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Growth of the Poisson–Stieltjes Integral in a Polydisc

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We describe the growth of the value $M_\infty(r_1, \dots r_n, v)=\max\{|v\,(z_1, \dots, z_n)| : |z_j| \le r_j\}, 0 \le r_j < 1$ in terms of the modulus of continuity of a measure μ, where the function v is represented by the Poisson–Stieltjes integral.

Key words: modulus of continuity, multiple Poisson kernel, Poisson–Stieltjes integral, polydisc.

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1. Introduction

Let $|z| = \max\{|z_j| : 1 \le j \le n\}$ be the polydisc norm in $\mathbb{C}^n, n \in \mathbb{N}$. Denote by $U^n = \{z \in \mathbb{C}^n : |z| < 1\}$ the unit polydisc with the distinguished boundary $T^n = \{z \in \mathbb{C}^n : |z_j| = 1, 1 \le j \le n\}$. For $z \in U^n$, $z_j = r_j e^{i\varphi_j}$, $w_j = e^{i\theta_j}$, $1 \le j \le n$ we write $\mathcal{P}(z, w) = \prod_{j=1}^{n} P_0(z_j, w_j)$, where

$$P_0\,(z_j, w_j) = \mathrm{Re}\frac{w_j + z_j}{w_j - z_j} = \frac{1 - r_j^2}{1 - 2r_j \cos\left(\varphi_j - \theta_j\right) + r_j^2}$$

is the Poisson kernel for the unit disc.

The function in U^n defined by the equality

$$P[d\mu](z) = \int\limits_{T^n} \mathcal{P}(z, w) d\mu(w) \tag{1}$$

is called the Poisson–Stieltjes integral of a finite (signed) Borel measure, provided that the total variation of the measure μ is finite, i.e. $|\mu|(T^n) < +\infty$. The function $P[d\mu]$ is n-harmonic in U^n, i.e. harmonic in each variable.

For a function u defined in U^n and $0 \le r < 1$ we denote $u_r(w) = u(rw)$, $w \in T^n$. Let $\|\cdot\|_p$, $p \ge 1$ denote the L^p-norm relatively to the normalized Lebesgue measure m_n on T^n, $m_n(T^n) = 1$.

The next theorem is known [1].

Theorem A. *If u is n-harmonic in U^n and $\|u_r\|_1$ is bounded as $r \to 1$, then there exists a unique measure μ on T^n such that $u = P[d\mu]$.*

Moreover, there exists a sequence $\rho_k \uparrow 1$ $(k \to +\infty)$ such that μ is a weak limit of the sequence of measures μ_{ρ_k} $(d\mu_{\rho_k}(w) = u(\rho_k w)\, dm_n(w))$ as $k \to +\infty$.

If u is nonnegative, then μ is nonnegative.

Note that under the assumptions of Theorem A the function u cannot be represented as the Poisson integral of a function from $L^1(T^n)$, i.e. μ can be singular, in general. Moreover, some other theorems, which are valid if $\|u_r\|_p$ is bounded for $p > 1$, fail to hold when $p = 1$ (see [2, Ch. 6]).

Let $U_\theta(\delta) = \{x \in [0; 2\pi] : |x - \theta| < \delta\}$, $\delta > 0$. For $\psi : [0; 2\pi] \to \mathbb{R}$, define the modulus of continuity

$$\omega(\delta, \theta; \psi) = \sup\{|\psi(x) - \psi(y)| : x, y \in U_\theta(\delta)\}, \omega(\delta; \psi) = \sup_{\theta \in [0; 2\pi]} \omega(\delta, \theta; \psi).$$

Given $\gamma \in (0; 1]$, we say that $\psi \in \Lambda_\gamma$ if $\omega(\delta; \psi) = O(\delta^\gamma)$ $(\delta \downarrow 0)$.

We are interested in the interplay between the growth of the Poisson–Stieltjes integral (1) and the properties of the measure μ. A background of results of this type is a classical theorem of G. Hardy and J. Littlewood [3], [4, Ch. 5] which states that for an analytic function in U the conditions $|f'(z)| = O((1 - |z|)^{\gamma - 1})$, $z \in U$, and $f(e^{i\theta}) \in \Lambda_\gamma \wedge f \in C(\bar{U})$ are equivalent for $\gamma \in (0, 1]$.

Let $M_\infty(r_1, \ldots r_n, v) = \max\{|v(z_1, \ldots, z_n)| : |z_j| \le r_j\}, 0 \le r_j < 1$, where

$$v(z_1, \ldots, z_n) = \int_{T^n} \mathcal{P}(z, w) d\mu(w).$$

The question on the growth of the value $M_\infty(r_1, \ldots r_n, v)$ in terms of the modulus of continuity of μ arises naturally. In [5] the following theorem is proved.

Theorem B. *Let u be a harmonic function in U, $0 < \gamma \le 1$. Then u has the form*

$$u(re^{i\varphi}) = \int_0^{2\pi} P(re^{i\varphi}, e^{it}) d\psi(t),$$

where ψ is of bounded variation on $[0; 2\pi]$ and $\psi \in \Lambda_\gamma$ if and only if

$$M_\infty(r,u) = O((1-r)^{\gamma-1}), \quad r \uparrow 1$$

and

$$\sup_{0<r<1} \|u_r\|_1 < +\infty. \tag{2}$$

The aim of the present paper is to generalize Theorem B on n-harmonic functions.

We write

$$\Pi_{a_1,\dots,a_n}^{b_1,\dots,b_n} = \{(e^{i\theta_1},\dots,e^{i\theta_n}) \in T^n : a_j \le \theta_j \le b_j\}.$$

We say that $\mu \in \Lambda_{\gamma_1,\dots,\gamma_n}(T^n), \gamma_1,\dots,\gamma_n \in (0;1)$ if

$$\sup_{a\in[-\pi;\pi]^n} |\mu| \left(\Pi_{a_1,\dots,a_n}^{a_1+\delta_1,\dots,a_n+\delta_n}\right) = O\left(\delta_1^{\gamma_1}\dots\delta_n^{\gamma_n}\right),\ 0<\delta_j<1. \tag{3}$$

Given $j \in \{1,\dots,n\}$, we say that $\mu \in \Lambda_\gamma^{(j)}(T^n)$ if

$$\sup_{a\in[-\pi;\pi]^n} |\mu| \left(\Pi_{a_1,\dots,a_n}^{a_1,\dots,a_j+\delta,\dots,a_n}\right) = O\left(\delta^{\gamma}\right),\ 0<\delta<1. \tag{4}$$

It is clear that $\Lambda_{\gamma_1,\dots,\gamma_n} \subset \bigcap_{j=1}^n \Lambda_{\gamma_j}^{(j)}$.

