$$\sum_{n=0}^{\infty} n \left|\sum_{k=n}^{\infty}(a_k - 1)\right|^2 < \infty.$$

Define c_n as

$$c_n = \sum_{k=n}^{\infty}(a_k - 1). \tag{3.1}$$

By assumption $c_n \in \ell_1^2$. Then $a_n - 1 = c_n - c_{n+1} \in \ell_1^2$. Define d_n as

$$d_n = \sum_{k=n}^{\infty}(a_k - 1)^2.$$

Then, by Lemma 2.4 of [7], $d_n \in \ell_1^2$. Therefore,

$$f_n = \sum_{k=n}^{\infty}(a_k^2 - 1) = d_n + 2c_n \in \ell_1^2.$$

In other words,

$$\sum_{n=0}^{\infty} n \left|\sum_{k=n}^{\infty}(a_k^2 - 1)\right|^2 < \infty.$$

Conversely, assume that

$$\sum_{n=0}^{\infty} n|\sum_{k=n}^{\infty}(a_k^2-1)|^2 < \infty.$$

Define f_n as

$$f_n = \sum_{k=n}^{\infty}(a_k^2-1).$$

By assumption $f_n \in \ell_1^2$. Then $a_n^2 - 1 = f_n - f_{n+1} \in \ell_1^2$. By Lemma 2.4 of [7],

$$\sum_{k=n}^{\infty}(a_k^2-1)^2 \in \ell_1^2.$$

Since

$$d_n := \sum_{k=n}^{\infty}(a_k-1)^2 < \sum_{k=n}^{\infty}(a_k-1)^2(a_k+1)^2 = \sum_{k=n}^{\infty}(a_k^2-1)^2 \in \ell_1^2,$$

we get that $d_n \in \ell_1^2$. Therefore,

$$c_n := \sum_{k=n}^{\infty}(a_k-1) = \frac{1}{2}(f_n - d_n) \in \ell_1^2$$

In other words,

$$\sum_{n=0}^{\infty} n\left|\sum_{k=n}^{\infty}(a_k-1)\right|^2 < \infty.$$

■

Proposition 3.3. $a, b \in (G) \Longrightarrow a, b \in (R) \Longrightarrow a, b \in (S)$.

P r o o f. The proof of the first implication is similar to the proof of Proposition 2.3 above. To prove the second implication take c_n (3.1). By (R) $c_n \in \ell_1^2$; therefore,

$$a_n - 1 = c_n - c_{n+1} \in \ell_1^2.$$

Then $a_n - 1 \in \ell^2$, which means (S). The proof for b_n is similar. ■

4. Direct Geronimus Relations

We assume that

$$w(\bar{t}) = w(t), \quad |t| = 1,$$

or equivalently, all Verblunsky parameters α_n are real numbers, and we assume that

$$w(t)m(dt) = w(t)\frac{dt}{2\pi it}$$

is a probability measure on $\mathbb{T}$. The image measure (on $[-2, 2]$) of this measure under the mapping $x = t + \frac{1}{t}$ is

$$w(t(x))\frac{dx}{\pi\sqrt{4 - x^2}}. \tag{4.1}$$

Therefore, it is also a probability measure on $[-2, 2]$.

For $\gamma_1, \gamma_2 = \pm 1$ we define four measures on $[-2, 2]$

$$\begin{aligned}\rho(x)dx =& c(\gamma_1, \gamma_2, w)\, w(t(x))\sqrt{(2 - x)^{\gamma_1}}\sqrt{(2 + x)^{\gamma_2}}\, dx\\ =& c(\gamma_1, \gamma_2, w)\, w(t(x))\sqrt{(2 - x)^{\gamma_1+1}}\sqrt{(2 + x)^{\gamma_2+1}}\frac{dx}{\sqrt{4 - x^2}},\end{aligned}$$

where the constants $c(\gamma_1, \gamma_2, w)$ are chosen such that the measures are probability measures. The later is possible since the measures are finite. For instance, if $\gamma_1 = \gamma_2 = -1$ and $c = \frac{1}{\pi}$, we get the above image measure (4.1). These four transforms are called the Szegő transforms.

The following equalities are known as the direct Geronimus relations (cf., [12, Ths. 13.1.7 and 13.2.1]).

