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Hyers–Ulam Stability of Ternary (σ, τ, ξ)-Derivations on C^*-Ternary Algebras

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Let q be a positive rational number and let A be a C^*-ternary algebra. Let σ, τ and ξ be linear maps on A. We prove the generalized Hyers–Ulam stability of Jordan ternary (σ, τ, ξ)-derivations, ternary (σ, τ, ξ)-derivations and Lie ternary (σ, τ, ξ)-derivations in A for the following Euler–Lagrange type additive mapping:

$$\Big(\sum_{i=1}^{n} f\big(\sum_{j=1}^{n} q(x_i - x_j)\big)\Big) + nf\big(\sum_{i=1}^{n} qx_i\big) = nq\sum_{i=1}^{n} f(x_i).$$

Key words: C^*-ternary algebra, Hyers–Ulam stability, ternary Banach algebra, Euler–Lagrange type additive mapping.

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1. Introduction

Ternary algebraic operations were considered in the XIX century by several mathematicians such as A. Cayley who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990. The comments on physical applications of ternary structures can be found in [1]. A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \to [xyz]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[xy[zwv]] = [x[wzy]v] = [[xyz]wv]$, and satisfies $\|[xyz]\| \leq \|x\|.\|y\|.\|z\|$ and $\|[xxx]\| = \|x\|^3$ (see [1, 2]). Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[xyz] := \langle x, y\rangle z$.

If a C^*-ternary algebra $(A, [...])$ has an identity, i.e. an element $e \in A$ such that $x = [xee] = [eex]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [xey]$ and $x^* := [exe]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[xyz] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra [3].

Let A be a C^*-ternary algebra and let σ, τ and ξ be linear maps on A. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a C^*-Jordan ternary (σ, τ, ξ)-derivation if

$$\delta([xxx]) = [\delta(x)\tau(x)\xi(x)] + [\sigma(x)\delta(x)\xi(x)] + [\sigma(x)\tau(x)\delta(x)]$$

for all $x \in A$. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a C^*-ternary (σ, τ, ξ)-derivation if

$$D([xyz]) = [D(x)\tau(y)\xi(z)] + [\sigma(x)D(y)\xi(z)] + [\sigma(x)\tau(y)D(z)]$$

for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $L : A \to A$ is called a C^*-Lie ternary (σ, τ, ξ)-derivation if

$$L([xyz]) = [L(x)yz]_{(\sigma,\tau,\xi)} + [L(y)xz]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x, y, z \in A$, where $[xyz]_{(\sigma,\tau,\xi)} = x\tau(y)\xi(z) - \sigma(z)\tau(y)x$.

The stability problem of functional equations originated from a question of Ulam [4] concerning the stability of group homomorphisms. Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers's theorem was generalized by Aoki [6] for additive mappings and by Th.M. Rassias [7] for linear mappings by considering an unbounded Cauchy difference as follows.

Theorem 1.1. *Let $f : E \longrightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$, then inequality (1.1) *holds for all $x, y \neq 0$, and* (1.2) *for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in real t for each fixed $x \in E$, then T is linear.*

Th.M. Rassias [8] during the 27th international Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [9], following the same approach as in Th.M. Rassias' [7], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [9], as well as by Th.M. Rassias and P. Šemrl [10], that one cannot prove a Th.M. Rassias type theorem when $p = 1$.

A generalization of the Th.M. Rassias theorem was obtained by Găvrtua [11] by replacing the unbounded Cauchy difference by the general control function in the spirit of Th.M. Rassias' approach.

On the other hand, J.M. Rassias [12] generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. During the last three decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and mappings (see [13–19]).

The purpose of the present paper is to study the generalized Hyers–Ulam stability of some functional equations on C^*-ternary algebras related to the Euler–Lagrange type additive mapping

$$\Big(\sum_{i=1}^{n} f(\sum_{j=1}^{n} q(x_i - x_j))\Big) + nf(\sum_{i=1}^{n} qx_i) = nq\sum_{i=1}^{n} f(x_i),$$

whose solution is said to be additive mapping of Euler–Lagrange type. The reader is referred to [20–22] for essential work in the subject.

In Sec. 2, we prove the generalized Hyers–Ulam stability of C^*-Jordan ternary (σ, τ, ξ)-derivations in C^*-ternary algebras for the Euler–Lagrange type additive mapping (see [23]).

In Sec. 3, we prove the generalized Hyers–Ulam stability of C^*-ternary (σ, τ, ξ)-derivations in C^*-ternary algebras for the Euler–Lagrange type additive mapping.

In Sec. 4, we prove the generalized Hyers–Ulam stability of C^*-Lie ternary (σ, τ, ξ)-derivations in C^*-ternary algebras for the Euler–Lagrange type additive mapping.

