

The Truncated Fourier Operator. General Results

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Let \mathcal{F} be the one dimensional Fourier–Plancherel operator and E be a subset of the real axis. The truncated Fourier operator is the operator \mathcal{F}_E of the form $\mathcal{F}_E = P_E \mathcal{F} P_E$, where $(P_E x)(t) = \mathbb{I}_E(t)x(t)$, and $\mathbb{I}_E(t)$ is the indicator function of the set E . In the presented work, the basic properties of the operator \mathcal{F}_E according to the set E are discussed. Among these properties there are the following ones:

- 1) the operator \mathcal{F}_E has a nontrivial null-space;
- 2) \mathcal{F}_E is strictly contractive;
- 3) \mathcal{F}_E is a normal operator;
- 4) \mathcal{F}_E is a Hilbert–Schmidt operator;
- 5) \mathcal{F}_E is a trace class operator.

Key words: truncated Fourier operator, normal operator, contractive operator, Hilbert–Schmidt operator, trace class operator.

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Let E be a measurable subset of the real axis \mathbb{R} . (The case $E = \mathbb{R}$ is not excluded). For $p \geq 1$, let $L^p(E)$ be the space of complex valued functions on E satisfying the condition $\int_E |x(t)|^p dt < \infty$. We mainly deal with the case $p = 2$, but episodically the case $p = 1$ is needed. The space $L^2(E)$, provided by the standard linear operations and the scalar product $\langle x, y \rangle_E$

$$\langle x, y \rangle_E = \int_E x(t) \overline{y(t)} dt, \quad x, y \in L^2(E), \quad (1)$$

is a Hilbert space. The norm in $L^2(E)$ is

$$\|x\|_E = \sqrt{\langle x, x \rangle_E}. \quad (2)$$

The Fourier operator \mathcal{F} is defined by the formula

$$(\mathcal{F}x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\xi} x(\xi) d\xi, \quad t \in \mathbb{R}. \quad (3)$$

One of central facts of the Fourier transform theory is the Parseval equality

$$\|(\mathcal{F}x)\|_{\mathbb{R}}^2 = \|x\|_{\mathbb{R}}^2, \quad \forall x \in L^2(\mathbb{R}). \quad (4)$$

This means that the Fourier operator \mathcal{F} is an isometric operator in $L^2(\mathbb{R})$. The next central fact of the Fourier transform theory is that the Fourier operator \mathcal{F} maps the space $L^2(\mathbb{R})$ onto the whole space $L^2(\mathbb{R})$, that is the \mathcal{F} is an *unitary operator* in $L^2(\mathbb{R})$. Moreover, the inverse operator \mathcal{F}^{-1} is determined by the formula

$$(\mathcal{F}^{-1}x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\xi} x(\xi) d\xi, \quad t \in \mathbb{R}. \quad (5)$$

R e m a r k 1. The integral in the right-hand side of (3) is a Lebesgue integral. It is well defined only if $x \in L^1(\mathbb{R})$. If $x \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then both the integral is well defined and the Parseval equality (4) holds. Thus, the operator \mathcal{F} can be defined originally by (3) only for $x \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The set of such x is dense in $L^2(\mathbb{R})$. Since the operator \mathcal{F} acts isometrically on this set, it can be extended by the continuity on the whole space $L^2(\mathbb{R})$. The same is related to the operator which appears in (5).

In this paper we deal with the truncated Fourier operator.

Definition 1. Let E be a measurable subset of the real axis, $0 < m(E) \leq \infty$. The operator $\mathcal{F}_E : L^2(E) \rightarrow L^2(E)$, is defined as

$$(\mathcal{F}_E x)(t) = \frac{1}{\sqrt{2\pi}} \int_E e^{it\xi} x(\xi) d\xi, \quad t \in E. \quad (6)$$

The operator $\mathcal{F}_E x$ is said to be the truncated Fourier operator, or in more detail, the Fourier operator truncated on the set E .

R e m a r k 2. If the set E is a set of finite Lebesgue measure: $\int_E dt < \infty$, then $L^2(E) \subset L^1(E)$ and the integral in (6) is well defined for every $x \in L^2(E)$.

The operator \mathcal{F}_E^* , which is the adjoint operator to \mathcal{F}_E with respect to the scalar product (1), is

$$(\mathcal{F}_E^* x)(t) = \frac{1}{\sqrt{2\pi}} \int_E e^{-it\xi} x(\xi) d\xi, \quad t \in E. \quad (7)$$

R e m a r k 3. The operator \mathcal{F}_E , acting in $L^2(E)$, may be naturally identified with the operator $P_E \mathcal{F} P_E$, acting in $L^2(\mathbb{R})$, where

$$(P_E x)(t) = \mathbb{I}_E(t)x(t), \tag{8}$$

$$\mathbb{I}_E(t) = \begin{cases} 1, & t \in E, \\ 0, & t \notin E. \end{cases} \tag{9}$$

Lemma 1. For any E , the operator \mathcal{F}_E is a contractive operator in $L^2(E)$

$$\|\mathcal{F}_E x\|_E^2 \leq \|x\|_E^2 \quad (\forall x \in L^2(E)). \tag{10}$$

P r o o f. Indeed, if

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_E e^{it\xi} x(\xi) d\xi, \quad (t \in \mathbb{R}), \tag{11}$$

then by the Parseval equality, $\int_{\mathbb{R}} |y(t)|^2 = \int_E |x(t)|^2$, hence $\int_E |y(t)|^2 \leq \int_E |x(t)|^2$. ■

Thus, the inequalities hold

$$0 \leq \mathcal{F}_E^* \mathcal{F}_E \leq I_E \text{ and } 0 \leq \mathcal{F} \mathcal{F}^* \leq I_E. \tag{12}$$

Here and in what follows I_E is the identity operator.

Theorem 1.

