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On Ideal Amenability of Banach Algebras

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Let $\mathfrak{A}$ be a Banach algebra. The Banach algebra $\mathfrak{A}$ is said to be ideally amenable if every continuous derivation from $\mathfrak{A}$ into $\mathcal{I}^*$ is inner, where $\mathcal{I}$ is a two-sided ideal of $\mathfrak{A}$. In this paper, we consider the ideal amenability of Banach algebras, and try to give some new results on the ideal amenability of Banach algebras and commutative Banach algebras.

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1. Introduction

Let $\mathfrak{A}$ be a Banach algebra. Let X be a Banach $\mathfrak{A}$-bimodule such that X is a Banach space and an $\mathfrak{A}$-bimodule, where the module operations

$$(a, x) \mapsto a.x \qquad \textit{and} \qquad (a, x) \mapsto x.a$$

from $\mathfrak{A} \times X$ into X are jointly continuous. We can define the right and the left actions of $\mathfrak{A}$ on the dual space X^* of X via

$$\langle x, \lambda.a\rangle = \langle a.x, \lambda\rangle \qquad \textit{and} \qquad \langle x, a.\lambda\rangle = \langle x.a, \lambda\rangle$$

for all $a \in \mathfrak{A}$, $x \in X$ and $\lambda \in X^*$. Similarly, the second dual X^{**} of X becomes a Banach $\mathfrak{A}$-bimodule (for more details see [1]). Thus, in particular, $\mathcal{I}$ is a Banach $\mathfrak{A}$-bimodule, and $\mathcal{I}^*$ is a dual $\mathfrak{A}$-bimodule for every closed two-sided ideal $\mathcal{I}$ in $\mathfrak{A}$. Let $\mathfrak{A}$ be a Banach algebra, and let X be a Banach $\mathfrak{A}$-bimodule. A derivation is a linear map $D : \mathfrak{A} \longrightarrow X$ such that

$$D(ab) = a.D(b) + D(a).b \qquad (a, b \in \mathfrak{A}).$$

The set of derivations from $\mathfrak{A}$ into X is denoted by $Z^1(\mathfrak{A}, X)$; it is a linear subspace of $\mathcal{L}(\mathfrak{A}, X)$, where $\mathcal{L}(\mathfrak{A}, X)$ is the space of all bounded linear mappings from $\mathfrak{A}$ into X. For $x \in X$, set $D_x : a \mapsto a.x - x.a, \mathfrak{A} \longrightarrow X$. Derivations of this form are termed inner derivations, and an inner derivation D_x is implemented by x; derivations which are not inner are called outer derivations. The set of inner derivations from $\mathfrak{A}$ into X is a linear subspace $N^1(\mathfrak{A}, X)$ of $Z^1(\mathfrak{A}, X)$. We consider the quotient space $\mathcal{H}^1(A, X) = Z^1(\mathfrak{A}, X)/N^1(\mathfrak{A}, X)$, it is called the first cohomology group of $\mathfrak{A}$ with coefficients in X. Clearly, $\mathcal{H}^1(A, X) = \{0\}$ if and only if every derivation from $\mathfrak{A}$ into X is inner.

The Banach algebra $\mathfrak{A}$ is called amenable if $\mathcal{H}^1(A, X^*) = \{0\}$, or, in other words, if every derivation from $\mathfrak{A}$ into every dual $\mathfrak{A}$-module is inner. The concept of amenability for the Banach algebra $\mathfrak{A}$ was introduced by Johnson in 1972 [2]. The Banach algebra $\mathfrak{A}$ is weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$. Of course, every amenable Banach algebra is weakly amenable. However, the class of weakly amenable Banach algebras is considerably larger than that of amenable Banach algebras. For example, the group algebra $L^1(G)$ is weakly amenable for each locally compact group G. The examples of weakly amenable, but not amenable, Banach function algebras are given in [3], where it is noted that the commutative Banach algebra $\mathfrak{A}$ is weakly amenable if and only if $\mathcal{H}^1(\mathfrak{A}, X) = \{0\}$ for each Banach $\mathfrak{A}$-module X.