Theorem 1. *Let μ be a finite Borel measure on T^n, $n \in \mathbb{N}, \gamma_1,\dots,\gamma_n \in (0;1)$. Then for the inequality*

$$\left|\int_{T^n} \mathcal{P}(z,w)\,d\mu(w)\right| \le C(\gamma_1,\dots,\gamma_n)(1-r_1)^{\gamma_1-1}\dots(1-r_n)^{\gamma_n-1}, \tag{5}$$

where $0 \le r_j < 1$, $1 \le j \le n$, to hold, it is necessary that $\mu \in \bigcap_{j=1}^n \Lambda_{\gamma_j}^{(j)}$, and sufficient that $\mu \in \Lambda_{\gamma_1,\dots,\gamma_n}(T^n)$.

The theorem is the best possible in the following sense.

Corollary 1. *Suppose that $\mu = \bigotimes_{j=1}^n \mu_j$, where μ_j is a finite Borel measure on T, $j \in \{1,\dots,n\}$, $n \in \mathbb{N}, \gamma_1,\dots,\gamma_n \in (0;1]$. In order for* (5) *to hold, it is necessary and sufficient to impose that $\mu \in \Lambda_{\gamma_1,\dots,\gamma_n}(T^n)$.*

To prove the corollary, it is sufficient to note that if $\mu = \bigotimes_{j=1}^n \mu_j$, then $\mu \in \Lambda_{\gamma_1,\dots,\gamma_n}(T^n)$ is equivalent to $\mu_j \in \Lambda_{\gamma_j}^{(j)}$, $1 \le j \le n$, i.e. $\Lambda_{\gamma_1,\dots,\gamma_n} = \bigcap_{j=1}^n \Lambda_{\gamma_j}^{(j)}$. Therefore, Corollary 1 follows from Theorem 1 if $\gamma_j \in (0,1)$, $1 \le j \le n$, or from Theorem B for the general case.

Combining Theorem 1 and Theorem A we obtain the following result.

Theorem 2. *Let u be an n-harmonic function in U^n, $\gamma_1, \dots, \gamma_n \in (0,1)$. If*

$$\sup_{0<r_j<1} \|u_r\|_1 < +\infty \tag{6}$$

and

$$M_\infty(r_1, \dots, r_n, u) = O\Big(\prod_{j=1}^{n}(1-r_j)^{\gamma_j - 1}\Big), \quad r_j \uparrow 1,$$

then there is a finite Borel measure μ on T^n, where $\mu \in \bigcap_{j=1}^n \Lambda_{\gamma_j}^{(j)}$, such that $u = P[d\mu]$. If $u = P[d\mu]$ for a finite Borel measure μ with $\mu \in \Lambda_{\gamma_1,\dots\gamma_n}$, then (6) *holds and $M_\infty(r_1, \dots, r_n, u) = O(\prod_{j=1}^n (1-r_j)^{\gamma_j-1}), \quad r_j \uparrow 1$.*

It is interesting to compare Theorem 2 with the results of W. Nestlerode and M. Stoll [6] originally proved for n-subharmonic functions. Given is an n-harmonic function u satisfying (6) Corollary 1 [6] that yields

$$\overline{\lim_{r_j \uparrow 1}}\, u(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n}) \prod_{j=1}^{n}(1-r_j) = 0$$

for all θ_j if and only if μ is continuous.

On the other hand, additional information on the continuity of μ allows us to state more on the growth of the n-harmonic function $P[d\mu]$.

R e m a r k 1. The authors do not know whether Theorem 1 is valid when $\gamma_j = 1$ for some $j \in \{1, \dots, n\}$. We can state only that if $\mu \in \Lambda_{1,\dots,1}$, then $P[d\mu]$ is bounded.

2. Preliminaries

We need the following notation. Let $\sigma_j \in \{0;1\}$, $1 \le j \le n$. If $\sigma_j = 0$, then $P_0(z_j, w_j)$ denotes the usual Poisson kernel, and if $\sigma_j = 1$, then

$$P_1(z_j, w_j) = \operatorname{Im}\frac{w_j + z_j}{w_j - z_j} = \frac{2r_j \sin(\varphi_j - \theta_j)}{1 - 2r_j \cos(\varphi_j - \theta_j) + r_j^2}$$

denotes the conjugated Poisson kernel.

Besides, we denote

$$v_{\sigma_1 \dots \sigma_n}(z_1, \dots, z_n) = \int_{T^n} P_{\sigma_1}(z_1, w_1) \cdots P_{\sigma_n}(z_n, w_n) d\mu(w),$$

$$v(z_1, \dots, z_n) = v_{\underbrace{0 \dots 0}_{n}}(z_1, \dots, z_n) = \int_{T^n} \mathcal{P}(z, w) d\mu(w),$$

$$S_0(z_j, w_j) = \frac{w_j + z_j}{w_j - z_j} = P_0(z_j, w_j) + iP_1(z_j, w_j),\ S_1(z_j, w_j) = \overline{S_0(z_j, w_j)},$$

$1 \le j \le n$.

We need some lemmas.

Lemma 1. *Let $|\mu|(T^n) < \infty$. If for some constant C_0*

$$|v_{0\dots 0}(z_1, \dots, z_n)| \le C_0 \prod_{j=1}^{n} (1 - |z_j|)^{\gamma_j - 1}, \quad z \in U^n,$$

then

$$|v_{\sigma_1 \dots \sigma_n}(z_1, \dots, z_n)| \le C(\gamma_1, \dots, \gamma_n) \prod_{j=1}^{n} (1 - |z_j|)^{\gamma_j - 1}, z \in U^n,\ \sigma_j \in \{0, 1\}, \quad (7)$$

$1 \le j \le n$, where $C(\gamma_1, \dots, \gamma_n)$ is a constant depending on $\gamma_1, \dots, \gamma_n$.

P r o o f. To prove the lemma we use the induction in $|\sigma| = \sigma_1 + \dots + \sigma_n$, i.e. the number of those σ_j's equal to 1.

Let $|\sigma| = 0$, then $|v_{0\dots 0}(z_1, \dots, z_n)| \le C_0 \prod_{j=1}^{n} (1 - |z_j|)^{\gamma_j - 1}$.

We now assume that (7) holds for every n-tuple $(\sigma_1, \dots, \sigma_n)$ of indices such that $|\sigma| \le k$, $k \in \{0, \dots, n-1\}$. It is sufficient to prove that from the estimate

$$|v_{\underbrace{1\dots 1}_{k} 0\dots 0}(z_1, \dots, z_n)|$$

$$= \left| \int_{T^n} P_1(z_1, w_1) \cdots P_1(z_k, w_k) P_0(z_{k+1}, w_{k+1}) \cdots P_0(z_n, w_n) d\mu(w) \right|$$

$$\le C_1 \prod_{j=1}^{n} (1 - |z_j|)^{\gamma_j - 1}$$

it follows that

$$|v_{\underbrace{1\dots 1}_{k+1} 0\dots 0}(z_1, \dots, z_n)|$$

$$= \left| \int_{T^n} P_1(z_1, w_1) \cdots P_1(z_{k+1}, w_{k+1}) P_0(z_{k+2}, w_{k+2}) \cdots P_0(z_n, w_n) d\mu(w) \right|$$

$$\le C_2 \prod_{j=1}^{n} (1 - |z_j|)^{\gamma_j - 1}.$$

Fix the points $z_1^0, \dots, z_k^0, z_{k+2}^0, \dots, z_n^0 \in U$. We write

$$h(z_{k+1}) = v_{\underbrace{1\ldots1}_{k}0\ldots0}\left(z_1^0, \dots, z_{k+1}, \dots, z_n^0\right),$$

$$\tilde{h}(z_{k+1}) = v_{\underbrace{1\ldots1}_{k+1}0\ldots0}\left(z_1^0, \dots, z_{k+1}, \dots, z_n^0\right).$$