Throughout this paper, assume that A is a C^*-ternary algebra with norm $\|.\|_A$, σ, τ and ξ are linear maps on A. Let q be a positive rational number. For a given mapping $f : A \to A$ and a given $\mu \in \mathbb{C}$, we define $D_\mu f : A^n \to A$ by

$$D_\mu f(x_1, \ldots, x_n) := \Big(\sum_{i=1}^{n} f(\sum_{j=1}^{n} q\mu(x_i - x_j))\Big) + nf(\sum_{i=1}^{n} q\mu x_i) - nq\mu\sum_{i=1}^{n} f(x_i)$$

for all $x_1, \ldots, x_n \in A$.

2. Stability of C^*-Jordan Ternary (σ, τ, ξ)-Derivations

In this section our aim is to establish the Hyers–Ulam stability of C^*-Jordan ternary (σ, τ, ξ)-derivations in C^*-ternary algebras for the Euler–Lagrange type additive mapping.

Theorem 2.1. *Let $n \in \mathbb{N}$. Assume that $r > 3$ if $nq > 1$ and that $0 < r < 1$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ such that*

$$\|D_\mu f(x_1, \ldots, x_n)\| \le \theta \sum_{j=1}^{n} \|x_j\|^r, \tag{2.1}$$

$$\|f([xxx]) - [f(x)h(x)k(x)] - [g(x)f(x)k(x)] - [g(x)h(x)f(x)]\| \le 3\theta\|x\|^r, \tag{2.2}$$

$$\|g(q\mu x_1 + \ldots + q\mu x_n) - q\mu g(x_1) - \ldots - q\mu g(x_n)\| \le \theta(\|x_1\|^r + \ldots + \|x_n\|^r), \tag{2.3}$$

$$\|h(q\mu x_1 + \ldots + q\mu x_n) - q\mu h(x_1) - \ldots . - q\mu h(x_n)\| \le \theta(\|x_1\|^r + \ldots + \|x_n\|^r), \tag{2.4}$$

$$\|k(q\mu x_1 + \ldots + q\mu x_n) - q\mu k(x_1) - \ldots - q\mu k(x_n)\| \le \theta(\|x_1\|^r + \ldots + \|x_n\|^r) \tag{2.5}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} | |\lambda| = 1\}$ and all $x_1, \ldots, x_n, x \in A$. Then there exist unique linear mappings $\sigma, \tau,$ and ξ from A to A and a unique C^*-Jordan ternary (σ, τ, ξ)-derivation $\delta : A \to A$ satisfying

$$\|g(x) - \sigma(x)\| \le \frac{n\theta}{(nq)^r - nq}\|x\|^r, \tag{2.6}$$

$$\|h(x) - \tau(x)\| \le \frac{n\theta}{(nq)^r - nq}\|x\|^r, \tag{2.7}$$

$$\|k(x) - \xi(x)\| \le \frac{n\theta}{(nq)^r - nq}\|x\|^r, \tag{2.8}$$

$$\|f(x) - \delta(x)\| \le \frac{\theta}{(nq)^r - nq}\|x\|^r \tag{2.9}$$

for all $x \in A$.

P r o o f. Letting $\mu = 1$ and $x_1 = \ldots = x_n = x$ in (2.1), we get

$$\|nf(nqx) - n^2 qf(x)\| \le n\theta\|x\|^r$$

for all $x \in A$. So

$$\|f(x) - nqf(\frac{x}{nq})\| \le \frac{\theta}{(nq)^r}\|x\|^r$$

for all $x \in A$. So

$$\begin{aligned} &\|(nq)^l f(\frac{x}{(nq)^l}) - (nq)^{l+m} f(\frac{x}{(nq)^{l+m}})\| \\ &\leq \sum_{j=l}^{l+m-1} \|(nq)^j f(\frac{x}{(nq)^j}) - (nq)^{j+1} f(\frac{x}{(nq)^{j+1}})\| \\ &\leq \frac{\theta}{(nq)^r} \sum_{j=l}^{l+m-1} \frac{(nq)^j}{(nq)^{rj}} \|x\|^r \end{aligned} \tag{2.10}$$

for all nonnegative integers m and l and all $x \in A$. It follows from (2.10) that the sequence $\{(nq)^m f(\frac{x}{(nq)^m})\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, then the sequence $\{(nq)^m f(\frac{x}{(nq)^m})\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{m\to\infty} (nq)^m f(\frac{x}{(nq)^m})$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.10), we get

$$\|f(x) - \delta(x)\| \leq \frac{\theta}{(nq)^r} \sum_{j=0}^{\infty} \frac{(nq)^j}{(nq)^{rj}} \|x\|^r$$

for all $x \in A$. So (2.9) holds for all $x \in A$.

It follows from (2.1) that

$$\begin{aligned} \|D_1\delta(x_1, \ldots, x_n)\| &= \lim_{m\to\infty} (nq)^m \|D_1 f(\frac{x_1}{(nq)^m}, \ldots, \frac{x_n}{(nq)^m})\| \\ &\leq \lim_{m\to\infty} \frac{(nq)^m \theta}{(nq)^{mr}} \sum_{j=1}^{n} \|x_j\|^r \end{aligned}$$

for all $x_1, \ldots, x_n \in A$. Hence,

$$D_1\delta(x_1, \ldots, x_n) = 0$$

for all $x_1, \ldots, x_n \in A$. By Lemma 3.1 of [24], the mapping $\delta : A \to A$ is Cauchy additive. By the same reasoning as in the proof of Theorem 2.1 of [25], the mapping $\delta : A \to A$ is linear.

