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Step-Initial Function to the MKdV Equation: Hyper-Elliptic Long-Time Asymptotics of the Solution

V. Kotlyarov and A. Minakov

Mathematical division, B.I. Verkin Institute for Low Temperature Physics and Engineering
47 Lenin Avenue, 61103 Kharkiv, Ukraine

E-mail: kotlyarov@ilt.kharkov.ua
minakov@ilt.kharkov.ua

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The modified Korteveg–de Vries equation on the line is considered. The initial function is a discontinuous and piece-wise constant step function, i.e. $q(x,0) = c_r$ for $x \geq 0$ and $q(x,0) = c_l$ for $x < 0$, where c_l, c_r are real numbers which satisfy $c_l > c_r > 0$. The goal of this paper is to study the asymptotic behavior of the solution of the initial-value problem as $t \to \infty$. Using the steepest descent method we deform the original oscillatory matrix Riemann–Hilbert problem to explicitly solvable model forms and show that the solution of the initial-value problem has different asymptotic behavior in different regions of the xt plane. In the regions $x < -6c_l^2 t + 12c_r^2 t$ and $x > 4c_l^2 t + 2c_r^2 t$ the main term of asymptotics of the solution is equal to c_l and c_r, respectively. In the region $(-6c_l^2 + 12c_r^2)t < x < (4c_l^2 + 2c_r^2)t$ the asymptotics of the solution takes the form of a modulated hyper-elliptic wave generated by an algebraic curve of genus 2.

Key words: modified Korteweg–de Vries equation, step-like initial value problem, Riemann–Hilbert problem, steepest descent method, modulated hyper-elliptic wave.

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1. Introduction

The inverse scattering transform method (IST) [1–3] used for initial-value problems for nonlinear integrable equations is proved to be very successful. It allows to obtain a large number of very interesting results in various areas of mathematics and physics. The IST method was further developed by P. Deift and X. Zhou [4–6]. They proposed to use the steepest descent method for solving the oscillatory matrix Riemann–Hilbert problems. This method appeared to give a nice possibility to study asymptotic behavior of the solutions of initial-value problems as well as many other problems of the theory of completely integrable nonlinear equations (sf. [7–26]), random matrix models, orthogonal polynomials and integrable statistical mechanics [27–29] without *a priori* assumptions.

The initial value problems for nonlinear integrable equations with step-like initial functions have a very long history. More about these problems can be found in [2, 30–46], and also in the references therein. Most results were obtained for the initial-value problems associated with self-adjoint Lax operators. First the step-like problems with non self-adjoint Lax operators were considered by Bikbaev [40, 41] and later by Novokshenov [43]. In their papers, the main attention was paid to the studying of the complex Whitham deformations, which allowed them to describe the long-time asymptotic behavior of the solution. However, for the present time, asymptotic formulas as well as their justifications seem to be not sufficiently clear and rigorous. Most recently an implementation of the rigorous RH scheme to the focusing nonlinear Schrödinger equation with non self-adjoint Lax operator was presented in [47–49].

In the short note [40], the initial-value problem

$$q_t + 6q^2 q_x + q_{xxx} = 0 \tag{1.1}$$

$$q(x,0) = q_0(x) \to \begin{cases} c_r, & x \to +\infty, \\ c_l, & x \to -\infty \end{cases} \tag{1.2}$$

was considered. In [40], the solution of the problem is described by a modulated two-gap solution of the mKdV equation that corresponds to the long-time dynamics of the compression wave when $-6c_l^2 t + 12c_r^2 t < x < 4c_l^2 t + 2c_r^2 t$ that has not been proved up to now. The goal of this paper is to justify this statement in a transparent form by using a suitable matrix Riemann–Hilbert problem and corresponding steepest descent method. The central point of the paper is to describe in an explicit form the so-called g-function mechanism which allows to deform the original oscillatory matrix Riemann–Hilbert problem to the solvable model forms. We emphasize that our formula for a hyper-elliptic wave is written in an explicit form via theta functions.

2. Jost Solutions of Lax Equations

To study the initial value problem (1.1)–(1.2) we use the Lax representation of the mKdV equation [2, 3] in the form of the over-determined system of differential equations

$$\Phi_x + ik\sigma_3\Phi = Q(x,t)\Phi, \tag{2.1}$$

$$\Phi_t + 4ik^3\sigma_3\Phi = \hat{Q}(x,t,k)\Phi, \tag{2.2}$$

where $\Phi = \Phi(x,t,k)$ is a 2×2 matrix-valued function,

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x,t) := \begin{pmatrix} 0 & q(x,t) \\ -q(x,t) & 0 \end{pmatrix},$$

$$\hat{Q}(x,t,k)$$
$$= 4k^2 Q(x,t,k) - 2ik(Q^2(x,t,k) + Q_x(x,t,k))\sigma_3 + 2Q^3(x,t,k) - Q_{xx}(x,t,k),$$

and $k \in \mathbb{C}$. Equations (2.1) and (2.2) are compatible if and only if the function $q(x,t)$ satisfies the mKdV equation (1.1). To apply the inverse scattering transform to the problem (1.1)–(1.2) we have to define the matrix valued Jost solutions of the Lax equations. We define them as the solutions of the compatible equations (2.1) and (2.2) satisfying the asymptotic conditions

$$\Phi_r(x,t,k) = E_r(x,t,k) + o(1), \quad x \to +\infty, \quad \operatorname{Im} k = 0, \tag{2.3}$$

$$\Phi_l(x,t,k) = E_l(x,t,k) + o(1), \quad x \to -\infty, \quad \operatorname{Im} k = 0. \tag{2.4}$$

Here $E_l(x,t,k)$, $E_r(x,t,k)$ are the solutions of the linear differential equations

$$E_x + ik\sigma_3 E = Q_c E,$$

$$E_t + 4ik^3\sigma_3 E = \hat{Q}_c(k)E,$$

where $c = c_l$ and $c = c_r$, respectively, and the constant matrix coefficients Q_c and $\hat{Q}_c(k)$ are as follows:

$$Q_c := \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \qquad \hat{Q}_c(k) = 4k^2 Q_c - 2ikQ_c^2\sigma_3 + 2Q_c^3.$$

We choose the solutions $E_l(x,t,k)$, $E_r(x,t,k)$ in the form

$$E_{l,r}(x,t,k) = \frac{1}{2}\begin{pmatrix} \varkappa_{l,r}(k) + \dfrac{1}{\varkappa_{l,r}(k)} & \varkappa_{l,r}(k) - \dfrac{1}{\varkappa_{l,r}(k)} \\ \varkappa_{l,r}(k) - \dfrac{1}{\varkappa_{l,r}(k)} & \varkappa_{l,r}(k) + \dfrac{1}{\varkappa_{l,r}(k)} \end{pmatrix} e^{-ixX_{l,r}(k)\sigma_3 - it\Omega_{l,r}(k)\sigma_3},$$

where

$$X_{l,r}(k) = \sqrt{k^2 + c_{l,r}^2}\,, \quad \Omega_{l,r}(k) = 2(2k^2 - c_{l,r}^2)X_{l,r}(k)\,, \quad \varkappa_{l,r}(k) = \sqrt[4]{\frac{k - ic_{l,r}}{k + ic_{l,r}}}\,. \tag{2.5}$$

The branches of the roots are fixed by the conditions $X_{l,r}(1) > 0$, $\varkappa_{l,r}(\infty) = 1$. Then the functions $X_{l,r}(k)$ and $\varkappa_{l,r}(k)$ are analytic in $\mathbb{C}\setminus[ic, -ic]$, where $c = c_l$ or $c = c_r$, respectively.

Solutions (2.3), (2.4) can be represented in the forms

$$\Phi_l(x,t,k) = E_l(x,t,k) + \int_{-\infty}^{x} K_l(x,y,t)E_l(y,t,k)dy, \qquad \operatorname{Im} k = 0, \tag{2.6}$$

$$\Phi_r(x,t,k) = E_r(x,t,k) + \int_{x}^{\infty} K_r(x,y,t)E_r(y,t,k)dy, \qquad \operatorname{Im} k = 0, \tag{2.7}$$

where the kernels $K_{l,r}(x,y,t)$ are sufficiently smooth and decrease to zero rapidly as $x+y \to \pm\infty$. Omitting the details of the proof of these representations, we formulate below the properties of the solutions.

