Journal of Mathematical Physics, Analysis, Geometry
2012, vol. 8, No. 2, pp. 177–189

On Geodesics of Tangent Bundle with Fiberwise Deformed Sasaki Metric over Kählerian Manifold

A. Yampolsky

Department of Mechanics and Mathematics, V.N. Karazin Kharkiv National University
4 Svobody Sq., Kharkiv 61077, Ukraine
E-mail: alexymp@gmail.com

Received February 24, 2011

We propose a fiber-wise deformation of the Sasaki metric on slashed and unit tangent bundles over the Kälerian manifold based on the Berger deformation of metric on a unit sphere. The geodesics of this metric have different projections on a base manifold for the slashed and unit tangent bundles in contrast to usual Sasaki metric. Nevertheless, the projections of geodesics of the unit tangent bundle over the locally symmetric Kählerian manifold still preserve the property to have all geodesic curvatures constant.

Key words: Sasaki metric, Kählerian manifold, tangent bundle, geodesics.

Mathematics Subject Classification 2000: 53B20, 53B25 (primary); 53B21 (secondary).

Introduction

Let (M, g) be a Riemannian manifold. Denote by TM and T_1M the tangent bundle and the unit tangent bundle of (M, g) with the Sasaki metric. It is easy to prove that if π is a bundle projection $\pi : TM \to M$ and $\Gamma(\sigma)$ is a non-vertical geodesic on TM or T_1M, then the projected curve $\gamma(\sigma) = (\pi \circ \Gamma)(\sigma)$ on M is the same. In other words, the non-vertical geodesic lines on TM or T_1M are generated by different vector fields along *the same* set of the curves on the base manifold. For the case of the base manifold of constant curvature, S. Sasaki [1] and K. Sato [2] gave a complete description of the curves and vector fields along them which generated non-vertical geodesics on T_1M^n and TM^n, respectively. They proved that the projected curves have constant (possibly zero) first and second geodesic curvatures while the others vanish. P. Nagy [3] generalized these results to the case of locally symmetric base manifold and proved that the projected curves have all geodesic curvatures constant.

The Sasaki metric weakly inherits the base manifold properties. Under most considerations it behaves just as a general Riemannian metric. That is why a number of authors proposed to deform the Sasaki metric in order to get some kind of "flexibility" of its properties (see [4–8] and others).

Using the concept of natural transformation of the Riemannian metric on the manifold to its tangent bundle, M.T.K. Abbassi and M. Sarih [6] proposed a much more general metric on the tangent and the unit tangent bundles which includes the Sasaki metric, the Cheeger–Gromoll metric and some others as partial cases. This metric uses some kind of "deformation" of the Sasaki metric in the direction of the "tangent bundle point".

In present paper we propose another natural way of deforming the Sasaki metric in the presence of almost complex structure J. If the base manifold (M, g) of dimension $2n$ is endowed with almost complex structure J, then the unit sphere S^{2n-1}_x in the tangent space T_xM carries the Hopf vector field $J\xi$, where ξ is a unit normal vector field on the sphere. Applying the Berger metric deformation to each tangent sphere, we get the unit tangent bundle over M with the Berger metric spheres as fibers. In a wider scope, one can deform the Sasaki metric on the slashed manifold $TM_0 := TM\backslash M$ in the direction of $J\xi$ such that the restriction of the deformed metric on the unit tangent bundle gives the construction described. The main result of the paper is the following.

Theorem 2.1. *Let Γ be a geodesic on the unit tangent bundle with the Berger-type deformed Sasaki metric over Kählerian locally symmetric manifold M and $\gamma = \pi \circ \Gamma$ be its projection to the base. Then all geodesic curvatures of γ are constant.*

If Γ is a geodesic on the slashed tangent bundle TM_0, then the projected curve $\gamma = \pi \circ \Gamma$ does not possess this property.

For the specific case of the Kählerian manifold of constant holomorphic curvature, Theorem 2.1 can be improved.

Theorem 2.2. *Let Γ be a geodesic of the unit tangent bundle with the Berger-type deformed Sasaki metric over Kählerian manifold M^{2n} $(n \geq 3)$ of constant holomorphic curvature. Then the geodesic curvatures of $\gamma = \pi \circ \Gamma$ are all constant and $k_6 = \cdots = k_{n-1} = 0$.*

1. Basic Properties of the Berger-Type Deformed Sasaki Metric

Let (M, g) be an n-dimensional Riemannian manifold with metric g. Denote by $\langle \cdot, \cdot \rangle$ a scalar product with respect to g.

