Journal of Mathematical Physics, Analysis, Geometry
2011, vol. 7, No. 4, pp. 333–351

On the Neumann Boundary Controllability for the Non-Homogeneous String on a Segment

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Received April 27, 2011

The control system $w_{tt} = w_{xx} - q(x)w$, $w_x(0,t) = u(t)$, $w_x(d,t) = 0$, $x \in (0,d)$, $t \in (0,T)$, $d > 0$, $0 < T \le d$ is considered. Here $q \in C^1[0,d]$, $q > 0$, $q'_+(0) = q'_-(d) = 0$, u is a control, $|u(t)| \le 1$ on $(0,T)$. The necessary and sufficient conditions of null-controllability and approximate null-controllability are obtained for this system. The controllability problems are considered in the modified Sobolev spaces. The controls that solve these problems are found explicitly. It is proved that among the solutions of the Markov trigonometric moment problem there are bang-bang controls solving the approximate null-controllability problem.

Key words: wave equation, controllability problem, Neumann control bounded by a hard constant, modified Sobolev space, Sturm–Liouville problem, tranformation operator.

Mathematics Subject Classification 2000: 93B05, 35B37, 35L05, 34B24.

1. Introduction

In the paper, the wave equation for a non-homogeneous string on a segment is considered. The velocity of the string is fixed at the right end point. At the left end point a control of the Neumann type is applied. The control is bounded by a hard constant. We assume that the potential q is not a constant, generally speaking. The time T is constrained ($T \in (0,d]$). The problems of null-controllability and approximate null-controllability are studied in the spaces $\mathcal{H}_Q^s$, $s \le 1$ (the modified Sobolev spaces of even periodic functions under the operator $(1 + D^2 + q(x))^{s/2}$ instead of $(1 + D^2)^{s/2}$, $D = -id/dx$).

Note that many results are available for controllability problems for hyperbolic partial differential equations (see [1–16] and others). We describe the papers most similar to ours.

The wave equation for a homogeneous string on a finite segment is well studied. The boundary L^p-controllability ($2 \le p \le \infty$) with the Dirichlet control for the equation was considered in [1–7] and others. In [8], the problem of exact null-controllability for the wave equation was considered in a bounded domain $\Omega \subset \mathbb{R}^m$ with the Neumann boundary control. The potential of the string was equal to zero. The problem was considered in $L^2(\Omega) \times \big(H_0^1(\Omega)\big)'$. The controllability results were obtained for the controls from a special class larger than L^2. Note that the potential q cannot be equal to zero in the present work.

The problems of boundary controllability for a non-homogeneous string were studied in a bounded domain $\Omega \subset \mathbb{R}^n$ in [9, 10] and on a segment in [11, 12]. In all these papers, the Dirichlet controls were extended on a part of a boundary and were of the class L^2 in [9, 10], of the class W_2^1 in [11], and of the class L^∞ in [12]. Note that in [11] the potential q was equal to a constant, and in [12], the potential was not equal to a constant, generally speaking. In [13], the problem of null- and approximate null-controllability for the wave equation was considered on a half-axis. The potential q was equal to a constant. The Neumann control at the point $x = 0$ was applied. The control was bounded by a hard constant. The problems were considered in the Sobolev spaces H_0^s, $s \le 1$. The necessary and sufficient conditions of null- and approximate null-controllability for the considered system were obtained. An explicit formula for the control was also found. Notice that if we put $q = \text{const} > 0$ in the present paper, then the results are close to those of [13]. Note also that the condition $q \ne \text{const}$ is essential distinction of the present paper from those where $q = \text{const}$, and it makes the study more complicated.

To study the controllability problems for the given wave equation we apply the method used in [12], namely, we apply the operators adjoint to the transformation operators for the Sturm–Liouville problem on a considered segment. We extend the transformation operators and their adjoints to $\mathcal{H}_Q^s$, $s \in \mathbb{R}$, and study them. As the investigation of the mentioned operators in $\mathcal{H}_Q^s$, $s \in \mathbb{R}$, is fundamental and very voluminous, it is carried out in Appendix. We also prove that $\mathcal{H}_Q^s$ is equivalent to the standard Sobolev space of even periodic functions. For the proof, the theorem (Hörmander, [14]) about a transformation of the Sobolev space under pseudo-differential operator is used. The application of the adjoint operators to the given control system allows to obtain an explicit formula for the control (that belongs to L^∞) and the necessary and sufficient conditions of null- and approximate null-controllability. In the paper, it is also proved that the control, the initial and the corresponding steering states of the control system are connected. This connection is described by the mentioned operators. We also prove that the system is (approximately) null-controllable from not an arbitrary initial state. The initial state of the string depends on its initial velocity, and this dependence is also described by the operators adjoint to the transformation operators. The distinctions of [12] from the present paper are the following: in [12], the control was of the Dirichlet type, therefore the control problem was considered in the modified Sobolev spaces of odd periodic functions under the operator $(1 + D^2 + q(x))^{s/2}$, $s \leq 0$. That is why the transformation operators considered in the present paper have different than in [12] kernels with different properties.

We also construct the bang-bang controls that solve the approximate null-controllability problem for $s < 1/2$. We prove that these controls are the solutions of the Markov trigonometric moment problem on $(0, T)$. Note that the construction of the bang-bang controls as the solutions of the Markov trigonometric moment problem on $(0, T)$ was applied in [15]. The authors studied the wave equation with $q = 0$ on a half-axis in the Sobolev spaces H_0^s, $s \leq 0$. A control of the Dirichlet type and of the class L^∞ was considered. The authors reduced the approximate null-controllability problem for $s < -1/2$ to the Markov trigonometric moment problem on $(0, T)$. We use their method of reducing. We should note that the distinctions of the present paper from [15] are the following: the potential q is not equal to zero, we consider the wave equation on a finite segment, and the control is of the Neumann type (therefore the bang-bang controls solve the approximate null-controllability problem for $s < 1/2$). Further, the Markov trigonometric moment problem can be solved by the algorithm given in [16]. In [13], the approximate null-controllability problem was studied for the wave equation with the potential $q = \text{const}$ on a half-axis. The equation was controlled by the control of the Neumann type. This problem was considered in H_0^s, $s \leq 1$. The bang-bang controls were constructed as the solutions of the Markov power moment problem on $(0, T)$. The bang-bang controls were proved to solve the approximate null-controllability problem for $s < 1/2$.

2. Notation

Consider the wave equation on a finite segment

$$w_{tt}(x,t) = w_{xx}(x,t) - q(x)w(x,t), \qquad x \in (0,d),\ t \in (0,T), \tag{2.1}$$

controlled by the boundary conditions

$$w_x(0,t) = u(t),\ w_x(d,t) = 0, \qquad t \in (0,T), \tag{2.2}$$

where $d > 0$, $0 < T \leq d$, and u is a control. We suppose that

$$q \in \mathcal{E}(0,d) = \left\{ r \in C^1[0,d] : r(x) > 0, r'_+(0) = r'_-(d) = 0 \right\},$$
$$u \in \mathcal{B}(0,T) = \left\{ v \in L^\infty(0,T) : |v(t)| \leq 1 \text{ a.e. on } (0,T) \right\}.$$

Consider control system (2.1), (2.2) with the initial conditions

$$\begin{pmatrix} w \\ w_t \end{pmatrix}(x,0) = \begin{pmatrix} w_0^0 \\ w_1^0 \end{pmatrix}(x) = w^0(x). \tag{2.3}$$

To introduce the spaces used in this work, we have to consider the Sturm–Liouville problem on the segment $(0,d)$

$$Gv \equiv -v''(x) + q(x)v(x) = \lambda^2 v(x), \quad v'(0) = v'(d) = 0, \qquad x \in (0,d), \tag{2.4}$$

where $q \in \mathcal{E}(0,d)$.

