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Blaschke Type Normalization on Light-Like Hypersurfaces

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In this paper we construct and study a Blaschke type normalization on the lightlike hypersurface immersions with 1-degenerate second fundamental form. We discuss basic examples and establish fundamental equations for this canonical transversal vector bundle. As an application, we characterize the Ricci flat 1-degenerate Blaschke immersions.

Key words: 1-degenerate lightlike hypersurface, Blaschke normalization (structure), null transversal vector field, characteristic vector field.

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1. Introduction

Before we indicate the aim of this article, let us recall some facts about the geometry of hypersurfaces in the Lorentzian spaces $\mathbb{R}_1^{n+2}$. In such spaces, due to the causal character of three categories (space-like, time-like and null) of the vector fields, there are three types of hypersurfaces M, namely, the Riemannian, the Lorentzian and the lightlike (null) ones. The induced metric g is a non-degenerate metric tensor field or a degenerate symmetric tensor field on M depending on whether M is of the first two types or the third one. As space-like and time-like hypersurfaces have a non-degenerate (semi)-Riemannian metric, one can consider all the fundamental intrinsic and extrinsic geometric notions. In particular, a well defined (up to sign) notion of the unit orthogonal vector field is known to lead to a canonical decomposition of the ambient tangent space $\mathbb{R}_1^{n+2}$ into two factors: tangent (to M) and orthogonal. Therefore, by respective projections, one has fundamental equations such as the Gauss, the Codazzi, the Weingarten equations,... along with the second fundamental form, sharp operator, induced connection, etc.

As for lightlike hypersurfaces, the normal bundle is a subbundle of the tangent one, the basic nuisance in studying their extrinsic geometry arises from the normalization problem.

Several authors considered this problem in various ways (Akivis–Goldberg [1, 2], Penrose [16], Katsuno [10], Dautcourt [7, 8], Rosca [18, 19], Carter [6], Taub [20], Larsen [14, 15], Pinl [17], ...). For the most part, these studies are specific for a given problem and a general theory is still desirable.

Following are two important attempts. In [11, 12, 13], Kupeli developed an approach using the factor vector bundle $TM^\star = TM/TM^\perp$, where $TM^\perp$ is the characteristic null line bundle, and used the canonical projection $\pi : TM \longrightarrow TM^\star$ in studying the intrinsic geometry of the degenerate semi-Riemannian manifolds. This approach switches the null geometry of the submanifold for a non-degenerate one. In 1991, Duggal and Bejancu [9] introduced a general geometric technique to deal with the above anomaly. Their approach is basically extrinsic in contrast to the intrinsic one developed by Kupeli, that is very close and consistent with the known theory of non-degenerate submanifolds. This approach introduces a non-degenerate screen distribution (or equivalently a null transversal vector bundle) so as to get three factors splitting the ambient tangent space and derive the main induced geometric objects such as second fundamental forms, sharp operators, induced connections, curvature, etc. Unfortunately, the screen distribution is not unique and there is no preferred one in general. As a consequence, it is a systematic task in this approach to study a dependence of the discussed structures and the induced geometric objects with respect not only to the screen distribution but also to the choice of the normalizing pair of null vectors.

Obviously, this situation is very close to the classical *affine differential geometry* in which the fundamental fact is the existence of the *Blaschke structure.* It is our aim in this article to introduce and study a natural analogue of the Blaschke structure for the class of lightlike hypersurfaces in the Lorentz spaces $\mathbb{R}_1^{n+2}, (n \geq 1)$. More precisely, for a 1-degenerate lightlike hypersurface immersion, we will first show the existence of a unique (up to sign) normalized null transversal vector field that is equiaffine and satisfies some apolarity condition. Thereafter we make a systematic study of the geometry of this structure.

The paper is organized as follows. In Section 2 we make a general set up on the lightlike hypersurfaces and establish some technical results. In Section 3 we introduce an admissible invariant metric volume form used in Section 4 to construct the Blaschke structure. Section 5 is devoted to some basic examples. In Section 6 we study the Blaschke fundamental equations and characterize the Ricci flat 1-degenerate Blaschke immersions.

2. Basic Facts on Lightlike Hypersurfaces

Consider a hypersurface M of an $(n+2)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant index $0 < \nu < n+2$. In the classical theory of non-degenerate hypersurfaces, the normal bundle has trivial intersection $\{0\}$ with the tangent bundle and plays an important role in the introduction of the main induced geometric objects on M. In case of lightlike (degenerate, null) hypersurfaces, the situation is totally different. The normal bundle $TM^\perp$ is a rank-one distribution on M: $TM^\perp \subset TM$ and then coincides with the so-called *radical distribution* $RadTM = TM \cap TM^\perp$. Hence, the induced metric tensor field g is degenerate on M and it has a constant rank n.

A complementary bundle of $TM^\perp$ in TM is a rank n non-degenerate distribution on M. It is called a *screen distribution* on M and is often denoted by $S(TM)$. A lightlike hypersurface with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^\perp$ lies in the tangent bundle, the following result is important in studying the extrinsic geometry of a lightlike hypersurface.

Theorem 2.1. [9] *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$ with a given screen distribution $S(TM)$. Then there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}|_M$ such that for any non-zero section ξ of $TM^\perp$ on a coordinate neighbourhood $\mathcal{U} \subset M$ there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying*

$$\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \qquad \forall\ W \in \Gamma(ST(M)|_{\mathcal{U}}). \tag{2.1}$$

Throughout the paper, all manifolds will be assumed to be smooth, connected and paracompact. We denote by $\Gamma(E)$ the $\mathcal{F}(M)$-module of the smooth sections of a vector bundle E over M, $\mathcal{F}(M)$ being the algebra of smooth functions on M. Also, by $\oplus_{Orth}$ and $\oplus$ we denote the orthogonal and non-orthogonal direct sum of two vector bundles. By Theorem 2.1, we may write down the following decompositions:

$$TM = S(TM) \oplus_{Orth} TM^\perp \tag{2.2}$$

and

$$T\overline{M}|_M = S(TM) \oplus_{Orth} (TM^\perp \oplus tr(TM)) = TM \oplus tr(TM). \tag{2.3}$$

As it is well known, we have the following:

Definition 2.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$ with a given screen distribution $S(TM)$. The induced connection, say ∇, on M is defined by*

$$\nabla_X Y = Q(\overline{\nabla}_X Y), \tag{2.4}$$

where $\overline{\nabla}_X Y$ denotes the Levi-Civita connection on $(\overline{M}, \overline{g})$ and Q is the projection onto TM with respect to the decomposition (2.3).

Remark 2.1. *Notice that the induced connection ∇ on M depends on both g and the specific given screen distribution $S(TM)$ on M . Also, a choice of the null line bundle $tr(TM)$ is equivalent to the choice of $S(TM)$.*

The projections Q and $I - Q$ give rise to the Gauss and the Weingarten formulas in the form

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall X, \in \Gamma(TM) \ \ \forall V \in \Gamma(tr(TM)). \tag{2.6}$$

Here, $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$. Hence, h is a $\Gamma(tr(TM))$- valued symmetric $F(M)$-bilinear form on $\Gamma(TM)$, A_V is an $F(M)$-linear operator on $\Gamma(TM)$, and ∇^t is a linear connection on the lightlike transversal vector bundle $tr(TM)$.