The matrices $\Phi_l(x,t,k)$ and $\Phi_r(x,t,k)$, defined by (2.6), (2.7) and their columns $\Phi_{lj}(x,t,k)$ and $\Phi_{rj}(x,t,k)$, $j = 1,2$, have the following properties:

1) determinants are equal to the identity matrix:
   $\det \Phi_{l,r}(x,t,k) = 1$;

2) analyticity:
   $\Phi_{r1}(x,t,k)$ is analytic in $k \in \mathbb{D}_{r-} := \mathbb{C}_- \setminus [0,-ic_r]$,
   $\Phi_{r2}(x,t,k)$ is analytic in $k \in \mathbb{D}_{r+} := \mathbb{C}_+ \setminus [0,ic_r]$,
   $\Phi_{l1}(x,t,k)$ is analytic in $k \in \mathbb{D}_{l+} := \mathbb{C}_+ \setminus [0,ic_l]$,
   $\Phi_{l2}(x,t,k)$ is analytic in $k \in \mathbb{D}_{l-} := \mathbb{C}_- \setminus [-ic_l,0]$;

3) continuity:
   $\Phi_{r1}(x,t,k)$ is continuous for $k \in \mathbb{D}_{r-} \cup (-ic_r,ic_r)_- \cup (-ic_r,ic_r)_+$,
   $\Phi_{r2}(x,t,k)$ is continuous for $k \in \mathbb{D}_{r+} \cup (-ic_r,ic_r)_- \cup (-ic_r,ic_r)_+$,
   $\Phi_{l1}(x,t,k)$ is continuous for $k \in \mathbb{D}_{l+} \cup (-ic_l,ic_l)_- \cup (-ic_l,ic_l)_+$,
   $\Phi_{l2}(x,t,k)$ is continuous for $k \in \mathbb{D}_{l-} \cup (-ic_l,ic_l)_- \cup (-ic_l,ic_l)_+$,
   where $(-ic_{l,r},ic_{l,r})_-$ and $(-ic_{l,r},ic_{l,r})_+$ are the left- and the right-hand sides of the interval $(-ic_{l,r},ic_{l,r})$;

4) symmetries:

$$\begin{aligned}
&\overline{\Phi_{22}(x,t,\overline{k})} = \Phi_{11}(x,t,k), && \Phi_{22}(x,t,-k) = \Phi_{11}(x,t,k),\\
&\overline{\Phi_{12}(x,t,\overline{k})} = -\Phi_{21}(x,t,k), && \Phi_{12}(x,t,-k) = -\Phi_{21}(x,t,k),\\
&\overline{\Phi_{jl}(x,t,-\overline{k})} = \Phi_{jl}(x,t,k), && j,l = \overline{1,2},
\end{aligned}$$

   where $\Phi(x,t,k)$ denotes $\Phi_l(x,t,k)$ or $\Phi_r(x,t,k)$;

5) large k asymptotics:

$$\left.\begin{array}{l} \Phi_{r1}(x,t,k)e^{+ikx+4ik^3t} \\ \Phi_{l2}(x,t,k)e^{-ikx-4ik^3t} \end{array}\right\} = 1 + O\left(\frac{1}{k}\right), \quad k \to \infty, \quad \operatorname{Im} k \le 0,$$

$$\left.\begin{array}{l}\Phi_{l1}(x,t,k)e^{+ikx+4ik^3t}\\ \Phi_{r2}(x,t,k)e^{-ikx-4ik^3t}\end{array}\right\} = 1 + O\left(\frac{1}{k}\right), \quad k\to\infty, \quad \operatorname{Im} k \geq 0;$$

6) jump:

$$\Phi_-(x,t,k) = \Phi_+(x,t,k)\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}, \quad k \in (ic, -ic),$$

where $\Phi(x,t,k)$ and c denote $\Phi_l(x,t,k)$ and c_l or $\Phi_r(x,t,k)$ and c_r, respectively, and $\Phi_\pm(x,t,k)$ are the non-tangential boundary values of matrix $\Phi(x,t,k)$ from the left $(-)$ and from the right $(+)$ of the downward-oriented interval $(-\mathrm{i}c, \mathrm{i}c)$.

The matrices $\Phi_l(x,t,k)$ and $\Phi_r(x,t,k)$ are solutions of equations (2.1) and (2.2). Hence they are linear dependent, i.e.,there exists the independent of x,t matrix

$$T(k) = \Phi_r^{-1}(x,t,k)\Phi_l(x,t,k), \quad k \in \mathbb{R}, \tag{2.8}$$

which is defined for those k for which $\operatorname{Im} X_r(k) = 0$. Some elements of this matrix have an extended domain of definition. Indeed, using (2.8), we can find

$$\begin{aligned} T_{11}(k) &= \det(\Phi_{l1}, \Phi_{r2}),\\ T_{21}(k) &= \det(\Phi_{r1}, \Phi_{l1}),\\ T_{12}(k) &= \det(\Phi_{l2}, \Phi_{r2}),\\ T_{22}(k) &= \det(\Phi_{r1}, \Phi_{l2}).\end{aligned}$$

Then the above properties of the solutions $\Phi_r(x,t,k)$ and $\Phi_l(x,t,k)$ imply:

- $T_{11}(k)$ is analytic in $k \in \mathbb{C}_+\backslash[0, ic_l]$ and has a continuous extension to $(0, ic_l)_- \bigcup (0, ic_l)_+$;
- $T_{22}(k)$ is analytic in $k \in \mathbb{C}_-\backslash[0, ic_l]$ and has a continuous extension to $(-ic_l, 0)_- \bigcup(-ic_l, 0)_+$;
- $T_{21}(k)$ is continuous in $k \in (-\infty, 0)\bigcup(0, -ic_l)_-\bigcup(-ic_l, 0)_+\bigcup(0, +\infty)$;
- $T_{12}(k)$ is continuous in $k \in (-\infty, 0)\bigcup(0, ic_l)_-\bigcup(ic_l, 0)_+\bigcup(0, +\infty)$,

where, as before, the signs $-$ and $+$ denote the left- and the right-hand sides of the intervals;

- $\overline{T_{22}(\overline{k})} = T_{11}(k), \quad T_{22}(-k) = T_{11}(k)$,
- $\overline{T_{12}(\overline{k})} = -T_{21}(k), \quad T_{12}(-k) = -T_{21}(k)$,
- $\overline{T_{jk}(-\overline{k})} = T_{jk}(k), \quad j,k = \overline{1,2}$.

Denote

$$a(k) = T_{11}(k),$$
$$b(k) = T_{21}(k).$$

Define the reflection coefficient

$$r(k) = \frac{b(k)}{a(k)}.$$

It has the property

$$\overline{r(-\overline{k})} = r(k).$$

The columns of the matrices Φ_l and Φ_r satisfy the following jump conditions:

7) $$\frac{(\Phi_{l1})_-(x,t,k)}{a_-(k)} - \frac{(\Phi_{l1})_+(x,t,k)}{a_+(k)} = f_1(k)\Phi_{r2}(x,t,k), \quad k \in (ic_r, ic_l);$$

8) $$\frac{(\Phi_{l2})_-(x,t,k)}{\overline{a_-(\overline{k})}} - \frac{(\Phi_{l2})_+(x,t,k)}{\overline{a_+(\overline{k})}} = f_2(k)\Phi_{r1}(x,t,k), \quad k \in (-ic_r, -ic_l),$$

where

$$f_1(k) = \frac{i}{a_-(k)a_+(k)}, \quad k \in (0, ic_l), \qquad f_2(k) = -\overline{f_1(\overline{k})}, \quad k \in (-ic_l, 0).$$