Let $\{\mu_n = \lambda_n^2\}_{n=1}^{\infty}$ be a set of eigenvalues, and $\{y_n(\lambda_n, x)\}_{n=1}^{\infty}$ be a system of the corresponding eigenfunctions of the operator $G \equiv -\left(\frac{d}{dx}\right)^2 + q(x)$. As it is known (see, e.g., [17]), the set of eigenvalues is countable, they are real, nonnegative and simple, and $\lambda_n \neq 0$, $n = \overline{1,\infty}$. It is also known that $\{y_n(\lambda_n, x)\}_{n=1}^{\infty}$ are real and form an orthonormal basis in $L^2[0,d]$.

Let $\mathcal{S}$ be the Schwartz space [18]

$$\mathcal{S} = \left\{ \varphi \in C^{\infty}(\mathbb{R}) : \forall m, l \in \mathbb{N} \cup 0 \, \exists C_{ml} > 0 \, \forall x \in \mathbb{R} \left| \varphi^{(m)}(x) \left(1 + |x|^2\right)^l \right| \leq C_{ml} \right\},$$

$\mathcal{S}'$ be the dual space.

A distribution $f \in \mathcal{S}'$ is said to be *odd* if $(f, \varphi(-x)) = -(f, \varphi(x))$, $\varphi \in \mathcal{S}$. A distribution $f \in \mathcal{S}'$ is said to be *even* if $(f, \varphi(-x)) = (f, \varphi(x))$, $\varphi \in \mathcal{S}$.

Let $\Omega : \mathcal{S}' \to \mathcal{S}'$ be the odd extension operator, $\Xi : \mathcal{S}' \to \mathcal{S}'$ be the even extension operator. Thus $(\Omega f)(x) = f(x) - f(-x)$, $(\Xi f)(x) = f(x) + f(-x)$ for $f \in \mathcal{S}'$. Let $\mathcal{T}_h$ be the translation operator: $\mathcal{T}_h \varphi(x) = \varphi(x+h)$, $\varphi \in \mathcal{S}$ and $(\mathcal{T}_h f, \varphi) = (f, \mathcal{T}_{-h}\varphi)$, $f \in \mathcal{S}'$, $\varphi \in \mathcal{S}$.

We assume that q and $y_n(\lambda_n, \cdot)$, $n = \overline{1,\infty}$, are defined on $\mathbb{R}$ and equal to zero on $\mathbb{R} \setminus [0,d]$. Denote $Q = \sum_{k \in \mathbb{Z}} \mathcal{T}_{2dk} \Xi q$, $\;Y_n(\lambda_n, \cdot) = \sum_{k \in \mathbb{Z}} \mathcal{T}_{2dk} \Xi y_n(\lambda_n, \cdot)$, $n = \overline{1,\infty}$. Since $q \in \mathcal{E}(0,d)$, then $Q \in C^1(\mathbb{R})$. Introduce the operator $D_Q^2 = Q(x) + D^2$, where $D = -id/dx$. Thus $D_Q^2 Y_n(\lambda_n, x) = \lambda_n^2 Y_n(\lambda_n, x)$, $n = \overline{1,\infty}$.

Consider the modified Sobolev spaces

$$\mathcal{H}_Q^s = \left\{ f \in \mathcal{S}' : f \text{ is even and } 2d\text{-periodic}, \ \left(1 + D_Q^2\right)^{s/2} f \in L^2_{loc}(\mathbb{R}) \right\}, \ s \in \mathbb{R},$$

with the norm $\|f\|_Q^s = \left(\int\limits_{-d}^{d} \left| \left(1 + D_Q^2\right)^{s/2} f(x) \right|^2 dx \right)^{1/2}$, and

$$\mathbb{H}_0^s = \left\{ f \in \mathcal{S}' : f(x) = \sum_{n=1}^{\infty} f_n e^{-i\lambda_n x} \mid \left\{ f_n (1 + \lambda_n^2)^{s/2} \right\}_{n=1}^{\infty} \in l^2 \right\}, \quad s \in \mathbb{R},$$

with the norm $\|f\|_0^s = \left(\sum_{n=1}^{\infty} \left|\left(1+\lambda_n^2\right)^{s/2} f_n\right|^2\right)^{1/2}$, where λ_n are the arithmetic roots of eigenvalues of the Sturm–Liouville problem (2.4). (For convenience in further reasoning we will number $\{\lambda_n\}_{n=1}^{\infty}$ in ascending order.)

We denote $\widetilde{\mathcal{H}}_Q^s = \mathcal{H}_Q^s \times \mathcal{H}_Q^{s-1}$ and use the norm

$$|||f|||_Q^s = \left(\left(\|f_1\|_Q^s\right)^2 + \left(\|f_2\|_Q^{s-1}\right)^2\right)^{1/2}$$

for $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \widetilde{\mathcal{H}}_Q^s$.

R e m a r k 2.1. In Lemma A.1, it is proved that $\mathcal{H}_Q^s$ and $H_{0,per}^s$ (the Sobolev space of even $2d$-periodic functions [19]), $s \in \mathbb{R}$, coincide as sets, have equivalent norms, and

$$\mathcal{H}_Q^s = \left\{ f \in \mathcal{S}' : f(x) = \sum_{n=1}^{+\infty} f_n Y_n(\lambda_n, x) \wedge \left\{ f_n (1+\lambda_n^2)^{s/2} \right\}_{n=1}^{+\infty} \in l^2 \right\}, \quad s \in \mathbb{R},$$

with the norm $\|f\|_Q^s = \left(\sum_{n=1}^{\infty} \left|f_n (1+\lambda_n^2)^{s/2}\right|^2\right)^{1/2}$.

R e m a r k 2.2. The series of exponentials in the definition of $\mathbb{H}_0^s$ converges with respect to the norm of the standard Sobolev space H_0^s. In fact, it is proved in Lemma A.2. If $f \in \mathbb{H}_0^s$ is even, then $f(x) = \sum_{n=1}^{\infty} f_n \cos \lambda_n x$. One can find the properties of the functions from $\mathbb{H}_0^s$ in Lemmas A.2–A.5.

Denote by $\mathcal{H}_Q^s(a,b)$ and $\mathbb{H}_0^s(a,b)$ the restrictions of the spaces $\mathcal{H}_Q^s$ and $\mathbb{H}_0^s$ to (a,b), respectively.

R e m a r k 2.3. It is proved in [12, Lemma A.1] that the system $\left\{e^{i\lambda_n x}\right\}_{n=1}^{\infty}$ is the Riesz basis in the space $L^2(-d,d)$. Hence $\mathcal{H}_Q^0(-d,d) = \left\{ f \in \mathbb{H}_0^0(-d,d) : f \text{ is even} \right\}$.

Further, throughout the paper we will assume that $s \leq 1$.

3. Definitions

Consider control system (2.1)–(2.3), where $w^0 \in \widetilde{\mathcal{H}}_Q^s(0,d)$. We consider the solution of (2.1)–(2.3) in the space $\mathcal{H}_Q^s$.

Extend $w(x,t)$ and $w^0(x)$ from the segment $(0,d)$ on the whole axis. Consider the even $2d$-periodic extensions (with respect to x)

$$W(\cdot,t) = \sum_{k\in\mathbb{Z}} \mathcal{T}_{2dk} \Xi w(\cdot,t), \quad W^0 = \sum_{k\in\mathbb{Z}} \mathcal{T}_{2dk} \Xi w^0, \quad t \in (0,T).$$

Obviously, $W^0 \in \widetilde{\mathcal{H}}_Q^s$, $W(\cdot,t) \in \mathcal{H}_Q^s$ $(t \in (0,T))$.