The function $\tilde{h}$ is harmonically conjugated to h. The function $F_{z_1^0,\dots,z_n^0}(z_{k+1}) = h(z_{k+1}) + i\tilde{h}(z_{k+1})$ is analytic in z_{k+1}. We estimate $|F_{z_1^0,\dots,z_n^0}(z_{k+1})|$. Using the arguments of Theorem 2.30 [7], we get

$$|F_{z_1^0,\dots,z_n^0}(z_{k+1}) - F_{z_1^0,\dots,z_n^0}(0)| \le 4 \int\limits_{1-r_{k+1}}^{1} t^{-1}\varphi\left(\frac{t}{2}\right) dt,$$

where $\varphi(t) = \max\limits_{\theta} \left|h\left((1-t)\, e^{i\theta}\right)\right|$.

Using the assumption of the induction, we have

$$|h| \le C_1(1-|z_{k+1}|)^{\gamma_{k+1}-1} \prod_{j\ne k+1} (1-|z_j^0|)^{\gamma_j-1}.$$

Then

$$\begin{aligned} &|F_{z_1^0,\dots,z_n^0}(z_{k+1}) - F_{z_1{}^0,\dots,z_n{}^0}(0)| \\ &\le 4 \int\limits_{1-r_{k+1}}^{1} t^{-1} C_1 \prod_{j\ne k+1} (1-|z_j^0|)^{\gamma_j-1}\left(\frac{t}{2}\right)^{\gamma_{k+1}-1} dt \\ &= C_3 \prod_{j\ne k+1} (1-|z_j^0|)^{\gamma_j-1}((1-r_{k+1})^{\gamma_{k+1}-1}-1). \end{aligned}$$

Hence,

$$|F_{z_1^0,\dots,z_n^0}(z_{k+1})| \le C_3 \prod_{j\ne k+1} (1-|z_j^0|)^{\gamma_j-1}(1-r_{k+1})^{\gamma_{k+1}-1} + |F_{z_1^0,\dots,z_n^0}(0)|.$$

Since $P_0(0,w_j) = 1$, $P_1(0,w_j) = 0$, we get

$$\left|F_{z_1^0,\dots,z_n^0}(0)\right| = |h(0)| \le C_1 \prod_{j\ne k+1} (1-|z_j^0|)^{\gamma_j-1}.$$

Then

$$\left|F_{z_1^0,\dots,z_n^0}(z_{k+1})\right| \le C_4 \prod_{j\ne k+1} (1-|z_j^0|)^{\gamma_j-1}(1-r_{k+1})^{\gamma_{k+1}-1}.$$

Therefore, $\tilde{h}(z_{k+1})$ admits the same estimate. Thus, the induction step is proved. Thereby,

$$|v_{\sigma_1\ldots\sigma_n}(z_1,\ldots,z_n)| \le C(\gamma_1,\ldots,\gamma_n)\prod_{j=1}^{n}(1-|z_j|)^{\gamma_j-1}, \quad 0\le |z_j|<1,\ 1\le j\le n.$$

This finishes the proof of the lemma.

Lemma 2.

$$\sum_{\substack{\sigma=(\sigma_1,\ldots,\sigma_m),\\ \sigma_j\in\{0;1\}}} S_{\sigma_1}(z_1,w_1)\cdots S_{\sigma_m}(z_m,w_m) = 2^m P_0(z_1,w_1)\cdots P_0(z_m,w_m),\ m\in\mathbb{N}.$$

The lemma can be proved by induction.

3. Proof of Theorem 1

S u f f i c i e n c y. For simplicity we assume that $n=2$. We also need the next notation [1].

Let $\alpha=(\alpha_1,\ \alpha_2)\in\mathbb{Z}_+^2$. The Cartesian product $I_1\times I_2$ of the semiopen intervals $I_j=(s_j,t_j]\subset[0;2\pi],\ j\in\{1,\ 2\}$ with $(t_1-s_1)\,2^{-\alpha_1}=(t_2-s_2)\,2^{-\alpha_2}$ is called an *α-block*.

We fix the numbers r_1 and r_2, $0\le r_1,\ r_2<1$. Then we can find the numbers $C(r_1)$ and $C(r_2)$, $1\le C(r_j)<2$, $j=1,2$, such that $\frac{\pi}{(1-r_j)C(r_j)}=2^{p_j}$, where p_j is an integer number.

Denote

$$x_{k,j}^- = 2^{k-1}(1-r_j)C(r_j),\ \ x_{k,j}^+ = 2x_{k,j}^-,\ \ k\in\mathbb{N},\ \ x_{0,j}^{\pm}=0,\ \ j\in\{1,2\}.$$

Let B_α be an α-block with the center at the point $\tilde{w}=\left(e^{i\varphi_1},e^{i\varphi_2}\right)$ and the lengthes of its sides $2x_{\alpha_1,1}^+$, $2x_{\alpha_2,2}^+$. Denote $s_j=\theta_j-\varphi_j$. Assume Q_α to be a set of all points $\left(e^{is_1},e^{is_2}\right)$, when (s_1,s_2) belongs to the α-block B_α, for which either $x_{\alpha_j,j}^-\le s_j<x_{\alpha_j,j}^+$ or $x_{\alpha_j,j}^-<-s_j\le x_{\alpha_j,j}^+$, $j=1,2$, holds.

Each Q_α is a union of 4 blocks, and $T^2=\bigcup\limits_{0\le\alpha_j\le p_j} Q_\alpha$ by the definitions of p_j and Q_α.

Using (3), we have

$$\begin{aligned}|\mu|(Q_\alpha) &\le |\mu|\left(\left\{w\in T^2:\ |\theta_j-\varphi_j|\le 2^{\alpha_j}(1-r_j)\,C(r_j)\right\}\right)\\ &\le \left(2^{\alpha_1+1}(1-r_1)\,C(r_1)\right)^{\gamma_1}\left(2^{\alpha_2+1}(1-r_2)\,C(r_2)\right)^{\gamma_2}\\ &= 2^{(\alpha_1+1)\gamma_1+(\alpha_2+1)\gamma_2}C^{\gamma_1}(r_1)\,C^{\gamma_2}(r_2)\,(1-r_1)^{\gamma_1}(1-r_2)^{\gamma_2}.\end{aligned}$$

As in [1] (proof of Theorem 3.1), we have

$$\left|\int\limits_{Q_\alpha} P_0\left(r_1e^{i\varphi_1}, e^{i\theta_1}\right)P_0\left(r_2e^{i\varphi_2}, e^{i\theta_2}\right)d\mu(e^{i\theta})\right| \le |\mu|\left(Q_\alpha\right)\cdot\prod_{j=1}^{2}P_0\left(r_je^{ix^-_{\alpha_j,j}},1\right),$$

$0 \le r_j < 1$.