3. The Basic Riemann–Hilbert Problem

The scattering relation (2.8) between the matrix-valued functions $\Phi_l(x,t,k)$ and $\Phi_r(x,t,k)$ and jump conditions 6, 7, 8 can be rewritten in terms of the Riemann–Hilbert problem. To do this, define the matrix-valued function

$$M(\xi,t,k) = \begin{cases} \left(\dfrac{\Phi_{l1}(x,t,k)}{a(k)}e^{it\theta(k,\xi)}, \Phi_{r2}(x,t,k)e^{-it\theta(k,\xi)}\right), \; k \in \mathbb{C}_+\backslash[0, ic_l], \\ \\ \left(\Phi_{r1}(x,t,k)e^{it\theta(k,\xi)}, \dfrac{\Phi_{l2}(x,t,k)}{\overline{a(\overline{k})}}e^{-it\theta(k,\xi)}\right), \; k \in \mathbb{C}_-\backslash[-ic_l, 0], \end{cases} \tag{3.1}$$

where $x = 12\xi t$ and $\theta(k,\xi) = 4k^3 + 12k\xi$ ($\xi = x/12t$). Below we restrict our consideration to the simplest shock problem where the initial function is discontinuous and piece-wise constant (pure step function):

$$q_0(x) = \begin{cases} c_r, & x \geq 0, \\ c_l, & x < 0. \end{cases} \tag{3.2}$$

Then

$$a(k) = \frac{1}{2}\left(\varkappa(k) + \frac{1}{\varkappa(k)}\right), \; b(k) = \frac{1}{2}\left(\varkappa(k) - \frac{1}{\varkappa(k)}\right), \; r(k) = \frac{\varkappa^2(k) - 1}{\varkappa^2(k) + 1} \quad (3.3)$$

are analytic in $k \in \mathbb{C} \setminus ([-ic_l, -ic_r] \cup [ic_l, ic_r])$, since the function $\varkappa(k) := \dfrac{\varkappa_l(k)}{\varkappa_r(k)}$ (see (2.5)) is analytic in this domain. The transition coefficient $a^{-1}(k)$ is bounded in $k \in \mathbb{C}_+ \setminus [ic_l, ic_r]$ because the function $a(k)$ equals zero nowhere, and hence the set of eigenvalues of the linear problem (2.1) is empty. We have that $f_1(k) \equiv f_2(k)$, and so we define $f(k) := f_1(k) = f_2(k)$. We also have

$$f(k) = r_-(k) - r_+(k), \quad k \in (-ic_l, -ic_r) \cup (ic_r, ic_l). \quad (3.4)$$

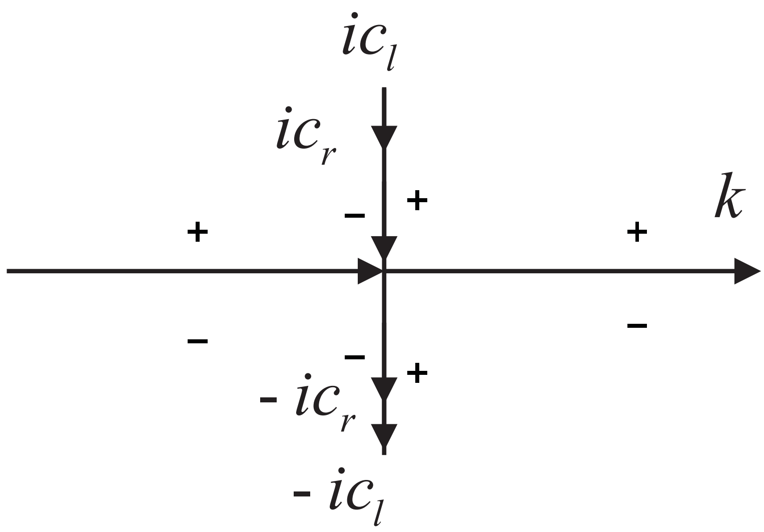

Fig. 1. Oriented contour Σ.

Let us define the oriented contour $\Sigma = \mathbb{R} \cup (ic_l, -ic_l)$ as in Fig. 1. Then the matrix (3.1) solves the next Riemann–Hilbert problem:

- the matrix-valued function $M(\xi, t, k)$ is analytic in the domain $\mathbb{C} \setminus \Sigma$;
- $M(\xi, t, k)$ is bounded in the neighborhood of the branching points ic_l, ic_r, $-ic_l$, $-ic_r$ and at the origin ($k = 0$);
- $M_-(\xi, t, k) = M_+(\xi, t, k) J(\xi, t, k), \quad k \in \Sigma \setminus \{0\}$,

where

$$J(\xi,t,k) = \begin{pmatrix} 1 & r(k)e^{-2it\theta(k,\xi)} \\ -r(k)e^{2it\theta(k,\xi)} & 1+|r(k)|^2 \end{pmatrix}, \qquad k\in\mathbb{R}\backslash\{0\}, \quad (3.5)$$

$$= \begin{pmatrix} 1 & 0 \\ f(k)e^{2it\theta(k,\xi)} & 1 \end{pmatrix}, \qquad k\in(ic_r, ic_l), \quad (3.6)$$

$$= \begin{pmatrix} 1 & f(k)e^{-2it\theta(k,\xi)} \\ 0 & 1 \end{pmatrix}, \qquad k\in(-ic_r, -ic_l), \quad (3.7)$$

$$= \begin{pmatrix} ir(k) & ie^{-2it\theta(k,\xi)} \\ f(k)e^{2it\theta(k,\xi)} & -ir(k) \end{pmatrix}, \qquad k\in(0, ic_r), \quad (3.8)$$

$$= \begin{pmatrix} -ir(k) & f(k)e^{-2it\theta(k,\xi)} \\ ie^{2it\theta(k,\xi)} & ir(k) \end{pmatrix}, \qquad k\in(0, -ic_r); \quad (3.9)$$

- $M(\xi,t,k) = I + O(k^{-1}), \quad k\to\infty,$

where $r(k) = -\overline{r(\bar{k})} = -r(-k)$ is given in (3.3), and $f(k)$ in (3.4).

If the initial function is arbitrary step-like, then $a(k)$ may have zeroes in the domain of analyticity. In this case the matrix $M(\xi,t,k)$ is meromorphic and residue relations between the columns of the matrix $M(\xi,t,k)$ must be added.

In what follows we suppose that the solution $q(x,t)$ of the shock problem (1.1)–(1.2) with the pure step initial function (3.2) does exist. The above Riemann–Hilbert problem gives $q(x,t)$ in the form

$$q(x,t) = 2i\lim_{k\to\infty} k[M(x/12t,t,k)]_{12}, \qquad (3.10)$$

where $[M(x/12t,t,k)]_{12}$ is the appropriate entry of the matrix $M(x/12t,t,k)$.

4. Long-Time Asymptotic Analysis of the Riemann–Hilbert Problem

The jump matrices $J(\xi,t,k)$ in (3.5)–(3.9) depend on $\exp\{\pm 2it\theta(k,\xi)\}$. The phase function $\theta(k,\xi)$ and the signature table of its imaginary part play a very important role. For a vanishing initial function the phase function $\theta(k,\xi)$ allows to use successfully the steepest descent method for oscillatory RH problem [5] when the conjugation contour Σ coincides with the real axis $\mathbb{R}$. For a non-vanishing initial function, the phase function $\theta(k,\xi)$ does not allow to carry out the asymptotic analysis of the RH problem because the contour Σ contains the segment $[ic_l, -ic_l]$ which imposes extra (bad) properties of the phase function (indeed, $e^{2it\theta(k,\xi)}$ grows exponentially). Therefore, we have to change the phase function $\theta(k,\xi)$ with a new one. In what follows we will use the phase function $g(k,\xi)$ which takes different forms in different regions.