Let $n \in \mathbb{N}$; the Banach algebra $\mathfrak{A}$ is called n-weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = \{0\}$. Dales, Ghahramani and Grønbæk brought the concept of the n-weak amenability of Banach algebras in [4]. The Banach algebra $\mathfrak{A}$ is called permanently weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = \{0\}$ for each $n \in \mathbb{N}$ (see [4]). The concept of the ideal amenability of Banach algebras was introduced by Gordji and Yazdanpanah in [5]. The Banach algebra $\mathfrak{A}$ is called ideally amenable if $\mathcal{H}^1(\mathfrak{A}, \mathcal{I}^*) = \{0\}$, and $\mathfrak{A}$ is n-$\mathcal{I}$-weakly amenable if $H^1(\mathfrak{A}, \mathcal{I}^{(n)}) = \{0\}$ for every closed two-sided ideal $\mathcal{I}$ of $\mathfrak{A}$. The ideal amenability of the group algebras $L^1(G)$, $L^\infty(G)$ and $M(G)$, where G is a locally compact group, are studied in [6]. In [7], the ideal amenability of abstract Segal algebras, Segal algebras and triangular Banach algebras are studied. In this paper, we continue to study [5–8] for the ideal amenability of Banach algebras.

2. General Results

For the Banach algebra $\mathfrak{A}$ we denote the character space of $\mathfrak{A}$ by $\Phi_{\mathfrak{A}}$.

Theorem 2.1. *Let $\mathfrak{A}$ be a Banach algebra, and $\varphi \in \Phi_{\mathfrak{A}}$ such that $ab = \varphi(a)b$ for each $a, b \in \mathfrak{A}$. Then $\mathfrak{A}$ is ideally amenable.*

P r o o f. Let $\mathcal{I}$ be a closed two-sided ideal of $\mathfrak{A}$, and let $D:\mathfrak{A}\longrightarrow\mathcal{I}^*$ be a continuous derivation. Then

$$\begin{aligned}\varphi(a)\langle c, Db\rangle &= \langle c, D(ab)\rangle = \langle c, a.D(b)+D(a).b\rangle\\ &= \langle ca, Db\rangle + \langle bc, Da\rangle\\ &= \varphi(c)\langle a, Db\rangle + \varphi(b)\langle c, Da\rangle \qquad (2.1)\end{aligned}$$

for each $a,b,c\in\mathfrak{A}$. Let $\lambda\in\mathcal{I}^*$, and let $\delta_\lambda:\mathfrak{A}\longrightarrow\mathcal{I}^*$ be the inner derivation specified by λ. Then

$$\begin{aligned}\langle b, \delta_\lambda(a)\rangle &= \langle b, a.\lambda-\lambda.a\rangle\\ &= \langle ba, \lambda\rangle - \langle ab, \lambda\rangle\\ &= \varphi(b)\langle a, \lambda\rangle - \varphi(a)\langle b, \lambda\rangle \qquad (2.2)\end{aligned}$$

for each $a,b\in\mathfrak{A}$. Choose $a_0\in\mathfrak{A}$ with $\varphi(a_0)=1$ and set $\lambda(a)=\langle a_0, Da\rangle$ for each $a\in\mathfrak{A}$. Then λ is a linear functional. By use of (2.1) and (2.2), we have

$$\begin{aligned}\langle b, \delta_\lambda(a)\rangle &= \varphi(b)\langle a, \lambda\rangle - \varphi(a)\langle b, \lambda\rangle\\ &= \varphi(b)\langle a_0, Da\rangle - \varphi(a)\langle a_0, Db\rangle\\ &= \varphi(a_0)\langle b, Da\rangle = \langle b, Da\rangle.\end{aligned}$$

Therefore, $D=\delta_\lambda$, and thus $\mathfrak{A}$ is $\mathcal{I}$-weakly amenable. ∎

In [5], Gordji and Yazdanpanah asked (*Question* 4.1): *If $\mathfrak{A}$ and $\mathfrak{B}$ are ideally amenable Banach algebras, then is $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ ideally amenable?* In [9], Mewomo answered to this question for a special case, when both $\mathfrak{A}$ and $\mathfrak{B}$ have a bounded approximate identity. In the theorem below we will give the answer to *Question* 4.1 of [5] in the following sense.

Theorem 2.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. Let $\varphi\in\Phi_{\mathfrak{A}}$, and $\psi\in\Phi_{\mathfrak{B}}$ such that*

$$ab=\varphi(a)b, \qquad cd=\psi(c)d$$

for each $a,b\in\mathfrak{A}$ and $c,d\in\mathfrak{B}$. Then $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is ideally amenable.