1. If $\text{mes } E > 0$, then no-one of the operators $\mathcal{F}_E = 0$ and $\mathcal{F}_E^* = 0$ equals zero: there exists $x \in L^2(E)$ for which both

$$\mathcal{F}_E x \neq 0 \quad \text{and} \quad \mathcal{F}_E^* x \neq 0. \tag{13}$$

2. If $\text{mes}(\mathbb{R} \setminus E) > 0$, then no-one of the operators \mathcal{F}_E and \mathcal{F}_E^* is isometric: there exists $x \in L^2(E)$ for which both

$$\|\mathcal{F}_E x\| < \|x\| \quad \text{and} \quad \|\mathcal{F}_E^* x\| < \|x\|. \tag{14}$$

P r o o f. Let $t_0 \in E$ is such that

$$\lim_{n \rightarrow \infty} \frac{m(E \cup [t_0 - 1/n, t_0 + 1/n])}{m([t_0 - 1/n, t_0 + 1/n])} = 1.$$

(Almost every point $t_0 \in E$ possesses this property). Let us set

$$x_n(\xi) = \frac{1}{m([t_0 - 1/n, t_0 + 1/n])} \begin{cases} 1, & \xi \in E \cap [t_0 - 1/n, t_0 + 1/n] \\ 0, & \xi \in E \setminus [t_0 - 1/n, t_0 + 1/n] \end{cases}.$$

It is clear that $x_n \in L^2(E) \forall n$, and that

$$\int_E x_n(\xi) e^{\pm i t \xi} d\xi \rightarrow e^{\pm i t t_0} \text{ as } n \rightarrow \infty, \text{ the limit is locally uniform on } \mathbb{R}.$$

If n is large enough, then $\int_{\mathbb{R} \setminus E} |\mathcal{F}x_n(t)|^2 dt > 0$, $\int_{\mathbb{R} \setminus E} |(\mathcal{F}^*x_n)(t)|^2 dt > 0$. Because x_n vanishes outside of E , $\int_E |x_n(t)|^2 dt = \int_{\mathbb{R}} |x_n(t)|^2 dt$. By the Parseval equality, $\int_{\mathbb{R}} |\mathcal{F}x_n(t)|^2 dt = \int_{\mathbb{R}} |(\mathcal{F}^*x_n)(t)|^2 dt = \int_E |x_n(t)|^2 dt$. Thus, the inequalities (14) hold for $x = x_n$ if n is large enough. ■

It is clear that if the set E is bounded, then the inequalities (13), (14) hold for any $x \in L^2(E)$, $x \neq 0$. Indeed, given $x \in L^2(E)$, $x \neq 0$, let y be determined from x according to (11). Since the set E is bounded, $y(t)$ is an entire function of t . Therefore the function $y(t)$ may vanish only in isolated points. In particular, $\int_E |y(t)|^2 dt > 0$, $\int_{\mathbb{R} \setminus E} |y(t)|^2 dt > 0$. The first inequality means that $\|\mathcal{F}x\| > 0$, the second one — that $\|\mathcal{F}x\| < \|x\|$. In [1, Prop. 5] it was shown that if $\text{mes } E < \infty$ (the set E may be unbounded), then the inequalities (14) hold for arbitrary $x \in L^2(E)$, $x \neq 0$.

Actually, the much more stronger statement takes place.

Theorem 2. *If $\text{mes } E < \infty$, then the inequalities*

$$\|\mathcal{F}_E x\|^2 \leq (1 - A^{-1} e^{-A(\text{mes } E)^2}) \|x\|^2, \quad \|\mathcal{F}_E^* x\|^2 \leq (1 - A^{-1} e^{-A(\text{mes } E)^2}) \|x\|^2 \tag{15}$$

hold for every $x \in L^2(E)$. Here A , $1 \leq A < \infty$, is an absolute constant: depends neither on x , nor on E .

P r o o f. In fact, Theorem 2 is a special case of the *Nazarov uncertainty principle*. In [2], F.L. Nazarov proved the remarkable inequality

$$\int_{\mathbb{R}} |y(t)|^2 dt \leq A e^{A(\text{mes } E)(\text{mes } F)} \left(\int_{\mathbb{R} \setminus E} |y(t)|^2 dt + \int_{\mathbb{R} \setminus F} |x(\xi)|^2 d\xi \right), \tag{16}$$

where $x \in L^2(\mathbb{R})$ is an arbitrary functions, y is the Fourier transform of x : $y(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it\xi} x(\xi) d\xi$, E and F are arbitrary measurable subsets of \mathbb{R} . If $F = E$ and

x is an arbitrary function vanishing outside of E , then the inequality (16) takes the form

$$\int_{\mathbb{R}} |y(t)|^2 dt \leq A e^{A(\text{mes } E)^2} \left(\int_{\mathbb{R}} |y(t)|^2 dt - \int_{\mathbb{E}} |y(t)|^2 dt \right).$$

Invoking the Parseval identity, we rewrite this inequality in the form

$$\int_{\mathbb{E}} |x(t)|^2 dt \leq A e^{A(\text{mes } E)^2} \left(\int_{\mathbb{E}} |x(t)|^2 dt - \int_{\mathbb{E}} |y(t)|^2 dt \right).$$

The latter inequality coincides with the first of the inequalities (15). ■

R e m a r k 4. As it is stated below, the equality for the Hilbert–Schmidt norm $\|\mathcal{F}_E\|_{\mathfrak{S}_2}$ holds $\|\mathcal{F}_E\|_{\mathfrak{S}_2} = \text{mes } E$. Since the Hilbert–Schmidt norm majorizes the operator norm, the inequalities hold

$$\|\mathcal{F}_E x\| \leq (\text{mes } E)\|x\|, \quad \|\mathcal{F}_E^* x\| \leq (\text{mes } E)\|x\|, \quad (17)$$

where E is an arbitrary measurable set and x is an arbitrary function from $L^2(E)$.

Both inequalities (15) and (17) are true. However, (17) is more precise for small values of $\text{mes } E$, and (15) — for large ones.