A. Construction of the phase function

1. Regions $\xi < -\frac{c_l^2}{2} + c_r^2$ **and** $\xi > \frac{c_l^2}{3} + \frac{c_r^2}{6}$**.** The asymptotic analysis used for studying the asymptotic behavior in these regions is similar to those given in [23, 25, 26]. Therefore we only mention that the suitable phase functions are given by the formulas

$$g(k,\xi) = 12\xi X_{c_l}(k) + 2(2k^2 - c_l^2)X_{c_l}(k), \quad \xi > \frac{c_l^2}{3} + \frac{c_r^2}{6}\,,$$

$$g(k,\xi) = 12\xi X_{c_r}(k) + 2(2k^2 - c_r^2)X_{c_r}(k), \quad \xi < -\frac{c_l^2}{2} + c_r^2\,,$$

where $X_c(k) = \sqrt{k^2 + c^2}$ is holomorphic outside the segment $[ic, -ic]$. We obtain the following

Theorem 4.1. *For $t \to \infty$ and $x > (4c_l^2 + 2c_r^2)t$ the solution of the problem* (1.1)–(1.2) *with the initial pure step function* (3.2) *takes the form*

$$q(x,t) = c_r + O(e^{-Ct}),$$

where $C > 0$ is some positive constant.

Theorem 4.2. *For $t \to \infty$ and $x < (-6c_l^2 + 12c_r^2)t$ the solution of the problem* (1.1)–(1.2) *with the initial pure step function* (3.2) *takes the form*

$$q(x,t) = c_l + O(t^{-1/2}).$$

In what follows we will deal only with the

2. Region $-\frac{c_l^2}{2} + c_r^2 < \xi < \frac{c_l^2}{3} + \frac{c_r^2}{6}$ $\left((-6c_l^2 + 12c_r^2)t < x < (4c_l^2 + 2c_r^2)t\right)$**.** In this region we use the function $g(k,\xi)$ with the following properties:
(1) $g(k,\xi)$ is analytic in the domain $k \in \mathbb{C} \setminus [ic_l, -ic_l]$;
(2) $\exists \lim_{k\to\infty} (g(k,\xi) - \theta(k,\xi)) = g_0(\xi) \in \mathbb{C}$;
(3) the set $\{k : \operatorname{Im} g(k,\xi) = 0\}$ divides the complex plane into four connected open sets and contains necessarily the set $\mathbb{R} \cup [ic_l, id] \cup [ic_r, -ic_r] \cup [-id, -ic_l]$, where $d = d(\xi) \in [ic_l, ic_r]$ is some function of ξ.

We look for such a function in the form

$$g(k,\xi) = \int\limits_{\mathrm{i}c_l}^{k} \frac{12k(k^2+\mu^2)(k^2+d^2)dk}{\mathrm{w}(k,\xi)}\,, \quad \mathrm{w}(k,\xi) = \sqrt{(k^2+c_l^2)(k^2+d^2)(k^2+c_r^2)}\,,$$

where function $\mathrm{w}(k,\xi)$ is positive on the positive part of the real axis and analytic in $k \in \mathbb{C}\setminus([ic_l, id] \cup [ic_r, -ic_r] \cup [-id, -ic_l])$. Here unknown numbers d and μ have to be determined as the functions of ξ. The integration contour is chosen to have no intersection with the segment $[ic_l, -ic_l]$. It is easy to see that $g(k,\xi) \in \mathbb{R}$ if k lies on the left- or right-hand side of the segment $[ic_l, id]$. To satisfy the requirement $g(k,\xi) \in \mathbb{R}$, if k lies on the left- or right-hand side of $[ic_r, -ic_r] \cup [-id, -ic_l]$, we have to choose such numbers μ and d that $\int_{ic_r}^{id} dg(k,\xi) = 0$ and $\int_{-id}^{-ic_r} dg(k,\xi) = 0$. Due to the symmetry of $dg(k,\xi)$ under the change of variable $k \mapsto -k$, we can see that the last two requirements are equivalent to each other and can be written as follows:

$$\int_{c_r}^{d} \frac{y(y^2-\mu^2)\sqrt{d^2-y^2}\,dy}{\sqrt{(c_l^2-y^2)(y^2-c_r^2)}} = 0. \tag{4.1}$$

This formula defines $\mu = \mu(d)$ as a strictly increasing function on the segment $[c_r, c_l]$, and

$$\mu(c_r) = c_r\ , \qquad \mu(c_l) = \sqrt{\frac{c_l^2+2c_r^2}{3}}. \tag{4.2}$$

Indeed, by expressing μ in d through formula (4.1) and then by taking the first derivative of $\mu^2(d)$, we find

$$\left(\mu^2\right)'_d = d\ \frac{\int_{c_r}^{d}\rho(y)h_1(y)dy\int_{c_r}^{d}\rho(y)h_2(y)dy - \int_{c_r}^{d}\rho(y)h_1(y)h_2(y)dy\int_{c_r}^{d}\rho(y)dy}{\left(\int_{c_r}^{d} h_1(y)h_2(y)\rho(y)dy\right)^2}\ ,$$

where we denote

$$\rho(y) = \frac{y}{\sqrt{(c_l^2-y^2)(d^2-y^2)(y^2-c_r^2)}}\ ,$$

$$h_1(y) = y^2,$$
$$h_2(y) = d^2 - y^2\ .$$

Let us note that all the three functions $\rho(.)$, $h_1(.)$, $h_2(.)$ are positive on the segment $[c_r, d]$, moreover, $h_1(.)$ is increasing on the segment and $h_2(.)$ is decreasing on the segment. Therefore we can use

Lemma 4.1. *Let $\rho(.)$, $h_1(.)$, $h_2(.)$ be positive functions on the segment $[a,b]\subset\mathbb{R}$ such that the integrals of their combinations $\rho(y)$, $\rho(y)h_1(y)$, $\rho(y)h_2(y)$, $\rho(y)h_1(y)h_2(y)$ are convergent in proper or improper sense. Let also assume that h_1 is the increasing function on the segment, and h_2 is decreasing function. Then*

$$\int_{c_r}^{d}\rho(y)h_1(y)dy\int_{c_r}^{d}\rho(y)h_2(y)dy-\int_{c_r}^{d}\rho(y)h_1(y)h_2(y)dy\int_{c_r}^{d}\rho(y)dy\geq 0.$$

We will prove this lemma in the Appendix.

Now we want to satisfy the requirement (2) $\lim\limits_{k\to\infty}(g(k,\xi)-\theta(k,\xi))\in\mathbb{C}$, which is fulfilled if $dg(k,\xi)-d\theta(k,\xi)=O(k^{-2})dk$, as $k\to\infty$, is fulfilled. Since $d\theta(k,\xi)=12(k^2+\xi)dk$ and $dg(k,\xi)=\left[12k^2-6(c_l^2+c_r^2-d^2-2\mu^2)+O(k^{-2})\right]dk$ as $k\to\infty$, we need

$$\xi+\frac{c_l^2+c_r^2}{2}=\mu^2+\frac{d^2}{2}. \tag{4.3}$$

Equations (4.2) yield that $\mu^2(d)+\dfrac{d^2}{2}-\dfrac{c_l^2+c_r^2}{2}$ varies over the segment $\left[-\dfrac{c_l^2}{2}+c_r^2,\dfrac{c_l^2}{3}+\dfrac{c_r^2}{6}\right]$ when d varies over the segment $[c_r,c_l]$. So, from (4.3) we get that for any $\xi\in\left[-\dfrac{c_l^2}{2}+c_r^2,\dfrac{c_l^2}{3}+\dfrac{c_r^2}{6}\right]$ there exists a single $d=d(\xi)\in[c_r,c_l]$ such that (4.1) and (4.3) are fulfilled. Equality (4.3) implies that $d=d(\xi)$ is a continuous function. Thus, the function $g(k,\xi)$ is completely defined and it has the property

(2a) $\lim\limits_{k\to\infty}(g(k,\xi)-\theta(k,\xi))=0$

which follows from the existence of the limit and the relations below:

$$g(k,\xi)-\theta(k,\xi)\in\mathrm{i}\mathbb{R}\ ,\quad k\in(ic_l,+i\infty)\ ;$$

$$g(k,\xi)-\theta(k,\xi)\in\mathbb{R}\ ,\quad k\in\mathbb{R}\ .$$

Besides,

(4) $g_-(k,\xi)+g_+(k,\xi)=0\ ,\quad k\in(ic_l,id)\cup(ic_r,-ic_r)\cup(-id,-ic_l)$;

(5) $g_-(k,\xi)-g_+(k,\xi)=B_g(\xi)\ ,\quad k\in(id,ic_r)\cup(-ic_r,-id)$, where

$$B_g(\xi)=2\int_{id}^{ic_l}dg_+(k,\xi)=2\int_{-id}^{-ic_l}dg_+(k,\xi)>0. \tag{4.4}$$

The signature table of the imaginary part of the function $g(k,\xi)$ is given in Fig. 2.