Also letting $\mu = 1$ and $x_1 = \ldots = x_n = x$ in (2.3), we get

$$\|g(qnx) - qng(x)\| \leq n\theta\|x\|^r$$

for all $x \in A$. So

$$\|g(x) - qng(\frac{x}{nq})\| \leq \frac{n\theta}{(nq)^r}\|x\|^r$$

for all $x \in A$. We easily prove that by induction that

$$\begin{aligned}
&\|(nq)^l g(\frac{x}{(nq)^l}) - (nq)^{l+m} g(\frac{x}{(nq)^{l+m}})\| \\
&\le \sum_{j=l}^{l+m-1} \|(nq)^j g(\frac{x}{(nq)^j}) - (nq)^{j+1} g(\frac{x}{(nq)^{j+1}})\| \\
&\le \frac{n\theta}{(nq)^r} \sum_{j=l}^{l+m-1} \frac{(nq)^j}{(nq)^{rj}} \|x\|^r \qquad (2.11)
\end{aligned}$$

for all nonnegative integers m and l with $x \in A$. It follows from (2.11) that the sequence $\{(nq)^m g(\frac{x}{(nq)^m})\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{(nq)^m g(\frac{x}{(nq)^m})\}$ converges. So one can define the mapping $\sigma : A \to A$ by

$$\sigma(x) := \lim_{m\to\infty} (nq)^m g(\frac{x}{(nq)^m})$$

for all $x \in A$. We easily prove by (2.3) that $\sigma(\mu x + \mu y) = \mu\sigma(x) + \mu\sigma(y)$ and by letting $l = 0$ and taking the limit $m \to \infty$ in (2.11), we get

$$\|g(x) - \sigma(x)\| \le \frac{n\theta}{(nq)^r} \sum_{j=0}^{\infty} \frac{(nq)^j}{(nq)^{rj}} \|x\|^r$$

for all $x \in A$. So (2.6) holds for all $x \in A$. Similarly, there exist linear mappings τ and ξ on A satisfying (2.7) and (2.8), respectively.

It follows from (2.2) that

$$\begin{aligned}
&\|\delta([xxx]) - [\delta(x)\tau(x)\xi(x)] - [\sigma(x)\delta(x)\xi(x)] - [\sigma(x)\tau(x)\delta(x)]\| \\
&= \lim_{m\to\infty} (nq)^{3m} \|f(\frac{[xxx]}{(nq)^{3m}}) - [f(\frac{x}{(nq)^m})h(\frac{x}{(nq)^m})k(\frac{x}{(nq)^m})] \\
&- [g(\frac{x}{(nq)^m})f(\frac{x}{(nq)^m})k(\frac{x}{(nq)^m})] - [g(\frac{x}{(nq)^m})h(\frac{x}{(nq)^m})f(\frac{x}{(nq)^m})]\| \\
&\le \lim_{m\to\infty} \frac{3(nq)^{3m}\theta}{(nq)^{mr}} (\|x\|^r) = 0
\end{aligned}$$

for all $x \in A$. So

$$\delta([xxx]) = [\delta(x)\tau(x)\xi(x)] + [\sigma(x)\delta(x)\xi(x)] + [\sigma(x)\tau(x)\delta(x)]$$

for all $x \in A$.

Now, let $\delta' : A \to A$ be another mapping satisfying (2.1) and (2.9). Then we have

$$\begin{aligned}\|\delta(x) - \delta'(x)\| &= (nq)^m \|\delta(\frac{x}{(nq)^m}) - \delta'(\frac{x}{(nq)^m})\| \\ &\leq (nq)^m \|\delta(\frac{x}{(nq)^m}) - f(\frac{x}{(nq)^m})\| + \|\delta'(\frac{x}{(nq)^m}) - f(\frac{x}{(nq)^m})\| \\ &\leq \frac{2(nq)^m \theta}{((nq)^r - nq)(nq)^{mr}} \|x\|^r,\end{aligned}$$

which tends to zero as $m \to \infty$ for all $x \in A$. So we can conclude that $\delta(x) = \delta'(x)$ for all $x \in A$. This proves the uniqueness property of δ. Thus the mapping $\delta : A \to A$ is a unique C^*-Jordan ternary (σ, τ, ξ)-derivation satisfying (2.9). Similarly, we can prove the uniqueness properties of σ, τ and ξ on A, and the proof of the theorem is complete. ■