R e m a r k 5. If the set E is not just of a set of finite measure, but a finite interval, then the estimate (15) can be refined for large values of $\text{mes } E$. For $E = [-l, l]$, the largest eigenvalue $\lambda_0(l)$ of the operator $\mathcal{F}_E^* \mathcal{F}_E$ coincides with the squares of norm $\|\mathcal{F}_E\|^2$. For $E = [-l, l]$, the operator $\mathcal{F}_E^* \mathcal{F}_E$ is the integral operator in $L^2([-l, l])$ of the form

$$(\mathcal{F}_E^* \mathcal{F}_E x)(t) = \frac{1}{\pi} \int_{[-l, l]} \frac{\sin l(t - \tau)}{t - \tau} x(\tau) d\tau. \quad (18)$$

The asymptotic behavior as $l \rightarrow \infty$ of the eigenvalue $\lambda_0(l)$ of the integral operator (18) was found by W.H.J. Fuchs in [3]

$$1 - \lambda_0(l) \approx 4\sqrt{\pi} l^{1/2} e^{-2l} (= 2\sqrt{2\pi}(\text{mes } E)^{1/2} e^{-\text{mes } E}).$$

Thus, for $E = [-l, l]$, the estimate holds, which is stronger than the estimate (15),

$$\|\mathcal{F}_E x\| \leq (1 - A(\varepsilon)e^{-(1+\varepsilon)\text{mes } E})\|x\|, \quad \|\mathcal{F}_E^* x\| \leq (1 - A(\varepsilon)e^{-(1+\varepsilon)\text{mes } E})\|x\|, \quad (19)$$

for every $x \in L^2(E)$. Here $\varepsilon > 0$ is arbitrary, and $A(\varepsilon) < \infty$ for any $\varepsilon > 0$. The value $A(\varepsilon)$ does not depend on l and x .

Theorem 1 claims that if the set E is bounded, then the null-spaces of each of operators $\mathcal{F}_E^* \mathcal{F}_E$, $\mathcal{F}_E \mathcal{F}_E^*$ are trivial — they consist of zero-vectors only.

Theorem 2 implies that if $\text{mes } E < \infty$, then the null-spaces of each of operators $I_E - \mathcal{F}_E^* \mathcal{F}_E$, $I - \mathcal{F}_E \mathcal{F}_E^*$ are trivial.

The following example shows that if $\text{mes } E = \infty$, then each of these null-spaces can be not only nontrivial, but even infinite dimensional.

Example 1. Let $K \subset \mathbb{R}$ be the interval

$$K = [-a, a], \text{ where } 0 < a < \sqrt{\frac{\pi}{2}}. \quad (20)$$

The set E is a "periodic" system of the intervals

$$E = \bigcup_{p \in \mathbb{Z}} (K + p\sqrt{2\pi}). \quad (21)$$

Let $u(t) \neq 0$ be a (smooth) function on \mathbb{R} such that

$$\text{supp } u \subseteq K. \quad (22)$$

The function $u(t)$ is representable in the form

$$u(t) = \int_{-\infty}^{\infty} e^{it\xi} v(\xi) d\xi, \quad (23)$$

where $v(\xi)$ is a fast decaying function. Let

$$y(t) = \sum_{p \in \mathbb{Z}} c_p u(t + p\sqrt{2\pi}), \quad (24)$$

where $\{c_p\}_{p \in \mathbb{Z}}$ is a summable sequence. From (21)–(24) it follows that

$$\text{supp } y \subseteq E. \quad (25)$$

Moreover,

$$y(t) = \int_{-\infty}^{\infty} e^{it\xi} v(\xi) \varphi(\xi) d\xi, \quad (26)$$

where

$$\varphi(\xi) = \sum_{p \in \mathbb{Z}} c_p e^{ip\xi\sqrt{2\pi}}, \quad -\infty < \xi < \infty. \quad (27)$$

The function φ is periodic,

$$\varphi(\xi + \sqrt{2\pi}) \equiv \varphi(\xi), \quad -\infty < \xi < \infty. \quad (28)$$

Let us invert the order of reasoning. Starting from a function $u(t)$ supported on K , (22), and $\sqrt{2\pi}$ -periodic function $\varphi(\xi)$, (28), we *define* the function $y(t)$ by (26), where $v(\xi)$ is determined from u by (23). Then the equality (24) holds, where $\{c_p\}_{p \in \mathbb{Z}}$ is the sequence of the Fourier coefficient by the originally given function φ : (27). Let a $\sqrt{2\pi}$ -periodic function $\varphi \neq 0$ satisfy the condition

$$\text{supp } \varphi \cap [-\sqrt{\pi/2}, -\sqrt{\pi/2}] \subseteq K, \tag{29}$$

where K is the same as before. Then

$$\text{supp } v(\xi)\varphi(\xi) \subseteq E. \tag{30}$$

If, moreover, the function φ is smooth, then the sequence $\{c_p\}_{p \in \mathbb{Z}}$, (27), is summable. Thus the function $y(t)$ is representable in the form (11), where $x(\xi) = \sqrt{2\pi}v(\xi)\varphi(\xi)$, $\text{supp } x \subseteq E$, $\text{supp } y \subseteq E$, therefore

$$\int_E |y(t)|^2 = \int_{\mathbb{R}} |y(t)|^2 = \int_E |x(\xi)|^2 d\xi,$$

i.e.,

$$\|\mathcal{F}_E x\|^2 = \|x\|^2. \tag{31}$$

Because of the freedom in the choice of $u(t)$ and $\varphi(\xi)$, the set of $x \in L^2(E)$ satisfying the condition (31) is an infinite dimensional subspace of $L^2(E)$.

Let $x_1(\xi) = x(\xi)e^{-ih\xi}$, where $h \in \mathbb{R}$, and

$$y_1(t) = \frac{1}{\sqrt{2\pi}} \int_E e^{it\xi} x_1(\xi) d\xi, \quad t \in \mathbb{R}.$$

Then $y_1(t) = y(t - h)$, $\text{supp } y_1 = h + \text{supp } y$. If $a < \frac{1}{2}\sqrt{\frac{\pi}{2}}$, then h can be chosen such that $(E + h) \cap E = \emptyset$. In this case, $y_1(t) = 0, \forall t \in E$, thus

$$\mathcal{F}_E x_1 = 0. \tag{32}$$

As before, the set of $x_1 \in L^2(E)$ satisfying the condition (32) is an infinite dimensional subspace of $L^2(E)$.