The following estimate of the Poisson kernel is known [1]:

$$P_0\left(re^{i\varphi}, e^{i\theta}\right) \le \left(\frac{\pi}{\varphi-\theta}\right)^2(1-r).$$

Applying this estimate to our case, we get

$$P_0\left(r_je^{ix^-_{\alpha_j,j}},1\right) \le \frac{\pi^2}{4^{\alpha_j-1}\left(1-r_j\right)C^2\left(r_j\right)}.$$

Thus,

$$\left|\int\limits_{Q_\alpha} P_0\left(z_1,w_1\right)P_0\left(z_2,w_2\right)d\mu\left(w\right)\right| \le 2^{(\alpha_1+1)\gamma_1+(\alpha_2+1)\gamma_2}\cdot C^{\gamma_1}\left(r_1\right)C^{\gamma_2}\left(r_2\right)$$

$$\times\left(1-r_1\right)^{\gamma_1}\left(1-r_2\right)^{\gamma_2}\cdot\frac{\pi^2}{4^{\alpha_1-1}\left(1-r_1\right)C^2\left(r_1\right)}\cdot\frac{\pi^2}{4^{\alpha_2-1}\left(1-r_2\right)C^2\left(r_2\right)}$$

$$\le C\left(\gamma_1,\ \gamma_2\right)\cdot 2^{\alpha_1(\gamma_1-2)}\cdot 2^{\alpha_2(\gamma_2-2)}\left(1-r_1\right)^{\gamma_1-1}\left(1-r_2\right)^{\gamma_2-1}.$$

Summing up over all α, we deduce

$$\left|\int\limits_{T^2}\mathcal{P}\left(z,w\right)d\mu\left(w\right)\right| = \left|\sum_\alpha\int\limits_{Q_\alpha}\mathcal{P}\left(z,w\right)d\mu\left(w\right)\right|$$

$$\le \sum_\alpha C\left(\gamma_1,\ \gamma_2\right)\cdot 2^{\alpha_1(\gamma_1-2)}2^{\alpha_2(\gamma_2-2)}$$

$$\times\left(1-r_1\right)^{\gamma_1-1}\left(1-r_2\right)^{\gamma_2-1} \le 4C\left(\gamma_1,\gamma_2\right)\left(1-r_1\right)^{\gamma_1-1}\left(1-r_2\right)^{\gamma_2-1}.$$

N e c e s s i t y. By Theorem A, we have that $\mu_{\rho_k}\left(d\mu_{\rho_k}\left(w\right) = u\left(\rho_k w\right)dm_n\left(w\right)\right)$ converges weakly to μ as functionals on $C\left(T^n\right)$, i.e.,

$$\lim_{k\to\infty}\int\limits_{T^2}\mathcal{P}\left(z,w\right)d\mu_{\rho_k}\left(w\right) = \int\limits_{T^2}\mathcal{P}\left(z,w\right)d\mu\left(w\right).$$

Using Lemma 1, we can see that for the analytic function

$$F\left(z_1,\, z_2\right) = \int\limits_{T^2} S_0\left(z_1, w_1\right) S_0\left(z_2, w_2\right) d\mu\left(w\right)$$

there is valid the inequality

$$\left|F\left(z_1,\, z_2\right)\right| \le C\left(\gamma_1,\, \gamma_2\right)\left(1-\left|z_1\right|\right)^{\gamma_1-1}\left(1-\left|z_2\right|\right)^{\gamma_2-1}.$$

From the notation in the previous section we have $\mathrm{Re}F\left(z_1,\, z_2\right) = v_{00}\left(z_1,\, z_2\right) - v_{11}\left(z_1,\, z_2\right)$. We denote

$$\mu^0_{\rho_k}\left(\theta_1,\, \theta_2\right) = \int_0^{\theta_1}\int_0^{\theta_2}\left(v_{00}\left(\rho_k e^{i\varphi_1},\, \rho_k e^{i\varphi_2}\right) - v_{11}\left(\rho_k e^{i\varphi_1},\, \rho_k e^{i\varphi_2}\right)\right) d\varphi_1 d\varphi_2. \quad (8)$$

In the sequel we will use the same notation for a function of the bounded variation defined on the $[0, 2\pi]^n$ and the (signed) Stieltjes measure on T^n generated by the function.

Define the analytic function $\Phi\left(z_1,\, z_2\right) = \int\limits_0^{z_1}\left(\int\limits_0^{z_2} F\left(\varsigma, \eta\right) d\eta\right) d\varsigma,\ z \in \bar{U}^2$.

For the arbitrary fixed φ_1, φ_2 and $0 < r_1' < r_1'' < 1,\ 0 < r_2' < r_2'' < 1$ we have

$$\left|\Phi\left(r_1'' e^{i\varphi_1}, r_2'' e^{i\varphi_2}\right) - \Phi\left(r_1' e^{i\varphi_1}, r_2' e^{i\varphi_2}\right)\right|$$

$$\le \int\limits_{r_1'}^{r_1''}\left(\int\limits_0^{r_2'}\left|F\left(\rho_1 e^{i\varphi_1}, \rho_2 e^{i\varphi_2}\right) d\rho_2\right| d\rho_1\right) + \int\limits_0^{r_1''}\left(\int\limits_{r_2'}^{r_2''}\left|F\left(\rho_1 e^{i\varphi_1}, \rho_2 e^{i\varphi_2}\right) d\rho_2\right| d\rho_1\right)$$

$$\le \int\limits_{r_1'}^{r_1''}\frac{C\left(\gamma_1, \gamma_2\right)}{\left(1-\rho_1\right)^{1-\gamma_1}}\left(-\frac{\left(1-\rho_2\right)^{\gamma_2}}{\gamma_2}\Bigg|_0^{r_2'}\right) d\rho_1$$

$$+ \int\limits_0^{r_1''}\frac{C\left(\gamma_1, \gamma_2\right)}{\left(1-\rho_1\right)^{1-\gamma_1}}\left(-\frac{\left(1-\rho_2\right)^{\gamma_2}}{\gamma_2}\Bigg|_{r_2'}^{r_2''}\right) d\rho_1$$

$$\le C\left(\gamma_1, \gamma_2\right)\frac{1}{\gamma_1\gamma_2}\left(\left(1-r_1'\right)^{\gamma_1} + \left(1-r_2'\right)^{\gamma_2}\right).$$

Therefore, by Cauchy's criterion, there exists

$$\lim_{\substack{r_1\uparrow 1\\ r_2\uparrow 1}} \Phi\left(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}\right) \equiv \Phi\left(e^{i\varphi_1}, e^{i\varphi_2}\right)$$

uniformly in φ_1, φ_2. Set $\tilde{\Phi}(\varphi_1, \varphi_2) \stackrel{\text{def}}{=} \Phi\left(e^{i\varphi_1}, e^{i\varphi_2}\right)$. Let us prove that $\tilde{\Phi} \in \Lambda^{(1)}_{\gamma_1} \cap \Lambda^{(2)}_{\gamma_2}$.