P r o o f. Without loss of generality, we suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are unital (see Proposition 1.14 of [5]). Let $e_{\mathfrak{A}}$ and $e_{\mathfrak{B}}$ be the unit elements of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Let $\mathcal{K}$ be a closed two-sided ideal of $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$, and let $D:\mathfrak{A}\widehat{\otimes}\mathfrak{B}\longrightarrow\mathcal{K}^*$ be a continuous derivation. For each $a,c,e\in\mathfrak{A}$ and $b,d,f\in\mathfrak{B}$ we have

$$\begin{aligned}\langle c\otimes d, D(ae\otimes bf)\rangle &= \langle c\otimes d, D((a\otimes b)(e\otimes f))\rangle\\ &= \langle c\otimes d, (a\otimes b).D(e\otimes f)\rangle + \langle c\otimes d, D(a\otimes b).(e\otimes f)\rangle\\ &= \langle ca\otimes db, D(e\otimes f)\rangle + \langle ec\otimes fd, D(a\otimes b)\rangle\\ &= \varphi(c)\psi(d)\langle a\otimes b, D(e\otimes f)\rangle + \varphi(e)\psi(f)\langle c\otimes d, D(a\otimes b)\rangle\\ &= \varphi(a)\psi(b)\langle c\otimes d, D(e\otimes f)\rangle. \qquad (2.3)\end{aligned}$$

Fix $b_0 \in \mathfrak{B}$ with $\psi(b_0) = 1$. Then from (2.3) we can write

$$\begin{aligned} \langle e_{\mathfrak{A}} \otimes e_{\mathfrak{B}}, D(a \otimes b)\rangle &= \langle e_{\mathfrak{A}} \otimes e_{\mathfrak{B}}, D((a \otimes b_0)(e_{\mathfrak{A}} \otimes b))\rangle \\ &= \varphi(a)\psi(b_0)\langle e_{\mathfrak{A}} \otimes e_{\mathfrak{B}}, D(e_{\mathfrak{A}} \otimes b)\rangle \\ &= \langle a \otimes b_0, D(e_{\mathfrak{A}} \otimes b)\rangle \end{aligned} \tag{2.4}$$

for each $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Hence, there exists $\lambda \in \mathcal{K}^*$ such that

$$\langle a \otimes b, \lambda\rangle = \langle a \otimes b_0, D(e_{\mathfrak{A}} \otimes b)\rangle \tag{2.5}$$

for each $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Let $\delta_\lambda : \mathfrak{A}\widehat{\otimes}\mathfrak{B} \longrightarrow \mathcal{K}^*$ be an inner derivation specified by λ. Take $a \in \mathfrak{A}$ and $b, c \in \mathfrak{B}$. Then

$$\begin{aligned} \langle a \otimes c, \delta_\lambda(e_{\mathfrak{A}} \otimes b)\rangle &= \langle a \otimes c, (e_{\mathfrak{A}} \otimes b).\lambda - \lambda.(e_{\mathfrak{A}} \otimes b)\rangle \\ &= \langle a \otimes c, (e_{\mathfrak{A}} \otimes b).\lambda\rangle - \langle a \otimes c, \lambda.(e_{\mathfrak{A}} \otimes b)\rangle \\ &= \langle a \otimes cb, \lambda\rangle - \langle a \otimes bc, \lambda\rangle \\ &= \langle a \otimes b_0, \psi(c)D(e_{\mathfrak{A}} \otimes b)\rangle - \langle a \otimes b_0, \psi(b)D(e_{\mathfrak{A}} \otimes c)\rangle \\ &= \langle a \otimes b_0, \psi(c)D(e_{\mathfrak{A}} \otimes b) - \psi(b)D(e_{\mathfrak{A}} \otimes c)\rangle \\ &= \langle a \otimes b_0, (e_{\mathfrak{A}} \otimes c).D(e_{\mathfrak{A}} \otimes b)\rangle \\ &= \langle a \otimes c, D(e_{\mathfrak{A}} \otimes b)\rangle. \end{aligned}$$

Therefore, $D(e_{\mathfrak{A}} \otimes b) = \delta_\lambda(e_{\mathfrak{A}} \otimes b)$ for each $b \in \mathfrak{B}$.