In this example, both

$$\text{mes}(E) = \infty, \quad \text{mes}(\mathbb{R} \setminus E) = \infty. \tag{33}$$

■

R e m a r k 6. In [1, Prop. 6] it was shown that if a set E satisfies the condition $\text{mes}(\mathbb{R} \setminus E) < \infty$, then the set of $x \in L^2(E)$ satisfying the equality (31) is an infinite dimensional subspace of $L^2(E)$.

We recall that the operator A acting in a Hilbert space is said to be *normal* if

$$A^*A = AA^*,$$

where A^* is the operator adjoint to the operator A .

Here and further

$$-E = \{t \in \mathbb{R} : -t \in E\}. \tag{34}$$

Lemma 2. *The truncated Fourier operator \mathcal{F}_E is normal if and only if the equality*

$$\int_{E \setminus (-E)} |y(t)|^2 dt = \int_{(-E) \setminus E} |y(t)|^2 dt \tag{35}$$

holds for every $y(t)$ of the form $y(t) = \frac{1}{\sqrt{2\pi}} \int_E e^{it\xi} x(\xi) d\xi$, $t \in \mathbb{R}$, where x runs over the whole space $L^2(E)$.

P r o o f. The condition $\mathcal{F}_E^* \mathcal{F}_E = \mathcal{F}_E \mathcal{F}_E^*$ is equivalent to the condition: the equality $\|\mathcal{F}_E x\|^2 = \|\mathcal{F}_E^* x\|^2$ holds for every $x \in L^2(E)$. If $x \in L^2(E)$, then $(\mathcal{F}_E x)(t) = y(t)$, $t \in E$, and $(\mathcal{F}_E^* x)(t) = y(-t)$, $t \in E$. Thus, the equality $\|\mathcal{F}_E x\|^2 = \|\mathcal{F}_E^* x\|^2$ takes the form $\int_E |y(t)|^2 dt = \int_E |y(-t)|^2 dt$. The latter equality is equivalent to the equality (35). ■

Definition 2. *The set E is said to be symmetric if*

$$\text{mes } \Delta(E, -E) = 0, \tag{36}$$

where $\Delta(E, -E)$ is the symmetric difference of the sets E and $-E$,

$$\Delta(E, -E) = (E \setminus (-E)) \cup ((-E) \setminus E).$$

Since $(E \setminus (-E)) \cap ((-E) \setminus E) = \emptyset$ and $\text{mes } (E \setminus (-E)) = \text{mes } ((-E) \setminus E)$, the condition (36) can be expressed in asymmetric form

$$\text{mes } (E \setminus (-E)) = 0.$$

Theorem 3. *If the set E is symmetric, then the operator \mathcal{F}_E is a normal operator.*

P r o o f. The theorem is an evident consequence of Lemma 2: the expressions in both sides of (35) are equal because both of them vanish. ■

Question 1. *Let the operator \mathcal{F}_E be normal. Is the set E symmetric?*

We can not answer this question in full generality. However, under some extra condition imposed on the set E the answer to this question is affirmative.

The set $S, S \subset \mathbb{R}$, is said to be *bounded*, *bounded from below* and *bounded from above*, respectively, if S is contained, respectively, in some bounded interval $[a, b]$, bounded from above interval $[a, +\infty)$ or bounded from below interval $(-\infty, b]$, where a, b are some finite numbers. (In the first case, $a < b$.) The set $S, S \subset \mathbb{R}$, is said to be *semi-bounded*, if S is either bounded from above, or is bounded from below. (In particular, every bounded set is semi-bounded.)

Theorem 4. *Assume that the following two conditions are satisfied:*

1. *The operator \mathcal{F}_E is normal;*
2. *The set $E \setminus (-E)$ is semi-bounded.*

Then the set E is symmetric.

Lemma 3. *Let $E, E \subset \mathbb{R}$ be a set of positive measure $\text{mes}(E) > 0$, and the set $S, S \subset \mathbb{R}$, is semi-bounded. Then the set of all functions of the form $y(t) = \frac{1}{\sqrt{2\pi}} \int_E e^{it\xi} x(\xi) d\xi, t \in S$, where x runs over $L^2(E)$, is dense in $L^2(S)$.*

P r o o f. Assume for definiteness that the set S is bounded from above, say $S \subseteq (-\infty, b]$, where $b < \infty$. If the set of all such y is not dense in $L^2(S)$, then there exists $v \in L^2(S), v \neq 0$, such that $\int_S v(t)\overline{y(t)} dt = 0$ for all $y(t)$. From this follows that $\int_S v(t)e^{-it\xi} dt = 0, \forall \xi \in E$. Since $S \subseteq (-\infty, b]$, the function $f(\xi) = e^{ib\xi} \int_S v(t)e^{-it\xi} dt, \xi \in \mathbb{R}$, belongs to the Hardy class H_+^2 . Since $v \in L^2(S)$ is non-zero, f is a non-zero function from H_+^2 . Moreover, $f(\xi) = 0$ for $\xi \in E$. However, the non-zero function from the Hardy class can not vanish on the set of positive measure. ■

P r o o f of Theorem 4. We show that if the set S is not symmetric, that is if $\text{mes}(E \setminus (-E)) > 0$, then the condition (35) is violated for some $y(t) = \frac{1}{\sqrt{2\pi}} \int_E e^{it\xi} x(\xi) d\xi$, where $x \in L^2(E)$. Then by Lemma 2, the operator \mathcal{F}_E is not normal.