Let P denote the projection morphism of $\Gamma(TM)$ onto $\Gamma(S(TM))$ with respect to the decomposition (2.2). We have

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + h^\star(X, PY) \ \ \forall X, Y \in \Gamma(TM), \tag{2.7}$$

$$\nabla_X U = -\overset{\star}{A}_U X + \nabla^{\star t}_X U, \quad \forall X, \in \Gamma(TM) \ \ \forall U \in \Gamma(TM^\perp). \tag{2.8}$$

Here $\overset{\star}{\nabla}_X PY$ and $\overset{\star}{A}_U X$ belong to $\Gamma(S(TM))$, $\overset{\star}{\nabla}$ and $\nabla^{\star t}$ are the linear connections on $S(TM)$ and $TM^\perp$, respectively. Then, $h^\star$ is a $TM^\perp$)-valued $F(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$, and $\overset{\star}{A}_U$ is a $\Gamma(S(TM))$-valued $F(M)$-linear operator on $\Gamma(TM)$. They are a second fundamental form and a shape operator of the screen distribution, respectively.

Equivalently, consider a normalizing pair ξ, N as in Theorem 2.1. Then (2.5) and (2.6) take the form

$$\overline{\nabla}_X Y = \nabla_X Y + B^N(X, Y)N, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}) \tag{2.9}$$

and

$$\overline{\nabla}_X N = -A_N X + \tau^N(X)N, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \tag{2.10}$$

where we put locally on $\mathcal{U}$

$$B^N(X, Y) = g(h(X, Y), \xi), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{2.11}$$

$$\tau^N(X) = \overline{g}(\nabla^t_X N, \xi), \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{2.12}$$

It is important to emphasize that the local second fundamental form B^N in (2.11) does not depend on the choice of the screen distribution.

We also define (locally) on $\mathcal{U}$ the following:

$$C^N(X, PY) = \overline{g}(h^\star(X, PY), N), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}). \tag{2.13}$$

Thus, one has for $X, Y \in \Gamma(TM|_{\mathcal{U}})$

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C^N(X, PY)\xi \tag{2.14}$$

and

$$\nabla_X N = -\overset{\star}{A}_\xi X - \tau^N(X)\xi. \tag{2.15}$$

It is straightforward to verify that for $X, Y \in \Gamma(TM)$

$$B^N(X, \xi) = 0, B^N(X, Y) = g(\overset{\star}{A}_\xi X, Y), \overset{\star}{A}_\xi \xi = 0, C^N(X, PY) = g(A_N X, Y) \tag{2.16}$$

The induced connection is torsion-free, but not necessarily g-metric, and we have for all tangent vector fields X, Y and Z in TM

$$(\nabla_X g)(Y, Z) = B^N(X, Y)\eta(Z) + B^N(X, Z)\eta(Y), \tag{2.17}$$

where η is a 1-form defined by

$$\eta(\cdot) = \overline{g}(N, \cdot). \tag{2.18}$$

From (2.17) it follows that ∇ is g-metric if and only if M is totally geodesic (i.e., $B^N = 0$). On the other hand, the linear connection $\overset{\star}{\nabla}$ is a metric connection on $S(TM)$.

The following lemma accounts for a relationship between the induced geometric objects described above with respect to the choice of two different normalizing pairs of the null vector fields as in Theorem 2.1.

Lemma 2.1. *Let $\{\xi, N\}$ be a normalizing pair as in Theorem 2.1 and make a change $\{\tilde{\xi}, \tilde{N}\}$ with $\tilde{N} = \phi N + \zeta$, where $\zeta \in \Gamma(TM)$ and $\phi \in C^\infty(M)^\star$. Then*

(a) $\tilde{\xi} = \frac{1}{\phi}\xi$,

(b) $2\phi\eta(\zeta) + ||\zeta||^2 = 0$,

(c) $B^{\tilde{N}}(X, Y) = \frac{1}{\phi}B^N(X, Y)$,

(d) $\widetilde{P} = PY - \frac{1}{\phi}g(\zeta, Y)\xi$,

(e) $C^{\widetilde{N}}(X, \widetilde{P}Y) = \phi C^N(X, PY) - g(\nabla_X \zeta, PY)$
$+ [\tau^N(X) + \dfrac{X \cdot \phi}{\phi} + \dfrac{1}{\phi} B^N(\zeta, X)] g(\zeta, Y),$

(f) $\widetilde{\nabla}_X Y = \nabla_X Y - \dfrac{1}{\phi} B^N(X, Y)\zeta,$

(g) $\tau^{\widetilde{N}} = \tau^N + d\ln|\phi| + \dfrac{1}{\phi} B^N(\zeta, \cdot),$

(h) $A_{\widetilde{N}} = \phi A_N - \nabla_{.}\zeta + [\tau^N + d\ln|\phi| + \dfrac{1}{\phi} B^N(\zeta, \cdot)]\zeta,$

(i) $\overset{\star}{A}_{\widetilde{\xi}} = \dfrac{1}{\phi} \overset{\star}{A}_{\xi} - \dfrac{1}{\phi^2} B^N(\zeta, \cdot)\xi,$

for all tangent vector fields X and Y.

P r o o f. The first two relations in items (a) and (b) are immediate consequences of the relations $\overline{g}(\widetilde{N}, \widetilde{\xi}) = 1$, $\overline{g}(\widetilde{N}, \widetilde{N}) = 0$ and $dim(Rad(TM)) = 1$. Writing the Gauss, respectively the Weingarten, formulas for both pairs $\{\xi, N\}$ and $\{\widetilde{\xi}, \widetilde{N}\}$, we obtain by identification the relations in items (c) and (f) (respectively, (g) and (h)). Now let $Y \in \Gamma(TM)$, we have

$$\begin{aligned} Y &= \widetilde{P}Y + \widetilde{\eta}(Y)\widetilde{\xi} \\ &= \widetilde{P}Y + \widetilde{\eta}(Y)(\frac{1}{\phi}\xi) \\ &= \widetilde{P}Y + \frac{1}{\phi}\widetilde{\eta}(Y)\xi. \end{aligned}$$