It is well known that at each point $Q = (q, \xi) \in TM$ the tangent space $T_Q TM$ splits into *vertical* and *horizontal* parts:

$$T_Q TM = \mathcal{H}_Q TM \oplus \mathcal{V}_Q TM.$$

The vertical part $\mathcal{V}_Q TM$ is tangent to the fiber, while the horizontal part is transversal to it. Denote by $(x^1, \dots, x^n; \xi^1, \dots, \xi^n)$ the natural induced local coordinate system on TM. Denote $\partial_i = \partial/\partial x^i$, $\partial_{n+i} = \partial/\partial \xi^i$. Then for $\tilde{X} \in T_Q TM^n$ we have

$$\tilde{X} = \tilde{X}^i \partial_i + \tilde{X}^{n+i} \partial_{n+i}.$$

Denote by $\pi : TM \to M$ the tangent bundle projection. The mapping π_* defines a point-wise linear isomorphism between $\mathcal{H}_Q TM$ and $T_q M$. Notice that $\ker \pi_*|_Q = \mathcal{V}_Q$.

The so-called *connection mapping* $K : T_Q TM \to T_q M$ acts on $\tilde{X}$ by

$$K\tilde{X} = (\tilde{X}^{n+i} + \Gamma^i_{jk} \xi^j \tilde{X}^k) \partial_i$$

and defines a point-wise linear isomorphism between $\mathcal{V}_Q TM$ and $T_q M$. Here Γ^i_{jk} are the Christoffel symbols of g. Notice that $\ker K|_Q = \mathcal{H}_Q$.

The images $\pi_* \tilde{X}$ and $K\tilde{X}$ are called *horizontal* and *vertical* projections of $\tilde{X}$, respectively. The operations inverse to the projections are called *lifts*. Namely, if $X \in T_q M^n$, then

$$X^h = X^i \partial_i - \Gamma^i_{jk} \xi^j X^k \partial_{n+i}$$

is in $\mathcal{H}_Q TM$ and is called the *horizontal lift* of X, and

$$X^v = X^i \partial_{n+i}$$

is in $\mathcal{V}_Q TM$ and is called the *vertical lift* of X.

Let $\tilde{X}, \tilde{Y} \in T_Q TM$. The standard *Sasaki metric* on TM is defined at each point $Q = (q, \xi) \in TM$ by the scalar product

$$\langle\langle \tilde{X}, \tilde{Y} \rangle\rangle\big|_Q = \langle \pi_* \tilde{X}, \pi_* \tilde{Y} \rangle\big|_q + \langle K\tilde{X}, K\tilde{Y} \rangle\big|_q.$$

The horizontal and vertical subspaces are mutually orthogonal with respect to the Sasaki metric. The Sasaki metric can be completely defined by a scalar product of various combinations of lifts by

$$\langle\langle X^h, Y^h \rangle\rangle = \langle X, Y \rangle, \quad \langle\langle X^h, Y^v \rangle\rangle = 0, \quad \langle\langle X^v, Y^v \rangle\rangle = \langle X, Y \rangle.$$

Let (M, g, J) be a Hermitian manifold of dimension $2n$ with an almost complex structure J, i.e. the $(1,1)$-tensor field satisfying $J^2 = -id$. Denote by TM_0

a slashed tangent bundle, i.e. the tangent bundle with zero section deleted. Define a fiber-wise *Berger-type deformation* of the Sasaki metric on TM_0 by

$$\begin{aligned} &\langle\langle X^h, Y^h\rangle\rangle = \langle X, Y\rangle, \\ &\langle\langle X^h, Y^v\rangle\rangle = 0, \\ &\langle\langle X^v, Y^v\rangle\rangle = \langle X, Y\rangle + \delta^2\langle X, J\xi\rangle\langle Y, J\xi\rangle, \end{aligned} \tag{1}$$

where δ is some constant.

In what follows we restrict the considerations to the case of the Kählerian base manifold. In this case J has no torsion and $\nabla J = 0$.

The following formulas are independent from the choice of the tangent bundle metric and are known as Dombrowski formulas.

Lemma 1.1. *At each point $(q, \xi) \in TM$ the brackets of lifts of vector fields from M to TM are*

$$[X^h, Y^h] = [X, Y]^h - \big(R(X, Y)\xi\big)^v, \quad [X^h, Y^v] = \big(\nabla_X Y\big)^v, \quad [X^v, Y^v] = 0,$$

where ∇ is the connection on M and R is its curvature tensor.