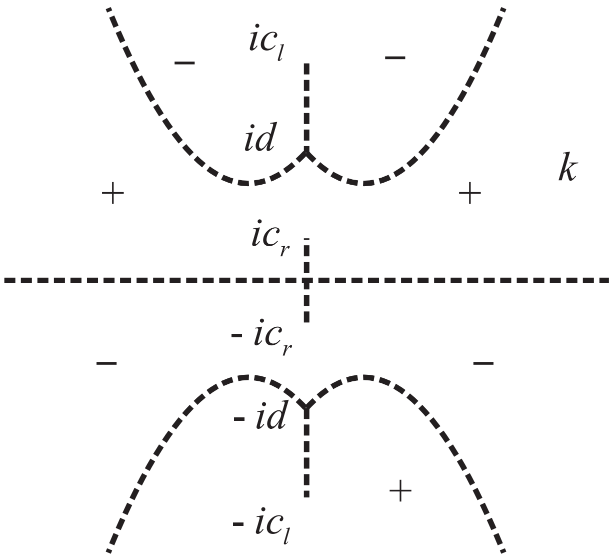

Fig. 2. The signature table of Im $g(k,\xi)$.

3. Changing of the phase function. As the phase function $\theta(k,\xi)$ is not suitable now, the Riemann–Hilbert problem for the matrix $M(\xi,t,k)$ has to be considered with a new phase function $g(k,\xi)$. Let us define the new matrix-function

$$M^{(1)}(\xi,t,k) = M(\xi,t,k)G^{(1)}(\xi,t,k),$$

where $G^{(1)}(\xi,t,k) = e^{it(g(k,\xi)-\theta(k,\xi))\sigma_3}$. Then the function $M^{(1)}(\xi,t,k)$ solves the RH problem

$$M_-^{(1)}(\xi,t,k) = M_+^{(1)}(\xi,t,k)J^{(1)}(\xi,t,k),\ k\in\Sigma_1 := \Sigma,\ M^{(1)}(\xi,t,k)\to I,\ k\to\infty,$$

where

$$J^{(1)}(\xi,t,k) = \begin{pmatrix} 1 & r(k)e^{-2itg(k,\xi)} \\ -r(k)e^{2itg(k,\xi)} & 1+|r(k)|^2 \end{pmatrix}, \qquad k\in\mathbb{R}\backslash\{0\}, \tag{4.5}$$

$$= \begin{pmatrix} e^{it(g_-(k,\xi)-g_+(k,\xi))} & 0 \\ f(k)e^{it(g_-(k,\xi)+g_+(k,\xi))} & e^{-it(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, \qquad k\in(ic_r, ic_l), \tag{4.6}$$

$$= \begin{pmatrix} e^{it(g_-(k,\xi)-g_+(k,\xi))} & f(k)\mathrm{e}^{-it(g_-(k,\xi)+g_+(k,\xi))} \\ 0 & e^{-it(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, \qquad k \in (-ic_r, -ic_l), \quad (4.7)$$

$$= \begin{pmatrix} ir(k)e^{it(g_-(k,\xi)-g_+(k,\xi))} & ie^{-it(g_-(k,\xi)+g_+(k,\xi))} \\ f(k)e^{it(g_-(k,\xi)+g_+(k,\xi))} & -\, r(k)e^{-it(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, \quad k \in (0, ic_r), \quad (4.8)$$

$$= \begin{pmatrix} -ir(k)e^{it(g_-(k,\xi)-g_+(k,\xi))} & f(k)e^{-it(g_-(k,\xi)+g_+(k,\xi))} \\ ie^{it(g_-(k,\xi)+g_+(k,\xi))} & ir(k)e^{-it(g_-(k,\xi)-g_+(k,\xi))} \end{pmatrix}, \; k \in (0, -ic_r). \quad (4.9)$$

4. Transferring of the jump contour from the real line. Define a decomposition of the k-complex plane into domains Ω_j, $j = 1, 2, 3, 4$ as shown in Fig. 3.

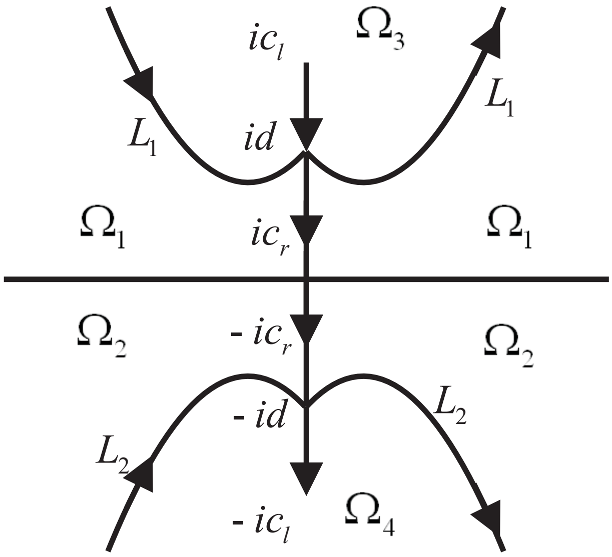

Fig. 3. The contour Σ_2.

Here the contour L_1 lies in the part of the complex plane, where $\mathrm{Im}g(k,\xi) > 0$, and L_2 lies in the part of the complex plane, where $\mathrm{Im}g(k,\xi) < 0$. The transformation below transfers the jump contour from the real line

$$M^{(2)}(\xi, t, k) = M^{(1)}(\xi, t, k)G^{(2)}(\xi, t, k),$$

where

$$G^{(2)}(\xi, t, k) = \begin{pmatrix} 1 & 0 \\ -r(k)e^{2itg(k,\xi)} & 1 \end{pmatrix}, \qquad k \in \Omega_1\ , \qquad (4.10)$$

$$= \begin{pmatrix} 1 & -r(k)e^{-2itg(k,\xi)} \\ 0 & 1 \end{pmatrix}, \qquad k \in \Omega_2\ , \qquad (4.11)$$

$$= I, \qquad k \in (\Omega_1 \cup \Omega_2)^C . \qquad (4.12)$$

$G^{(2)}$ — transformation leads to the RH-problem

$$M_-^{(2)}(\xi,t,k) = M_+^{(2)}(\xi,t,k)J^{(2)}(\xi,t,k)\ , \quad k\in\Sigma_2, \quad M^{(2)}(\xi,t,k)\to I\ , \quad k\to\infty\ ,$$

where

$$J^{(2)}(\xi,t,k) = \left(G_+^{(2)}\right)^{-1} J^{(1)}(\xi,t,k)G_-^{(2)}(\xi,t,k).$$

Taking into account the definition of $G^{(2)}$ (4.10)–(4.12), $J^{(1)}$ (4.5)–(4.9) and the property $a^2(k)-b^2(k)=1$, we get

$$\begin{aligned}
J^{(2)}(\xi,t,k) &= J^{(1)}(\xi,t,k), && k\in(ic_l,id)\cup(-id,-ic_l),\\
&= e^{it(g_-(k,\xi)-g_+(k,\xi))\sigma_3} = e^{itB_g(\xi)\sigma_3}, && k\in(ic_r,id)\cup(-id,-ic_r),\\
&= \begin{pmatrix} 0 & i\\ i & 0\end{pmatrix} e^{it(g_-(k,\xi)+g_+(k,\xi))\sigma_3} = \begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}, && k\in(-ic_r,ic_r),\\
&= G^{(2)}(\xi,t,k), && k\in L_1,\\
&= \left(G^{(2)}\right)^{-1}(\xi,t,k), && k\in L_2.
\end{aligned}$$