We claim that $D(a \otimes e_{\mathfrak{B}}) = \delta_\lambda(a \otimes e_{\mathfrak{B}})$ for each $a \in \mathfrak{A}$. Choose $a_0 \in \mathfrak{A}$ with $\varphi(a_0) = 1$. Then from (2.5) we can write

$$\begin{aligned} \langle a \otimes b, \lambda\rangle &= \langle a \otimes b_0, D(e_{\mathfrak{A}} \otimes b)\rangle \\ &= \varphi(a)\langle e_{\mathfrak{A}} \otimes e_{\mathfrak{B}}, D(e_{\mathfrak{A}} \otimes b)\rangle \\ &= \langle a_0 \otimes b, D(a \otimes e_{\mathfrak{B}})\rangle \end{aligned} \tag{2.6}$$

for each $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Now by (2.6), we have

$$\begin{aligned} \langle c \otimes b, \delta_\lambda(a \otimes e_{\mathfrak{B}})\rangle &= \langle c \otimes b, (a \otimes e_{\mathfrak{B}}).\lambda - \lambda.(a \otimes e_{\mathfrak{B}})\rangle \\ &= \langle c \otimes b, (a \otimes e_{\mathfrak{B}}).\lambda\rangle - \langle c \otimes b, \lambda.(a \otimes e_{\mathfrak{B}})\rangle \\ &= \langle ca \otimes b, \lambda\rangle - \langle ac \otimes b, \lambda\rangle \\ &= \langle a_0 \otimes b, \varphi(c)D(a \otimes e_{\mathfrak{B}})\rangle - \langle a \otimes b_0, \varphi(a)D(c \otimes e_{\mathfrak{B}})\rangle \\ &= \langle a_0 \otimes b, \varphi(c)D(a \otimes e_{\mathfrak{B}}) - \varphi(a)D(c \otimes e_{\mathfrak{B}})\rangle \\ &= \langle a_0 \otimes b, (c \otimes e_{\mathfrak{B}}).D(a \otimes e_{\mathfrak{B}})\rangle \\ &= \langle c \otimes b, D(a \otimes e_{\mathfrak{B}})\rangle. \end{aligned}$$

Hence, $D(a \otimes e_{\mathfrak{B}}) = \delta_\lambda(a \otimes e_{\mathfrak{B}})$ for each $a \in \mathfrak{A}$. Then for each $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ we have

$$\begin{aligned} D(a \otimes b) &= D((a \otimes e_{\mathfrak{B}})(e_{\mathfrak{A}} \otimes b)) \\ &= (a \otimes e_{\mathfrak{B}}).D(e_{\mathfrak{A}} \otimes b) + D(a \otimes e_{\mathfrak{B}}).(e_{\mathfrak{A}} \otimes b) \\ &= (a \otimes b).\lambda - \lambda.(a \otimes b) = \delta_\lambda(a \otimes b). \end{aligned}$$

So, $D = \delta_\lambda$ on $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$. Thus $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ is ideally amenable. ∎

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and $\vartheta : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a bounded homomorphism. $\mathfrak{B}^{(n)}$ (the nth dual of $\mathfrak{B}$) can be regarded as an $\mathfrak{A}$-bimodule under the module actions

$$a.b^{(n)} = \vartheta(a).b^{(n)}, \quad b^{(n)}.a = b^{(n)}.\vartheta(a) \qquad (a \in \mathfrak{A}, b^{(n)} \in \mathfrak{B}^{(n)}).$$

Also the first and second transposes of ϑ, $\vartheta^* : \mathfrak{B}^* \longrightarrow \mathfrak{A}^*$ and $\vartheta^* : \mathfrak{A}^{**} \longrightarrow \mathfrak{B}^{**}$ are $\mathfrak{B}$-module morphisms. We can continue this to the nth transpose of ϑ.

Theorem 2.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are two-sided ideals of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Let $\mathcal{F} : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a bounded epimorphism, $\theta : \mathcal{I} \longrightarrow \mathcal{J}$ and $\varphi : \mathcal{J} \longrightarrow \mathcal{I}$ be bounded homomorphisms such that $\theta \circ \varphi = id_{\mathcal{J}}$ ($id_{\mathcal{J}}$ is the identity on $\mathcal{J}$). Then:*

(i) *Let $D : \mathfrak{B} \longrightarrow \mathcal{J}^{(2n-1)}$ be a continuous derivation. Then $\mathcal{D} := (\theta^{(2n-1)} \circ D \circ \mathcal{F}) : \mathfrak{A} \longrightarrow \mathcal{I}^{(2n-1)}$ is a continuous derivation, $n \in \mathbb{N}$.*