Proposition 4.1 (Direct Geronimus Relations)**.** *Jacobi parameters a_n and b_n of the measure $\rho(x)dx$ are expressed in terms of the Verblunsky parameters of the measure $w(t)m(dt)$ as follows:*

In the case $\gamma_1 = \gamma_2 = -1$ they are

$$\begin{aligned}[a_{n+1}^{(e)}]^2 =& (1 - \alpha_{2n-1})(1 - \alpha_{2n}^2)(1 + \alpha_{2n+1}),\\ b_{n+1}^{(e)} =& (1 - \alpha_{2n-1})\alpha_{2n} - (1 + \alpha_{2n-1})\alpha_{2n-2}.\end{aligned}$$

In the case $\gamma_1 = \gamma_2 = 1$ they are

$$\begin{aligned}[a_{n+1}^{(o)}]^2 =& (1 + \alpha_{2n+1})(1 - \alpha_{2n+2}^2)(1 - \alpha_{2n+3}),\\ b_{n+1}^{(o)} =& -(1 + \alpha_{2n+1})\alpha_{2n+2} + (1 - \alpha_{2n+1})\alpha_{2n}.\end{aligned}$$

In the case $\gamma_1 = -\gamma_2 = \pm 1$ *they are*

$$\begin{aligned}[a_{n+1}^{(\pm)}]^2 =&(1 \pm \alpha_{2n})(1 - \alpha_{2n+1}^2)(1 \mp \alpha_{2n+2}),\\ b_{n+1}^{(\pm)} =& \mp (1 \pm \alpha_{2n})\alpha_{2n+1} - \pm(1 \mp \alpha_{2n})\alpha_{2n-1}.\end{aligned}$$

Here $n = 0, 1, \ldots,$ $\alpha_{-1} = -1$.

Proposition 4.2.

$$\begin{aligned}\alpha \in \ell_1^2 \Longrightarrow&\ \ a, b \in (R),\\ \alpha \in (K) \Longrightarrow&\ \ a, b \in (G).\end{aligned}$$

P r o o f. We consider case (e), other ones are similar. By Proposition 4.1

$$\begin{aligned}[a_{n+1}^{(e)}]^2 =&(1 - \alpha_{2n-1})(1 - \alpha_{2n}^2)(1 + \alpha_{2n+1}) = 1 - \alpha_{2n-1} + \alpha_{2n+1}\\ &-\alpha_{2n}^2 - \alpha_{2n-1}\alpha_{2n+1} + \alpha_{2n-1}\alpha_{2n}^2 - \alpha_{2n+1}\alpha_{2n}^2 + \alpha_{2n-1}\alpha_{2n}^2\alpha_{2n+1}.\end{aligned}$$

Let $\alpha_n \in \ell_1^2$. Consider

$$\sum_{k=n}^{\infty}(a_{k+1}^2 - 1) = -\alpha_{2n-1} + \sum_{k=n}^{\infty} \ldots$$

All terms in the sum on the right are either "quadratic" in α or dominated by terms "quadratic" in α. By Lemma 2.4 of [7], the sequence on the right is in ℓ_1^2. Therefore,

$$\sum_{k=n}^{\infty}(a_{k+1}^2 - 1) \in \ell_1^2,$$

meaning that $a_n \in (R)$.

Let $\alpha_n \in (K)$, then

$$\begin{aligned}|a_{n+1}^2 - 1| \le& | - \alpha_{2n-1} + \alpha_{2n+1}|\\ &+|\alpha_{2n}|^2 + |\alpha_{2n-1}\alpha_{2n+1}| + |\alpha_{2n-1} - \alpha_{2n+1}||\alpha_{2n}|^2 + |\alpha_{2n-1}\alpha_{2n}^2\alpha_{2n+1}|\\ &\le C(| - \alpha_{2n-1} + \alpha_{2n+1}| + |\alpha_{2n}|^2 + |\alpha_{2n-1}|^2 + |\alpha_{2n+1}|^2).\end{aligned}$$

By Proposition 2.3 $\alpha \in (K) \Longrightarrow \alpha \in \ell_1^2$. Therefore, $a_n \in (G)$.

The proofs for b_n are similar. ■

5. Inverse Geronimus Relations

Since there are four direct Szegő transforms (Geronimus relations), there are, respectively, four inverse Szegő transforms (inverse Geronimus relations).