We first present the proof assuming that the set $E \setminus (-E)$ is bounded. If the set $E \setminus (-E)$ is bounded, then the set

$$S \stackrel{\text{def}}{=} (E \setminus (-E)) \cup ((-E) \setminus E) \tag{37}$$

is bounded as well. We define the function $g(t)$ as

$$g(t) = 1, \quad t \in (E \setminus (-E)), \quad g(t) = 0, \quad t \in ((-E) \setminus E).$$

Since the sets $(E \setminus (-E))$ and $((-E) \setminus E)$ do not intersect, the definition of the function g is not contradictory. According to Lemma 3, for every number $\varepsilon > 0$ there exists a function $y(t) = \frac{1}{\sqrt{2\pi}} \int_E e^{it\xi} x(\xi) d\xi$ such that

$$\int_S |g(t) - y(t)|^2 dt \leq \varepsilon^2 \text{mes}(E \setminus (-E)).$$

Since $g(t) = 1$ on $(E \setminus (-E))$,

$$\int_{E \setminus (-E)} |y(t)|^2 dt \geq (1 - \varepsilon)^2 \text{mes}(E \setminus (-E)).$$

Since $g(t) = 0$ on $(-E) \setminus E$,

$$\int_{(-E) \setminus E} |y(t)|^2 dt \leq \varepsilon^2 \text{mes}(E \setminus (-E)).$$

Choosing $\varepsilon < 1/2$, we find $y(t) = \frac{1}{\sqrt{2\pi}} \int_E e^{it\xi} x(\xi) d\xi$ for which the equality (35) is violated.

If the set $E \setminus (-E)$ is semi-bounded but not bounded, then the set S , (37), is not bounded "in both directions". We assume for definiteness that the set $E \setminus (-E)$ is bounded from above, say* $(E \setminus (-E)) \subset (-\infty, b]$, where $b < \infty$. We construct such a function $y(t)$ of the form (11) for which

$$\int_{E \setminus (-E)} |y(t)|^2 dt < \int_{(-E) \setminus E} |y(t)|^2 dt. \tag{38}$$

Since the set $E \setminus (-E)$ is bounded from above but not bounded,

$$\text{mes}(((-E) \setminus (E)) \cap (b, \infty)) > 0.$$

Therefore, there exists a finite interval $[p, q]$, $[p, q] \in (b, \infty)$, such that

$$\text{mes}([p, q] \cap ((-E) \setminus E)) > 0.$$

Let

$$S = (-\infty, q] \cap ((E \setminus (-E)) \cup ((-E) \setminus E)),$$

$$g(t) = \begin{cases} 0, & \text{if } t \in S, \quad -\infty < t < p, \\ 1, & \text{if } t \in S, \quad p \leq t \leq q. \end{cases}$$

*Strictly speaking, this condition should be formulated as $\text{mes}((E \setminus (-E)) \cap (b, \infty)) = 0$.

Clearly,

$$(E \setminus (-E)) \subset S \cap (-\infty, p), \quad S \cap [p, q] = ((-E) \setminus E) \cap [p, q], \quad (39)$$

and

$$\int_{E \setminus (-E)} |g(t)|^2 dt < \int_{[p, q] \cap ((-E) \setminus E)} |g(t)|^2 dt. \quad (40)$$

(The left-hand side of this inequality is equal to zero, and the right-hand side is equal to the strictly positive number $\text{mes}([p, q] \cap ((-E) \setminus E))$.) By Lemma 3, for any $\varepsilon > 0$ there exists a function y of the form (11) such that

$$\int_S |y(t) - g(t)|^2 dt < \varepsilon^2. \quad (41)$$

If ε is small enough, then from (39)–(41) it follows that

$$\int_{E \setminus (-E)} |y(t)|^2 dt < \int_{((-E) \setminus E) \cap [p, q]} |y(t)|^2 dt,$$

and all the more the inequality (38) holds. ■

R e m a r k 7. If $\text{mes } E < \infty$, then the operators $\mathcal{F}_E^* \mathcal{F}_E$ and $\mathcal{F}_E \mathcal{F}_E^*$ are integral operators with the kernels $\int_E e^{i\omega(t-s)} d\omega$, $\int_E e^{-i\omega(t-s)} d\omega$, $t, s \in E$. The normality of the operator \mathcal{F}_E means that these kernels coincide, that is

$$\int_E e^{i\omega\tau} d\omega = \int_E e^{-i\omega\tau} d\omega, \quad \forall \tau \in E - E.$$

Thus, the normality of the operator \mathcal{F}_E is equivalent to the equality

$$\int_E \sin \omega\tau d\omega = 0, \quad \forall \tau \in E - E. \quad (42)$$

Since $\sin \omega\tau$ is an odd function of ω ,

$$\int_E \sin \omega\tau d\omega = \int_{E \setminus (-E)} \sin \omega\tau d\omega.$$

Thus, to obtain the negative answer to Question 1, it is enough to construct an asymmetric set E of finite positive measure

$$E \cap (-E) = \emptyset, \quad 0 < \text{mes } E < \infty \quad (43)$$

such that

$$\int_E \sin \omega \tau d\omega = 0, \quad \forall \tau \in E - E. \quad (44)$$

It is known that if $\text{mes } E > 0$, then the set $E - E$ contains some interval $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$. In particular, the condition (44) implies that

$$\int_E \sin \omega \tau d\omega = 0, \quad \forall \tau \in [-\varepsilon, \varepsilon] \text{ for some } \varepsilon > 0. \quad (45)$$

We remark that in the paper [4] the set $E, 0 < \text{mes } E < \infty$, was constructed such that

$$\int_E e^{i\omega\tau} d\omega = 0, \quad \forall \tau \in [-\varepsilon, \varepsilon] \text{ for some } \varepsilon > 0.$$

The methods used in the paper [4] (see also [5]) might help to obtain the negative answer to Question 1.

Theorem 5. *Assume that the Lebesgue measure of the set E is finite. Then:*

1. *The operator $\mathcal{F}_E^* \mathcal{F}_E$ is an integral operator*

$$(\mathcal{F}_E^* \mathcal{F}_E x)(t) = \int_E K_E(t, s) x(s) ds \quad (46)$$

with the kernel

$$K_E(t, s) = \int_E e^{i\xi(t-s)} d\xi, \quad t, s \in E. \quad (47)$$

2. *The operator $\mathcal{F}_E^* \mathcal{F}_E$ is a trace class operator*

$$\text{trace } \mathcal{F}_E^* \mathcal{F}_E = (\text{mes } E)^2, \quad (48)$$

i.e., the operator \mathcal{F}_E belongs to the class \mathfrak{S}_2 of Hilbert-Schmidt operators

$$\|\mathfrak{F}_E\|_{\mathfrak{S}_2} = \text{mes } E. \quad (49)$$

In particular, the operator \mathcal{F}_E is a compact operator.