Then

$$\begin{aligned} \widetilde{P}Y &= Y - \frac{1}{\phi}\widetilde{\eta}(Y)\xi \\ &= Y - \frac{1}{\phi}\overline{g}(\widetilde{N}, Y)\xi \\ &= Y - \frac{1}{\phi}\overline{g}(\phi N + \zeta, Y)\xi \\ &= Y - \frac{1}{\phi}[\phi\eta(Y) + g(\zeta, Y)]\xi \\ &= Y - \eta(Y)\xi - \frac{1}{\phi}g(\zeta, Y)\xi \\ &= PY - \frac{1}{\phi}g(\zeta, Y)\xi \end{aligned}$$

and the item (d) is derived. By using the definition of $C^{\widetilde{N}}$, we have

$$\begin{aligned} C^{\widetilde{N}}(X, \widetilde{P}Y) &= \overline{g}(A_{\widetilde{N}}X, \widetilde{P}Y) \\ &\overset{(h)}{=} \overline{g}(\phi A_N X - \nabla_X \zeta + [\tau^N(X) \\ &\quad + X \cdot (\ln|\phi|) + \frac{1}{\phi} B^N(\zeta, X)]\zeta, \widetilde{P}Y) \\ &\overset{(d)}{=} \overline{g}(\phi A_N X - \nabla_X \zeta + [\tau^N(X) \\ &\quad + X \cdot (\ln|\phi|) + \frac{1}{\phi} B^N(\zeta, X)]\zeta, PY - \frac{1}{\phi} g(\zeta, Y)\xi) \\ &\overset{(d)}{=} \overline{g}(\phi A_N X - \nabla_X \zeta + [\tau^N(X) \\ &\quad + X \cdot (\ln|\phi|) + \frac{1}{\phi} B^N(\zeta, X)]\zeta, PY) \\ &= C^{\widetilde{N}}(X, PY) = \phi C^N(X, PY) - g(\nabla_X \zeta, PY) \\ &\quad + [\tau^N(X) + \frac{X \cdot \phi}{\phi} + \frac{1}{\phi} B^N(\zeta, X)] g(\zeta, Y) \end{aligned}$$

which establishes relation (e). Finally, we have

$$\widetilde{\nabla}_X \widetilde{\xi} = - \overset{\star}{A}_{\widetilde{\xi}} X - \tau^{\widetilde{N}}(X)\widetilde{\xi}.$$

But using (f), we get

$$\begin{aligned} \widetilde{\nabla}_X \widetilde{\xi} &= \nabla_X \widetilde{\xi} - \frac{1}{\phi} B^N(X, \widetilde{\xi})\zeta \\ &= \nabla_X \widetilde{\xi} \\ &\overset{(a)}{=} \nabla_X (\frac{1}{\phi}\xi) \\ &= -\frac{X \cdot (\phi)}{\phi^2}\xi + \frac{1}{\phi}(- \overset{\star}{A}_{\xi} X - \tau^N(X)\xi) \\ &= -[\frac{X \cdot (\phi)}{\phi^2} + \frac{1}{\phi}\tau^N(X)]\xi - \frac{1}{\phi} \overset{\star}{A}_{\xi} X. \end{aligned}$$

Identifying the above two expressions of $\widetilde{\nabla}_X \widetilde{\xi}$, we get

$$\begin{aligned} - \overset{\star}{A}_{\widetilde{\xi}} X &= \frac{1}{\phi} \overset{\star}{A}_{\xi} X + [\frac{X \cdot (\phi)}{\phi^2} + \frac{1}{\phi}\tau^N(X)]\xi - \frac{1}{\phi}\tau^{\widetilde{N}}(X)\xi \\ &= \frac{1}{\phi} \overset{\star}{A}_{\xi} X + [\frac{X \cdot (\phi)}{\phi^2} + \frac{1}{\phi}\tau^N(X)]\xi - \frac{1}{\phi}[\tau^N(X) + \frac{X \cdot (\phi)}{\phi} + \frac{1}{\phi} B^N(\zeta, X)]\xi \\ &= \frac{1}{\phi} \overset{\star}{A}_{\xi} X - \frac{1}{\phi^2} B^N(\zeta, X)\xi, \end{aligned}$$

and we obtain the relation in (i), which completes the proof. ∎

3. An Invariant Metric Volume Form

Consider a lightlike hypersurface immersion $f : M \longrightarrow \mathbb{R}^{n+2}_1$, and let g denote the metric tensor field induced on $f(M)$. We have

$$g(X, Y)|_x = \langle f_\star X, f_\star Y\rangle|_{f(x)}$$

for any X, Y tangent to M, where $\langle , \rangle := \overline{g}$ denotes the Lorentz metric on $\mathbb{R}^{n+2}_1$, and $f_\star$ denotes the tangent map. In the sequel, we identify M and $f(M)$ and write x and M instead of $f(x)$ and $f(M)$. Also, throughout the text, we consider on $\mathbb{R}^{n+2}_1$ a parallel volume form given by the standard determinant det.

Let N be a null transversal vector field on M. As B^N is degenerate, it is not possible to define a volume form ω_{B^N} relative to B^N in a usual way. By item (c) in Lemma 2.1, it is remarkable that the rank of second fundamental form B^N is invariant under the change of transversal null vector field N . We define this invariant as a rank of the lightlike hypersurface immersion. Now consider the case when B^N has the maximal rank n (or equivalently has nullity degree 1). In this case we say that the lightlike hypersurface immersion is 1-degenerate, which we assume from now on.

The following range of indices will be in the order: $i, j, \dots = 0, \dots, n$; $a, b, \dots = 1, \dots, n$ and $\alpha, \beta, \dots = 0, \dots, n+1$. Let $Q = TM/TM^{\perp}$ denote the factor bundle by the characteristic line bundle, and for $\overline{X}, \overline{Y} \in \Gamma(Q)$ set

$$\overline{B^N}(\overline{X}, \overline{Y}) = B^N(X, Y). \tag{3.1}$$

As $B(\xi\ ,\ ) = 0$, $\overline{B^N}$ is well defined. Furthermore, it is non-degenerate on Q. Let $\mathbb{F}$ denote the frame bundle of M. Then each point of $\mathbb{F}$ is $(x, e_0, \dots, e_n)$, where $(e_0, \dots, e_n)$ is a basis of T_xM with (without loss of generality) e_0 generating the characteristic null space $Rad(T_xM)$. Consider some patch with coordinates (x^i) so that the coordinate vector fields ∂_i form a local basis of TM , with $Rad(TM) = span\partial_0 = \xi$. These coordinate systems are called $\mathbb{F}$-admissible coordinate systems and a passage of a coordinate system (x^i) in a coordinate system (y^i) with $\dfrac{\partial y^c}{\partial x^0} = 0$, and $y^0 = \varepsilon x^0 + \lambda$ $(\varepsilon = \pm 1, \lambda \in \mathbb{R})$ is called an admissible coordinate change. At each point in the domain of such an admissible coordinate system, the matrix of B^N with respect to (∂_i) has the form

$$(B^N_{ab}) = \begin{pmatrix} 0 & \cdots\cdots & 0 \\ \vdots & & \\ \vdots & \overline{B^N_{ab}} & \\ 0 & & \end{pmatrix}, \tag{3.2}$$

where $\overline{B^N_{ab}} = \overline{B^N}(\partial_a, \partial_b) = B^N(\partial_a, \partial_b) = B^N_{ab}$ are the entries of the invertible rank n matrix of $\overline{B^N}$ with respect to the (local) frame $(\overline{\partial}_a)$ of $\Gamma(Q)$.