Note that

$$\mu^0_{\rho_k}(\theta_1+h, \theta_2) - \mu^0_{\rho_k}(\theta_1, \theta_2) = \operatorname{Re} \int\limits_0^{\theta_1+h} \int\limits_0^{\theta_2} F\left(\rho_k e^{i\varphi_1}, \rho_k e^{i\varphi_2}\right) d\varphi_1 d\varphi_2.$$

Let $h_1 \in (0;1)$. We write

$$\varsigma_0 = e^{i\varphi_1};\ \ \varsigma_1 = (1-h_1)\, e^{i\varphi_1};\ \ \varsigma_2 = (1-h_1)\, e^{i(\varphi_1+h_1)};\ \ \varsigma_3 = e^{i(\varphi_1+h_1)}.$$

Fixing z_2 and using Cauchy's theorem, we get

$$\Phi(\varsigma_3, z_2) - \Phi(\varsigma_0, z_2) = \int\limits_0^{\varsigma_3} \int\limits_0^{z_2} F(\varsigma, \eta)\, d\varsigma d\eta - \int\limits_0^{\varsigma_0} \int\limits_0^{z_2} F(\varsigma, \eta)\, d\varsigma d\eta$$

$$= \int\limits_0^{z_2} \left(\int\limits_{\varsigma_0}^{\varsigma_3} F(\varsigma, \eta)\, d\varsigma \right) d\eta = \int\limits_0^{z_2} \left(\left(\int\limits_{[\varsigma_0;\varsigma_1]} + \int\limits_{\varsigma_1}^{\varsigma_2} + \int\limits_{[\varsigma_2;\varsigma_3]} \right) F(\varsigma, \eta)\, d\varsigma \right) d\eta,$$

$$\left| \Phi\left(e^{i(\varphi_1+h_1)}, z_2\right) - \Phi\left(e^{i\varphi_1}, z_2\right) \right| \le \int\limits_{[0,z_2]} \left| \int\limits_{\varsigma_0}^{\varsigma_3} F(\varsigma, \eta)\, d\varsigma \right| |d\eta|. \tag{9}$$

For sufficiently small $h_1 > 0$, we have

$$\left| \int\limits_{[\varsigma_2;\varsigma_3]} F(\varsigma, \eta)\, d\varsigma \right| = \left| \int\limits_{1-h_1}^{1} F\left(e^{i(\varphi_1+h_1)}t, \eta\right) e^{i(\varphi_1+h_1)} dt \right|$$

$$\le \int\limits_{1-h_1}^{1} \left| F\left(e^{i(\varphi_1+h_1)}t, \eta\right) \right| dt$$

$$\le C(\gamma_1, \gamma_2) \int\limits_{1-h_1}^{1} (1-t)^{\gamma_1-1} (1-|\eta|)^{\gamma_2-1}\, dt \le C(\gamma_1, \gamma_2)\, h_1^{\gamma_1} (1-|\eta|)^{\gamma_2-1}. \tag{10}$$

Analogously,

$$\left| \int\limits_{[\varsigma_0;\varsigma_1]} F(\varsigma, \eta)\, d\varsigma \right| \le C(\gamma_1, \gamma_2)\, h_1^{\gamma_1} (1-|\eta|)^{\gamma_2-1}. \tag{11}$$

Finally,

$$\left|\int_{\varsigma_1}^{\varsigma_2} F(\varsigma,\eta)\,d\varsigma\right| = \left|\int_0^{h_1} F\left((1-h_1)\,e^{i(\varphi_1+h_1)},\eta\right)(1-h_1)\,ie^{i(\varphi_1+h_1)}d\varphi_1\right|$$

$$\le C(\gamma_1,\gamma_2)\,(1-|\eta|)^{\gamma_2-1}\,h_1^{\gamma_1}.$$

Then from (9)–(11) and the last inequality we obtain

$$\left|\Phi\left(e^{i(\varphi_1+h_1)},z_2\right)-\Phi\left(e^{i\varphi_1},z_2\right)\right| \le \int_{[0;z_2]} C(\gamma_1,\gamma_2)h_1^{\gamma_1}\,(1-|\eta|)^{\gamma_2-1}\,|d\eta|$$

$$= C(\gamma_1,\gamma_2)\,h_1^{\gamma_1}\int_0^{|z_2|}(1-s)^{\gamma_2-1}\,ds \le \frac{C(\gamma_1,\gamma_2)}{\gamma_2}h_1^{\gamma_1}\,.$$

In a similar way, one can deduce the following estimate:

$$\left|\Phi\left(z_1,e^{i(\varphi_2+h_2)}\right)-\Phi\left(z_1,e^{i\varphi_2}\right)\right| \le C(\gamma_1,\gamma_2)\,h_2^{\gamma_2}, \quad h_2\in(0;1)\,.$$

Thus, $\tilde\Phi \in \Lambda_{\gamma_1}^{(1)} \cap \Lambda_{\gamma_2}^{(2)}$. Consequently, $\tilde\Phi(\varphi_1,\varphi_2) \overset{\text{def}}{=} \Phi\left(e^{i\varphi_1},e^{i\varphi_2}\right)$ is a continuous function on $[0;\ 2\pi]^2$.

For $\rho_k \in (0;1)$ we denote

$$\lambda_{\rho_k}^0(\theta_1,\theta_2) \overset{\text{def}}{=} \int_0^{\theta_1}\int_0^{\theta_2} F\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)d\varphi_1 d\varphi_2$$

$$= \int_0^{\theta_1}\int_0^{\theta_2} F\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)\frac{d\left(\rho_k e^{i\varphi_1}\right)}{i\rho_k e^{i\varphi_1}}\cdot\frac{d\left(\rho_k e^{i\varphi_2}\right)}{i\rho_k e^{i\varphi_2}}\,.$$

Firstly, we calculate the internal integral

$$\int_0^{\theta_1} F\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)\frac{d\left(\rho_k e^{i\varphi_1}\right)}{i\rho_k e^{i\varphi_1}} = \frac{1}{i\rho_k e^{i\theta_1}}\cdot\frac{\partial\Phi}{\partial z_2}\left(\rho_k e^{i\theta_1},\rho_k e^{i\varphi_2}\right)$$

$$-\frac{1}{i\rho_k}\cdot\frac{\partial\Phi}{\partial z_2}\left(\rho_k,\rho_k e^{i\varphi_2}\right) + \int_0^{\theta_1}\frac{\partial\Phi}{\partial z_2}\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)\frac{d\varphi_1}{\rho_k e^{i\varphi_1}}.$$