Proposition 5.1 (Inverse Geronimus Relations)**.** *Let the spectrum of J $\sigma(J) \subseteq [-2,2]$. Let P_n and Q_n be the monic orthogonal polynomials of the first and the second kind, respectively, for J with the parameters a_n and b_n. We define $F_n(\pm 2)$ as follows:*

- *for the case (e)*

$$R_n(-2) = P_n(-2), \quad R_n(2) = P_n(2);$$

- *for the case (o)*

$$R_n(-2) = P_n(-2) + \frac{Q_n(-2)}{m(-2)}, \quad R_n(2) = P_n(2) + \frac{Q_n(2)}{m(2)};$$

- *for the case $(+)$*

$$R_n(-2) = P_n(-2), \quad R_n(2) = P_n(2) + \frac{Q_n(2)}{m(2)};$$

- *for the case $(-)$*

$$R_n(-2) = P_n(-2) + \frac{Q_n(-2)}{m(-2)}, \quad R_n(2) = P_n(2).$$

We define

$$A_n = -\frac{R_{n+1}(-2)}{R_n(-2)}, \quad B_n = \frac{R_{n+1}(2)}{R_n(2)}.$$

Then α_n for the inverse Geronimus relations are computed as follows:

- *for the case (e)*

$$\alpha_{2n} = \frac{A_n - B_n}{A_n + B_n}, \quad \alpha_{2n-1} = 1 - \frac{A_n + B_n}{2};$$

- *for the case (o)*

$$-\alpha_{2n+2} = \frac{A_n - B_n}{A_n + B_n}, \quad -\alpha_{2n+1} = 1 - \frac{A_n + B_n}{2};$$

- *for the case* (+)

$$-\alpha_{2n+1} = \frac{A_n - B_n}{A_n + B_n}, \quad -\alpha_{2n} = 1 - \frac{A_n + B_n}{2};$$

- *for the case* (−)

$$\alpha_{2n+1} = \frac{A_n - B_n}{A_n + B_n}, \quad \alpha_{2n} = 1 - \frac{A_n + B_n}{2}.$$

We define the "right" inverse Szegő transform by following Ryckman.

Definition 5.2. *Let $m(z)$ be the m-function of a Jacobi matrix J:*

$$m(z) = \int_{-2}^{2} \frac{\rho(x)\,dx}{x - z}\,.$$

The "right" inverse Szegő transform for the J is

- *(e) if both $m(-2)$ and $m(2)$ are infinite,*
- *(o) if both $m(-2)$ and $m(2)$ are finite,*
- *(+) if $m(-2)$ is infinite and $m(2)$ is finite,*
- *(−) if $m(-2)$ is finite and $m(2)$ is infinite.*

R e m a r k 5.3. In what follows we will use the function

$$F_n(z) := P_n(z) + \frac{Q_n(z)}{m(z)}.$$

In general,

$$R_n(\pm 2) \neq F_n(\pm 2). \tag{5.1}$$

However, for the "right" inverse Szegő transform

$$R_n(\pm 2) = F_n(\pm 2).$$

Theorem 5.4. *Let a, b be Jacobi parameters such that the corresponding Jacobi matrix does not have a discrete spectrum, then*

$$a, b \in (R) \Longrightarrow \alpha \in \ell_1^2,$$
$$a, b \in (G) \Longrightarrow \alpha \in (K),$$

where α are defined by the "right" inverse Szegő transform.

R e m a r k 5.5. The first implication was proved by E. Ryckman in [7, 8] (Cor. 5.8 below), the second is a result of this note. As it was shown in [5] for the class (G) and in [7], for the class (R), a Jacobi matrix of the classes may have at most finitely many eigenvalues outside $[-2, 2]$. To apply the inverse Szegő transform we need to assume that there is no discrete spectrum.

R e m a r k 5.6. The following example from B. Simon's book [12] (Example 13.1.3 Revisited on page 876) shows that the assertion of Theorem 5.4 may fail if one chooses an inverse Szegő transform, which is not the "right" one. Namely, let $a_n = 1, b_n = 0$ for $n \geq 1$. Then the spectral measure and m-function are

$$\rho(dx) = \frac{\sqrt{4-x^2}}{2\pi}\chi_{[-2,2]}\,dx, \qquad m(z) = \frac{\sqrt{z^2-4}-z}{2}.$$

Therefore, both $m(\pm 2)$ are finite. In this case the "right" inverse Szegő transform is (o), and the corresponding $\alpha_n = 0$ is in (K).