Theorem 2.2. *Let $n \in \mathbb{N}$. Assume that $0 < r < 1$ if $nq > 1$ and that $r > 3$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.1)–(2.5)*. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-Jordan ternary (σ, τ, ξ)-derivation $\delta : A \to A$ satisfying*

$$\|g(x) - \sigma(x)\| \leq \frac{n\theta}{nq - (nq)^r} \|x\|^r, \tag{2.12}$$

$$\|h(x) - \tau(x)\| \leq \frac{n\theta}{nq - (nq)^r} \|x\|^r, \tag{2.13}$$

$$\|k(x) - \xi(x)\| \leq \frac{n\theta}{nq - (nq)^r} \|x\|^r, \tag{2.14}$$

$$\|f(x) - \delta(x)\| \leq \frac{\theta}{nq - (nq)^r} \|x\|^r \tag{2.15}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x_1 = \ldots = x_n = x$ in (2.1), we get

$$\|nf(nqx) - n^2 q f(x)\| \leq n\theta \|x\|^r$$

for all $x \in A$. So

$$\|f(x) - \frac{1}{nq} f(nqx)\| \leq \frac{\theta}{nq} \|x\|^r$$

for all $x \in A$. So

$$\begin{aligned}&\|\frac{1}{(nq)^l}f((nq)^l x) - \frac{1}{(nq)^{l+m}}f((nq)^{l+m}x)\|\\ &\le \sum_{j=l}^{l+m-1}\|\frac{1}{(nq)^j}f((nq)^j x) - \frac{1}{(nq)^{j+1}}f((nq)^{j+1}x)\|\\ &\le \frac{\theta}{nq}\sum_{j=l}^{l+m-1}\frac{(nq)^{rj}}{(nq)^j}\|x\|^r \qquad (2.16)\end{aligned}$$

for all nonnegative integers m and l with $x \in A$. It follows from (2.16) that the sequence $\{\frac{1}{(nq)^m}f((nq)^m x)\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{\frac{1}{(nq)^m}f((nq)^m x)\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{m\to\infty}\frac{1}{(nq)^m}f((nq)^m x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.16), we get

$$\|f(x) - \delta(x)\| \le \frac{\theta}{nq}\sum_{j=0}^{\infty}\frac{(nq)^{rj}}{(nq)^j}\|x\|^r$$

for all $x \in A$. So (2.15) holds for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. ■

Theorem 2.3. *Let $n \in \mathbb{N}$. Assume that $r > 1$ if $nq > 1$ and that $0 < nr < 1$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.3)–(2.5) *such that*

$$\|D_\mu f(x_1, \ldots, x_n)\| \le \theta \prod_{j=1}^{n}\|x_j\|^r \qquad (2.17)$$

$$\|f([xxx]) - [f(x)h(x)k(x)] - [g(x)f(x)k(x)] - [g(x)h(x)f(x)]\| \le \theta\|x\|^{3r} \qquad (2.18)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}||\lambda| = 1\}$ and all $x_1, \ldots, x_n, x \in A$. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-Jordan ternary (σ, τ, ξ)-derivation $\delta : A \to A$ satisfying (2.6)–(2.8) *such that*

$$\|f(x) - \delta(x)\| \le \frac{\theta}{n((nq)^{nr} - nq)}\|x\|^{nr} \qquad (2.19)$$

for all $x \in A$.

P r o o f. Letting $\mu = 1$ and $x_1 = \ldots = x_n = x$ in (2.17), we get

$$\|nf(nqx) - n^2 qf(x)\| \le \theta \|x\|^{nr}$$

for all $x \in A$. So

$$\|f(x) - nqf(\frac{x}{nq})\| \le \frac{\theta}{n(nq)^{nr}} \|x\|^{nr}$$

for all $x \in A$. So

$$\begin{aligned}
&\|(nq)^l f(\frac{x}{(nq)^l}) - (nq)^{l+m} f(\frac{x}{(nq)^{l+m}})\| \\
&\le \sum_{j=l}^{l+m-1} \|(nq)^j f(\frac{x}{(nq)^j}) - (nq)^{j+1} f(\frac{x}{(nq)^{j+1}})\| \\
&\le \frac{\theta}{n(nq)^{nr}} \sum_{j=l}^{l+m-1} \frac{(nq)^j}{(nq)^{nrj}} \|x\|^{nr} \qquad (2.20)
\end{aligned}$$

for all nonnegative integers m and l with $x \in A$. It follows from (2.20) that the sequence $\{(nq)^m f(\frac{x}{(nq)^m})\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{(nq)^m f(\frac{x}{(nq)^m})\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{m \to \infty} (nq)^m f(\frac{x}{(nq)^m})$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.20), we get (2.19). The proof of uniqueness property of δ, σ, τ and ξ is similar to the proof of Theorem 2.1.