It is easy to see that control problem (2.1)–(2.3) is equivalent to the following problem:

$$W_{tt}(x,t) = W_{xx}(x,t) - Q(x)W(x,t) - 2u(t)\sum_{k\in\mathbb{Z}} \mathcal{T}_{2dk}\delta(x), \quad x \in \mathbb{R},\ t \in (0,T), \tag{3.1}$$

$$W(x,0) = W_0^0(x), \quad W_t(x,0) = W_1^0(x), \quad x \in \mathbb{R}, \tag{3.2}$$

where δ is the Dirac distribution. Consider this system with the steering conditions

$$W(x,T) = W_0^T(x), \quad W_t(x,T) = W_1^T(x), \quad x \in \mathbb{R}, \tag{3.3}$$

where $W^T = \begin{pmatrix} W_0^T \\ W_1^T \end{pmatrix} \in \widetilde{\mathcal{H}}_Q^s$. We consider the solution of (3.1)–(3.3) in $\mathcal{H}_Q^s$.

Let $T > 0$, $w^0 \in \widetilde{\mathcal{H}}_Q^s(0,d)$. Denote by $\mathcal{R}_T(w^0)$ the set of the states $W^T \in \widetilde{\mathcal{H}}_Q^s$ for which there exists a control $u \in \mathcal{B}(0,T)$ such that problem (3.1)–(3.3) has a unique solution in $\mathcal{H}_Q^s$.

Definition 3.1. *A state $w^0 \in \widetilde{\mathcal{H}}_Q^s(0,d)$ is called null-controllable at a given time $T > 0$ if 0 belongs to $\mathcal{R}_T(w^0)$ and approximately null-controllable at a given time $T > 0$ if 0 belongs to the closure of $\mathcal{R}_T(w^0)$ in $\widetilde{\mathcal{H}}_Q^s$.*

Definition 3.2. *Denote by $\mathrm{S}_T : \widetilde{\mathcal{H}}_Q^p \to \widetilde{\mathcal{H}}_Q^p$, $p \in \mathbb{R}$, the operator*

$$(\mathrm{S}_T f)(x) = \sum_{n=1}^{\infty} \begin{pmatrix} \cos\lambda_n T & \frac{\sin\lambda_n T}{\lambda_n} \\ -\lambda_n \sin\lambda_n T & \cos\lambda_n T \end{pmatrix} \begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix} Y_n(\lambda_n, x),$$

where $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \sum_{n=1}^{\infty} \begin{pmatrix} f_n^1 \\ f_n^2 \end{pmatrix} Y_n(\lambda_n, x)$, $D(\mathrm{S}_T) = R(\mathrm{S}_T) = \widetilde{\mathcal{H}}_Q^p$.

The operator S_T is linear and continuous (the proof of these facts is similar to the proof of [12, Lemma A.3]). It was also proved there that

$$|||S_T f|||_Q^s \le C_S\, |||f|||_Q^s\,, \tag{3.4}$$

where $C_S^2 = 2\max\{2, 2T^2+1\}$.

Definition 3.3. *Denote by $\partial^{-1} : \mathbb{H}_0^p \to \mathbb{H}_0^{p+1}$, $p \in \mathbb{R}$, the operator*

$$\partial^{-1} f = i \sum_{n=1}^{\infty} f_n \frac{1}{\lambda_n} e^{-i\lambda_n x},$$

where $f = \sum_{n=1}^{\infty} f_n e^{-i\lambda_n x}$, $D(\partial^{-1}) = \mathbb{H}_0^p$, $R(\partial^{-1}) = \mathbb{H}_0^{p+1}$.

In Lemma A.6, ∂^{-1} is proved to be continuous.

Further we define the transformation operators for the Sturm–Liouville problem on a segment. As it is known from [20], the integral operator $(\mathcal{K}f)(x) = f(x) + \int_0^x \mathtt{K}(x,t;0)f(t)\,dt$ transfers the solution of the Cauchy problem $y'' + \lambda^2 y = 0$, $y(0) = 1$, $y'(0) = 0$ on $[-d, d]$ to the solution of the Cauchy problem $y'' - q(x)y + \lambda^2 y = 0$, $y(0) = 1$, $y'(0) = 0$ on $[-d, d]$. According to [20], the operator $\mathcal{K}$ has the inverse one that we denote by $\mathcal{L}$ (see Appendix B for details). In the present paper we determine these operators for $p \in \mathbb{R}$ in the following spaces: $\mathcal{K} : \mathbb{H}_0^{-p}(-d, d) \longrightarrow \mathcal{H}_Q^{-p}(-d, d)$, $\mathcal{L} : \mathcal{H}_Q^{-p}(-d, d) \longrightarrow \mathbb{H}_0^{-p}(-d, d)$, where $D(\mathcal{K}) = \left\{f \in \mathbb{H}_0^{-p}(-d, d) : f \text{ is even}\right\}$, $D(\mathcal{L}) = \mathcal{H}_Q^{-p}(-d, d)$, $R(\mathcal{K}) = D(\mathcal{L})$, $R(\mathcal{L}) = D(\mathcal{K})$ (see Lemma B.1). We prove that these operators are linear and isometric in Lemma B.2. In Definition B.1 (Appendix B), we also determine the adjoint operators $\mathcal{K}^* : \mathcal{H}_Q^p(-d, d) \longrightarrow \mathbb{H}_0^p(-d, d)$, $\mathcal{L}^* : \mathbb{H}_0^p(-d, d) \longrightarrow \mathcal{H}_Q^p(-d, d)$, where $D(\mathcal{K}^*) = \mathcal{H}_Q^p(-d, d)$, $D(\mathcal{L}^*) = \{f \in \mathbb{H}_0^p(-d, d) : f \text{ is even}\}$, $R(\mathcal{K}^*) = D(\mathcal{L}^*)$, $R(\mathcal{L}^*) = D(\mathcal{K}^*)$. They are also linear and isometric. The properties of the operators $\mathcal{K}$, $\mathcal{L}$, $\mathcal{K}^*$, $\mathcal{L}^*$ are studied in Appendix B.

4. Controllability Conditions

The following two theorems are the main result of the paper. Theorem 4.1 gives the description of the set $\mathcal{R}_T(w^0)$, and Theorem 4.2 gives the necessary and sufficient conditions of null-controllability and approximate null-controllability for the initial state of (2.1)–(2.3).

Theorem 4.1. *Let* $0 < T \le d$, $w^0 \in \widetilde{\mathcal{H}}_Q^s(0, d)$, $s \le 1$. *Then*

$$\mathcal{R}_T(w^0) = \left\{ \mathrm{S}_T \left[W^0 - \sum_{k\in\mathbb{Z}} \mathcal{T}_{2kd} \mathcal{L}^* \begin{pmatrix} -\partial^{-1}\Omega U \\ \Xi U \end{pmatrix} \right] : u \in \mathcal{B}(0, T) \right\}, \tag{4.1}$$

where $U = u$ *on* $[0, T]$ *and* $U = 0$ *on* $\mathbb{R} \setminus [0, T]$.

P r o o f. The proof of the theorem is similar to that of [12, Theorem 3.1]. Therefore we only give its scheme. According to Lemma A.1, we have

$$W(x, t) = \sum_{n=1}^{\infty} w_n(t) Y_n(\lambda_n, x), \quad \sum_{k\in\mathbb{Z}} \mathcal{T}_{2dk}\delta(x) = \sum_{n=1}^{\infty} \delta_n Y_n(\lambda_n, x), \tag{4.2}$$

$$W_0^\gamma(x) = \sum_{n=1}^{\infty} w_{0n}^\gamma Y_n(\lambda_n, x), \qquad W_1^\gamma(x) = \sum_{n=1}^{\infty} w_{1n}^\gamma Y_n(\lambda_n, x), \qquad \gamma = 0, T, \tag{4.3}$$

where $w_n(t) = (w(\cdot, t), y_n(\lambda_n, \cdot))$, $\delta_n = \frac{1}{2}(\delta, Y_n(\lambda_n, \cdot)) = \frac{1}{2}$, $w_{0n}^\gamma = (w_0^\gamma(\cdot), y_n(\lambda_n, \cdot))$, $w_{1n}^\gamma = (w_1^\gamma(\cdot), y_n(\lambda_n, \cdot))$, $\gamma = 0, T$, $n = \overline{1, \infty}$.