Then a routine computation yields

$$\int_0^{\theta_1}\int_0^{\theta_2} F\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)\frac{d\left(\rho_k e^{i\varphi_1}\right)}{i\rho_k e^{i\varphi_1}}\cdot\frac{d\left(\rho_k e^{i\varphi_2}\right)}{i\rho_k e^{i\varphi_2}}$$

$$=\int_0^{\theta_2}\frac{1}{i\rho_k e^{i\theta_1}}\cdot\frac{\partial\Phi}{\partial z_2}\left(\rho_k e^{i\theta_1},\rho_k e^{i\varphi_2}\right)\frac{d\left(\rho_k e^{i\varphi_2}\right)}{i\rho_k e^{i\varphi_2}}-\int_0^{\theta_2}\frac{1}{i\rho_k}\cdot\frac{\partial\Phi}{\partial z_2}\left(\rho_k,\rho_k e^{i\varphi_2}\right)\frac{d\left(\rho_k e^{i\varphi_2}\right)}{i\rho_k e^{i\varphi_2}}$$

$$+\int_0^{\theta_2}\int_0^{\theta_1}\frac{\partial\Phi}{\partial z_2}\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)\frac{d\varphi_1}{\rho_k e^{i\varphi_1}}\cdot\frac{d\left(\rho_k e^{i\varphi_2}\right)}{i\rho_k e^{i\varphi_2}}$$

$$=-\frac{\Phi\left(\rho_k e^{i\theta_1},\rho_k e^{i\theta_2}\right)}{\rho_k^2 e^{i\theta_1}e^{i\theta_2}}+\frac{\Phi\left(\rho_k e^{i\theta_1},\rho_k\right)}{\rho_k^2 e^{i\theta_1}}+\frac{1}{i\rho_k^2 e^{i\theta_1}}\int_0^{\theta_2}\Phi\left(\rho_k e^{i\theta_1},\rho_k e^{i\varphi_2}\right)\frac{d\varphi_2}{e^{i\varphi_2}}$$

$$+\frac{\Phi\left(\rho_k,\rho_k e^{i\theta_2}\right)}{\rho_k^2 e^{i\theta_2}}-\frac{1}{\rho_k^2}\Phi\left(\rho_k,\rho_k\right)-\frac{1}{i\rho_k^2}\int_0^{\theta_2}\Phi\left(\rho_k,\rho_k e^{i\varphi_2}\right)\frac{d\varphi_2}{e^{i\varphi_2}}$$

$$+\frac{1}{i\rho_k^2 e^{i\theta_2}}\int_0^{\theta_1}\frac{\Phi\left(\rho_k e^{i\varphi_1},\rho_k e^{i\theta_2}\right)d\varphi_1}{e^{i\varphi_1}}-\frac{1}{i\rho_k^2}\int_0^{\theta_1}\frac{\Phi\left(\rho_k e^{i\varphi_1},\rho_k\right)d\varphi_1}{e^{i\varphi_1}}$$

$$+\frac{1}{\rho_k^2}\int_0^{\theta_1}\int_0^{\theta_2}\frac{\Phi\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)d\varphi_1 d\varphi_2}{e^{i\varphi_1}e^{i\varphi_2}}.$$

Since $\Phi(z_1,z_2)$ is continuous in $\bar{U}^2$ and, consequently, uniformly continuous, we have

$$\Phi\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)\underset{\varphi_1,\varphi_2}{\rightrightarrows}\Phi\left(e^{i\varphi_1},e^{i\varphi_2}\right)$$

as $\rho_k\uparrow 1$.

Therefore, we have

$$\lambda^0_{\rho_k}(\theta_1,\theta_2)\rightrightarrows-\frac{\Phi\left(e^{i\theta_1},e^{i\theta_2}\right)}{e^{i\theta_1}e^{i\theta_2}}+\frac{\Phi\left(e^{i\theta_1},1\right)}{e^{i\theta_1}}+\frac{1}{ie^{i\theta_1}}\int_0^{\theta_2}\frac{\Phi\left(e^{i\theta_1},e^{i\varphi_2}\right)d\varphi_2}{e^{i\varphi_2}}$$

$$+\frac{\Phi\left(1,e^{i\theta_2}\right)}{e^{i\theta_2}}-\Phi(1,1)-\frac{1}{i}\int_0^{\theta_2}\frac{\Phi\left(1,e^{i\varphi_2}\right)d\varphi_2}{e^{i\varphi_2}}+\frac{1}{ie^{i\theta_2}}\int_0^{\theta_1}\frac{\Phi\left(e^{i\varphi_1},e^{i\theta_2}\right)d\varphi_1}{e^{i\varphi_1}}$$

$$-\frac{1}{i}\int_0^{\theta_1}\frac{\Phi\left(e^{i\varphi_1},1\right)d\varphi_1}{e^{i\varphi_1}}+\int_0^{\theta_1}\int_0^{\theta_2}\frac{\Phi\left(e^{i\varphi_1},e^{i\varphi_2}\right)d\varphi_1 d\varphi_2}{e^{i\varphi_1}e^{i\varphi_2}}\equiv\lambda^0\left(\theta_1,\theta_2\right)$$

as $\rho_k\uparrow 1$.

Since $\tilde{\Phi}\in\Lambda_{\gamma_1}^{(1)}\cap\Lambda_{\gamma_2}^{(2)}$, the last equality yields that $\lambda^0\in\Lambda_{\gamma_1}^{(1)}\cap\Lambda_{\gamma_2}^{(2)}$.

On the other hand, $\mu^0\left(\theta_1,\theta_2\right)=\lim_{\rho_k\uparrow 1}\mu_{\rho_k}^0\left(\theta_1,\theta_2\right)$, $\mu_{\rho_k}^0\left(\theta_1,\theta_2\right)=\mathrm{Re}\lambda_{\rho_k}^0\left(\theta_1,\theta_2\right)$. Thus, $\mu^0\in\Lambda_{\gamma_1}^{(1)}\cap\Lambda_{\gamma_2}^{(2)}$.

We now consider the function

$$F_1\left(z_1,z_2\right)=\int_{T^2}S_0\left(z_1,w_1\right)S_1\left(z_2,w_2\right)d\mu\left(w\right)$$

which is analytic in z_1 and antianalytic in z_2 in U^2.

Note that $\mathrm{Re}F_1\left(z_1,z_2\right)=v_{00}\left(z_1,z_2\right)+v_{11}\left(z_1,z_2\right)$.

Denote

$$\mu_{\rho_k}^1\left(\theta_1,\theta_2\right)=\int_0^{\theta_1}\int_0^{\theta_2}\left(v_{00}\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)+v_{11}\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)\right)d\varphi_1 d\varphi_2. \qquad (12)$$

Define the function $K\left(z_1,z_2\right)=\int_0^{z_1}\left(\int_0^{z_2}F_1\left(\varsigma,\eta\right)d\bar{\eta}\right)d\varsigma$, $z\in\bar{U}^2$. Obviously, $K\left(z_1,z_2\right)$ is analytic in z_1 and antianalytic as a function of z_2.