It follows from (2.18) that

$$\begin{aligned}
&\|\delta([xxx]) - [\delta(x)\tau(x)\xi(x)] - [\sigma(x)\delta(x)\xi(x)] - [\sigma(x)\tau(x)\delta(x)]\| \\
&= \lim_{m \to \infty} (nq)^{3m} \|f(\frac{[xxx]}{(nq)^{3m}}) - [f(\frac{x}{(nq)^m})h(\frac{x}{(nq)^m})k(\frac{x}{(nq)^m})] \\
&- [g(\frac{x}{(nq)^m})f(\frac{x}{(nq)^m})k(\frac{x}{(nq)^m})] - [g(\frac{x}{(nq)^m})h(\frac{x}{(nq)^m})f(\frac{x}{(nq)^m})]\| \\
&\le \lim_{m \to \infty} \frac{(nq)^{3m}\theta}{(nq)^{3mr}} (\|x\|^{3r}) = 0
\end{aligned}$$

for all $x \in A$. So

$$\delta([xxx]) = [\delta(x)\tau(x)\xi(x)] + [\sigma(x)\delta(x)\xi(x)] + [\sigma(x)\tau(x)\delta(x)]$$

for all $x \in A$, and the proof of the theorem is complete. ∎

Theorem 2.4. *Let $n \in \mathbb{N}$. Assume that $r > 1$ if $nq < 1$ and that $0 < nr < 1$ if $nq > 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.3)–(2.5), (2.17) *and* (2.18). *Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*J-ordan ternary (σ, τ, ξ)-derivation $\delta : A \to A$ satisfying* (2.12)–(2.14) *such that*

$$\|f(x) - \delta(x)\| \leq \frac{\theta}{n(nq - (nq)^{nr})}\|x\|^{nr} \tag{2.21}$$

for all $x \in A$.

P r o o f. Letting $\mu = 1$ and $x_1 = \ldots = x_n = x$ in (2.17), we get

$$\|nf(nqx) - n^2 qf(x)\| \leq \theta\|x\|^{nr}$$

for all $x \in A$. So

$$\|f(x) - \frac{1}{nq}f(nqx)\| \leq \frac{\theta}{n^2 q}\|x\|^{nr}$$

for all $x \in A$. So

$$\begin{aligned}
&\|\frac{1}{(nq)^l}f((nq)^l x) - \frac{1}{(nq)^{l+m}}f((nq)^{l+m}x)\| \\
&\leq \sum_{j=l}^{l+m-1}\|\frac{1}{(nq)^j}f((nq)^j x) - \frac{1}{(nq)^{j+1}}f((nq)^{j+1}x)\| \\
&\leq \frac{\theta}{n^2 q}\sum_{j=l}^{l+m-1}\frac{(nq)^{nrj}}{(nq)^j}\|x\|^{nr}
\end{aligned} \tag{2.22}$$

for all nonnegative integers m and l with $x \in A$. It follows from (2.22) that the sequence $\{\frac{1}{(nq)^m}f((nq)^m x)\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{\frac{1}{(nq)^m}f((nq)^m x)\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{m\to\infty}\frac{1}{(nq)^m}f((nq)^m x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.22), we get (2.21). The proof of uniqueness property of δ, σ, τ and ξ is similar to the proof of Theorem 2.1.

It follows from (2.18) that

$$\begin{aligned}
&\|\delta([xxx]) - [\delta(x)\tau(x)\xi(x)] - [\sigma(x)\delta(x)\xi(x)] - [\sigma(x)\tau(x)\delta(x)]\| \\
&= \lim_{m\to\infty} \frac{1}{(nq)^{3m}} \|f((nq)^{3m}[xxx]) - [f((nq)^m x)h((nq)^m x)k((nq)^m x)] \\
&- [g((nq)^m x)f((nq)^m x)k((nq)^m x)] - [g((nq)^m x)h((nq)^m x)f((nq)^m x)]\| \\
&\le \lim_{m\to\infty} \frac{(nq)^{3mr}\theta}{(nq)^{3m}} (\|x\|^{3r}) = 0
\end{aligned}$$

for all $x \in A$. So

$$\delta([xxx]) = [\delta(x)\tau(x)\xi(x)] + [\sigma(x)\delta(x)\xi(x)] + [\sigma(x)\tau(x)\delta(x)]$$

for all $x \in A$, and the proof of the theorem is complete. ■

3. Stability of C^*-Ternary (σ, τ, ξ)-Derivations

We prove the generalized Hyers–Ulam stability of C^*-ternary (σ, τ, ξ)-derivations in C^*-ternary algebras for the Euler–Lagrange type additive mapping.

Theorem 3.1. *Let $n \in \mathbb{N}$. Assume that $r > 3$ if $nq > 1$ and that $0 < r < 1$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.1) *and* (2.3)–(2.5) *such that*

$$\|f([xyz]) - [f(x)h(y)k(z)] - [g(x)f(y)k(z)] - [g(x)h(y)f(z)]\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{3.1}$$

for all $x, y, z \in A$. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-ternary (σ, τ, ξ)-derivation $D : A \to A$ satisfying (2.6)–(2.8) *such that*

$$\|f(x) - D(x)\| \le \frac{\theta}{(nq)^r - nq} \|x\|^r. \tag{3.2}$$

P r o o f. By the same reasoning as in the proof of Theorem 2.1, there exist unique linear mappings σ, τ and ξ on A and a unique linear mapping $D : A \to A$ satisfying (2.6)–(2.8) and (3.2). The mapping $D : A \to A$ is defined by

$$D(x) := \lim_{m\to\infty} (nq)^m f(\frac{x}{(nq)^m})$$

for all $x \in A$.