Let us define a metric volume form on M relative to B^N by

$$\omega_{B^N} = \sqrt{|det(\overline{B^N_{ab}})|}dx^0 \wedge dx^1 \wedge \cdots dx^n. \tag{3.3}$$

The $(n+1)$-form ω_{B^N} is indeed invariant under admissible coordinate changes. Let (y^j) be the admissible coordinates of another chart intersecting the one of (x^i)'s . In terms of the y-coordinates, the volume form is

$$|det(\frac{\partial x^a}{\partial y^c}\frac{\partial x^b}{\partial y^d}\overline{B^N_{ab}})|^{\frac{1}{2}}dy^0 \wedge dy^1 \wedge \cdots dy^n,$$

and noting that $\dfrac{\partial y^c}{\partial x^0} = 0$, $dy^0 = \varepsilon dx^0$ with $\varepsilon = \pm 1$ and $dy^c = \dfrac{\partial y^c}{\partial x^a}dx^a$, this becomes

$$|det(\frac{\partial x^a}{\partial y^c})|\sqrt{|det(\overline{B^N_{ab}})|}\varepsilon dx^0 \wedge det(\frac{\partial y^c}{\partial x^a})dx^1 \wedge \cdots dx^n,$$

that is equal to

$$\pm\varepsilon\sqrt{|det(\overline{B^N_{ab}})|}dx^0 \wedge dx^1 \wedge \cdots dx^n.$$

Hence, by appropriate choice of orientation, we get $\pm\varepsilon = 1$, and ω_{B^N} is invariant under admissible coordinate changes.

Remark 3.1. *Starting with a null transversal vector field N, make a change $N = \phi N + \zeta$. Then, by item (c) in Lemma 2.1, we have*

$$det(\overline{B^{\tilde{N}}_{ab}}) = \phi^{-n}det(\overline{B^N_{ab}}),$$

hence we have

$$\omega_{B^{\tilde{N}}} = \phi^{-\frac{n}{2}}\omega_{B^N}. \tag{3.4}$$

4. The Blaschke Structure on the 1-Degenerate M

Let θ be an arbitrary volume form in the neighbourhood $\mathcal{U}$ of a point x. An admissible basis $(\partial_0, \ldots, \partial_n)$ of T_xM is said to be unimodular for θ if $\theta(\partial_0, \ldots, \partial_n) = 1$. For a lightlike hypersurface immersion $f : M \longrightarrow \mathbb{R}^{n+2}_1$, let N be a null transversal vector field and consider the parallel volume form on $\mathbb{R}^{n+2}_1$ given by the standard determinant *det*. In addition to the induced geometric objects discussed in Section , we set

$$\theta^N(X_0, \ldots, X_n) = det[f_\star(X_0), \ldots, f_\star(X_n), N]. \tag{4.1}$$

Then θ^N is a volume form on M called the induced volume form (with respect to N). Now, for an admissible basis $(\partial_0, \dots, \partial_n)$ of T_xM, consider the matrix $(\overline{B^N_{ab}})$ of $\overline{B^N}$. We have

$$det(\overline{B^N_{ab}}) = \begin{vmatrix} 1 & 0 & \dots\dots & 0 \\ 0 & & & \\ \vdots & & & \\ \vdots & & \overline{B^N_{ab}} & \\ 0 & & & \end{vmatrix} = \begin{vmatrix} 1 & 0 & \dots\dots & 0 \\ 0 & & & \\ \vdots & & & \\ \vdots & & B^N_{ab} & \\ 0 & & & \end{vmatrix}$$

as $\overline{B^N_{ab}} = B^N_{ab}$ using (3.1). Let

$$E^N_a = \begin{pmatrix} 0 \\ B^N_{1a} \\ \vdots \\ \vdots \\ B^N_{na} \end{pmatrix} \quad \text{and} \quad E^N_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

Then,

$$det(\overline{B^N_{ab}}) = det(E^N_0, E^N_1, \dots, E^N_n) = \psi(B^N) det(\partial_0, \dots, \partial_n).$$

As there exists a non-vanishing function ρ (independent of $(\partial_0, \dots, \partial_n)$) such that $det(\partial_0, \dots, \partial_n) = \rho\theta^N(\partial_0, \dots, \partial_n)$, it follows that

$$det(\overline{B^N_{ab}}) = \psi(B^N)\rho\theta^N(\partial_0, \dots, \partial_n).$$

Hence, if we restrict on a unimodular admissible basis for θ^N, then the determinant of the matrix $(\overline{B^N_{ab}})$ is independent of the choice of unimodular admissible basis $(\partial_0, \dots, \partial_n)$ for θ^N. We denote this number $\psi(B^N)\rho$ by $det_{\theta^N} B^N$ and call it *the determinant of* B^N *relative to* θ^N.

Remark 4.1. *It follows that for an arbitrary admissible basis* $(\partial_0, \dots, \partial_n)$ *we have*

$$det(\overline{B^N_{ab}}) = det_{\theta^N} B^N \theta^N(\partial_0, \dots, \partial_n). \tag{4.2}$$

The following lemma shows what is the behaviour of $det_{\theta^N} B^N$ with respect to a change in null transversal vector field.

Lemma 4.1. *In the lightlike hypersurface immersion* $f : M \longrightarrow \mathbb{R}^{n+2}_1$, *suppose we change a null transversal vector field* N *to* $\widetilde{N} = \phi N + \zeta$. *Then,*

$$det_{\theta^{\widetilde{N}}} B^{\widetilde{N}} = \phi^{-(n+2)}(det_{\theta^N} B^N). \tag{4.3}$$

P r o o f. Using (4.1) for $\widetilde{N}$ we find $\theta^{\widetilde{N}} = \phi\theta^N$ Then, $(\partial_0, \dots, \partial_n)$ being a unimodular admissible basis for θ^N, it follows that $(\partial_0, \phi^{-1}\partial_1, \dots, \partial_n)$ is a unimodular admisible basis for unimodular admissible basis for $\theta^{\widetilde{N}}$. Hence, we obtain

$$det_{\theta^{\tilde{N}}} B^N = \phi^{-2}(det_{\theta^N} B^N). \tag{4.4}$$

On the other hand, by item (c) in Lemma 2.1, we have $B^{\widetilde{N}} = \dfrac{1}{\phi} B^N$ and

$$det_{\theta^N} B^{\widetilde{N}} = \phi^{-n}(det_{\theta^N} B^N). \tag{4.5}$$

Finally, we get

$$\begin{aligned} det_{\theta^{\tilde{N}}} B^{\widetilde{N}} &\overset{(4.4)}{=} \phi^{-2}(det_{\theta^N} B^{\widetilde{N}}) \\ &\overset{(4.5)}{=} \phi^{-2}\phi^{-n}(det_{\theta^N} B^N) \\ &= \phi^{-(n+2)}(det_{\theta^N} B^N). \end{aligned}$$

■

Now, it is our aim to achieve, by an appropriate choice of the null transversal vector field N , the following two goals:

(B_1) (∇^N, θ^N) is an equiaffine structure, i.e $\nabla^N \theta^N = 0$,

(B_2) θ^N coincides with the volume element ω_{B^N} relative to the 1-degenerate second fundamental form B^N.