5. Next transformation. The function $f(k)$ has the analytic continuation $\hat f(k) := \dfrac{1}{a(k)b(k)}$ from the intervals $(ic_l,ic_r)\cup(-ic_r,-ic_l)$. Thus we can factorize the jump matrix $J^{(2)}(\xi,t,k)$ on the intervals $(ic_l,id)\cup(-id,-ic_l)$ as follows:

$$\begin{aligned}
&J^{(2)}(\xi,t,k)\\
&= F_+^{-\sigma_3}(k,\xi)\begin{pmatrix} 1 & \dfrac{F_+^2(k,\xi)e^{-2itg_+(k,\xi)}}{\hat f_+(k)}\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}\\
&\times\begin{pmatrix} 1 & \dfrac{-F_-^2(k,\xi)e^{-2itg_-(k,\xi)}}{\hat f_-(k)}\\ 0 & 1\end{pmatrix} F_-^{\sigma_3}(k,\xi)\ , \quad k\in(ic_l,id),
\end{aligned}\tag{4.13}$$

$$\begin{aligned}
&= F_+^{-\sigma_3}(k,\xi)\begin{pmatrix} 1 & 0\\ \dfrac{e^{2itg_+(k,\xi)}}{F_+^2(k,\xi)\hat f_+(k)} & 1\end{pmatrix}\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}\\
&\times\begin{pmatrix} 1 & 0\\ \dfrac{-e^{2itg_-(k,\xi)}}{F_-^2(k,\xi)\hat f_-(k)} & 1\end{pmatrix} F_-^{\sigma_3}(k,\xi)\ , \quad k\in(-ic_l,-id).
\end{aligned}\tag{4.14}$$

Direct calculations show that the above is possible if:
- $F(k,\xi)$ is analytic outside the segment $[ic_l, -ic_l]$;
- $F(k,\xi)$ does not vanish in the complex plane with the cut along $[ic_l, -ic_l]$;
- $F(k,\xi)$ satisfies the jump relations

$$F_+(k,\xi)F_-(k,\xi) = \begin{cases} -if(k), & k \in (ic_l, id), \\ \dfrac{i}{f(k)}, & k \in (-id, -ic_l), \\ 1\ , & k \in (ic_r, -ic_r), \end{cases}$$

$$F_+(k,\xi) = F_-(k,\xi)e^{i\Delta(\xi)}\ , \quad k \in (id, ic_r) \cup (-ic_r, -id),$$

where $\Delta(\xi)$ is some function of ξ, which has to be determined;
- $F(k,\xi)$ is bounded at the infinity;
- $F(k,\xi)a(k)$ is bounded in a small neighborhood of the point ic_l;
- $F(k,\xi)a^{-1}(k)$ is bounded in a small neighborhood of the point $-ic_l$;
- $F(k,\xi)$ is bounded in small neighborhoods of the points $\pm id$, $\pm ic_r$, 0.

To solve this conjugation problem we use the function

$$\mathrm{w}(k) = \sqrt{(k^2+c_l^2)(k^2+d^2)(k^2+c_r^2)}.$$

Let us note that

$$-if(k) = \frac{1}{a_-(k)a_+(k)} > 0.$$

The jump relations on F can be rewritten in the form:

$$\left[\frac{\log F(k,\xi)}{\mathrm{w}(k,\xi)}\right]_+ - \left[\frac{\log F(k,\xi)}{\mathrm{w}(k,\xi)}\right]_- = \frac{-\log\left(a_-(k)a_+(k)\right)}{\mathrm{w}_+(k,\xi)}\ , \quad k \in (ic_l, id)\,,$$

$$\left[\frac{\log F(k,\xi)}{\mathrm{w}(k,\xi)}\right]_+ - \left[\frac{\log F(k,\xi)}{\mathrm{w}(k,\xi)}\right]_- = \frac{\log\left(a_-(k)a_+(k)\right)}{\mathrm{w}_+(k,\xi)}\ , \quad k \in (-id, -ic_l)\,,$$

$$\left[\frac{\log F(k,\xi)}{\mathrm{w}(k,\xi)}\right]_+ - \left[\frac{\log F(k,\xi)}{\mathrm{w}(k,\xi)}\right]_- = \frac{i\Delta(\xi)}{\mathrm{w}(k,\xi)}\ , \quad k \in (id, ic_r) \cup (-ic_r, -id).$$

The function

$$F(k,\xi) = \exp\left\{\frac{\mathrm{w}(k)}{2\pi i}\int\limits_{ic_l}^{id}\frac{-\log\left(a_+(s)a_-(s)ds\right)}{(s-k)\mathrm{w}_+(s)}\right\}\exp\left\{\frac{\mathrm{w}(k)}{2\pi i}\int\limits_{-id}^{-ic_l}\frac{\log\left(a_+(s)a_-(s)ds\right)}{(s-k)\mathrm{w}_+(s)}\right\}$$

$$\times\exp\left\{\frac{i\Delta(\xi)\mathrm{w}(k)}{2\pi i}\int\limits_{id}^{ic_r}\frac{ds}{(s-k)\mathrm{w}(s)}\right\}\exp\left\{\frac{i\Delta(\xi)\mathrm{w}(k)}{2\pi i}\int\limits_{-ic_r}^{-id}\frac{ds}{(s-k)\mathrm{w}(s)}\right\}$$

satisfies the first, the second and the third properties. To make $F(k,\xi)$ bounded at the infinity we have to expand $F(k,\xi)$ in series when $k \to \infty$. Since for the integer n

$$\int\limits_{ic_l}^{id} \frac{-s^n \log\left(a_+(s)a_-(s)\right) ds}{\mathrm{w}_+(s)} = (-1)^{n+1} \int\limits_{-id}^{-ic_l} \frac{s^n \log\left(a_+(s)a_-(s)\right) ds}{\mathrm{w}_+(s)} ,$$

$$\int\limits_{id}^{ic_r} \frac{s^n ds}{\mathrm{w}(s)} = (-1)^{n+1} \int\limits_{-ic_r}^{-id} \frac{s^n ds}{\mathrm{w}(s)} ,$$

then the behavior of $\log F(k,\xi)$ at the infinity is described by the asymptotic formula

$$\log F(k,\xi) = 2k \left(\frac{1}{2\pi i} \int\limits_{ic_l}^{id} \frac{s \log\left(a_+(s)a_-(s)\right) ds}{\mathrm{w}_+(s,\xi)} - \frac{-i\Delta(\xi)}{2\pi i} \int\limits_{id}^{ic_r} \frac{s ds}{\mathrm{w}(s,\xi)} \right) + o(1).$$

So, we put

$$\Delta(\xi) = -i \int\limits_{ic_l}^{id} \frac{s \log\left(a_+(s)a_-(s)ds\right)}{\mathrm{w}_+(s,\xi)} \left(\int\limits_{id}^{ic_r} \frac{s ds}{\mathrm{w}(s,\xi)} \right)^{-1} . \tag{4.15}$$

It is easy to check that $\Delta(\xi) \in \mathbb{R}$. Thus the function $F(k,\xi)$ satisfies all the requirements.