(ii) *Let $D : \mathfrak{B} \longrightarrow \mathcal{J}^{(2n)}$ be a continuous derivation. Then $\mathcal{D} := (\varphi^{(2n)} \circ D \circ \mathcal{F}) : \mathfrak{A} \longrightarrow \mathcal{I}^{(2n)}$ is a continuous derivation, $n \in \mathbb{N}$.*

(iii) *In cases* (i) *and* (ii)*, if $\mathcal{D}$ is inner, then D is also inner.*

(iv) *If $\mathfrak{A}$ is n-ideally amenable, then $\mathfrak{B}$ is also n-ideally amenable.*

P r o o f. (i) Let D be a continuous derivation. For each $a, b \in \mathfrak{A}$ we have

$$\begin{aligned} \mathcal{D}(ab) &= (\theta^{(2n-1)} \circ D \circ \mathcal{F})(ab) \\ &= \theta^{(2n-1)} \circ D(\mathcal{F}(a)\mathcal{F}(b)) \\ &= \theta^{(2n-1)}(a.D(\mathcal{F}(b)) + D(\mathcal{F}(a)).b) \\ &= a.\theta^{(2n-1)}(D(\mathcal{F}(b)) + \theta^{(2n-1)}(D(\mathcal{F}(a)).b \\ &= a.\mathcal{D}(b) + \mathcal{D}(a).b. \end{aligned} \tag{2.7}$$

Let $\mathcal{D} = \mathcal{D}_\lambda$ be an inner derivation from $\mathfrak{A}$ into $\mathcal{I}^{(2n-1)}$ specified by $\lambda \in \mathcal{I}^{(2n-1)}$. Take $b \in \mathfrak{B}$, therefore there exists $a \in \mathfrak{A}$ such that $\mathcal{F}(a) = b$. For each

$G \in \mathcal{J}^{(2n-1)}$ we have

$$\begin{aligned}
\langle D(b), G\rangle &= \langle D(\mathcal{F}(a)), G\rangle = \langle D(\mathcal{F}(a)), \theta^{(2n-2)} \circ \varphi^{(2n-2)}(G)\rangle \\
&= \langle \theta^{(2n-1)} \circ D(\mathcal{F}(a)), \varphi^{(2n-2)}(G)\rangle \\
&= \langle \mathcal{D}(a), \varphi^{(2n-2)}(G)\rangle = \langle a.\lambda - \lambda.a, \varphi^{(2n-2)}(G)\rangle \\
&= \langle \varphi^{(2n-1)}(a.\lambda - \lambda.a), G\rangle \\
&= \langle b.\varphi^{(2n-1)}(\lambda) - \varphi^{(2n-1)}(\lambda).b, G\rangle. \qquad (2.8)
\end{aligned}$$

By similar arguments, statements (ii) and (iii) hold. Part (iv) follows trivially from (iii). ∎

3. Commutative Banach Algebras

The ideal amenability of commutative Banach algebras is studied in [8], and the authors referred to obtained many useful results for this case. In this section, we study the ideal amenability of commutative Banach algebras and special classes of them, namely, ℓ^1-convolution and Lipschitz algebras.

Theorem 3.1. *Let $\mathfrak{A}$ be a commutative Banach algebra. If there exists a closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that*

(i) *$\mathfrak{B}$ is dense in $\mathfrak{A}$;*

(ii) *every derivation from $\mathfrak{B}$ into the dual of every closed ideal of $\mathfrak{A}$ is inner,*

then $\mathfrak{A}$ is ideally amenable.

P r o o f. Let $\mathcal{I}$ be a closed two-sided ideal of $\mathfrak{A}$, and let D be a continuous derivation from $\mathfrak{A}$ into $\mathcal{I}^*$. Define $D|_{\mathfrak{B}} = D'$. Since every derivation from $\mathfrak{B}$ into the dual of every closed ideal of $\mathfrak{A}$ is inner and $\mathcal{I}^*$ is a symmetric $\mathfrak{B}$-bimodule, then by [3], $D' = 0$.