We prove the following result.

Theorem 4.1. *Let $f : M \longrightarrow \mathbb{R}_1^{n+2}$ be a 1-degenerate lightlike hypersurface (isometric) immersion. For each point $x_0 \in M$ there is a null transversal vector field defined in a neighbourhood of x_0 satisfying conditions (B_1) and (B_2) above, such a null transversal vector field is unique up to sign and gives rise to a normalization of the null characteristic vector field.*

P r o o f. Start by a tentative null transversal N , and note that by (4.2) in Remark 4.1 we have

$$\omega_{B^N} = |det_{\theta^N} B^N|^{\frac{1}{2}} |\theta^N((\partial_0, \dots, \partial_n))|^{\frac{1}{2}} dx^0 \wedge dx^1 \dots \wedge dx^n.$$

It follows that $\theta^N = \omega_{B^N}$ if and only if $|det_\theta^N B^N|^{\frac{1}{2}} = 1$. Now make the change $\widetilde{N} = \phi N + \zeta$, $\zeta \in \Gamma(TM)$. Then, by Lemma 4.1, we have $det_{\theta^{\tilde{N}}} B^{\widetilde{N}} = \phi^{-(n+2)}(det_{\theta^N} B^N)$. Hence to realize $\omega_{B^{\tilde{N}}} = \theta^{\widetilde{N}}$, it is necessary and sufficient to set

$$det_{\theta^{\tilde{N}}} B^{\widetilde{N}} = 1 = \phi^{-(n+2)}(det_{\theta^N} B^N),$$

that is

$$|\phi| = |det_{\theta^N} B^N)|^{\frac{1}{n+2}}. \qquad (4.6)$$

On the other hand, let D denote the flat connection on $\mathbb{R}_1^{n+2}$. From the standard parallel volume form det, we have

$$0 = (D_X det)(X_0, \dots, X_n, N) = (\nabla_X^N \theta^N)(X_{0,\dots,X_n}) - \tau^N(X)\theta^N(X_0, \dots, X_n)$$

for all basis $(X_1, \dots, X_n)$ and X is tangent to M. It follows that the equiaffine condition is equivalent to $\tau^N = 0$, that is $D_X N$ is tangent to M. Hence, in item (g) in Lemma 2.1, ϕ being chosen as in (4.6), we have to choose ζ such that $\tau^{\widetilde{N}} = 0$. But,

$$\tau^{\widetilde{N}} = 0 \iff B^N(\zeta, \dot{}) = -\phi\tau^N - d\phi. \qquad (4.7)$$

B^N is 1-degenerate with characteristic direction ξ, then the last equality in (4.7) determines ζ up to the characteristic component. But from item (b) in Lemma 2.1 we have $2\phi\eta(\zeta) + ||\zeta||2 = 0$. So, we only need a non-characteristic component from (4.7), and we use the above relation to complete ζ.

Let $\zeta = \zeta^0\xi + \zeta^a\partial_a$. From (4.7), we have

$$-\phi\tau^N(\xi) - \xi(\phi) = 0,$$

and

$$\zeta^a B^N_{ac} = -\phi\tau^N(\partial_c) - \partial_c(\phi).$$

As $B^N_{ab} = \overline{B^N}_{ab}$, $(a, b = 1 \dots, n)$, we have

$$\begin{aligned} \zeta^a \overline{B^N}_{ab} &= \zeta^a B^N_{ac} \\ &= -\phi\tau^N(\partial_c) - \partial_c(\phi). \end{aligned}$$

Hence

$$\zeta^a = -\overline{B^N}^{ac}[\phi\tau^N(\partial_c) + \partial_c(\phi)]. \qquad (4.8)$$

Then we have

$$\|\zeta\|^2 = \overline{g}_{ab}\overline{B^N}^{ac}\overline{B^N}^{be}[\phi\tau^N(\partial_c) + \partial_c(\phi)][\phi\tau^N(\partial_e) + \partial_e(\phi)],$$

and by item (b) in Lemma2.1, the null component of ζ is given by

$$\eta(\zeta) = -\frac{1}{2\phi}\left(\overline{g}_{ab}\overline{B^N}^{ac}\overline{B^N}^{be}[\phi\tau^N(\partial_c) + \partial_c(\phi)][\phi\tau^N(\partial_e) + \partial_e(\phi)]\right). \qquad (4.9)$$

Then ζ is determined by (4.8) and (4.9).

Now, we show that the null transversal vector field is unique up to sign. Suppose N and $\widetilde{N} = \phi N + \zeta$ satisfy (B_1) and (B_2). It follows that

$$|det_{\theta^N} B^N| = 1 = |det_{\theta^{\tilde{N}}} B^{\widetilde{N}}|,$$

that is, using (4.6), $|\phi| = 1$ or $|\phi| = -1$. But condition (B_1) for both N and $\widetilde{N}$ leads to $\tau^N = \tau^{\widetilde{N}} = 0$. Since $\phi = \pm 1$, relation (4.8) leads to $\zeta^a = 0$ for all $a = 1, \ldots, n$ and $\|\zeta\| = 0$. Then, as $\phi \neq 0$, it follows item (b) in Lemma 2.1 that $\eta(\zeta) = 0$. Finally, we obtain $\zeta = 0$, $\phi = \pm 1$ and $\widetilde{N} = \pm N$, which completes the proof. ∎

Definition 4.1. *A null transversal vector field satisfying* (B_1) *and* (B_2) *is called Blaschke null transversal vector field. Locally it is uniquely determined up to sign. For each point* $x \in M$*, the line through* x *in the direction of the Blaschke null transversal vector* N_x *is independent of the choice of the sign for* N *and is called Blaschke null transverse through* x*. The triplet* (∇^N, B^N, A_N) *is called the Blaschke structure on the* 1*-degenerate lightlike hypersurface* (M, g)*. The later with this structure will be denoted* (M, g, N_{Bla})*. The unique null vector field* ξ *with* $\langle \xi, N \rangle = 1$ *is called the Blaschke normalized null characteristic (radical) vector field.*

5. Some Examples

Beyond all physical considerations, the null cone $\wedge_0^{n+2} \subset \mathbb{R}_1^{n+2}$ is one of the most important manifolds with lightlike metric. In fact, as we know from [5], the null cone is, up to homogeneous Riemannian factor, the only homogeneous lightlike manifold on which a Lie group with finite center acts faithfully, isometrically and non-properly. In this interest, the following example considers the case of the lightlike cone $\wedge_0^3$ in the Minkowski space $\mathbb{R}_1^4$. This example can easily be generalized to $\wedge_0^{n+2} \subset \mathbb{R}_1^{n+2}$. Our second example is concerned with more general Monge hypersurfaces.