It follows from (3.1) that

$$\begin{aligned}
&\|D([xyz]) - [D(x)\tau(y)\xi(z)] - [\sigma(x)D(y)\xi(z)] - [\sigma(x)\tau(y)D(z)]\| \\
&= \lim_{m\to\infty}(nq)^{3m}\|f(\frac{[xyz]}{(nq)^{3m}}) - [f(\frac{x}{(nq)^m})h(\frac{y}{(nq)^m})k(\frac{z}{(nq)^m})] \\
&- [g(\frac{x}{(nq)^m})f(\frac{y}{(nq)^m})k(\frac{z}{(nq)^m})] - [g(\frac{x}{(nq)^m})h(\frac{y}{(nq)^m})f(\frac{z}{(nq)^m})]\| \\
&\le \lim_{m\to\infty}\frac{(nq)^{3m}\theta}{(nq)^{mr}}(\|x\|^r + \|y\|^r + \|z\|^r) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$D([xyz]) = [D(x)\tau(y)\xi(z)] + [\sigma(x)D(y)\xi(z)] + [\sigma(x)\tau(y)D(z)]$$

for all $x, y, z \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. ■

Theorem 3.2. *Let $n \in \mathbb{N}$. Assume that $0 < r < 1$ if $nq > 1$ and that $r > 3$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.1), (2.3)–(2.5) *and* (3.1). *Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-ternary (σ, τ, ξ)-derivation $D : A \to A$ satisfying* (2.12)–(2.14) *such that*

$$\|f(x) - D(x)\| \le \frac{\theta}{nq - (nq)^r}\|x\|^r. \tag{3.3}$$

P r o o f. By the same reasoning as in the proof of Theorem 2.2, there exist unique linear mappings σ, τ and ξ on A and a unique linear mapping $D : A \to A$ satisfying (2.12)–(2.14) and (3.3). The mapping $D : A \to A$ is defined by

$$D(x) := \lim_{m\to\infty}\frac{1}{(nq)^m}f((nq)^m x)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1. ■

Theorem 3.3. *Let $n \in \mathbb{N}$. Assume that $r > 1$ if $nq > 1$ and that $0 < nr < 1$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.3)–(2.5) *and* (2.17) *such that*

$$\|f([xyz]) - [f(x)h(y)k(z)] - [g(x)f(y)k(z)] - [g(x)h(y)f(z)]\| \le \theta\|x\|^r\|y\|^r\|z\|^r \tag{3.4}$$

for all $x, y, z \in A$. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-ternary (σ, τ, ξ)-derivation $D : A \to A$ satisfying (2.6)–(2.8) *such that*

$$\|f(x) - D(x)\| \leq \frac{\theta}{n((nq)^{nr} - nq)}\|x\|^{nr}. \tag{3.5}$$

P r o o f. By the same reasoning as in the proof of Theorem 2.3, there exist unique linear mappings σ, τ and ξ on A and a unique linear mapping $D : A \to A$ satisfying (2.6)–(2.8) and (3.5). The mapping $D : A \to A$ is defined by

$$D(x) := \lim_{m\to\infty} (nq)^m f(\frac{x}{(nq)^m})$$

for all $x \in A$.

It follows from (3.4) that

$$\begin{aligned}
&\|D([xyz]) - [D(x)\tau(y)\xi(z)] - [\sigma(x)D(y)\xi(z)] - [\sigma(x)\tau(y)D(z)]\|\\
&= \lim_{m\to\infty} (nq)^{3m}\|f(\frac{[xyz]}{(nq)^{3m}}) - [f(\frac{x}{(nq)^m})h(\frac{y}{(nq)^m})k(\frac{z}{(nq)^m})]\\
&- [g(\frac{x}{(nq)^m})f(\frac{y}{(nq)^m})k(\frac{z}{(nq)^m})] - [g(\frac{x}{(nq)^m})h(\frac{y}{(nq)^m})f(\frac{z}{(nq)^m})]\|\\
&\leq \lim_{m\to\infty} \frac{(nq)^{3m}\theta}{(nq)^{3mr}}(\|x\|^r.\|y\|^r.\|z\|^r) = 0
\end{aligned}$$

for all $x \in A$. So

$$D([xyz]) = [D(x)\tau(y)\xi(z)] + [\sigma(x)D(y)\xi(z)] + [\sigma(x)\tau(y)D(z)]$$

for all $x, y, z \in A$, and the proof of the theorem is complete. ∎

Theorem 3.4. *Let $n \in \mathbb{N}$. Assume that $r > 1$ if $nq < 1$ and that $0 < nr < 1$ if $nq > 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.3)–(2.5), (2.17) *and* (3.4). *Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-ternary (σ, τ, ξ)-derivation $D : A \to A$ satisfying* (2.12)–(2.14) *such that*

$$\|f(x) - \delta(x)\| \leq \frac{\theta}{n(nq - (nq)^{nr})}\|x\|^{nr}. \tag{3.6}$$

P r o o f. By the same reasoning as in the proof of Theorem 2.4, there exist unique linear mappings σ, τ and ξ on A and a unique linear mapping $D : A \to A$ satisfying (2.12)–(2.14) and (3.6). The mapping $D : A \to A$ is defined by

$$D(x) := \lim_{m\to\infty} \frac{1}{(nq)^m} f((nq)^m x)$$

for all $x \in A$.