Substituting (4.2) and (4.3) into (3.1)–(3.3), we obtain

$$w_n''(t) + w_n(t)\lambda_n^2 = -u(t), \qquad t \in [0,T], \quad n = \overline{1,\infty}. \tag{4.4}$$

$$w_n(0) = w_{0n}^0, \quad \frac{\partial w_n}{\partial t}(0) = w_{1n}^0, \qquad w_n(T) = w_{0n}^T, \quad \frac{\partial w_n}{\partial t}(T) = w_{1n}^T. \tag{4.5}$$

It is easy to see that system (4.4), (4.5) is equivalent to the linear system ($n = \overline{1,\infty}$)

$$v_n'(t) = \mathtt{A}_n v_n(t) + \mathtt{b}_n(t), \quad t \in [0,T],$$
$$v_n(0) = \begin{pmatrix} w_{0n}^0 \\ w_{1n}^0 \end{pmatrix}, \ v_n(T) = \begin{pmatrix} w_{0n}^T \\ w_{1n}^T \end{pmatrix},$$

where $v_n = \begin{pmatrix} w_n \\ w_n' \end{pmatrix}$, $\mathtt{A}_n = \begin{pmatrix} 0 & 1 \\ -\lambda_n^2 & 0 \end{pmatrix}$, $\mathtt{b}_n(t) = \begin{pmatrix} 0 \\ -u(t) \end{pmatrix}$. Thus we have for $n = \overline{1,\infty}$

$$\begin{pmatrix} \cos\lambda_n T & \frac{\sin\lambda_n T}{\lambda_n} \\ -\lambda_n \sin\lambda_n T & \cos\lambda_n T \end{pmatrix} \begin{pmatrix} w_{0n}^0 + \int_0^T \frac{\sin\lambda_n t}{\lambda_n} u(t)\,dt \\ w_{1n}^0 - \int_0^T \cos\lambda_n t \cdot u(t)\,dt \end{pmatrix} = \begin{pmatrix} w_{0\,n}^T \\ w_{1\,n}^T \end{pmatrix}. \tag{4.6}$$

Since $u \in \mathcal{B}(0,T) \subset L^2(0,d)$, then $U \in L^2(-d,d)$. Using Lemmas A.3, A.4 and A.6, we conclude that $U \in \mathbb{H}_0^0(-d,d)$, $\Omega U \in \mathbb{H}_0^0(-d,d)$, $\Xi U \in \mathbb{H}_0^0(-d,d)$, $(\Omega U)' = \Xi U' \in \mathbb{H}_0^{-1}(-d,d)$. Hence we can apply the operator $\mathcal{L}^*$ to ΞU and $\Xi U'$. Reasoning similarly to [12], we get

$$\int_0^T \frac{\sin\lambda_n t}{\lambda_n} u(t)\,dt = \frac{1}{2\lambda_n^2}\left((\Omega U)', \cos\lambda_n t\right) = \frac{1}{\lambda_n^2}\left(\mathcal{L}^*\Xi U', y_n\right),$$
$$\int_0^T \cos\lambda_n t u(t)\,dt = \frac{1}{2}\left(\Xi U, \cos\lambda_n t\right) = \left(\mathcal{L}^*\Xi U, y_n\right).$$

Denote $u_n = (\mathcal{L}^*\Xi U, y_n)$, $u_n' = (\mathcal{L}^*\Xi U', y_n)$. Thus (4.6) is equivalent to the equality

$$\begin{pmatrix} \cos\lambda_n T & \frac{\sin\lambda_n T}{\lambda_n} \\ -\lambda_n \sin\lambda_n T & \cos\lambda_n T \end{pmatrix} \begin{pmatrix} w_{0n}^0 + \frac{u_n'}{\lambda_n^2} \\ w_{1n}^0 - u_n \end{pmatrix} = \begin{pmatrix} w_{0\,n}^T \\ w_{1\,n}^T \end{pmatrix}, \qquad n = \overline{1,\infty}. \tag{4.7}$$

Due to Lemma A.1, we get

$$\sum_{k\in\mathbb{Z}} \mathcal{T}_{2kd}\left(\mathcal{L}^*\Xi U\right)(x) = \sum_{n=1}^{\infty} u_n Y_n(\lambda_n, x), \quad \sum_{k\in\mathbb{Z}} \mathcal{T}_{2kd}\left(\mathcal{L}^*\Xi U'\right)(x) = \sum_{n=1}^{\infty} u_n' Y_n(\lambda_n, x).$$

Therefore, using Definition 3.3, from (4.7) we obtain

$$S_T\left[W^0-\sum_{k\in\mathbb{Z}}\mathcal{T}_{2kd}\mathcal{L}^*\begin{pmatrix}-\partial^{-2}\Xi U'\\ \Xi U\end{pmatrix}\right]=W^T.$$

Since $\Xi U'=(\Omega U)'=\partial\Omega U$, we obtain (4.1). The theorem is proved.

Theorem 4.2. *Let $0<T\le d$, $w^0\in\widetilde{\mathcal{H}}^s_Q(0,d)$, $s\le 1$. Then the following statements are equivalent:*

(i) *the state w^0 is null-controllable at the time T;*

(ii) *the state w^0 is approximately null-controllable at the time T;*

(iii) *the conditions below hold*

$$\operatorname{supp} w_1^0\subset[0,T], \tag{4.8}$$

$$w_1^0\in L^\infty(0,d) \text{ and } \left|\left(\mathcal{K}^*\Xi w_1^0\right)(x)\right|\le 1 \text{ a.e. on } [-d,d], \tag{4.9}$$

$$\Xi w_0^0=-\mathcal{L}^*\partial^{-1}\left(\operatorname{sign} t\mathcal{K}^*\Xi w_1^0\right). \tag{4.10}$$

In addition, the control u is given by the formula

$$u(t)=w_1^0(t)+\int_t^T K(x,t;0)w_1^0(x)\,dx,\quad t\in[0,T]. \tag{4.11}$$

The proof of this theorem is rather similar to the proof of [12, Theorem 3.2], but formula (4.1) is used instead of the corresponding formula in [12], as well as Lemmas A.3, A.5, B.2, B.4, B.5, and the Riesz theorem.

R e m a r k 4.1. We have from (4.11) that there exists $U>0$ such that $|u|\le U$ on $(0,T)$ iff there exists $V>0$ such that $\left|w_1^0\right|\le V$ on $(0,d)$.

R e m a r k 4.2. Let $q(x)\equiv q=\text{const}>0$ on $(0,d)$. Hence, $Q(x)\equiv q$ on $(-d,d)$. Find the kernels $\mathtt{L}(x,t;0)$ and $\mathtt{K}(x,t;0)$ of the transformation operators on $(-d,d)\times(-d,d)$. We have $\mathtt{K}(x,t;0)=\mathtt{K}(x,t)+\mathtt{K}(x,-t)$, $\mathtt{L}(x,t;0)=\mathtt{L}(x,t)+\mathtt{L}(x,-t)$, where $\mathtt{K}(x,t)$ and $\mathtt{L}(x,t)$ are the solutions of the following systems (see Appendix B):

$$\mathtt{K}_{xx}(x,t)-\mathtt{K}_{tt}(x,t)=q\mathtt{K}(x,t),\qquad \mathtt{L}_{xx}(x,t)-\mathtt{L}_{tt}(x,t)=-q\mathtt{L}(x,t),$$

$$\mathtt{K}(x,x)=\frac{1}{2}qx,\quad \mathtt{K}(x,-x)=0,\qquad \mathtt{L}(x,x)=-\frac{1}{2}qx,\quad \mathtt{L}(x,-x)=0.$$