3. *The trace norm of the operator \mathcal{F}_E satisfies the condition*

$$(\text{mes } E)^2 \leq \|\mathfrak{F}_E\|_{\mathfrak{S}_1} \leq \infty. \quad (50)$$

P r o o f.

1. The representation (46), (47) is a direct consequence of the equalities (6) and (7) and of the rule for calculation the kernel of the product of two integral operators in terms of their kernels.

2. The kernel $K_E(t, s)$, (47), is positive definite, bounded and uniformly continuous for $(t, s) \in E \times E$. As it claimed in [6, Chap. 3, Sec. 10], from these properties of the kernel of an integral operator it follows that this operator is a trace class operator, and that its trace is equal to the integral $\int_E K_E(t, t) dt$. (See the last paragraph of Section 10 of the quoted reference.)

3. The equality (48) means that

$$\sum_{1 \leq j < \infty} (s_j(\mathcal{F}_E))^2 = (\text{mes } E)^2, \tag{51}$$

where $s_j(\mathcal{F}_E)$ are the singular values of the operator \mathcal{F}_E . In view of (12),

$$s_j(\mathcal{F}_E) \leq 1, \quad 1 \leq j < \infty.$$

Thus,

$$\sum_{1 \leq j < \infty} s_j(\mathcal{F}_E) \geq (\text{mes } E)^2. \tag{52}$$

To study under which conditions the operator \mathcal{F}_E , or what is the same (see Remark 3) the operator $P_E \mathcal{F}_E P_E$, belongs to the trace class \mathfrak{S}_1 , we have to consider the more general operator

$$\mathcal{F}_{S_1, S_2} = P_{S_2} \mathcal{F} P_{S_1}, \tag{53}$$

where $S_1, S_2 \subset \mathbb{R}$ are measurable sets, and for the set $S, S \subset \mathbb{R}$, the operator $P_S : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is defined as

$$(P_S x)(t) = \mathbb{I}_S(t)x(t), \text{ where } \mathbb{I}_S(t) = 1, t \in S, \mathbb{I}_S(t) = 0, t \notin S. \tag{54}$$

■

Theorem 6. *If the truncated Fourier operator \mathcal{F}_E is a Hilbert-Schmidt operator $\mathcal{F}_E \in \mathfrak{S}_2$, then the set E is of finite measure, and the equality (49) holds.*

P r o o f. The equality (49) was obtained under the assumption that $\text{mes } E < \infty$. If $\text{mes } E = \infty$, the kernel $K_E(t, s)$, (47), is not well defined, and the reasoning used in the proof of Theorem 5 is not applicable. Consider the set $E_n = E \cap [-n, n]$ and the operator \mathcal{F}_{E_n} . The operator \mathcal{F}_{E_n} can be identified with the operator $P_{E_n} \mathcal{F}_E P_{E_n}$ (see Remark 3), where P_{E_n} is the orthoprojector in $L^2(\mathbb{R})$: $(P_{E_n} x)(t) = \mathbb{I}_{E_n}(t)x(t)$. Therefore, $\|\mathcal{F}_{E_n}\|_{\mathfrak{S}_2} \leq \|\mathcal{F}_E\|_{\mathfrak{S}_2}$. On the other hand, the set E_n is of finite measure, and the formula (49) is applicable to E_n : $\text{mes } E_n = \|\mathcal{F}_{E_n}\|_{\mathfrak{S}_2}$. Thus, for every n , $\text{mes } E_n \leq \|\mathcal{F}_E\|_{\mathfrak{S}_2}$. Turning n to infinity, we obtain that $\text{mes } E \leq \|\mathcal{F}_E\|_{\mathfrak{S}_2} < \infty$. ■

Lemma 4. *Assume that S_1, S_2 are bounded measurable sets*

$$S_1, S_2 \subseteq [-R, R] \quad \text{for some } R \in (0, \infty). \quad (55)$$

Then the operator \mathcal{F}_{S_1, S_2} belongs to the trace class \mathfrak{S}_1 , and its trace norm $\|\mathcal{F}_{S_1, S_2}\|_{\mathfrak{S}_1}$ admits the estimate

$$\|\mathcal{F}_{S_1, S_2}\|_{\mathfrak{S}_1} \leq (\text{mes } S_1)^{1/2} \cdot (\text{mes } S_2)^{1/2} \cdot e^{R^2}. \quad (56)$$

P r o o f. The operator \mathcal{F}_{S_1, S_2} is an integral operator in the space $L^2(\mathbb{R})$ with the kernel $k(t, \xi) = \mathbb{I}_{S_2}(t)e^{it\xi} \mathbb{I}_{S_1}(\xi)$, which is the sum of the rank-one kernels

$$k(t, \xi) = \sum_{0 \leq j < \infty} i^j k_j(t, \xi), \quad k_j(t, \xi) = \frac{1}{j!} \cdot \mathbb{I}_{S_2}(t)t^j \cdot \xi^j \mathbb{I}_{S_1}(\xi).$$

The one-dimensional integral operator $K_j \geq 0$ with the kernel $k_j(t, \xi)$ admits the estimate

$$\|K_j\|_{\mathfrak{S}_1} \leq \frac{1}{j!} \cdot \|\mathbb{I}_{S_2}(t)t^j\|_{L^2(\mathbb{R})} \cdot \|\xi^j \mathbb{I}_{S_1}(\xi)\|_{L^2(\mathbb{R})}.$$

Since $S_1 \subset [-R, R], S_2 \subset [-R, R]$,

$$\|\mathbb{I}_S(t)t^j\|_{L^2(\mathbb{R})} \leq (\text{mes } S)^{1/2} R^j, \quad S = S_1, S_2.$$

Therefore,

$$\|\mathcal{F}_{S_1, S_2}\|_{\mathfrak{S}_1} \leq \sum_{0 \leq j < \infty} \|K_j\|_{\mathfrak{S}_1} \leq (\text{mes } S_1)^{1/2} \cdot (\text{mes } S_2)^{1/2} \cdot \sum_{0 \leq j < \infty} \frac{R^{2j}}{j!}.$$

■

The estimate (56) shows that if the set E is bounded, then the operator \mathcal{F}_E is a trace class operator. However, this estimate does not work if the set E is unbounded.