Denote by $\tilde\nabla$ the Levi–Civita connection of metric (1). The following lemma contains the Kowalski-type formulas [9] and it is the main tool for further considerations.

Lemma 1.2. *Let (M, g, J) be a Kählerian manifold. The Levi–Civita connection of the Berger-type deformed Sasaki metric (1) on the slashed tangent bundle TM_0 is completely defined by*

$$\begin{aligned} &\tilde\nabla_{X^h} Y^h = \big(\nabla_X Y\big)^h - \tfrac{1}{2}\big(R(X, Y)\xi\big)^v, \\ &\tilde\nabla_{X^h} Y^v = \tfrac{1}{2}\Big(R(\xi, Y)X + \delta^2\,\langle Y, J\xi\rangle\, R(\xi, J\xi)X\Big)^h + \Big(\nabla_X Y\Big)^v, \\ &\tilde\nabla_{X^v} Y^h = \tfrac{1}{2}\Big(R(\xi, X)Y + \delta^2\,\langle X, J\xi\rangle\, R(\xi, J\xi)Y\Big)^h \\ &\tilde\nabla_{X^v} Y^v = \delta^2\Big(\,\langle X, J\xi\rangle\, JY + \langle Y, J\xi\rangle\, JX - \\ &\qquad\qquad \tfrac{\delta^2}{1+\delta^2|\xi|^2}\big(\,\langle Y, \xi\rangle\,\langle X, J\xi\rangle + \langle X, \xi\rangle\,\langle Y, J\xi\rangle\,\big)J\xi\Big)^v, \end{aligned}$$

where ∇ is the Levi-Civita connection on M and R is its curvature tensor.

P r o o f. The proof is based on the following rules of differentiations:

$$\begin{aligned} &X^h\langle\langle Y^h, Z^h\rangle\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \\ &X^h\langle\langle Y^v, Z^v\rangle\rangle = \langle\langle(\nabla_X Y)^v, Z^v\rangle\rangle + \langle\langle Y^v, (\nabla_X Z)^v\rangle\rangle, \\ &X^v\langle\langle Y^h, Z^h\rangle\rangle = 0, \\ &X^v\langle\langle Y^v, Z^v\rangle\rangle = \delta^2\Big(\langle Y, JX\rangle\langle Z, J\xi\rangle + \langle Y, J\xi\rangle\langle Z, JX\rangle\Big). \end{aligned} \tag{2}$$

$(2)_1$: Indeed, keeping in mind (1), we have

$$X^h\langle\langle Y^h, Z^h\rangle\rangle = X^h\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

$(2)_2$: In a similar way,

$$\begin{aligned}X^h\langle\langle Y^v, Z^v\rangle\rangle &= X^h\Big(\langle Y, Z\rangle + \delta^2\langle Y, J\xi\rangle\langle Z, J\xi\rangle\Big)\\ &= \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \delta^2 X^h\Big(\langle Y, J\xi\rangle\langle Z, J\xi\rangle\Big).\end{aligned}$$

As M is Kählerian and hence $\nabla_X J = 0$, we have

$$\begin{aligned}X^h\langle Y, J\xi\rangle &= -X^h\langle JY, \xi\rangle = -X^i\partial_i\langle JY, \xi\rangle + \Gamma^s_{jk}\xi^j X^k\partial_{n+s}\langle JY, \xi\rangle\\ &= -X^i\langle J\nabla_i Y, \xi\rangle - X^i\xi^k\langle JY, \Gamma^s_{ki}\partial_s\rangle + \Gamma^s_{ki}\xi^k X^i\langle JY, \partial_s\rangle = \langle\nabla_X Y, J\xi\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}X^h\langle\langle Y^v, Z^v\rangle\rangle &= \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \delta^2\langle\nabla_X Y, J\xi\rangle\langle Z, J\xi\rangle\\ &\delta^2\langle Y, J\xi\rangle\langle\nabla_X Z, J\xi\rangle = +\langle\langle(\nabla_X Y)^v, Z^v\rangle\rangle + \langle\langle Y^v, (\nabla_X Z)^v\rangle\rangle.\end{aligned}$$

$(2)_3$: We have $X^v\langle\langle Y^h, Z^h\rangle\rangle = X^v\langle Y, Z\rangle = X^i\partial_{n+i}\langle Y, Z\rangle = 0$.