5.1. Blaschke structure on the lightcone $\wedge_0^3$. Let us conider the lightcone $\wedge_0^3$ as the immersion

$$\begin{array}{rcl} f : M = \mathbb{R}^3 \setminus \{0\} & \longrightarrow & \mathbb{R}_1^4 \\ (x, y, z) & \longmapsto & \left[x, y, z, \varepsilon(x^2 + y^2 + z^2)^{\frac{1}{2}}\right], \varepsilon = \pm 1. \end{array}$$

Locally, $\wedge_0^3$ is the graph $t = \varepsilon(x^2 + y^2 + z^2)^{\frac{1}{2}}$ and it is an obvious fact that this is a lightlike hypersurface immersion.

Let us take

$$N = x\partial_x + y\partial_y + z\partial_z - t\partial_t$$

as a tentative null transversal vector field. The induced volume form θ^N is thus given by

$$\theta^N(u, v, w) = det\left[f_\star u, f_\star v, f_\star w, N\right].$$

Let $p_0 = (x_0, y_0, z_0) \in M$. We may assume (without loss of generality) that $x_0 \neq 0$ as $p \neq 0$. Then there is a neighbourhood $\mathcal{U}$ of p_0 such that $x \neq 0$ on $\mathcal{U}$. Then, let

$$\begin{aligned} r &= \frac{1}{2t^2}(x\partial_x + y\partial_y + z\partial_z), \\ u_1 &= -\frac{1}{x}(y\partial_x - x\partial_y), \\ u_2 &= \frac{1}{t}(z\partial_x - x\partial_z). \end{aligned}$$

To see that (r, u_1, u_2) is a unimodular basis for θ^N, it is easy to check that

$$\begin{aligned} f_\star r &=: e_0 = \quad \xi = \frac{1}{2t^2}(x\partial_x + y\partial_y + z\partial_z + t\partial_t), \\ f_\star u_1 &=: e_1 = \quad -\frac{1}{x}(y\partial_x - x\partial_y), \\ f_\star u_2 &=: e_2 = \quad \frac{1}{t}(z\partial_x - x\partial_z), \end{aligned}$$

and then, $\theta^N(r, u_1, u_2)) = det[f_\star r, f_\star u_1, f_\star u_2, N] = 1$. We have also

$$\langle e_0, e_0\rangle = \langle e_0, e_1\rangle = \langle e_0, e_2\rangle = \langle e_1, e_N\rangle = \langle e_2, N\rangle = 0, \text{ and } \langle e_0, N\rangle = 1.$$

Hence (e_0, e_1, e_2) is an admissible basis on $f(\mathcal{U})$. Now, D being the flat Levi–Civita connection on $\mathbb{R}^4_1$, by direct calculation we have

$$\begin{aligned} D_r e_0 &= 0, \\ D_r e_1 &= 0, \\ D_r e_2 &= 0, \\ D_{u_1} e_0 &= \frac{1}{2t^2} e_1, \\ D_{u_1} e_1 &= -\frac{y}{x^3}(y\partial_x - x\partial_y) - \frac{1}{x^2}(x\partial_x + y\partial_y), \end{aligned}$$

which shows that the transversal component of $D_{u_1} e_1$ is $-\dfrac{x^2+y^2}{2x^2t^2}N$. Also,

$$D_{u_1} e_2 = \frac{y}{tx}\partial_z,$$

then its transversal component is $\frac{yz}{2xt^3}N$. We also obtain

$$\begin{aligned} D_{u_2}e_0 &= 0, \\ D_{u_2}e_1 &= \frac{yz}{2xt^3}, \\ D_{u_2}e_2 &= -\frac{x^2+z^2}{2t^4}, \end{aligned}$$

It follows that the second fundamental form B^N is given with respect to the unimodular admissible basis (r, u_1, u_2) on M by

$$B^N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{x^2+y^2}{2x^2t^2} & \frac{yz}{2xt^3} \\ 0 & \frac{yz}{2xt^3} & -\frac{x^2+z^2}{2t^4} \end{pmatrix}, \tag{5.1}$$

which shows that $\wedge_0^3$ is a 1-degenerate lightlike hypersurface in $\mathbb{R}_1^4$ with

$$det_{\theta^N} B^N = \left(\frac{1}{\sqrt{2}t}\right)^4. \tag{5.2}$$

Hence, we obtain

$$|\phi| = \frac{1}{\sqrt{2}t}. \tag{5.3}$$

In the sequel, we choose

$$\phi = \frac{1}{\sqrt{2}t}. \tag{5.4}$$

Now we compute τ^N. By similar calculations as above, we get

$$\begin{aligned} D_r N &= \frac{1}{t}N, \\ D_{u_1} N &= e_1, \\ D_{u_2} N &= e_2. \end{aligned}$$

Hence, for all X tangent to $\mathcal{U} \subset M$, $\tau^N(f_\star X) = \frac{1}{2t^2}\langle X, N\rangle$, i.e.

$$\tau^N = \frac{1}{2t^2}\eta. \tag{5.5}$$

It follows (4.8), (5.4) and (5.5), that $\zeta^a = 0$, $a = 1, 2$ as $\eta(e_1) = \eta(e_2) = 0$ and $e_1 \cdot \phi = e_2 \cdot \phi = 0$. We also get from (4.9), $\eta(\zeta) = 0$ and then $\zeta = 0$. Finally, we obtain the Blaschke null transversal vector field along $\wedge_0^3$

$$\widetilde{N} = \frac{1}{\sqrt{2}t}(x\partial_x + y\partial_y + z\partial_z - t\partial_t). \tag{5.6}$$

Remark 5.1. *This enables a canonical Blaschke normalization of the null characteristic (radical) vector field along $\wedge_0^3$ as follows:*

$$\widetilde{\xi} = \frac{1}{\sqrt{2}t}(x\partial_x + y\partial_y + z\partial_z + t\partial_t). \tag{5.7}$$

5.2. Monge surfaces in $\mathbb{R}_1^3$. Consider the graph M of the function F, $x = F(y, z)$ as the immersion $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}_1^3$ given by $(y, z) \longmapsto (F(y, z), y, z) \in \mathbb{R}_1^3$ with $F \in C^\infty(\Omega)$. M is lightlike if and only if

$$(F'_y)^2 + (F'_z)^2 = 1. \tag{5.8}$$

In this case, using ∂_y and ∂_z for coordinate vector fields on $\mathbb{R}^2$, we have

$$\begin{aligned} f_\star(\partial_y) &= (F'_y, 1, 0), \\ f_\star(\partial_z) &= (F'_z, 0, 1), \end{aligned}$$

and the null characteristic (radical) distribution is spanned by the null vector field

$$\xi = \partial_x + F'_y\partial_y + F'_z\partial_z. \tag{5.9}$$

Set

$$N = -\partial_x + F'_y\partial_y + F'_z\partial_z. \tag{5.10}$$

As $\langle \xi, N\rangle = 2$ and $\langle N, N\rangle = 0$, let us take N as a tentative null transversal vector field along f. The induced volume form (by the standard determinant) is given by

$$\theta^N(u, v) = det(f_\star u, f_\star v, N).$$

A unimodular frame field for θ^N is then given by (ξ, W) with

$$W = \frac{1}{2}\left(F'_z\partial_y - F'_y\partial_z\right);$$

in particular, (ξ, W, N) is an admissible frame field on $\mathbb{R}_1^3$ along $f(M)$, according to decomposition (2.3), and we have

$$\langle \xi, \xi\rangle = \langle \xi, W\rangle = \langle W, N\rangle = 0 \text{ and } \langle \xi, N\rangle = 2.$$