Using factorizations (4.13) and (4.14) and the transformation

$$M^{(3)}(\xi,t,k) = M^{(2)}(\xi,t,k) G^{(3)}(\xi,t,k) ,$$

where

$$\begin{aligned} G^{(3)}(\xi,t,k) &= F^{-\sigma_3}(k,\xi) \begin{pmatrix} 1 & \dfrac{F^2(k,\xi)e^{-2itg(k,\xi)}}{\hat{f}(k)} \\ 0 & 1 \end{pmatrix} , && k \in \Omega_5 \cup \Omega_7 , \\ &= F^{-\sigma_3}(k,\xi) \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2itg(k,\xi)}}{F^2(k,\xi)\hat{f}(k)} & 1 \end{pmatrix} , && k \in \Omega_6 \cup \Omega_8 , \\ &= F^{-\sigma_3} , && k \notin (\Omega_5 \cup \Omega_6 \cup \Omega_7 \cup \Omega_8) , \end{aligned}$$

we obtain the RH problem

$$M_-^{(3)}(\xi,t,k) = M_+^{(3)}(\xi,t,k) J^{(3)}(\xi,t,k), \quad k \in \Sigma_3, \quad M^{(3)}(\xi,t,k) \to I , \quad k \to \infty.$$

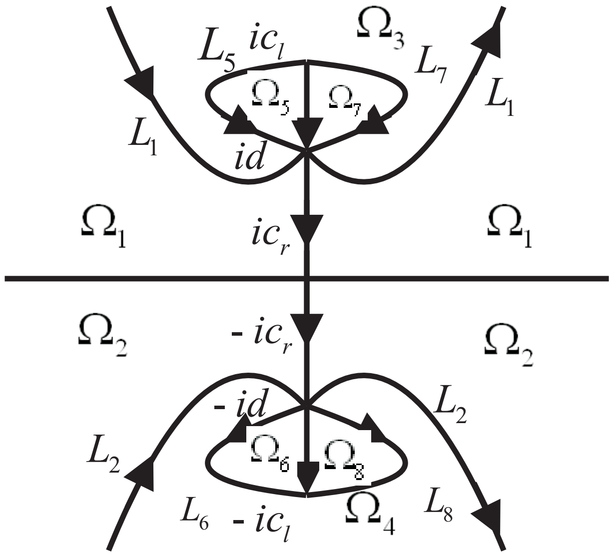

Fig. 4. The contour Σ_3.

The jump matrix $J^{(3)}(\xi,t,k) = \left(G_+^{(3)}\right)^{-1} J^{(2)}(\xi,t,k)G^{(3)}(\xi,t,k)$ is

$$J^{(3)}(\xi,t,k) = \begin{pmatrix} 1 & 0 \\ -r(k)F^{-2}(k,\xi)e^{2itg(k,\xi)} & 1 \end{pmatrix}, \qquad k \in L_1,$$

$$= \begin{pmatrix} 1 & r(k)F^{2}(k,\xi)e^{-2itg(k,\xi)} \\ 0 & 1 \end{pmatrix}, \qquad k \in L_2,$$

$$J^{(3)}(\xi,t,k) = \begin{pmatrix} 1 & \dfrac{F^2(k,\xi)e^{-2itg(k,\xi)}}{\hat f(k)} \\ 0 & 1 \end{pmatrix}, \qquad k \in L_7,$$

$$= \begin{pmatrix} 1 & \dfrac{-F^2(k,\xi)e^{-2itg(k,\xi)}}{\hat f(k)} \\ 0 & 1 \end{pmatrix}, \qquad k \in L_5,$$

$$J^{(3)}(\xi,t,k) = \begin{pmatrix} 1 & 0 \\ \dfrac{e^{2itg(k,\xi)}}{F^2(k,\xi)\hat f(k)} & 1 \end{pmatrix}, \qquad k \in L_8,$$

$$= \begin{pmatrix} 1 & 0 \\ \dfrac{-e^{2itg(k,\xi)}}{F^2(k,\xi)\hat f(k)} & 1 \end{pmatrix}, \qquad k \in L_6,$$

$$J^{(3)}(\xi,t,k) = e^{(itB_g(\xi)+i\Delta(\xi))\sigma_3}\,, \qquad k \in (id, ic_r) \cup (-ic_r, -id)\,,$$
$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\,, \qquad k \in (ic_l, id) \cup (ic_r, -ic_r) \cup (-id, -ic_l).$$

6. Model problem. Now we consider a model problem $M_-^{(mod)}(\xi,t,k) = M_+^{(mod)}(\xi,t,k)J^{(mod)}(\xi,t,k)$, $M^{(mod)}(\xi,t,k) \to I$ as $k \to \infty$, where

$$J^{(mod)}(\xi,t,k) = \begin{cases} \begin{pmatrix} e^{itB_g(\xi)+\mathrm{i}\Delta(\xi)} & 0 \\ 0 & e^{-itB_g(\xi)-i\Delta(\xi)} \end{pmatrix}, & k \in (id, ic_r) \cup (-ic_r, -id)\,, \\ \\ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & k \in (ic_l, id) \cup (ic_r, -ic_r) \cup (-id, -ic_l). \end{cases} \tag{4.16}$$

To solve the model problem (4.16), we introduce the Riemann surface X, which is given by

$$\mathrm{w}^2(k) = (k^2 + c_l^2)(k^2 + d^2)(k^2 + c_r^2).$$

We will use a realization of this algebraic curve as the two-sheet Riemann surface. The upper and lower sheets of the surface are two complex planes merged along the cuts $[ic_l, id]$, $[ic_r, -ic_r]$ and $[-id, -ic_l]$. On the upper sheet of this surface $\mathrm{w}(1) > 0$. The basis $\{\boldsymbol{a}_1, \boldsymbol{b}_1, \boldsymbol{a}_2, \boldsymbol{b}_2\}$ of cycles of this Riemann surface is as follows. The $\boldsymbol{a}_1$-cycle starts from the right-hand side of the cut $[ic_l, id]$ on the upper sheet, goes to the right-hand side of the cut $[ic_r, -ic_r]$, proceeds to the lower sheet and then returns to the starting point. The $\boldsymbol{b}_1$-cycle is a closed counter clock-wise oriented simple loop around the cut $[ic_l, id]$. The $\boldsymbol{a}_2$-cycle starts from the right-hand side of the cut $[id, -id]$, $[ic_r, -ic_r]$ on the upper sheet, goes to the right-hand side of the cut $[-id, -ic_l]$, proceeds to the lower sheet and then returns to the starting point. The $\boldsymbol{b}_2$-cycle is a closed counter clock-wise oriented simple loop around the segment $[ic_l, -ic_r]$. The basis

$$d\omega = \begin{pmatrix} d\omega_1 \\ d\omega_2 \end{pmatrix}$$

of the normalized holomorphic differentials on X has the form

$$d\omega_1 = \pi i \frac{kdk}{\mathrm{w}(k)} \left(\int_{\boldsymbol{a}_1} \frac{kdk}{\mathrm{w}(k)} \right)^{-1} + \pi i \frac{dk}{\mathrm{w}(k)} \left(\int_{\boldsymbol{a}_1} \frac{dk}{\mathrm{w}(k)} \right)^{-1},$$

$$d\omega_2 = \pi i \frac{kdk}{\mathrm{w}(k)} \left(\int\limits_{\boldsymbol{a}_1} \frac{kdk}{\mathrm{w}(k)} \right)^{-1} - \pi i \frac{dk}{\mathrm{w}(k)} \left(\int\limits_{\boldsymbol{a}_1} \frac{dk}{\mathrm{w}(k)} \right)^{-1} .$$

Then

$$\int\limits_{\boldsymbol{a}_j} d\omega_l = 2\pi i \delta_{jl}, \qquad B = B(\xi) = \left(\left(B_{jl} = \int\limits_{b_j} d\omega_l \right) \right) = \begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix},$$

where

$$B_1 := \int\limits_{\boldsymbol{b}_1} d\omega_1 \ , \quad B_2 := \int\limits_{\boldsymbol{b}_1} d\omega_2 \ , \text{ and } B_1 < B_2 < 0 \ ,$$

and theta function

$$\Theta(z) = \Theta(z|B(\xi)) = \sum_{m\in\mathbb{Z}^2} \exp\left\{ \frac{1}{2} \left(B(\xi)m, m\right) + (z, m) \right\} \ , \quad z \in \mathbb{C}^2,$$

has the property

$$\Theta(z + 2\pi i n + B(\xi)l) = \Theta(z) \exp\left\{ -\frac{1}{2} \left(B(\xi)l, l\right) - (z, l) \right\}, \quad n \in \mathbb{Z}^2, l \in \mathbb{Z}^2.$$