Set $a \in \mathfrak{A}$. Then for each $\varepsilon > 0$ there exists $b \in \mathfrak{B}$ such that $\|a-b\|_{\mathfrak{A}} < \frac{\varepsilon}{\|D\|+1}$. Then we have

$$\begin{aligned}
\|Da\| &= \|Da - D'b\| = \|D(a-b)\| \\
&\leq \|D\|\|a-b\|_{\mathfrak{A}} < \varepsilon.
\end{aligned}$$

Since $a \in \mathfrak{A}$ and $\varepsilon > 0$ are arbitrary, then $D = 0$. Thus $\mathfrak{A}$ is $\mathcal{I}$-weakly amenable, and the proof is complete. ∎

Theorem 3.2. *Let $\mathfrak{A}$ be a commutative Banach algebra, and $\mathcal{I}$ be an unital two-sided ideal of $\mathfrak{A}$. If $\mathfrak{A}$ is $\mathcal{I}$-weakly amenable, then the closed linear span of $\mathfrak{A}\mathcal{I}\mathfrak{A}$ is dense in $\mathcal{I}$.*

P r o o f. Let $\mathfrak{B}$ be a closed linear span of $\mathfrak{A}\mathcal{I}\mathfrak{A}$ such that $\mathfrak{B} \neq \mathcal{I}$. Choose $a \in \mathcal{I}$ such that $a \notin \mathfrak{B}$. Put $\mathfrak{B}_1 := \{b + a'a : b \in \mathfrak{B},\ a' \in \mathcal{I},\ a' \notin \mathfrak{B}\}$. Define $\varphi : \mathfrak{B}_1 \longrightarrow \mathcal{I}$ by $\varphi(b + a'a) = a'$. Thus, φ is a homomorphism. We can also extend φ to a homomorphism $\lambda_1 : \mathcal{I} \longrightarrow \mathcal{I}$, where $\lambda_1|_{\mathfrak{B}} = 0$ and $\lambda_1 \neq 0$.

Since $a \in \mathcal{I}$ and $a \notin \mathfrak{B}$, then there exists nonzero $\lambda_2 \in \mathcal{I}^*$ such that $\lambda_2|_{\mathfrak{B}} = 0$. Now define the mapping $D : \mathfrak{A} \longrightarrow \mathcal{I}^*$ by

$$D(a) = \lambda_1(a).\lambda_2$$

for all $a \in \mathfrak{A}$. From the definition of λ_1 and λ_2, D is linear. Since λ_2 on $\mathfrak{B}$ is zero, then D is a nonzero derivation from $\mathfrak{A}$ into $\mathcal{I}^*$. But $\mathfrak{A}$ is $\mathcal{I}$-weakly amenable, consequently, this is a contradiction. ∎

Let Λ be a non-empty totally ordered set, and regard it as a semigroup by defining the product of two elements to be their maximum. The resulting semigroup, which we denote by $\Lambda_\vee$, is a semilattice. We may then form the ℓ^1-convolution algebra $\ell^1(\Lambda_\vee)$. For every $t \in \Lambda_\vee$ we denote the point mass concentrated at t by e_t. The definition of multiplication in $\ell^1(\Lambda_\vee)$ ensures that $e_s e_t = e_{\max(s,t)}$ for all s and t.

The semilattice $\Lambda_\vee$ is a commutative semigroup in which every element is idempotent. If we denote the set of idempotent elements of $\Lambda_\vee$ by $E(\Lambda_\vee)$, then $E(\Lambda_\vee) = \Lambda_\vee$. The ℓ^1-convolution algebras of semilattices provide interesting examples of commutative Banach algebras.

Proposition 3.3. *Let Λ be a totally ordered set. Then $\ell^1(\Lambda_\vee)$ is n-ideally amenable, $n \in \mathbb{N}$.*

P r o o f. Since $E(\Lambda_\vee) = \Lambda_\vee$, then by Proposition 2.2 of [10], $\ell^1(\Lambda_\vee)$ is weakly amenable. The Banach algebra $\ell^1(\Lambda_\vee)$ is a commutative Banach algebra. Then by Theorem 2.1 of [8], the weak amenability of $\ell^1(\Lambda_\vee)$ implies its n-ideal amenability. ∎

Let K be a compact metric space with metric d, and take α such that $0 < \alpha \leq 1$. Then $Lip_\alpha K$ is a space of the complex-valued functions f on K such that

$$p_\alpha(f) = \sup\{\frac{|f(x) - f(y)|}{d(x,y)^\alpha} : x, y \in K, x \neq y\}$$

is finite. For $f \in Lip_\alpha K$, set

$$\|f\|_\alpha = |f|_K + p_\alpha(f).$$

Then $(Lip_\alpha K, \|f\|_\alpha)$ is a Banach algebra on K. A function $f \in lip_\alpha K$ if

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \longrightarrow 0 \quad \textit{as } d(x,y) \longrightarrow 0,$$

where $lip_\alpha K$ is a close subalgebra of $Lip_\alpha K$ (for more details see [11] and [12]). Many results on the amenability and weak amenability of Lipschitz algebras are given in [3].