Note that

$$\mu_{\rho_k}^1\left(\theta_1,\theta_2+h\right)-\mu_{\rho_k}^1\left(\theta_1,\theta_2\right)=\mathrm{Re}\int_0^{\theta_1}\int_0^{\theta_2+h}F_1\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)d\varphi_1 d\varphi_2.$$

Let $h_3\in\left(0;1\right)$, we write

$$\vartheta=e^{i\varphi_2},\ \ \vartheta_1=\left(1-h_3\right)e^{i\varphi_2},\ \ \vartheta_2=\left(1-h_3\right)e^{i\left(\varphi_2+h_3\right)},\ \ \vartheta_3=e^{i\left(\varphi_2+h_3\right)}.$$

Fixing z_1 and using the counterpart of Cauchy's theorem for antianalytic functions, we get

$$K\left(z_1,\vartheta_3\right)-K\left(z_1,\vartheta_0\right)=\int_0^{z_1}\left(\int_0^{\vartheta_3}F_1\left(\varsigma,\eta\right)d\bar{\eta}-\int_0^{\vartheta_0}F_1\left(\varsigma,\eta\right)d\bar{\eta}\right)d\varsigma$$

$$=\int_0^{z_1}\left(\int_{\vartheta_0}^{\vartheta_3}F_1\left(\varsigma,\eta\right)d\bar{\eta}\right)d\varsigma=\int_0^{z_1}\left(\left(\int_{[\vartheta_0;\vartheta_1]}+\int_{\vartheta_1}^{\vartheta_2}+\int_{[\vartheta_2;\vartheta_3]}\right)F_1\left(\varsigma,\eta\right)d\bar{\eta}\right)d\varsigma,$$

$$\left|K\left(z_1,\vartheta_3\right)-K\left(z_1,\vartheta_0\right)\right|\le\int\limits_{[0;z_1]}\left|\int\limits_{\vartheta_0}^{\vartheta_3}F_1\left(\varsigma,\eta\right)d\bar\eta\right|\left|d\varsigma\right|.$$

The rest of the proof of the relation $\tilde K\in\Lambda^{(1)}_{\gamma_1}\cap\Lambda^{(2)}_{\gamma_2}$, where $\tilde K(\theta_1,\theta_2)=K(e^{i\theta_1},e^{i\theta_2})$, is similar to that of $\tilde\Phi\in\Lambda^{(1)}_{\gamma_1}\cap\Lambda^{(2)}_{\gamma_2}$, so we omit it.

For $\rho_k\in(0;1)$ we denote

$$\lambda^1_{\rho_k}\left(\theta_1,\theta_2\right)\overset{\text{def}}{=}\int\limits_0^{\theta_1}\int\limits_0^{\theta_2}F_1\left(\rho_k e^{i\varphi_1},\rho_k e^{i\varphi_2}\right)d\varphi_1 d\varphi_2.$$

As above, we have

$$\lambda^1_{\rho_k}(\theta_1,\theta_2)\rightrightarrows\frac{K\left(e^{i\theta_1},e^{i\theta_2}\right)}{e^{i\theta_1}e^{i\theta_2}}-\frac{K\left(e^{i\theta_1},1\right)}{e^{i\theta_1}}-\frac{1}{e^{i\theta_1}}\int\limits_0^{\theta_2}\frac{K\left(e^{i\theta_1},e^{i\varphi_2}\right)d\varphi_2}{e^{i\varphi_2}}-\frac{K\left(1,e^{i\theta_2}\right)}{e^{i\theta_2}}$$

$$+K\left(1,1\right)+\frac{1}{i}\int\limits_0^{\theta_2}\frac{K\left(1,e^{i\varphi_2}\right)d\varphi_2}{e^{i\varphi_2}}-\frac{1}{ie^{i\theta_2}}\int\limits_0^{\theta_1}\frac{K\left(e^{i\varphi_1},e^{i\theta_2}\right)d\varphi_1}{e^{i\varphi_1}}$$

$$+\frac{1}{i}\int\limits_0^{\theta_1}\frac{K\left(e^{i\varphi_1},1\right)d\varphi_1}{e^{i\varphi_1}}-\int\limits_0^{\theta_1}\int\limits_0^{\theta_2}\frac{K\left(e^{i\varphi_1},e^{i\varphi_2}\right)d\varphi_1 d\varphi_2}{e^{i\varphi_1}e^{i\varphi_2}}\equiv\lambda^1\left(\theta_1,\theta_2\right)$$

as $\rho_k\uparrow 1$.

On the other hand, $\mu^1\left(\theta_1,\theta_2\right)=\lim\limits_{\rho_k\uparrow 1}\mu^1_{\rho_k}\left(\theta_1,\theta_2\right)$, $\mu^1_{\rho_k}\left(\theta_1,\theta_2\right)=\mathrm{Re}\lambda^1_{\rho_k}\left(\theta_1,\theta_2\right)$. Thus, $\mu^1\in\Lambda^{(1)}_{\gamma_1}\cap\Lambda^{(2)}_{\gamma_2}$. Using (8) and (12), we obtain

$$\mu=\frac{1}{2}\left(\mu^0+\mu^1\right)\in\Lambda^{(1)}_{\gamma_1}\cap\Lambda^{(2)}_{\gamma_2}.$$

For the case of $n>2$ we denote

$$\mu^{(\sigma_2,\dots,\sigma_n)}_{\rho_k}\left(\theta_1,\dots,\theta_n\right)=\mathrm{Re}\int\limits_0^{\theta_1}\dots\int\limits_0^{\theta_n}F_{(\sigma_2,\dots,\sigma_n)}\left(\rho_k e^{i\varphi_1},\dots,\rho_k e^{i\varphi_n}\right)d\varphi_1\dots d\varphi_n,$$

where

$$F_{(\sigma_2,\dots,\sigma_n)}\left(z_1,\dots,z_n\right)=\int\limits_{T^n}S_0\left(z_1,w_1\right)S_{\sigma_2}\left(z_2,w_2\right)\cdots S_{\sigma_n}\left(z_n,w_n\right)d\mu\left(w\right).$$

Using Lemma 1, we get

$$\left|\mathrm{Re} F_{(\sigma_2,\ldots,\sigma_n)}\left(z_1,\ldots,z_n\right)\right| \leq C\left(\gamma_1,\ldots,\gamma_n\right)\left(1-|z_1|\right)^{\gamma_1-1}\cdots\left(1-|z_n|\right)^{\gamma_n-1}.$$

Lemma 2 yields

$$\sum_{\substack{\sigma=(0,\sigma_2,\ldots,\sigma_n)\\ \sigma_j\in\{0;1\}}} \int\limits_{T^n} S_0\left(z_1,w_1\right) S_{\sigma_2}\left(z_2,w_2\right)\cdots S_{\sigma_n}\left(z_n,w_n\right) d\mu\left(w\right)$$
$$= 2^{n-1}\int\limits_{T^n} P_0\left(z_1,w_1\right)\cdots P_0\left(z_n,w_n\right) d\mu\left(w\right). \tag{13}$$

To prove that $\mu_{\rho_k}^{(\sigma_2,\ldots,\sigma_n)}\left(\theta_1,\ldots,\theta_n\right)$ belongs to the class $\bigcap_{j=1}^{n}\Lambda_{\gamma_j}^{(j)}$, one can use the arguments similar to those used in the case of $n=2$.