Theorem 7. *Let the set $E, E \subset \mathbb{R}$, satisfies the condition*

$$\sum_{j \in \mathbb{Z}} (\text{mes } (E_j))^{1/2} < \infty, \quad (57)$$

where

$$E_j = E \cap [j - 1/2, j + 1/2], \quad j \in \mathbb{Z}. \quad (58)$$

Then the operator \mathcal{F}_E is a trace class operator.

The following lemma is a modification of Lemma 4:

Lemma 5. Let $E, E \subset \mathbb{R}$, be a measurable set, and the operator $\mathcal{F}_{E_p, E_q} = P_{E_q} \mathcal{F} P_{E_p}$ is defined by (53)-(54), with $S_1 = E_p, S_2 = E_q$.
Then

$$\|\mathcal{F}_{E_p, E_q}\|_{\mathfrak{S}_1} \leq e^{1/4} \cdot (\text{mes } E_p)^{1/2} \cdot (\text{mes } E_q)^{1/2}, \quad \forall p, q \in \mathbb{Z}. \quad (59)$$

P r o o f. For $p = 0, q = 0$, the estimate (59) is the special case of Lemma 4 corresponding to $S_1 = E_0, S_2 = E_0, R = 1/2$. The general case of arbitrary integers p and q can be reduced to the case $p = 0, q = 0$ by means of translation. The sets $S_1 = -p + E_p$ and $S_2 = -q + E_q$ are contained in the interval $[-1/2, 1/2]$, and the operator \mathcal{F}_{E_p, E_q} is related to the operator \mathcal{F}_{S_1, S_2} by the equality $\mathcal{F}_{E_p, E_q} = U_q \mathcal{F}_{S_1, S_2} U_p$, where $U_r, r \in \mathbb{Z}$, are the unitary operators $(U_r x)(t) = e^{irt} x(t)$. ■

P r o o f of Theorem 7. Identifying the operator \mathcal{F}_E with the operator $P_E \mathcal{F} P_E$, we represent it as the double sum

$$\mathcal{F}_E = \sum_{p \in \mathbb{Z}, q \in \mathbb{Z}} P_{E_q} \mathcal{F} P_{E_p},$$

hence

$$\|\mathcal{F}_E\|_{\mathfrak{S}_1} \leq \sum_{p \in \mathbb{Z}, q \in \mathbb{Z}} \|P_{E_q} \mathcal{F} P_{E_p}\|_{\mathfrak{S}_1}.$$

Applying the estimate (59) to the summand in the right-hand side, we obtain that

$$\|\mathcal{F}_E\|_{\mathfrak{S}_1} \leq e^{1/4} \left(\sum_{j \in \mathbb{Z}} (\text{mes } E_j)^{1/2} \right)^2, \quad (60)$$

where E_j is defined in (58). ■

R e m a r k 8. Since $\sum a_j \leq (\sum a_j^{1/2})^2$ for non-negative a_j ,

$$\text{mes } E = \sum_{j \in \mathbb{Z}} \text{mes } E_j \leq \left(\sum_{j \in \mathbb{Z}} (\text{mes } E_j)^{1/2} \right)^2.$$

Thus the set E , for which the expression in the right-hand side of (60) is finite automatically, satisfies the condition $\text{mes } E < \infty$. However, there are the sets E of finite measure for which the expression in the right-hand side of (60) is infinite. For example, $E = \bigcup_{1 \leq j < \infty} [j - j^{-2}, j + j^{-2}]$.

It should be mentioned that Theorem 7 is related to some results by M.Sh. Birman and M.Z. Solomyak [7, Theorem 11.1], and may be considered as a special case of their result. However, our presentation is more direct and simple.

Theorem 7 is precise.

Theorem 8 (B. Simon, [8, Prop. 4.7]). *Let the operator \mathcal{F}_E be a trace class operator. Then the set E satisfies the condition (57), (58).*

P r o o f. We identify the operator \mathcal{F}_E with the operator $P_E \mathcal{F} P_E$. (See (8), (9), and Remark 3.) Since the Fourier–Plancherel operator \mathcal{F} is unitary, the operator $\mathcal{F}^{-1} \mathcal{F}_E = \mathcal{F}^{-1} P_E \mathcal{F} P_E$ is a trace class operator as well

$$\mathcal{F}^{-1}(P_E \mathcal{F} P_E) \in \mathfrak{S}_1. \tag{61}$$

We are to deduce from (61) that the set E satisfies the condition (57), (58). According to Theorem 6, the condition (61) implies that the set E is of finite measure $\text{mes } E < \infty$ (see (50)). Therefore, the function

$$h_E(t) = \frac{1}{2\pi} \int_E e^{-i\xi t} d\xi, \quad t \in \mathbb{R}, \tag{62}$$

is well-defined and continuous on \mathbb{R} . The operator

$$\mathcal{C}_e = (\mathcal{F}^{-1} P_E \mathcal{F}) P_E \tag{63}$$

can be represented as the product of the multiplication and the convolution operators

$$(\mathcal{C}_E x)(t) = \int_{\mathbb{R}} h_E(t - \xi) \mathbb{I}_E(\xi) x(\xi) d\xi. \tag{64}$$

We assume that the operator \mathcal{C}_E is a trace class operator acting in the space $L^2(\mathbb{R})$. We have to derive from here that the set E satisfies the condition (57), (58).