For this example (e) is not the "right" inverse Szegő transform, the corresponding sequence of Verblunsky parameters $\alpha_{2n} = 0$, $\alpha_{2n-1} = -(n+1)^{-1}$, $n \geq 0$, is not in ℓ^2_1.

The key tool in proving Theorem 5.4 is the following asymptotics of the so-called small solution of the Jacobi equation.

Theorem 5.7 (E. Ryckman [7, 8])**.** *Let*

$$F_n(z) = P_n(z) + \frac{Q_n(z)}{m(z)},$$

where P_n and Q_n are as above. In other words, $F_n(z)$ is a solution of the equation

$$F_{k+1}(z) + (b_{k+1} - z)F_k(z) + a_k^2 F_{k-1}(z) = 0 \tag{5.2}$$

with the initial conditions

$$F_{-1}(z) = -\frac{1}{m(z)}, \quad F_0(z) = 1.$$

Let $a, b \in (R)$, then

$$\frac{F_k(\pm 2)}{F_{k-1}(\pm 2)} = \pm 1 + \epsilon_k(\pm 2), \qquad \epsilon_k(\pm 2) \in \ell^2_1.$$

Corollary 5.8 (E. Ryckman [7, 8])**.** *Let $a, b \in (R)$, then*

$$\alpha_n \in \ell^2_1,$$

where α_n are defined by the "right" inverse Szegő transform.

P r o o f. Since we use the "right" inverse Szegő transform, (5.1) holds and we have, by Theorem 5.7, that

$$A_n = -\frac{R_{n+1}(-2)}{R_n(-2)} = -\frac{F_{n+1}(-2)}{F_n(-2)} = 1 + \ell_1^2$$

and

$$B_n = \frac{R_{n+1}(2)}{R_n(2)} = \frac{F_{n+1}(2)}{F_n(2)} = 1 + \ell_1^2.$$

Hence, $A_n - B_n$ and $1 - \frac{1}{2}(A_n + B_n)$ are in ℓ_1^2. By the inverse Geronimus relations we get that $\alpha_n \in \ell_1^2$, (which is verified separately for even and odd indices). ■

We will prove the second assertion in Theorem 5.4 as a corollary of the next proposition.

Proposition 5.9. *Let $a, b \in (G)$, then*

$$\frac{F_k(\pm 2)}{F_{k-1}(\pm 2)} = \pm 1 + \epsilon_k(\pm 2), \tag{5.3}$$

where

$$\sum_{k=0}^{\infty} k|\epsilon_k(\pm 2) - \epsilon_{k+1}(\pm 2)| < \infty.$$

P r o o f. We show it for $z = 2$; for $z = -2$ the proof is analogous. We use again equation (5.2). From there we have

$$\frac{F_{k+1}(z)}{F_k(z)} + (b_{k+1} - z) + a_k^2 \frac{F_{k-1}(z)}{F_k(z)} = 0. \tag{5.4}$$

We substitute (5.3) into (5.4) to get

$$(1 + \epsilon_{k+1}) + (b_{k+1} - 2) + a_k^2 \frac{1}{1 + \epsilon_k} = 0.$$

Transform it

$$(1 + \epsilon_{k+1}) + (b_{k+1} - 2) + a_k^2 \left(1 - \epsilon_k + \frac{\epsilon_k^2}{1 + \epsilon_k}\right) = 0,$$

$$(1 + \epsilon_{k+1}) + (b_{k+1} - 2) + \left(1 - \epsilon_k + \frac{\epsilon_k^2}{1 + \epsilon_k}\right) + (a_k^2 - 1)\left(1 - \epsilon_k + \frac{\epsilon_k^2}{1 + \epsilon_k}\right) = 0.$$

Since $\epsilon_k \to 0$, we get from here that

$$|\epsilon_{k+1} - \epsilon_k| \le C(|b_{k+1}| + |\epsilon_k|^2 + |a_k^2 - 1|).$$