It is proved in [12] that

$$\mathtt{K}(x,t;0) = qx\frac{I_1\left(\sqrt{q(x^2-t^2)}\right)}{\sqrt{q(x^2-t^2)}}, \qquad |t|<|x|, \qquad \mathtt{K}(x,t;0)=0, \quad |t|\ge|x|;$$

$$\mathtt{L}(x,t;0) = -qx\frac{J_1\left(\sqrt{q(x^2-t^2)}\right)}{\sqrt{q(x^2-t^2)}}, \qquad |t|<|x|, \qquad \mathtt{L}(x,t;0)=0, \quad |t|\ge|x|,$$

where $J_m(z)$ is the Bessel function, $I_m(z) = i^{-m}J_m(iz)$ is the modified Bessel function, $m \in \mathbb{Z}$. Thus in the case of $q = \text{const}$, we get

$$u(t) = w_1^0(t) + q\int_t^T \frac{xI_1\left(\sqrt{q(x^2-t^2)}\right)}{\sqrt{q(x^2-t^2)}} w_1^0(x)\,dx, \qquad t\in(0,T),$$

$$w_1^0(t) = u(t) - q\int_t^T \frac{xJ_1\left(\sqrt{q(x^2-t^2)}\right)}{\sqrt{q(x^2-t^2)}} u(x)\,dx, \qquad t\in(0,T).$$

One can see that if $\left|w_0^0\right| \le C_w$ on $(0,d)$, then $|u| \le C_w I_0(\sqrt{q}T)$ and if $|u|\le C_u$ on $(0,T)$, then $\left|w_0^0\right| \le C_u\left(1+\sqrt{q}T\right)$. Note that similar formulas were obtained for the semi-infinite string with $q = \text{const}$ in [13].

E x a m p l e 4.1. Assume that $d = 24$, $T = 20$, $q(x) \equiv q > 0$, $x \in (0, 24)$. Let $w_1^0(x) = \frac{x^2}{3\cdot 20^2 I_0(20\sqrt{q})}$ on $(0, 20)$, $w_1^0(x) = 0$ on $(20, 24)$. Let $w_0^0(x)$ such that $\Xi w_0^0 = -\mathcal{L}^*\partial^{-1}\left(\operatorname{sign} x\mathcal{K}^*\Xi w_1^0\right)$ on $(-24, 24)$. Thus, $\left|w_1^0(x)\right| \le \frac{20^2}{3\cdot 20^2 I_0(20\sqrt{q})} = \frac{1}{3I_0(20\sqrt{q})}$. Due to Remark 4.2, we get the estimate $\left|\mathcal{K}^*\Xi w_1^0(x)\right| \le \frac{1}{3I_0(20\sqrt{q})} I_0(20\sqrt{q}) = \frac{1}{3}$. Consequently, proposition (iii) of Theorem 4.2 holds. Hence the state w^0 is null-controllable at the time $T = 20$. From the formula for a control obtained in Remark 4.2, we get

$$u(t) = \frac{20^2 qI_0(\sqrt{q(20^2-t^2)}) - 2\sqrt{q(20^2-t^2)}I_1(\sqrt{q(20^2-t^2)})}{3\cdot 20^2 qI_0(20\sqrt{q})}, \ t\in(0,20).$$

5. The Moment Problem

We consider the Markov trigonometric moment problem on $(0, T)$ in this section. The main goal of the section is to prove that among the solutions of the moment problem there are bang-bang solutions of the approximate null-controllability problem for $s < 1/2$. As stated in Introduction, we use the methods from [15] and [16] to prove the results of this section. The proofs of theorems have the same scheme as the proofs of the corresponding theorems in [12].

Consider control system (2.1)–(2.3). Let $0 < T \le d$, $w^0 \in \widetilde{\mathcal{H}}^s_Q(0,d)$, $s \le 1$. Assume that proposition (iii) of Theorem 4.2 holds. Put

$$\omega_m = \int_0^T e^{i\frac{\pi m x}{d}} \left(\mathcal{K}^* \Xi w^0_1\right)(x)\, dx, \qquad m = \overline{-\infty, \infty}. \tag{5.1}$$

The problem of determination of a function $u \in \mathcal{B}(0,T)$ such that

$$\int_0^T e^{i\frac{\pi m x}{d}} u(x)\, dx = \omega_m, \qquad m = \overline{-\infty, \infty}, \tag{5.2}$$

for a given $\{\omega_m\}_{m=-\infty}^{\infty}$ and $T > 0$ is called the Markov trigonometric moment problem on $(0,T)$ for the infinite sequence $\{\omega_m\}_{m=-\infty}^{\infty}$.

Consider (5.2) for a finite set of m. The problem of determination of a function $u \in \mathcal{B}(0,T)$ such that

$$\int_0^T e^{i\frac{\pi m x}{d}} u(x)\, dx = \omega_m, \qquad m = \overline{-M, M}, \qquad M \in \mathbb{N}, \tag{5.3}$$

for a given $\{\omega_m\}_{m=-M}^{M}$ and $T > 0$ is called the Markov trigonometric moment problem on $(0,T)$ for the finite sequence $\{\omega_m\}_{m=-M}^{M}$.

The following theorem is proved similarly to [12, Theorem 5.1].

Theorem 5.1. *Let $0 < T \le d$, $w^0 \in \widetilde{\mathcal{H}}^s_Q(0,d)$, $s \le 1$. Assume that proposition (iii) of Theorem 4.2 holds. Define the sequence $\{\omega_m\}_{m=-\infty}^{\infty}$ by (5.1). Then the state w^0 is null-controllable at the time T iff the Markov trigonometric moment problem (5.2) has a unique solution on $(0,T)$. Moreover, this solution is of the form (4.11).*

R e m a r k 5.1. Since the system $\{e^{-i\frac{\pi m x}{d}}\}_{m=-\infty}^{\infty}$ forms an orthonormal basis in $L^2(-d,d)$, then the moment problem (5.2) has a unique solution in $L^2(-d,d)$. If $T < d$, then the system $\{e^{-i\frac{\pi m x}{d}}\}_{m=-\infty}^{\infty}$ is complete in $L^2(-T,T)$, and the solution is unique, if exists. Therefore the solution of (5.2) is in $\mathcal{B}(0,T)$, and it is unique, if exists, and coincides with $\mathcal{K}^* \Xi w^0_1$. Thus Theorem 5.1 is close to Theorem 4.2.

R e m a r k 5.2. It is evident that u of the form (4.11) is a solution of problem (5.3), but it is not unique.

Theorem 5.2. *Let* $0 < T \le d$, $w^0 \in \widetilde{\mathcal{H}}^s_Q(0,d)$, $s < 1/2$. *Assume that proposition* (*iii*) *of Theorem* 4.2 *holds. Define the sequence* $\{\omega_m\}_{m=-\infty}^{\infty}$ *by* (5.1)*. Let some* $M \in \mathbb{N}$*. If* $u_M \in \mathcal{B}(0,T)$ *is the solution of the Markov trigonometric moment problem* (5.3)*, then the state* w^0 *is approximately null-controllable at the time* T*, and the following estimate is valid:*

$$\left|\left|\left|W^T\right|\right|\right|^s_Q \le \frac{2^{\frac{5}{2}}\pi^{s-1}C_{\mathrm{S}}\sqrt{C^2_{\partial^{-1}}+1}PTM^{s-\frac{1}{2}}}{d^s\sqrt{-2s+1}}, \tag{5.4}$$

where W *is the corresponding solution of control system* (3.1)–(3.3)*,* $C_{\mathrm{S}} > 0$ *is the constant from estimate* (3.4)*,* $C_{\partial^{-1}} > 0$ *is the constant from Lemma A*.6*,* $P > 0$ *is the constant from Remark A*.1*.*