Let

$$\mu_{\rho_k}\left(\theta_1,\ldots,\theta_n\right)=\int\limits_0^{\theta_1}\ldots\int\limits_0^{\theta_n} v_{0\ldots 0}\left(\rho_k e^{i\varphi_1},\ldots,\rho_k e^{i\varphi_n}\right) d\varphi_1\ldots d\varphi_n.$$

From (13) it follows that

$$\mu_{\rho_k}\left(\theta_1,\ldots,\theta_n\right)=\frac{1}{2^{n-1}}\sum_{(\sigma_2,\ldots,\sigma_n)}\mu_{\rho_k}^{(\sigma_2,\ldots,\sigma_n)}\left(\theta_1,\ldots,\theta_n\right)$$

$$=\mathrm{Re}\sum_{\sigma=(0,\sigma_2,\ldots,\sigma_n)}\int\limits_0^{\theta_1}\ldots\int\limits_0^{\theta_n}\left(\int\limits_{T^n} S_0\left(z_1,w_1\right)\cdots S_{\sigma_n}\left(z_n,w_n\right) d\mu\left(w\right)\right)\frac{d\varphi_1\ldots d\varphi_n}{2^{n-1}}$$

$$=\int\limits_0^{\theta_1}\ldots\int\limits_0^{\theta_n}\left(\int\limits_{T^n} P_0\left(z_1,w_1\right)\cdots P_0\left(z_n,w_n\right) d\mu\left(w\right)\right) d\varphi_1\ldots d\varphi_n.$$

Since $\mu^{(\sigma_2,\ldots,\sigma_n)}\left(\theta_1,\ldots,\theta_n\right)=\lim\limits_{\rho_k\uparrow 1}\mu_{\rho_k}^{(\sigma_2,\ldots,\sigma_n)}\left(\theta_1,\ldots,\theta_n\right)$, we have

$$\mu=\frac{1}{2^{n-1}}\sum_{(\sigma_2,\ldots,\sigma_n)}\mu^{(\sigma_2,\ldots,\sigma_n)}.$$

Thus, $\mu\in\bigcap_{j=1}^{n}\Lambda_{\gamma_j}^{(j)}$. The theorem is proved.

P r o o f of Remark 1.

Note that $\mu \in \Lambda_{1,1}$ if $|\mu| \left(\prod_{a_1,a_2}^{a_1+\delta_1,a_2+\delta_2}\right) \le C\delta_1\delta_2$. Thereby,

$$|\mu(a_1+\delta_1,a_2+\delta_2)-\mu(a_1+\delta_1,a_2)-(\mu(a_1,a_2+\delta_2)-\mu(a_1,a_2))| \le C\delta_1\delta_2. \tag{14}$$

Since μ is absolutely continuous in a_1 and a_2, we have that for any a_1 and a_2 there exists $\mu'_{a_1}(a_1,a_2)$ almost everywhere in a_1 for any a_2, and $\mu'_{a_2}(a_1,a_2)$ almost everywhere in a_2 for any a_1.

We divide the left-hand side of (14) by δ_2 and then take the limit

$$\lim_{\delta_2\to 0}\left(\frac{\mu(a_1+\delta_1,a_2+\delta_2)-\mu(a_1+\delta_1,a_2)}{\delta_2}-\frac{\mu(a_1,a_2+\delta_2)-\mu(a_1,a_2)}{\delta_2}\right)$$
$$=\mu'_{a_2}(a_1+\delta_1)-\mu'_{a_2}(a_1),\quad \forall a_2\in E_1,\ mesE_1=2\pi.$$

Then from (14) we get $\left|\mu'_{a_2}(a_1+\delta_1)-\mu'_{a_2}(a_1)\right| \le C\delta_1$.

Consequently, $\mu'_{a_2}(a_1,a_2)$ is absolutely continuous in a_1 for all $a_2 \in E_1$. Hence, there exists $\mu''_{a_2a_1}(a_1,a_2)$,

$$\mu''_{a_2a_1}(a_1,a_2)=\lim_{\delta_1\to 0}\left(\frac{\mu'_{a_2}(a_1+\delta_1)-\mu'_{a_2}(a_1)}{\delta_1}\right),$$

almost everywhere in a_1, and $\left|\mu''_{a_2a_1}(a_1,a_2)\right| \le C$.

Conversely, there is $E_2 \subset [0;2\pi]$, $mesE_2 = 2\pi$ such that for all $a_1 \in E_2$ there exists $\mu''_{a_1a_2}(a_1,a_2)$ almost everywhere, and $\left|\mu''_{a_1a_2}(a_1,a_2)\right| \le C$. So, the class $\Lambda_{1,1}$ consists of functions that are integrals of bounded functions.

Thus,

$$\left|\int_{T^2}\mathcal{P}(z,w)\,d\mu(w)\right| = \left|\int_T\int_T \mathcal{P}(z,w)\,\mu''_{w_1w_2}(w_1,w_2)dw_1dw_2\right|$$
$$\le C\int_{T^2}\mathcal{P}(z,w)\,dw = C.$$

E x a m p l e. In this example we show that $\Lambda_{\gamma_1,\dots,\gamma_n} \ne \bigcap_{j=1}^n \Lambda_{\gamma_j}^{(j)}$. Let $n=2$ and $\gamma_1=\gamma_2=1$. Define the function

$$\mu(\theta,\varphi)=\begin{cases} 0, & \theta=\varphi;\\ (\theta-\varphi)^2\cos\frac{1}{\theta-\varphi}, & \theta\ne\varphi.\end{cases}$$

By the definition of the class $\Lambda_{\gamma_j}^{(j)}$, we have that $\mu \in \Lambda_1^{(1)}$ if

$$\sup_\theta|\mu(\theta+\delta_1,\varphi)-\mu(\theta,\varphi)| \le C\delta_1.$$

Moreover,

$$\mu'_\theta(\theta,\varphi) = \begin{cases} 2(\theta-\varphi)\cos\frac{1}{\theta-\varphi} + \sin\frac{1}{\theta-\varphi}, & \theta\neq\varphi; \\ 0, & \theta=\varphi. \end{cases}$$

Therefore, $\mu\in\Lambda_1^{(1)}$. In a similar way, one can deduce that $\mu\in\Lambda_1^{(2)}$.

On the other hand, one can show that $|\mu|\left(\Pi_{\varphi,\varphi}^{\varphi+\delta,\varphi+\delta}\right)\geq c\delta$. Thus, $\mu\notin\Lambda_{1,1}$. Hence, $\Lambda_{\gamma_1,\ldots,\gamma_n}\neq\bigcap_{j=1}^n\Lambda_{\gamma_j}^{(j)}$.

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