By assumption, $a, b \in (G)$, then, by Proposition 3.3, $a, b \in (R)$. Therefore, by Theorem 5.7, $\epsilon_k \in \ell_1^2$ Hence, we get

$$\sum_{k=0}^{\infty} k|\epsilon_{k+1} - \epsilon_k| < \infty.$$

■

P r o o f (of the second assertion in Theorem 5.4). As in Corollary 5.8

$$A_n = -\frac{R_{n+1}(-2)}{R_n(-2)} = -\frac{F_{n+1}(-2)}{F_n(-2)} = -1 + \epsilon_{n+1}(-2)$$

and

$$B_n = \frac{R_{n+1}(2)}{R_n(2)} = \frac{F_{n+1}(2)}{F_n(2)} = 1 + \epsilon_{n+1}(2).$$

Therefore,

$$A_n - A_{n-1} = \epsilon_{n+1}(-2) - \epsilon_n(-2), \qquad B_n - B_{n-1} = \epsilon_{n+1}(2) - \epsilon_n(2).$$

Hence, by Proposition 5.9,

$$\sum_{n=0}^{\infty} n(|A_n - A_{n-1}| + |B_n - B_{n-1}|) < \infty.$$

Consider

$$\begin{aligned} &\left|\frac{A_{n+1} - B_{n+1}}{A_{n+1} + B_{n+1}} - \frac{A_n - B_n}{A_n + B_n}\right| = 2\left|\frac{A_{n+1}B_n - A_nB_{n+1}}{(A_{n+1} + B_{n+1})(A_n + B_n)}\right| \\ =&2\left|\frac{(A_{n+1} - A_n)B_n - A_n(B_{n+1} - B_n)}{(A_{n+1} + B_{n+1})(A_n + B_n)}\right| \\ \le& C(|A_{n+1} - A_n| + |B_{n+1} - B_n|). \end{aligned}$$

Also

$$\left|\left(1 - \frac{A_n + B_n}{2}\right) - \left(1 - \frac{A_n + B_n}{2}\right)\right| \le C(|A_{n+1} - A_n| + |B_{n+1} - B_n|).$$

Hence, by Inverse Geronimus relations we get that $\alpha_n \in (K)$, (which is verified separately for even and odd indices). ■

References

[1] *L.D Faddeev*, On the Relation Between S-Matrix and Potential for the One-Dimensional Schrödinger Operator. — *Dokl. Akad. Nauk SSSR* **121** (1958). (Russian)

[2] *P. Deift and E. Trubowitz*, Inverse Scattering on the Line. — *Comm. Pure Appl. Math.* **32** (1979), No. 2, 121–251.

[3] *V.A. Marchenko*, Operatory Shturma-Liuvillya i Ikh Prilozheniya. (Russian) [Sturm–Liouville Operators and Their Applications] Izd-vo Naukova Dumka, Kiev, 1977.

[4] *G.Sh. Guseinov*, The Determination of an Infinite Jacobi Matrix from the Scattering Data. — *Soviet. Math. Dokl.* **17** (1976), 596–600.

[5] *G.Sh. Guseinov*, The Scattering Problem for an Infinite Jacobi Matrix. — *Izv. Akad. Nauk ArmSSR. Ser. Mat.* **12** (1977), No. 5, 365–379. (Russian)

[6] *E. Ryckman*, A Spectral Equivalence for Jacobi Matrices. — *J. Approx. Theory* **146** (2007), No. 2, 252–266.

[7] *E. Ryckman*, A Strong Szegő Theorem for Jacobi Matrices. — *Comm. Math. Phys.* **271** (2007), No. 3, 791–820.

[8] *E. Ryckman*, Erratum: A Strong Szegő Theorem for Jacobi Matrices. — *Comm. Math. Phys.* **275** (2007), No. 2, 581–585.

[9] *E. Ryckman*, Two Spectral Equivalences for Jacobi Matrices. PhD Theses, UCLA, Los Angeles, 2007.

[10] *J. Geronimo and K. Case*, Scattering Theory and Polynomials Orthogonal on the Unit Circle. — *J. Math. Phys.* **20** (1979), No. 2, 299–310.

[11] *J. Geronimo and K. Case*, Scattering Theory and Polynomials Orthogonal on the Real Line. — *Trans. Amer. Math. Soc.* **258** (1980), No. 2, 467–494.

[12] *B. Simon*, Orthogonal Polynomials on the Unit Circle. Part 1: Classical Theory; Part 2: Spectral Theory. AMS Colloquium Series, AMS, Providence, RI, 2005.