The value $h(0) = \frac{1}{2\pi} \text{mes } E$ is strictly positive, and the function h is continuous. Therefore there exists $\delta > 0$ such that

$$\text{Re } h(t) > \frac{1}{4\pi} \text{mes } E, \quad t \in [-\delta, \delta]. \tag{65}$$

Since the operator \mathcal{C}_E is a trace class operator in $L^2(\mathbb{R})$, for any two systems* $\{\varphi_m\}_{m \in M}$ and $\{\psi_m\}_{m \in M}$, orthonormal in $L^2(\mathbb{R})$, the inequality

$$\sum_{m \in M} |\langle \mathcal{C}_E \varphi_m, \psi_m \rangle_{L^2_E}| \leq \|\mathcal{C}_E\|_{\mathfrak{S}_1} < \infty \tag{66}$$

holds. We obtain the information concerning the set E by choosing the systems $\{\varphi_j\}_{j \in M}$ and $\{\psi_j\}_{j \in M}$ in an appropriate way.

* $M \subseteq \mathbb{Z}$ is an indexing set.

Let

$$Q_{m,\delta} = [m\delta - \delta/2, m\delta + \delta/2), \quad m \in \mathbb{Z}. \quad (67)$$

The intervals $Q_{m,\delta}$, $m \in \mathbb{Z}$, form the partition of the real axis. Let

$$E_{m,\delta} = E \cap Q_{m,\delta}, \quad m \in \mathbb{Z}. \quad (68)$$

We will prove that for chosen δ ,

$$\sum_{m \in \mathbb{Z}} (\text{mes } E_{m,\delta})^{1/2} < \infty. \quad (69)$$

Let $M = \{m \in \mathbb{Z} : \text{mes } E_{m,\delta} > 0\}$. For $m \in M$, we set

$$\varphi_m(t) = (\text{mes } Q_{m,\delta})^{-1/2} \cdot \mathbb{I}_{Q_{m,\delta}}(t), \quad (70a)$$

$$\psi_m(t) = (\text{mes } E_{m,\delta})^{-1/2} \cdot \mathbb{I}_{E_{m,\delta}}(t). \quad (70b)$$

The systems $\{\varphi_m\}_{m \in M}$ and $\{\psi_m\}_{m \in M}$, defined by (70), are orthonormal. Let us calculate and estimate the scalar product $\langle \mathcal{C}_E \varphi_m, \psi_m \rangle_{L^2(\mathbb{R})}$. According to (63) and (70),

$$\begin{aligned} \langle \mathcal{C}_E \varphi_m, \psi_m \rangle_{L^2(\mathbb{R})} &= (\text{mes } Q_{m,\delta})^{-1/2} (\text{mes } E_{m,\delta})^{-1/2} \iint_{\substack{t \in Q_{m,\delta} \\ \xi \in E_{m,\delta}}} h(t - \xi) \mathbb{I}_{E_{m,\delta}}(\xi) d\xi. \end{aligned} \quad (71)$$

According to (67), for $t \in Q_{m,\delta}$, $\xi \in Q_{m,\delta}$, the inequality $|t - \xi| \leq \delta$ holds. Together with (65), this yields

$$\text{Re } h(t - \xi) \geq c > 0 \quad \text{for } t \in Q_{m,\delta}, \xi \in Q_{m,\delta}, \text{ where } c = \frac{1}{4\pi} \text{mes } E. \quad (72)$$

Invoking (71), we obtain

$$\begin{aligned} |\langle \mathcal{C}_E \varphi_m, \psi_m \rangle_{L^2(\mathbb{R})}| &\geq \text{Re } \langle \mathcal{C}_E \varphi_m, \psi_m \rangle_{L^2(\mathbb{R})} \\ &= (\text{mes } Q_{m,\delta})^{-1/2} (\text{mes } E_{m,\delta})^{-1/2} \iint_{\substack{t \in Q_{m,\delta} \\ \xi \in E_{m,\delta}}} \text{Re } h(t - \xi) \mathbb{I}_{E_{m,\delta}}(\xi) d\xi. \end{aligned} \quad (73)$$

Finally, taking into account that $\text{mes } Q_{m,\delta} = \delta$, we get

$$|\langle \mathcal{C}_E \varphi_m, \psi_m \rangle_{L^2(\mathbb{R})}| \geq c \delta (\text{mes } E_{m,\delta})^{1/2}. \quad (74)$$

From here and (66), the condition (69) follows.

The condition (69) is almost what we need. We obtain the condition for any δ satisfying the condition (65). We need the condition (69) for $\delta = 1$. (For $\delta = 1$, the condition (57) is needed.)

Actually, if the condition (69) is fulfilled for *some* positive δ , then it is fulfilled for *any* positive δ . We show this in the generality which we need. Because we may diminish δ without violating the condition (65), we choose δ of the form

$$\delta = \frac{1}{N}, \quad N \text{ is a positive integer which is large enough.} \quad (75)$$

If a_1, a_2, \dots, a_N are non-negative numbers, then

$$(a_1 + a_2 + \dots + a_N)^{1/2} \leq \sum_{1 \leq k \leq N} a_k^{1/2}.$$

It is clear that for $\delta = 1/N$, either the sets $Q_{j,1}$ and $Q_{m,\delta}$ do not intersect, or the set $Q_{m,\delta}$ is contained in $Q_{j,1}$. Moreover, the total number of the sets $Q_{m,\delta}$ contained in $Q_{j,1}$ is equal to N . Thus,

$$\text{mes}(E \cap Q_{j,1}) = \sum_{m: Q_{m,\delta} \subset Q_{j,1}} \text{mes}(E \cap Q_{m,\delta}),$$

and the sum in the right-hand side contains precisely N summands. Therefore, for every $j \in \mathbb{Z}$,

$$(\text{mes}(E \cap Q_{j,1}))^{1/2} \leq \sum_{m: Q_{m,\delta} \subset Q_{j,1}} (\text{mes}(E \cap Q_{m,\delta}))^{1/2},$$

and

$$\sum_{j \in \mathbb{Z}} (\text{mes}(E \cap Q_{j,1}))^{1/2} \leq \sum_{m \in \mathbb{Z}} (\text{mes}(E \cap Q_{j,\delta}))^{1/2}. \quad (76)$$

■

Question 2. *For which sets E the operator \mathcal{F}_E is compact?*

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