$(2)_4$: Finally,

$$X^v\langle Y, J\xi\rangle = X^i\partial_{n+i}\langle Y, J\xi\rangle = \langle Y, JX\rangle,$$

and therefore

$$\begin{aligned}X^v\langle\langle Y^v, Z^v\rangle\rangle &= X^v\Big(\langle Y, Z\rangle + \delta^2\langle Y, J\xi\rangle\langle Z, J\xi\rangle\Big)\\ &= \delta^2\Big(\langle Y, JX\rangle\langle Z, J\xi\rangle + \langle Y, J\xi\rangle\langle Z, JX\rangle\Big).\end{aligned}$$

Now we can prove the lemma relatively easy by applying Lemma 1.1 and the Kozsul formula for the Levi–Civita connection

$$\begin{aligned}2\langle\nabla_A B, C\rangle &= A\langle B, C\rangle + B\langle A, C\rangle - C\langle A, B\rangle\\ &+ \langle[A, B], C\rangle + \langle[C, A], B\rangle - \langle[B, C], A\rangle.\end{aligned}$$

- Take $A = X^h, B = Y^h, C = Z^h$. Then

$$2\langle\langle\tilde\nabla_{X^h} Y^h, Z^h\rangle\rangle = 2\langle\nabla_X Y, Z\rangle = 2\langle\langle\,(\nabla_X Y)^h, Z^h\rangle\rangle.$$

  Take $A = X^h, B = Y^h, C = Z^v$. Then

$$\begin{aligned}2\langle\langle\tilde\nabla_{X^h} Y^h, Z^v\rangle\rangle &= -Z^v\langle\langle X^h, Y^h\rangle\rangle + \langle\langle[X^h, Y^h], Z^v\rangle\rangle\\ &= -\langle\langle(R(X, Y)\xi)^v, Z^v\rangle\rangle.\end{aligned}$$

  Hence

$$\tilde\nabla_{X^h} Y^h = \big(\nabla_X Y\big)^h - \frac{1}{2}\big(R(X, Y)\xi\big)^v.$$

- Take $A = X^h, B = Y^v, C = Z^h$. Then

$$
\begin{aligned}
2\big\langle\big\langle\tilde\nabla_{X^h}Y^v, Z^h\big\rangle\big\rangle &= \big\langle\big\langle[Z^h, X^h], Y^v\big\rangle\big\rangle \\
&= \big\langle\big\langle(R(X,Z)\xi)^v, Y^v\big\rangle\big\rangle = \big\langle R(X,Z)\xi, Y\big\rangle + \delta^2\big\langle R(X,Z)\xi, J\xi\big\rangle\big\langle Y, J\xi\big\rangle \\
&= \big\langle R(\xi,Y)X, Z\big\rangle + \delta^2\big\langle Y, J\xi\big\rangle\big\langle R(\xi, J\xi)X, Z\big\rangle \\
&= \big\langle\big\langle\big(R(\xi,Y)X + \delta^2\big\langle Y, J\xi\big\rangle R(\xi, J\xi)X\big)^h, Z^h\big\rangle\big\rangle.
\end{aligned}
$$

Take $A = X^h, B = Y^v, C = Z^v$. Then by (2), we have

$$
\begin{aligned}
2\big\langle\big\langle\tilde\nabla_{X^h}Y^v, Z^v\big\rangle\big\rangle &= X^h\big\langle\big\langle Y^v, Z^v\big\rangle\big\rangle + \big\langle\big\langle[X^h, Y^v], Z^v\big\rangle\big\rangle \\
&+ \big\langle\big\langle[Z^v, X^h], Y^v\big\rangle\big\rangle = \big\langle\big\langle(\nabla_X Y)^v, Z^v\big\rangle\big\rangle + \big\langle\big\langle Y^v, (\nabla_X Z)^v\big\rangle\big\rangle \\
&+ \big\langle\big\langle(\nabla_X Y)^v, Z^v\big\rangle\big\rangle - \big\langle\big\langle(\nabla_X Z)^v, Y^v\big\rangle\big\rangle = 2\big\langle\big\langle(\nabla_X Y)^v, Z^v\big\rangle\big\rangle.
\end{aligned}
$$