Now, by straightforward calculation, one sees that the matrix of the local second fundamental form B^N with respect to the unimodular admissible basis ξ, W is given by

$$B^N = \begin{pmatrix} 0 & 0 \\ & \\ 0 & -\frac{1}{8}(F"_{yy} + F"_{zz}) \end{pmatrix}, \tag{5.11}$$

which shows that the lightlike surface M is 1-degenerate provided $\Delta F = F"_{yy} + F"_{zz}$ be everywhere non-zero on M, which we assume from now on. Then, the determinant of B^N relative to θ^N is given by $det_{\theta^N} B^N = -\frac{1}{8}(F"_{yy} + F"_{zz})$. Hence, we obtain

$$|\phi| = \left[|\frac{1}{8}(F"_{yy} + F"_{zz})| \right]^{\frac{1}{3}} = \frac{1}{2}\left(|\Delta F|\right)^{\frac{1}{3}}.$$

As ΔF is continuous and nowhere vanishing, we may assume $|\phi| = \frac{1}{2}(\Delta F)^{\frac{1}{3}}$ and choose for the sequel

$$\phi = \frac{1}{2}(\Delta F)^{\frac{1}{3}}.$$

Now, by standard calculations and using differentiation of relation (5.8), one finds $\tau^N = 0$. Set

$$L(F) = F'_z F^{(3)}_{yyy} + F'_z F^{(3)}_{yzz} - F'_y F^{(3)}_{yyz} - F'_y F^{(3)}_{zzz}.$$

Then, using (4.8), (4.9), the above expression of ϕ and $\tau^N = 0$, we obtain the Blaschke null transversal vector field

$$\widetilde{N} = \frac{1}{2}(\Delta F)^{\frac{1}{3}} N - \frac{2}{3}(\Delta F)^{-\frac{5}{3}} L(F) W - \frac{1}{72}(\Delta F)^{-\frac{10}{3}} L(F)^2 \xi. \tag{5.12}$$

Remark 5.2. *For $\Delta F > 0$, the canonical Blaschke normalization of the null characteristic (radical) vector field along M is as follows:*

$$\widetilde{\xi} = [\Delta F]^{-\frac{1}{3}} (\partial_x + F'_y \partial_y + F'_z \partial_z). \tag{5.13}$$

Also, if $L(F) = 0$, then,

$$\widetilde{N} = \frac{1}{2}[\Delta F]^{\frac{1}{3}} (-\partial_x + F'_y \partial_y + F'_z \partial_z). \tag{5.14}$$

6. Blaschke Fundamental Equations

Consider a Blaschke 1-degenerate (M, g, N_{Bla}). The following theorem summarizes and accounts fundamental equations on this normalization.

Theorem 6.1. *For the Blaschke structure (M, g, N_{Bla}), with Blaschke null transversal N, we have the following:*

$$g(R(X,Y)Z, PW) = B(Y,Z)C(X,PW) - B(X,Z)C(Y,PW), \tag{6.1}$$

$$(\nabla_X B)(Y,Z) = (\nabla_Y B)(X,Z), \tag{6.2}$$

$$(\nabla_X C)(Y,PZ) = (\nabla_Y C)(X,PZ), \tag{6.3}$$

$$\eta(R(X,Y)Z) = 0, \tag{6.4}$$

$$B(\overset{\star}{A}_\xi X, Y) = B(X, \overset{\star}{A}_\xi Y), \tag{6.5}$$

$$C(\overset{\star}{A}_\xi X, Y) = C(X, \overset{\star}{A}_\xi Y), \tag{6.6}$$

$$\theta = \omega_B, \tag{6.7}$$

$$\nabla_{\omega_B} = 0 \tag{6.8}$$

for X, Y, Z tangent to M, where ξ is the (Blaschke) normalized characteristic (null) vector field.

P r o o f. The last two equalities are a part of the Blaschke conditions. To show $B(\overset{\star}{A}_\xi X, Y) = B(X, \overset{\star}{A}_\xi Y)$, use (2.16) and the symmetry of B. Now recall the following equations [9] using the local Gauss–Codazzi equations from the general setting:

$$\begin{aligned} \langle \overline{R}(X,Y)Z, \xi\rangle &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + \tau(X)B(Y,Z) - \tau(Y)B(X,Z), \end{aligned} \tag{6.9}$$

$$\begin{aligned} \langle \overline{R}(X,Y)Z, PW\rangle &= \langle \overline{R}(X,Y)Z, PW\rangle + B(X,Z)C(Y,PW) \\ &\quad - B(Y,Z)C(X,PW), \end{aligned} \tag{6.10}$$

$$\begin{aligned}\langle\overline{R}(X,Y)\xi,N\rangle &= \langle R(X,Y)\xi,N\rangle = C(\overset{\star}{A}_\xi X,Y) - C(\overset{\star}{A}_\xi Y,X)\\ &\quad -2d\tau(X,Y), \qquad (6.11)\end{aligned}$$

$$\langle\overline{R}(X,Y)Z,N\rangle = \langle R(X,Y)Z,N\rangle, \qquad (6.12)$$

$$\begin{aligned}\langle\overline{R}(X,Y)PZ,N\rangle &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ)\\ &\quad +\tau(Y)C(X,PZ) - \tau(X)C(Y,PZ). \qquad (6.13)\end{aligned}$$

Also, we see so far that the equiaffine condition is equivalent to $\tau = 0$. Finally, as the target space in the Blaschke immersion is the flat $\mathbb{R}^{n+2}_1$, set in the above equations, $\overline{R} = 0$ and $\tau = 0$ and the proof is complete. ∎

Corollary 6.1. *For the Blaschke structure (M^{n+1}, g, N_{Bla}) with the Blaschke null transversal N , we have the following:*

(i) *$C = 0$ if and only if $R = 0$.*

(2) *$Ric(X,Y) = B(X,Y)trA_N - B(A_NX,Y)$ and if $n > 1$, $Ric = 0$ if and only if $C = 0$.*

(iii) *For totally geodesic (M,g), if the Blaschke screen is totally umbilical in M with $C = \lambda g$, then $\lambda = cte$ with*