P r o o f. Since proposition (iii) of Theorem 4.2 holds, then there exists the solution $\widetilde{u} \in \mathcal{B}(0,T)$ of the controllability problem of system (2.1)–(2.3). Put $\widetilde{U} = \widetilde{u}$ on $[0,T]$, $\widetilde{U} = 0$ on $\mathbb{R} \setminus [0,T]$. Then $\Xi w^0 = \mathcal{L}^* \begin{pmatrix} -\partial^{-1}\Omega\widetilde{U} \\ \Xi\widetilde{U} \end{pmatrix}$. Let $u_M \in \mathcal{B}(0,T)$ be the solution of the Markov trigonometric moment problem (5.3) on $(0,T)$ for the finite sequence $\{\omega_m\}_{m=-M}^{M}$. Put $U_M = u_M$ on $[0,T]$, $U_M = 0$ on $\mathbb{R} \setminus [0,T]$. From (4.1) we have

$$W^T = S_T \sum_{k\in\mathbb{Z}} \mathcal{T}_{2kd}\mathcal{L}^* \begin{pmatrix} -\partial^{-1}\Omega\left(\widetilde{U} - U_M\right) \\ \Xi\left(\widetilde{U} - U_M\right) \end{pmatrix},$$

where W is the solution of (3.1)–(3.3). Using (3.4) and Lemmas A.4, A.6 and B.3, we obtain

$$\left|\left|\left|W^T\right|\right|\right|^s_Q \le 2C_S\sqrt{C^2_{\partial^{-1}}+1}\left\|\left(\widetilde{U} - U_M\right)\right\|^{s-1}_0. \tag{5.5}$$

Since $\widetilde{U}$ and U_M belong to $L^\infty(-d,d)$, then we can consider the following series expansions on $(-d,d)$: $\widetilde{U}(x) = \frac{1}{d}\sum_{m=-\infty}^{\infty}\omega_m e^{-i\frac{\pi m x}{d}}$, $U_M(x) = \frac{1}{d}\sum_{m=-\infty}^{\infty}\nu_m e^{-i\frac{\pi m x}{d}}$, where

$$\omega_m = \int_{-d}^{d} e^{i\frac{\pi m x}{d}}\widetilde{U}(x)\,dx = \int_0^T e^{i\frac{\pi m x}{d}}\left(\mathcal{K}^*\Xi w^0_1\right)(x)\,dx,$$

$$\nu_m = \int_{-d}^{d} e^{i\frac{\pi m x}{d}}U_M(x)\,dx = \int_0^T e^{i\frac{\pi m x}{d}}u(x)\,dx.$$

Hence,

$$\widetilde{U}(x) - U_M(x) = \frac{1}{d}\sum_{m=-\infty}^{\infty}\left(\omega_m - \nu_m\right)e^{-i\frac{\pi m x}{d}}. \tag{5.6}$$

Using Remark A.1 and reasoning similarly to the proof of [12, Theorem 5.2], we obtain the estimate

$$\left\| \widetilde{U} - U_M \right\|_0^{s-1} \leq \frac{2\sqrt{2}\pi^{s-1} PTM^{s-\frac{1}{2}}}{d^s \sqrt{-2s+1}}, \qquad s < 1/2, \; P > 0. \tag{5.7}$$

Substituting (5.7) into (5.5), we obtain (5.4).

It is clear from (5.4) that $\left|\left|\left| W^T \right|\right|\right|_Q^s \to 0$ as $M \to \infty$. Thus the state w^0 is approximately null-controllable at the given time T. The theorem is proved.

Consider the set of bang-bang controls. Denote

$$\begin{aligned} \mathcal{B}_M(0,T) = \{ u_M \in \mathcal{B}(0,T) | \exists T_* \in (0,T) : (|u_M| = 1 \text{ a.e. on } (0,T_*)), \\ (u_M = 0 \text{ a.e. on } (T_*,T)) \\ (u_M \text{ has not more than } M \text{ discontinuities on } (0,T_*)) \} . \end{aligned}$$

Theorem 5.3. *Let* $0 < T \leq d$, $w^0 \in \widetilde{\mathcal{H}}_Q^s(0,d)$, $s < 1/2$. *Assume that assertion* (*iii*) *of Theorem 4.2 holds. Define the sequence* $\{\omega_m\}_{m=-\infty}^{\infty}$ *by* (5.1). *Then for all* $\varepsilon > 0$ *there exists* $M \in \mathbb{N}$ *such that for this* M *there exists a solution* $u_M \in \mathcal{B}_M(0,T)$ *of the Markov trigonometric moment problem* (5.3). *The number* M *is defined from the condition*

$$\frac{2^{\frac{5}{2}} \pi^{s-1} C_{\mathrm{S}} \sqrt{C_{\partial^{-1}}^2 + 1} PTM^{s-\frac{1}{2}}}{d^s \sqrt{-2s+1}} < \varepsilon.$$

Moreover, control system (2.1)–(2.3) *is approximate null-controllable at the time* T *(the estimate* $\left|\left|\left| W^T \right|\right|\right|_Q^s \leq \varepsilon$ *is valid).*

The proof of this theorem is absolutely similar to the proof of [12, Theorem 5.3].

A. The Spaces $\mathcal{H}_Q^s$ and $\mathbb{H}_0^s(-d,d)$, $s \in \mathbb{R}$

Lemma A.1. *The spaces* $\mathcal{H}_Q^s$ *and* $H_{0,per}^s$ (*the Sobolev space of even* $2d$*-periodic functions,* [19]), $s \in \mathbb{R}$, *coincide as sets, have equivalent norms, and*

$$\mathcal{H}_Q^s = \left\{ f \in \mathcal{S}' : f(x) = \sum_{n=1}^{+\infty} f_n Y_n(\lambda_n, x) \text{ and } \left\{ f_n (1+\lambda_n^2)^{s/2} \right\}_{n=1}^{+\infty} \in l^2 \right\}, \; s \in \mathbb{R}, \tag{A.1}$$

where $f_n = \frac{1}{2} (f, Y_n(\lambda_n, \cdot))$ *on* $(-d,d)$, *with the norm*

$$\|f\|_Q^s = \left(\sum_{n=1}^{\infty} \left| \left(1+\lambda_n^2\right)^{s/2} f_n \right|^2 \right)^{1/2}. \tag{A.2}$$

P r o o f. We have to prove that

$$\left(1+|D|^2\right)^{s/2} f \in L^2(\mathbb{R}) \quad \text{iff} \quad \left(1+|D|^2+Q(x)\right)^{s/2} f \in L^2(\mathbb{R}).$$

We use the following theorem from [14, Chap. 18]:

Theorem A.1 ((Hörmander, [14])). *If $a(x,\xi) \in S^m$, then $a(x,D)$ is a continuous operator from H_0^s to H_0^{s-m} for all s, where for $m \in \mathbb{R}$*

$$S^m = S^m(\mathbb{R}^n \times \mathbb{R}^n)$$
$$= \left\{a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \forall \alpha, \beta \ |\partial_\beta^\alpha a(x,\xi)| \le C_\beta^\alpha (1+|\xi|)^{m-|\alpha|}\right\},$$
$$\partial_\beta^\alpha a(x,\xi) = \partial_\xi^\alpha \partial_x^\beta a(x,\xi), \qquad x, \xi \in \mathbb{R}^n.$$

Let us denote for $x, \xi \in \mathbb{R}$

$$\widetilde{S}^m = \widetilde{S}^m(\mathbb{R}\times\mathbb{R}) = \left\{a \in C_x(\mathbb{R}) \cap C_\xi^\infty(\mathbb{R}) : \forall \alpha \ |\partial_\xi^\alpha a(x,\xi)| \le C_\alpha (1+|\xi|)^{m-\alpha}\right\}.$$

Analyzing the proof of Theorem A.1 we conclude that it remains true for $a(x,\xi) \in \widetilde{S}^m$. Thus, if we prove that $a(x,\xi) = \left(1+\xi^2+Q(x)\right)^{s/2} \in \widetilde{S}^m$, then we will have that $a(x,D)$ is a continuous operator from H_0^s to H_0^{s-m} for all s.