So, we see that

$$
\tilde\nabla_{X^h}Y^v = \frac{1}{2}\Big(R(\xi,Y)X + \delta^2\,\langle Y, J\xi\rangle\, R(\xi, J\xi)X\Big)^h + \Big(\nabla_X Y\Big)^v.
$$

- Take $A = X^v, B = Y^h, C = Z^h$. Then

$$
\begin{aligned}
2\big\langle\big\langle\tilde\nabla_{X^v}Y^h, Z^h\big\rangle\big\rangle &= X^v\big\langle\big\langle Y^h, Z^h\big\rangle\big\rangle + \big\langle\big\langle[X^v, Y^h], Z^h\big\rangle\big\rangle + \big\langle\big\langle[Z^h, X^v], Y^h\big\rangle\big\rangle \\
&- \big\langle\big\langle[Y^h, Z^h], X^v\big\rangle\big\rangle = \big\langle\big\langle(R(Y,Z)\xi)^v, X^v\big\rangle\big\rangle = \big\langle R(Y,Z)\xi, X\big\rangle \\
&+ \delta^2\big\langle R(Y,Z)\xi, J\xi\big\rangle\big\langle X, J\xi\big\rangle = \big\langle R(\xi,X)Y, Z\big\rangle + \delta^2\big\langle X, J\xi\big\rangle\big\langle R(\xi, J\xi)Y, Z\big\rangle \\
&= \big\langle\big\langle(R(\xi,X)Y + \delta^2\big\langle X, J\xi\big\rangle R(\xi, J\xi)Y)^h, Z^h\big\rangle\big\rangle.
\end{aligned}
$$

Take $A = X^v, B = Y^h, C = Z^v$. Then

$$
\begin{aligned}
2\big\langle\big\langle\tilde\nabla_{X^v}Y^h, Z^v\big\rangle\big\rangle &= Y^h\big\langle\big\langle Z^v, X^v\big\rangle\big\rangle + \big\langle\big\langle[X^v, Y^h], Z^v\big\rangle\big\rangle - \big\langle\big\langle[Y^h, Z^v], X^v\big\rangle\big\rangle \\
&= \big\langle\big\langle(\nabla_Y Z)^v, X^v\big\rangle\big\rangle + \big\langle\big\langle Z^v, (\nabla_Y X)^v\big\rangle\big\rangle \\
&- \big\langle\big\langle(\nabla_Y X)^v, Z^v\big\rangle\big\rangle - \big\langle\big\langle(\nabla_Y Z)^v, X^v\big\rangle\big\rangle = 0.
\end{aligned}
$$

So, we have

$$
\tilde\nabla_{X^v}Y^h = \frac{1}{2}\Big(R(\xi,X)Y + \delta^2\,\langle X, J\xi\rangle\, R(\xi, J\xi)Y\Big)^h.
$$

- Take $A = X^v, B = Y^v, C = Z^h$. Then we have

$$
\begin{aligned}
2\big\langle\big\langle\tilde\nabla_{X^v}Y^v, Z^h\big\rangle\big\rangle &= -Z^h\big\langle\big\langle X^v, Y^v\big\rangle\big\rangle + \big\langle\big\langle[Z^h, X^v], Y^v\big\rangle\big\rangle \\
&- \big\langle\big\langle[Y^v, Z^h], X^v\big\rangle\big\rangle = -\big\langle\big\langle(\nabla_Z X)^v, Y^v\big\rangle\big\rangle - \big\langle\big\langle X^v, (\nabla_Z Y)^v\big\rangle\big\rangle \\
&+ \big\langle\big\langle(\nabla_Z X)^v, Y^v\big\rangle\big\rangle + \big\langle\big\langle(\nabla_Z Y)^v, X^v\big\rangle\big\rangle = 0.
\end{aligned}
$$

Finally, take $A = X^v, B = Y^v, C = Z^v$. Then

$$
\begin{aligned}
2\langle\langle \tilde\nabla_{X^v} Y^v, Z^v\rangle\rangle &= X^v\langle\langle Y^v, Z^v\rangle\rangle + Y^v\langle\langle X^v, Z^v\rangle\rangle - Z^v\langle\langle X^v, Y^v\rangle\rangle \\
&= \delta^2\Big(\langle Y, JX\rangle\langle Z, J\xi\rangle + \langle Y, J\xi\rangle\langle Z, JX\rangle + \langle X, JY\rangle\langle Z, J\xi\rangle \\
&\quad + \langle X, J\xi\rangle\langle Z, JY\rangle - \langle X, JZ\rangle\langle Y, J\xi\rangle - \langle X, J\xi\rangle\langle Y, JZ\rangle\Big) \\
&= 2\delta^2\Big(\langle Y, J\xi\rangle\langle JX, Z\rangle + \langle X, J\xi\rangle\langle JY, Z\rangle\Big).
\end{aligned}
$$