P r o o f. Let $p \in M$ and assume $C = 0$ at p. Then, by (6.1) in Theorem 6.1, $g(R(X,Y)Z,PW) = 0$ for all tangent vectors X, Y, Z and W; i.e. $R(X,Y)Z \in Rad(TM)$. But $\eta(R(X,Y)Z) = 0$ from (6.4). It follows that $R = 0$. Conversely, if $R = 0$, then $B(Y,Z)C(X,PW) = B(X,Z)C(Y,PW)$ using (6.1). At p, as B is symmetric and real valued, consider a quasi-orthonormal basis $(\xi, e_1, \ldots, e_n)$ with respect to B such that $(e_1, \ldots, e_n)$ spans $S(T_pM)$. Then, for i,j,k with $k \neq i$, we have $B_{kk}C_{kj} = 0$ and $C_{ij} = 0$ for all i and j that is $C = 0$, which proves (i). Now, the following formula of the Ricci tensor is known [4]:

$$Ric(X,Y) = \overline{Ric}(X,Y) - \eta(\overline{R}(\xi,Y)X) + B(X,Y)trA_N - B(A_NX,Y). \quad (6.14)$$

Setting $\overline{R} = 0$, we obtain the expression in item (ii). Henceforth, it is immediate that if $C = 0$, then $Ric = 0$. Suppose $Ric = 0$. Then, $B(X,Y)trA_N - B(A_NX,Y) = 0$, i.e. $B(trA_NX - A_NX, Y) = 0$ for all Y. As B is 1-degenerate with null direction $\langle\xi\rangle$, we have $trA_NX - A_NX \in Rad(T_pM)$. Then $A_NX = (trA_N)PX$ for all X. Hence, we get $trA_N = n(trA_N)$. It follows that if $n > 1$, we obtain $trA_N = 0$, that is $C = 0$, and (ii) is proved. Let $C = \lambda g$. We have

$$\begin{aligned}(\nabla_X C)(Y,PZ) &= \nabla_X(C(Y,PZ)) - C(\nabla_X Y,PZ) - C(Y,\overset{\star}{\nabla}_X PZ)\\ &= X\cdot[\lambda g(Y,Z)] - \lambda g(\nabla_X Y,PZ) - \lambda g(Y,\overset{\star}{\nabla}_X PZ)\\ &= (X\cdot\lambda)g(Y,PZ) + \lambda(\nabla_X g)(Y,PZ).\end{aligned}$$

Then, by (6.3), we have

$$\begin{aligned} 0 &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ &= X \cdot \lambda)g(Y, PZ) - Y \cdot \lambda)g(X, PZ) \\ &\quad +\lambda[(\nabla_X g)(Y, PZ) - \nabla_Y g)(X, PZ)]. \end{aligned}$$

But $\nabla_X g)(Y, PZ) = \nabla_Y g)(X, PZ) = 0$ as M is totally geodesic, and we get

$$g((X \cdot \lambda)Y - (Y \cdot \lambda)X, PZ) = 0,$$

which means that

$$(X \cdot \lambda)PY - (Y \cdot \lambda)PX = 0$$

for all X, $Y \in \Gamma(TM|_{\mathcal{U}})$, which shows that $X \cdot \lambda = 0$ on $\mathcal{U} \subset M$ and the proof is complete. ■

Corollary 6.2. *In Blaschke normalization with $n > 1$, the induced Ricci is flat if and only if the Blaschke screen is integrable with totally geodesic leaves (in M^{n+1}) that are parallel along the null characteristic orbits.*

P r o o f. This is immediate from item (ii) in Corollary 6.1 as $C = 0$ is a rephrasing of the fact that the screen distribution is integrable with totally geodesic leaves, parallel along the null characteristic orbits. Note that the later condition reads $C(\xi, PY) = 0$ for all Y while the former means that $C(PX, PY)$ $= 0$, for all X, Y. ■

References

[1] *M.A. Akivis and V.V. Goldberg,* On Some Methods of Construction of Invariant Normalizations of Lightlike Hypersurfaces. — *Diff. Geom. Appl.* **12** (2000), 121–143.

[2] *M.A. Akivis and V.V. Goldberg,* Lightlike Hypersurfaces on Manifolds Endowed with a Conformal Structure of Lorentzian Signature. — *Acta Appl. Math.* **57** (1999), 155–185.

[3] *C. Atindogbe, J.-P. Ezin, and J. Tossa*, Pseudo-Inversion of Degenerate Metrics. — *Int. J. Math. and Math. Sci.* **55** (2003), 3479–3501.

[4] *C. Atindogbe and K.L. Duggal,* Conformal Screen on Lightlike Hypersurfaces. — *Int. J. Pure and Appl. Math.* **11** (2004), No. 4, 421–442.

[5] *E. Bekkara, C. Frances, and A. Zeghib*, On Lightlike Geometry: Isometric Actions and Rigidity Aspects. — *C. R. Acad. Sci. Paris* **343** (2006), No. 5, 317–321.

[6] *B. Carter*, Killing Horizons and Orthogonally Transitive Groups in Space-time. — *J. Math. Phys.* **10** (1969), 70-81.

[7] *G. Dautcourt*, Characteristic Hypersurfaces in General Relativity. — *J. Math. Phys.* **8** (1967), 1492–1501.

[8] *G. Dautcourt*, Zur Differentialgeometrie Singularen Riemannschen Raune. — *Math. Nachr.* **36** (1968), 311–322.

[9] *K.L. Duggal and A. Bejancu*, Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Kluwer Acad. Publishers, Dordrecht, 1996.

[10] *K. Katsuno*, Null Hypersurfaces in Lorentzian Manifolds. — *Math. Proc. Cab. Phil. Soc.* **88** (1980), 175–182.

[11] *D.N. Kupeli*, Degenerate Submanifolds in Semi-Riemannian Geometry. — *Geometry Dedicata* **24** (1987), 337–361.

[12] *D.N. Kupeli*, On Null Submanifolds in Spacetimes. — *Geometry Dedicata* **23** (1987), 33–51.

[13] *D.N. Kupeli*, Singular Semi-Riemannian Geometry. Kluwer Academic Publishers, Dordrecht, 1996.

[14] *J.C. Larsen*, Singular Semi-Riemannian Geometry. — *J. Geom. Phys.* **9** (1992), No. 1, 3–23.

[15] *J.C. Larsen*, Submanifold Geometry. — *J. Geom. Anal.* **4** (1994), No. 2, 189–205.

[16] *R. Penrose*, The Twistor Geometry of Light Rays. — *Geometry and Physics, Classical Quantum Gravity* (1997), 14 (1A), A299–A323.

[17] *M. Pinl*, Zur Theorie del Halbisotrope Flachen im R4. — *J. Geom. Phys.* **9** (1992), No. 1, 3-23.

[18] *R. Rosca*, Sur les Hypersurfaces de Défaut 1 Incluses dans une Variété Lorentzienne. — *C. R. Acad. Sci. Paris Sér. A-B* **292** (1971), A393–A396.

[19] *R. Rosca*, On Null Hypersurfaces of a Lorentziann Manifolds. — *Tensor N. S.* **23** (1972), 66–74.

[20] *A.H. Taub*, Singular Hhypersurfaces in General Relativity. — *Illinois J. Math* **1** (1957), 378–388.