Proposition 3.4. *Let K be an infinite compact metric space, and let $\alpha \in (0,1)$. Then $lip_\alpha K$ is 2-ideally amenable.*

P r o o f. The algebra $lip_\alpha K$ is Arens regular, and $(lip_\alpha K)^{**}$ is semisimple, and so by Corollary 2.4 of [8], $lip_\alpha K$ is 2-ideally amenable. ■

In the following theorem $\mathbb{T}$ is a group of complex numbers of modulus one,

$$\mathbb{T} = \{z \in \mathbb{C} : \quad |z| = 1\},$$

such that it is isomorphic to $\mathbb{R}/\mathbb{Z}$, and $\mathbb{I}$ is the closed interval $[0,1]$.

Proposition 3.5. *Let K be a compact metric space, and let $\alpha \in (0,1)$. Then*

(i) *If K is $\mathbb{T}$, and $\alpha > \frac{1}{2}$, then $lip_\alpha \mathbb{T}$ is not ideally amenable. Furthermore, $lip_\alpha \mathbb{T}$ is not $2k+1$-ideally amenable for any $k \in \mathbb{Z}^+$.*

(ii) *If K is $\mathbb{I}$, then $Lip_\alpha \mathbb{I}$ is not ideally amenable.*

P r o o f. (i) By Theorem 3.11 of [3], $lip_\alpha \mathbb{T}$ is not weakly amenable, and by Theorem 2.1 of [8], $lip_\alpha \mathbb{T}$ is not ideally amenable. (ii) By Proposition 9.2 of [11], there are nonzero continuous point derivations on $Lip_\alpha \mathbb{I}$, and by Proposition 1.3 of [4], $Lip_\alpha \mathbb{I}$ is not weakly amenable, then by theorem 2.1 of [8], $Lip_\alpha \mathbb{I}$ is not ideally amenable. ■

References

[1] *F.F. Bonsall and J. Duncan*, Complete Normed Algebras. Springer–Verlag, Berlin, 1973.

[2] *B.E. Johnson*, Cohomology in Banach Algebras. — *Mem. Amer. Math. Soc.* **127**, Providence, R.I., USA, 1972.

[3] *W.G. Bade, P.C. Curtis, and H.G. Dales*, Amenability and Weak Amenability for Beurling and Lipschitz Algebras. — *Proc. London Math. Soc.* **55** (1987), No. 2, 359–377.

[4] *H.G. Dales, F. Ghahramani, and N. Grønbæk*, Derivations into Iterated Duals of Banach Algebras. — *Studia Math.* **128** (1998), No. 1, 19–54.

[5] *M.E. Gordji and T. Yazdanpanah*, Derivations into Duals of Ideals of Banach Algebras. — *Proc. Indian Acad. Sci.* **114** (2004), No. 4, 339–403.

[6] *M.E. Gordji and S.A.R. Hosseiniun*, Ideal Amenability of Banach Algebras on Locally Compact Groups. — *Proc. Indian Acad. Sci.* **115** (2005), No. 3, 319–325.

[7] *A. Jabbari*, Ideal Amenability of Various Classes of Banach Algebras. — *Acta Math. Vietnam* **36** (2011), No. 3, 651–658.

[8] *M.E. Gordji and R. Memarbashi*, Derivations into n-th Duals of Ideals of Banach Algebras. — *Bull. Iran. Math. Soc.* **34** (2008), No. 1, 59–71.

[9] *O.T. Mewomo*, On Ideal Amenability in Banach Algebras. — *An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.).* **56** (2010), No. 2, 273–278.

[10] *H.G. Dales, A.T.M. Lau, and D. Strauss*, Banach Algebras on Semigroups and their Compactifications. —- *Mem. Amer. Math. Soc.* **205** (2010), No. 966.

[11] *D.R. Sherbert*, The Structure of Ideals and Point Derivations in Banach Algebras of Lipschitz Functions. — *Trans. Amer. Math. Soc.* **111** (1964), 240–272.

[12] *N. Weaver*, Lipschitz Algebras. World Scientific Publishing Co. Pte Ltd., 1999.