It follows from (3.4) that

$$\begin{aligned}
&\|D([xyz]) - [D(x)\tau(y)\xi(z)] - [\sigma(x)D(y)\xi(z)] - [\sigma(x)\tau(y)D(z)]\| \\
&= \lim_{m\to\infty} \frac{1}{(nq)^{3m}} \|f((nq)^{3m}[xyz]) - [f((nq)^m x)h((nq)^m y)k((nq)^m z)] \\
&- [g((nq)^m x)f((nq)^m y)k((nq)^m z)] - [g((nq)^m x)h((nq)^m y)f((nq)^m z)]\| \\
&\le \lim_{m\to\infty} \frac{(nq)^{3mr}\theta}{(nq)^{3m}} (\|x\|^r.\|y\|^r.\|z\|^r) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$D([xyz]) = [D(x)\tau(y)\xi(z)] + [\sigma(x)D(y)\xi(z)] + [\sigma(x)\tau(y)D(z)]$$

for all $x \in A$, and the proof of the theorem is complete. ■

4. Stability of C^*-Lie Ternary (σ, τ, ξ)-Derivations

We are going to study the stability of C^*-Lie ternary (σ, τ, ξ)-derivations in C^*-ternary algebras, associated with the generalized Hyers–Ulam for the Euler–Lagrange type additive mapping.

Theorem 4.1. *Let $n \in \mathbb{N}$. Assume that $r > 3$ if $nq > 1$ and that $0 < r < 1$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying (2.1) and (2.3)–(2.5) such that*

$$\|f([xyz]) - [f(x)yz]_{(g,h,k)} - [f(y)xz]_{(g,h,k)} - [f(z)yx]_{(g,h,k)}\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{4.1}$$

for all $x, y, z \in A$. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-Lie ternary (σ, τ, ξ)-derivation $L : A \to A$ satisfying (2.6)–(2.8) such that

$$\|f(x) - L(x)\| \le \frac{\theta}{(nq)^r - nq} \|x\|^r. \tag{4.2}$$

P r o o f. By the same reasoning as in the proof of Theorem 2.1, there exist unique linear mappings σ, τ and ξ on A and a unique linear mapping $L : A \to A$ satisfying (2.6)–(2.8) and (4.2). The mapping $L : A \to A$ is defined by

$$L(x) := \lim_{m\to\infty} (nq)^m f(\frac{x}{(nq)^m})$$

for all $x \in A$.

It follows from (4.1) that

$$\begin{aligned}
&\|L([xyz]) - [L(x)yz]_{(\sigma,\tau,\xi)} - [L(y)xz]_{(\sigma,\tau,\xi)} - [L(z)yx]_{(\sigma,\tau,\xi)}\| \\
&= \lim_{m\to\infty} (nq)^{3m} \|f(\frac{[xyz]}{(nq)^{3m}}) - [f(\frac{x}{(nq)^m})\frac{y}{(nq)^m}\frac{z}{(nq)^m}]_{(g,h,k)} \\
&- [f\frac{y}{(nq)^m})\frac{x}{(nq)^m}\frac{z}{(nq)^m}]_{(g,h,k)} - [f(\frac{z}{(nq)^m})\frac{y}{(nq)^m}\frac{x}{(nq)^m}]_{(g,h,k)}\| \\
&\le \lim_{m\to\infty} \frac{(nq)^{3m}\theta}{(nq)^{mr}}(\|x\|^r + \|y\|^r + \|z\|^r) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$L([xyz]) = [L(x)yz]_{(\sigma,\tau,\xi)} + [L(y)xz]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x, y, z \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. ■

Theorem 4.2. *Let $n \in \mathbb{N}$. Assume that $0 < r < 1$ if $nq > 1$ and that $r > 3$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.1), (2.3)–(2.5) *and* (4.1)*. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-Lie ternary (σ, τ, ξ)-derivation $L : A \to A$ satisfying* (2.12)–(2.14) *such that*

$$\|f(x) - L(x)\| \le \frac{\theta}{nq - (nq)^r}\|x\|^r. \tag{4.3}$$

P r o o f. By the same reasoning as in the proof of Theorem 2.2, there exist unique linear mappings σ, τ and ξ on A and a unique linear mapping $L : A \to A$ satisfying (2.1), (2.3)–(2.5). The mapping $L : A \to A$ is defined by

$$L(x) := \lim_{m\to\infty} \frac{1}{(nq)^m} f((nq)^m x)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 4.1. ■