Since $q \in \mathcal{E}(0,d)$, then $Q \in C^1(\mathbb{R})$, and there exist $m_q > 0$, $M_q < \infty$ such that $m_q \le Q \le M_q$ on $\mathbb{R}$. It is easy to prove that

$$\sqrt{\frac{\widehat{m}}{2}}(1+|\xi|) \le \sqrt{1+\xi^2+Q} \le \sqrt{2\widehat{M}}(1+|\xi|),$$

where $\widehat{m} = \min\{m_q, 1\}$, $\widehat{M} = \max\{M_q, 1\}$.

Using these two inequalities, one can easily prove that $\left(1+\xi^2+Q(x)\right)^{s/2} \in \widetilde{S}^s$. Therefore, $\left(1+|D|^2+Q(x)\right)^{s/2}$ is the continuous operator from H_0^s to $H_0^0 = L^2(\mathbb{R})$. This implies that $f \in H_0^s$ iff $\left(1+|D|^2+Q(x)\right)^{s/2} f \in L^2(\mathbb{R})$ as required.

We prove representation (A.1) with norm (A.2) for the space $\mathcal{H}_Q^s$ in the same way as in the proof of [12, Lemma A.2].

It is easy to prove that

$$H_{0,per}^s = \left\{ f \in \mathcal{S}' : f(x) = \sum_{n=1}^{+\infty} f_n^1 \cos\frac{\pi n x}{d} \text{ and } \left\{f_n^1 (1+n^2)^{s/2}\right\}_{n=1}^{+\infty} \in l^2 \right\}, \ s \in \mathbb{R},$$

where $f_n^1 = \frac{1}{d}\left(f, \cos\frac{\pi n x}{d}\right)$ on $(-d,d)$ with the norm

$$\|f\|_0^s = \left(\sum_{n=1}^{\infty} \left|\left(1+n^2\right)^{s/2} f_n^1\right|^2\right)^{1/2}. \tag{A.3}$$

It is known [21, chap. V] that $\lambda_n = n\frac{\pi}{d} + \varepsilon_n$, where $\varepsilon_n = O\left(\frac{1}{n}\right)$, $n = \overline{1,\infty}$. It follows easily from here that norms (A.2) and (A.3) are equivalent. The lemma is proved.

The proofs of the following four lemmas are absolutely similar to the proofs of the corresponding lemmas in [12, Appendix A].

Lemma A.2. *$f \in \mathbb{H}_0^s$, $s \in \mathbb{R}$ iff $\left(1 + D^2\right)^{s/2} f \in \mathbb{H}_0^0$.*

Lemma A.3. *$f \in \mathbb{H}_0^0(-d,d)$ iff $f \in L^2(-d,d)$.*

Lemma A.4. *Let $g \in \mathbb{H}_0^s$, $s \in \mathbb{R}$. Then we have $\Omega g \in \mathbb{H}_0^s$, $\Xi g \in \mathbb{H}_0^s$ and $\|\Omega g\|_0^s = \|\Xi g\|_0^s = 2\|g\|_0^s$.*

Lemma A.5. *$\mathbb{H}_0^s$ is dense in $\mathbb{H}_0^0$, $s \geq 0$.*

R e m a r k A.1. It follows from Lemma A.3 that there exist $P, P_1 > 0$ such that $\|f\|_0^0 \leq P\|f\|_{L^2}$ and $\|f\|_{L^2} \leq P_1\|f\|_0^0$ for $f \in \mathbb{H}_0^0(-d,d)$.

Lemma A.6. *The operator $\partial^{-1} : \mathbb{H}_0^s \to \mathbb{H}_0^{s+1}$, $s \in \mathbb{R}$ (see Definition 3.3) is linear and continuous. If g is odd, then $\partial^{-1}g$ is even and if g is even, then $\partial^{-1}g$ is odd.*

P r o o f. Let $s \in \mathbb{R}$. Let us denote by $\partial = \frac{d}{dx} : \mathbb{H}_0^{s+1} \to \mathbb{H}_0^s$ the operator of differentiation, $D(\partial) = \mathbb{H}_0^{s+1}$, $R(\partial) = \mathbb{H}_0^s$. We have $(\partial g)(x) = -i\sum_{n=1}^{\infty} \lambda_n g_n e^{-i\lambda_n x}$ for $g(x) = \sum_{n=1}^{\infty} g_n e^{-i\lambda_n x}$. Using the trivial inequality $\lambda_n < \sqrt{1+\lambda_n^2}$, we get $\|\partial g\|_0^s < \|g\|_0^{s+1}$ for $g \in \mathbb{H}_0^{s+1}$. Thus the operator ∂ is linear and continuous. It is also obvious that if g is odd, then ∂g is even and if g is even, then ∂g is odd.

Denote by $\widehat{\partial} : \mathbb{H}_0^s \to \mathbb{H}_0^{s+1}$ the operator $\left(\widehat{\partial} g\right)(x) = i\sum_{n=1}^{\infty} \frac{g_n}{\lambda_n} e^{-i\lambda_n x}$, where $g \in \mathbb{H}_0^s$, $g(x) = \sum_{n=1}^{\infty} g_n e^{-i\lambda_n x}$, $D\left(\widehat{\partial}\right) = \mathbb{H}_0^s$, $R\left(\widehat{\partial}\right) = \mathbb{H}_0^{s+1}$. It also transfers odd functions to even and vice versa. We have $\widehat{\partial}\partial f = f$ and $\partial\widehat{\partial} g = g$ for $f \in \mathbb{H}_0^{s+1}$, $g \in \mathbb{H}_0^s$. Therefore, $\widehat{\partial} = \partial^{-1}$, $D(\partial^{-1}) = \mathbb{H}_0^s$, $R(\partial^{-1}) = \mathbb{H}_0^{s+1}$. Due to the inverse operator theorem, we have that there exists $C_{\partial^{-1}} > 0$ such that $\left\|\partial^{-1} g\right\|_0^{s+1} \leq C_{\partial^{-1}} \|g\|_0^s$. The lemma is proved.

B. The Transformation Operators for the Sturm–Liouville Problem on a Segment and Their Adjoints

We make a quotation of definitions and properties of the transformation operators from [20]. Denote $\widetilde{y}_n(\lambda_n, \cdot) = \Xi y_n(\lambda_n, \cdot)$, $n = \overline{1,\infty}$. It is evident that $\widetilde{y}_n(\lambda_n, x)$ satisfies the following Cauchy problem for $n = \overline{1,\infty}$:

$$-\widetilde{y}\,''_n(\lambda_n, x) + Q(x)\widetilde{y}_n(\lambda_n, x) = \lambda_n^2 \widetilde{y}_n(\lambda_n, x), \qquad x \in (-d,d),$$
$$\widetilde{y}_n(\lambda_n, 0) = 1, \qquad \widetilde{y}\,'_n(\lambda_n, 0) = 0.$$

Due to [20], we have

$$\widetilde{y}_n(\lambda_n, x) = \mathcal{K}(\cos\lambda_n t)(x) = \cos\lambda_n x + \int\limits_0^x \mathtt{K}(x,t;0)\cos\lambda_n t\,dt, \qquad n = \overline{1,\infty},$$

$$\cos\lambda_n x = \mathcal{L}(\widetilde{y}_n(\lambda_n, t))(x) = \widetilde{y}_n(\lambda_n, x) + \int\limits_0^x \mathtt{L}(x,t;0)\widetilde{y}_n(\lambda_n, t)\,dt, \quad n = \overline{1,\infty},$$

where $\mathtt{K}(x,t;0) = \mathtt{K}(x,t) + \mathtt{K}(x,-t)$, $\mathtt{L}(x,t;0) = \mathtt{L}(x,t) + \mathtt{L}(x,-t)$. Under the condition $Q \in C^1[-d,d]$ the continuous functions $\mathtt{K}(x,t)$ and $\mathtt{L}(x,t)$ are the solutions of the following systems on $[-d,d] \times [-d,d]$:

$$\mathtt{K}_{xx}(x,t) - \mathtt{K}_{tt}(x,t) = Q(x)\mathtt{K}(x,t), \qquad \mathtt{L}_{xx}(x,t) - \mathtt{L}_{tt}(x,t) = -Q(x)\mathtt{L}(x,t),$$

$$\mathtt{K}(x,x) = \frac{1}{2}\int\limits_0^x Q(\xi)\,d\xi, \qquad \mathtt{L}(x,x) = -\frac{1}{2}\int\limits_0^x Q(\xi)\,d\xi,$$

$$\mathtt{K}(x,-x) = 0, \qquad \mathtt{L}(x,-x) = 0.$$

It is also proved in [20] that the kernels $\mathtt{K}(x,t;0)$, $\mathtt{L}(x,t;0)$ are bounded functions with respect to the both arguments on $[-d,d] \times [-d,d]$ and $\mathtt{K}(x,t) = \mathtt{L}(x,t) = 0$ for $|t| \geq |x|$.