Now we introduce the Abel map on X

$$A : X \to \mathbb{C}^2 / \left(2\pi i \mathbb{Z}^2 + B(\xi)\mathbb{Z}^2\right) \ , \qquad A(P) = \int\limits_{ic_l}^{P} d\omega \tag{4.17}$$

and the functions $\varphi(k,\xi), \psi(k,\xi) : \{\text{the first sheet of the X}\} \to \mathbb{C}$

$$\begin{aligned} \varphi_j(k,\xi) &= \frac{\Theta(A(k) - A(D_j) - K - (itB_g(\xi) + i\Delta(\xi))\,(1,1)^T)}{\Theta(A(k) - A(D_j) - K)}, \\ \psi_j(k,\xi) &= \frac{\Theta(-A(k) - A(D_j) - K - (itB_g(\xi) + i\Delta(\xi))\,(1,1)^T)}{\Theta(-A(k) - A(D_j) - K)}, \end{aligned} \qquad j = 1, 2. \tag{4.18}$$

Here $D_1 = P_1 + P_2$ is the divisor consisting of two points on the lower sheet,

$$P_1 = i\sqrt{\frac{c_l c_r d}{c_l + c_r - d}} \qquad \text{and} \qquad P_2 = -i\sqrt{\frac{c_l c_r d}{c_l + c_r - d}},$$

$D_2 = \tau D_1$ lies on the upper sheet, and $A(D_1) = -A(D_2)$. The vector K is the Riemann constant of the surface X, $B_g(\xi)$ and $\Delta(\xi)$ are defined in (4.4) and

(4.15). The integration contour in (4.17) is taken from the upper sheet and it does not intersect the interval $(-ic_l, ic_l)$. The functions (4.18) have the following properties:

$$\left.\begin{aligned}\varphi_{j+}(k,\xi) &= \psi_{j-}(k,\xi),\\ \psi_{j+}(k,\xi) &= \varphi_{j-}(k,\xi),\end{aligned}\right. \qquad k \in (ic_l, id) \cup (ic_r, -ic_r) \cup (-id, -ic_l).$$

$$\left.\begin{aligned}\varphi_{j-}(k,\xi) &= \varphi_{j+}(k,\xi)e^{itB_g(\xi)+\mathrm{i}\Delta(\xi)},\\ \psi_{j-}(k,\xi) &= \psi_{j+}(k,\xi)e^{-itB_g(\xi)-\mathrm{i}\Delta(\xi)},\end{aligned}\right. \qquad k \in (id, ic_r) \cup (-ic_r, -id).$$

Define a function

$$\gamma(k) = \gamma(k,\xi) = \sqrt[4]{\frac{k-ic_l}{k-id}}\sqrt[4]{\frac{k-ic_r}{k+ic_r}}\sqrt[4]{\frac{k+id}{k+ic_l}},$$

which is analytic outside the union of segments $[ic_l, id] \cup [ic_r, -ic_r] \cup [-id, -ic_l]$ and satisfies the jump conditions

$$\gamma_-(k,\xi) = i\gamma_+(k,\xi), \qquad k \in [ic_l, id] \cup [ic_r, -ic_r] \cup [-id, -ic_l].$$

Then the solution of the model problem (4.16) can be written as follows:

$$M^{(mod)}(\xi,t,k) = \begin{pmatrix} M_{11}^{(mod)}(\xi,t,k) & M_{12}^{(mod)}(\xi,t,k) \\ M_{21}^{(mod)}(\xi,t,k) & M_{22}^{(mod)}(\xi,t,k) \end{pmatrix},$$

$$\begin{aligned}
M_{11}^{(mod)}(\xi,t,k) &= \frac{1}{2}\left(\gamma(k,\xi)+\frac{1}{\gamma(k,\xi)}\right)\frac{\varphi_1(k,\xi)}{\varphi_1(\infty,\xi)},\\
M_{12}^{(mod)}(\xi,t,k) &= \frac{1}{2}\left(\gamma(k,\xi)-\frac{1}{\gamma(k,\xi)}\right)\frac{\psi_1(k,\xi)}{\varphi_1(\infty,\xi)},\\
M_{21}^{(mod)}(\xi,t,k) &= \frac{1}{2}\left(\gamma(k,\xi)-\frac{1}{\gamma(k,\xi)}\right)\frac{\varphi_2(k,\xi)}{\psi_2(\infty,\xi)},\\
M_{22}^{(mod)}(\xi,t,k) &= \frac{1}{2}\left(\gamma(k,\xi)+\frac{1}{\gamma(k,\xi)}\right)\frac{\psi_2(k,\xi)}{\psi_2(\infty,\xi)}.
\end{aligned}$$

Then, by the formula (3.10),

$$\begin{aligned}
q_{mod}(x,t) :=& \lim_{k\to\infty} 2ik\left(M^{(mod)}\left(\frac{x}{12t},t,k\right)-I\right)_{21}\\
&= (c_l - d(\xi) + c_r)\frac{\Theta(A(\infty)+A(D_1)-K-(itB_g(\xi)+i\Delta(\xi))(1,1)^T)}{\Theta(A(\infty)+A(D_1)-K)}\\
&\quad\times\frac{\Theta(-A(\infty)+A(D_1)-K)}{\Theta(-A(\infty)+A(D_1)-K-(itB_g(\xi)+i\Delta(\xi))(1,1)^T)}.
\end{aligned}$$

Theorem 4.3. *Let $t \to \infty$. Then in the region $(12c_r^2 - 6c_l^2)t < x < (4c_l^2 + 2c_r^2)t$ the solution of the problem* (1.1)–(1.2) *with initial pure step function takes the form of a modulated hyper-elliptic wave*

$$q(x,t) = (c_l - d(\xi) + c_r)\frac{\Theta(A(\infty) + A(D_1) - K - (itB_g(\xi) + i\Delta(\xi))(1,1)^T)}{\Theta(A(\infty) + A(D_1) - K)}$$
$$\times \frac{\Theta(-A(\infty) + A(D_1) - K)}{\Theta(-A(\infty) + A(D_1) - K - (itB_g(\xi) + i\Delta(\xi))(1,1)^T)} + O(t^{-1/2}).$$

5. Appendix

Lemma. *Let $\rho(.)$, $h_1(.)$, $h_2(.)$ be positive functions on the segment $[a,b] \subset \mathbb{R}$ such that the integrals of their products ρ, $\rho(y)h_1(y)$, $\rho(y)h_2(y)$, $\rho(y)h_1(y)h_2(y)$ are convergent in proper or improper sense. Let also h_1 be an increasing function on the segment, and h_2 be a decreasing function. Then*

$$\int_{c_r}^{d} \rho(y)h_1(y)dy \int_{c_r}^{d} \rho(y)h_2(y)dy - \int_{c_r}^{d} \rho(y)h_1(y)h_2(y)dy \int_{c_r}^{d} \rho(y)dy \geq 0.$$

P r o o f. If all integrals are proper, then we can approximate them by partial sums. By substituting them into the input inequality instead of integrals, we obtain the inequality

$$\frac{1}{N^2}\sum_{n=1}^{N} \rho(y_n)h_1(y_n)\sum_{m=1}^{N} \rho(y_m)h_2(y_m) - \frac{1}{N^2}\sum_{n=1}^{N} \rho(y_n)h_1(y_n)h_2(y_n)\sum_{m=1}^{N} \rho(y_m) \geq 0.$$

Let us multiply both sides on N^2 and multiply the expressions in parenthesis

$$\sum_{n,\ m} \rho(y_n)h_1(y_n)\rho(y_m)h_2(y_m) - \sum_{n,\ m} \rho(y_n)\rho(y_m)h_1(y_n)h_2(y_n) \geq 0.$$

We can see that the terms of series, when $n = m$, disappear. Thus we obtain

$$\sum_{n<m} \rho(y_n)\rho(y_m)\left(h_1(y_n)h_2(y_m) + h_1(y_m)h_2(y_n)\right)$$
$$- \sum_{n<m} \rho(y_n)\rho(y_m)\left(h_1(y_n)h_2(y_n) + h_1(y_m)h_2(y_m)\right) \geq 0.$$

Then the above is equivalent to

$$\sum_{n<m} \rho(y_n)\rho(y_m)\left(h_1(y_n) - h_1(y_m)\right)\left(h_2(y_m) - h_2(y_n)\right) \geq 0,$$

which is evidently true.

If the integrals are improper, then we can approximate them by proper ones for which the lemma is proven.

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