Theorem 4.3. *Let $n \in \mathbb{N}$. Assume that $r > 1$ if $nq > 1$ and that $0 < nr < 1$ if $nq < 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.3)–(2.5) *and* (2.17) *such that*

$$\|f([xyz]) - [f(x)yz]_{(g,h,k)} - [f(y)xz]_{(g,h,k)} - [f(z)yx]_{(g,h,k)}\| \le \theta\|x\|^r\|y\|^r\|z\|^r \tag{4.4}$$

for all $x, y, z \in A$. Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-Lie ternary (σ, τ, ξ)-derivation $L : A \to A$ satisfying (2.6)–(2.8) *such that*

$$\|f(x) - L(x)\| \leq \frac{\theta}{n((nq)^{nr} - nq)} \|x\|^{nr}. \tag{4.5}$$

P r o o f. By the same reasoning as in the proof of Theorem 2.3, there exist unique linear mappings σ, τ and ξ on A and a unique linear mapping $L : A \to A$ satisfying (2.6)–(2.8) and (4.5). The mapping $L : A \to A$ is defined by

$$L(x) := \lim_{m\to\infty} (nq)^m f(\frac{x}{(nq)^m})$$

for all $x \in A$.

It follows from (4.4) that

$$\begin{aligned}
&\|L([xyz]) - [L(x)yz]_{(\sigma,\tau,\xi)} - [L(y)xz]_{(\sigma,\tau,\xi)} - [L(z)yx]_{(\sigma,\tau,\xi)}\| \\
&= \lim_{m\to\infty} (nq)^{3m} \|f(\frac{[xyz]}{(nq)^{3m}}) - [f(\frac{x}{(nq)^m})\frac{y}{(nq)^m}\frac{z}{(nq)^m}]_{(g,h,k)} \\
&- [f(\frac{y}{(nq)^m})\frac{x}{(nq)^m}\frac{z}{(nq)^m}]_{(g,h,k)} - [f(\frac{z}{(nq)^m})\frac{y}{(nq)^m}\frac{x}{(nq)^m}]_{(g,h,k)}\| \\
&\leq \lim_{m\to\infty} \frac{(nq)^{3m}\theta}{(nq)^{3mr}} (\|x\|^r.\|y\|^r.\|z\|^r) = 0
\end{aligned}$$

for all $x \in A$. Hence,

$$L([xyz]) = [L(x)yz]_{(\sigma,\tau,\xi)} + [L(y)xz]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x, y, z \in A$, and the proof of the theorem is complete. ■

Theorem 4.4. *Let $n \in \mathbb{N}$. Assume that $r > 1$ if $nq < 1$ and that $0 < nr < 1$ if $nq > 1$. Let θ be a positive real number, and let $f : A \to A$ be an odd mapping for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ satisfying* (2.3)–(2.5), (2.17) *and* (4.4). *Then there exist unique linear mappings σ, τ, and ξ from A to A and a unique C^*-ternary (σ, τ, ξ)-derivation $D : A \to A$ satisfying* (2.12)–(2.14) *such that*

$$\|f(x) - \delta(x)\| \leq \frac{\theta}{n(nq - (nq)^{nr})} \|x\|^{nr}. \tag{4.6}$$

P r o o f. By the same reasoning as in the proof of Theorem 2.4, there exist unique linear mappings σ, τ and ξ on A and a unique linear mapping $L : A \to A$ satisfying (2.12)–(2.14) and (4.6). The mapping $L : A \to A$ is defined by

$$L(x) := \lim_{m\to\infty} \frac{1}{(nq)^m} f((nq)^m x)$$

for all $x \in A$.

It follows from (4.4) that

$$\begin{aligned}
&\|L([xyz]) - [L(x)yz]_{(\sigma,\tau,\xi)} - [L(y)xz]_{(\sigma,\tau,\xi)} - [L(z)yx]_{(\sigma,\tau,\xi)}\| \\
&= \lim_{m\to\infty} \frac{1}{(nq)^{3m}} \|f((nq)^{3m}[xyz]) - [f((nq)^m x)(nq)^m y(nq)^m z]_{(g,h,k)} \\
&- [f((nq)^m y)(nq)^m x(nq)^m z)]_{(g,h,k)} - [f((nq)^m z)(nq)^m y(nq)^m x]_{(g,h,k)}\| \\
&\le \lim_{m\to\infty} \frac{(nq)^{3mr}\theta}{(nq)^{3m}} (\|x\|^r.\|y\|^r.\|z\|^r) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$L([xyz]) = [L(x)yz]_{(\sigma,\tau,\xi)} + [L(y)xz]_{(\sigma,\tau,\xi)} + [L(z)yx]_{(\sigma,\tau,\xi)}$$

for all $x \in A$, and the proof of the theorem is complete. ■

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