We determine the transformation operators in the spaces $\mathbb{H}_0^{-s}(-d,d)$ and $\mathcal{H}_Q^{-s}(-d,d)$ via series in the present paper.

Lemma B.1.

$$\mathcal{K} : \mathbb{H}_0^{-s}(-d,d) \longrightarrow \mathcal{H}_Q^{-s}(-d,d), \quad \mathcal{L} : \mathcal{H}_Q^{-s}(-d,d) \longrightarrow \mathbb{H}_0^{-s}(-d,d), \quad s \in \mathbb{R},$$

where $D(\mathcal{K}) = \{f \in \mathbb{H}_0^{-s}(-d,d) : f \text{ is even}\}$, $D(\mathcal{L}) = \mathcal{H}_Q^{-s}(-d,d)$, $R(\mathcal{K}) = D(\mathcal{L})$, $R(\mathcal{L}) = D(\mathcal{K})$.

P r o o f. Let $s \in \mathbb{R}$. Let $\psi \in \mathcal{H}_Q^{-s}(-d,d)$, $\varphi \in \mathbb{H}_0^{-s}(-d,d)$, φ be even. Hence, $\psi(x) = \sum_{n=1}^{\infty} \psi_n \Xi y_n(\lambda_n, x)$ and $\left\{\left(1+\lambda_n^2\right)^{\frac{-s}{2}}\psi_n\right\}_{n=1}^{\infty} \in l^2$, $\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \cos\lambda_n x$ and $\left\{\left(1+\lambda_n^2\right)^{\frac{-s}{2}}\varphi_n\right\}_{n=1}^{\infty} \in l^2$. Applying the transformation operators, we obtain

$$(\mathcal{K}\varphi)(x) = \sum_{n=1}^{\infty} \varphi_n \mathcal{K}(\cos\lambda_n t)(x) = \sum_{n=1}^{\infty} \varphi_n \widetilde{y}_n(\lambda_n, x) = \widetilde{\psi}(x),$$

$$(\mathcal{L}\psi)(x) = \sum_{n=1}^{\infty} \psi_n \mathcal{L}(\Xi y_n(\lambda_n, t))(x) = \sum_{n=1}^{\infty} \psi_n \cos\lambda_n t = \widetilde{\varphi}(x).$$

Evidently, $\widetilde{\varphi} \in \mathbb{H}_0^{-s}(-d,d)$ and $\widetilde{\psi} \in \mathcal{H}_Q^{-s}(-d,d)$. The lemma is proved.

Lemma B.2. *The operators $\mathcal{K}$ and $\mathcal{L}$ are linear and isometric on their domains.*

P r o o f. Let $s \in \mathbb{R}$ and $\varphi \in \mathbb{H}_0^{-s}(-d,d)$ be even. Hence, $\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \cos\lambda_n x$ and $\left\{\left(1+\lambda_n^2\right)^{\frac{-s}{2}} \varphi_n\right\}_{n=1}^{\infty} \in l^2$. According to Lemma B.1, $\mathcal{K}\varphi \in \mathcal{H}_Q^{-s}(-d,d)$ and $(\mathcal{K}\varphi)(x) = \sum_{n=1}^{\infty} \varphi_n \widetilde{y}_n(\lambda_n, x)$. Write down the norms: $\left(\|\varphi\|_0^{-s}\right)^2 = \sum_{n=1}^{\infty} \left|\varphi_n \left(1+\lambda_n^2\right)^{-s/2}\right|^2$, $\left(\|\mathcal{K}\varphi\|_Q^{-s}\right)^2 = \sum_{n=1}^{\infty} \left|\varphi_n \left(1+\lambda_n^2\right)^{-s/2}\right|^2 = \left(\|\varphi\|_0^{-s}\right)^2$. Hence $\mathcal{K}$ is isometric from $\mathbb{H}_0^{-s}(-d,d)$ to $\mathcal{H}_Q^{-s}(-d,d)$. One can see that $\mathcal{L}$ is also isometric from $\mathcal{H}_Q^{-s}(-d,d)$ to $\mathbb{H}_0^{-s}(-d,d)$.

The linearity of the operators is obvious. The lemma is proved.

Definition B.1. *Define by $\mathcal{K}^*$ and $\mathcal{L}^*$ the adjoint operators for $\mathcal{K}$ and $\mathcal{L}$: $(\mathcal{K}^* f, \varphi) = (f, \mathcal{K}\varphi)$, $(\mathcal{L}^* g, \psi) = (g, \mathcal{L}\psi)$, where $f \in D(\mathcal{K}^*) = \mathcal{H}_Q^s(-d,d)$, $\varphi \in D(\mathcal{K})$, $g \in D(\mathcal{L}^*) = \{f \in \mathbb{H}_0^s(-d,d) : f \text{ is even}\}$, $\psi \in D(\mathcal{L})$, $s \in \mathbb{R}$.*

Thus, $\mathcal{K}^* : \mathcal{H}_Q^s(-d,d) \to \mathbb{H}_0^s(-d,d)$, $\mathcal{L}^* : \mathbb{H}_0^s(-d,d) \to \mathcal{H}_Q^s(-d,d)$, and they are linear and isometric. Moreover, $(\mathcal{K}^* f)(x) = \sum_{n=1}^{\infty} f_n \cos\lambda_n x$, $(\mathcal{L}^* g)(x) = \sum_{n=1}^{\infty} g_n \widetilde{y}_n(\lambda_n, x)$. Evidently, $R(\mathcal{K}^*) = D(\mathcal{L}^*)$, $R(\mathcal{L}^*) = D(\mathcal{K}^*)$. It is also obvious that $\mathcal{K}^* f$ and $\mathcal{L}^* g$ are even if f and g are even.

Lemma B.3. *Let $f \in \mathbb{H}_0^s(-d,d) \times \mathbb{H}_0^{s-1}(-d,d)$, $s \in \mathbb{R}$, f be even. Then $|||\mathcal{L}^* f|||_Q^s = |||f|||_0^s$.*

The proof is trivial, taking into account that $\mathcal{L}^*$ is isometric.

The following two lemmas are proved similarly to Lemmas B.4 and B.5 of [12, Appendix B] by changing odd functions into even functions and taking into account that $\mathtt{L}(x,t;0)$ and $\mathtt{K}(x,t;0)$ are even on t and odd on x.

Lemma B.4. *Let $f \in L^2(-d,d)$ be even and $\operatorname{supp} f \subset [-T,T]$. Then $\operatorname{supp}(\mathcal{L}^* f) \subset [-T,T]$ and $(\mathcal{L}^* f)(t) = f(t) + \int_{|t|}^{T} \mathtt{L}(x,t;0) f(x)\,dx$.*

Lemma B.5. *Let $f \in L^2(-d,d)$ be even and $\operatorname{supp} f \subset [-T,T]$. Then $\operatorname{supp}(\mathcal{K}^* f) \subset [-T,T]$ and $(\mathcal{K}^* f)(t) = f(t) + \int_{|t|}^{T} \mathtt{K}(x,t;0) f(x)\,dx$.*

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