Thus, we see that

$$\langle\langle \tilde\nabla_{X^v} Y^v, Z^v\rangle\rangle = \delta^2\Big(\langle Y, J\xi\rangle\langle JX, Z\rangle + \langle X, J\xi\rangle\langle JY, Z\rangle\Big).$$

On the other hand,

$$\langle\langle (JY)^v, Z^v\rangle\rangle = \langle JY, Z\rangle + \delta^2\langle Y, \xi\rangle\langle Z, J\xi\rangle$$

and

$$\langle\langle (J\xi)^v, Z^v\rangle\rangle = \langle J\xi, Z\rangle + \delta^2\langle Z, J\xi\rangle|\xi|^2 = (1+\delta^2|\xi|^2)\langle Z, J\xi\rangle.$$

Therefore,

$$\langle Z, J\xi\rangle = \frac{1}{1+\delta^2|\xi|^2}\langle\langle (J\xi)^v, Z^v\rangle\rangle$$

and, as a consequence,

$$
\begin{aligned}
\langle JY, Z\rangle &= \langle\langle (JY)^v, Z^v\rangle\rangle - \delta^2\langle Y, \xi\rangle\frac{1}{1+\delta^2|\xi|^2}\langle\langle (J\xi)^v, Z^v\rangle\rangle \\
&= \langle\langle (JY)^v - \frac{\delta^2}{1+\delta^2|\xi|^2}\langle Y, \xi\rangle (J\xi)^v, Z^v\rangle\rangle.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\langle\langle \tilde\nabla_{X^v} Y^v, Z^v\rangle\rangle = \delta^2\langle\langle \Big[\langle X, J\xi\rangle\Big(JY - \frac{\delta^2}{1+\delta^2|\xi|^2}\langle Y, \xi\rangle J\xi\Big) \\
+ \langle Y, J\xi\rangle\Big(JX - \frac{\delta^2}{1+\delta^2|\xi|^2}\langle X, \xi\rangle J\xi\Big)\Big]^v, Z^v\rangle\rangle.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\tilde\nabla_{X^v} Y^v = \delta^2\Big(\langle X, J\xi\rangle JY + \langle Y, J\xi\rangle JX \\
- \frac{\delta^2}{1+\delta^2|\xi|^2}\big(\langle Y, \xi\rangle\langle X, J\xi\rangle + \langle X, \xi\rangle\langle Y, J\xi\rangle\big)J\xi\Big)^v.
\end{aligned}
$$

The lemma is proved. ■

2. Geodesics of the Berger-Type Deformed Sasaki Metric

Consider a non-vertical curve Γ on the tangent bundle with metric (1). Geometrically, $\Gamma = \{x(\sigma), \xi(\sigma)\}$, where $x(\sigma)$ is a curve on M and $\xi(\sigma)$ is a vector field along this curve. Let σ be an arc length parameter on Γ. Then $\Gamma' = \big(\frac{dx}{d\sigma}\big)^h + (\nabla_{\frac{dx}{d\sigma}}\xi)^v$. Introduce the notations $x' = \frac{dx}{d\sigma}$ and $\xi' = \nabla_{\frac{dx}{d\sigma}}\xi$. Then

$$\Gamma' = (x')^h + (\xi')^v.$$

Now we can easily derive the differential equations of geodesic lines of metric (1).

Lemma 2.1. *Let (M, g, J) be a Kählerian manifold and TM_0 be its slashed tangent bundle with the Berger-type deformed Sasaki metric. The curve $\Gamma = \{x(\sigma), \xi(\sigma)\}$ is a geodesic on TM_0 if and only if $x(\sigma)$ and $\xi(\sigma)$ satisfy*

$$\begin{aligned} &x'' + \mathcal{R}(\xi, \xi')x' = 0 \\ &\xi'' + 2\delta^2\big\langle \xi', J\xi\big\rangle\Big(J\xi' - \tfrac{\delta^2}{1+\delta^2|\xi|^2}\big\langle\xi', \xi\big\rangle J\xi\Big) = 0, \end{aligned} \tag{3}$$

where $\mathcal{R}(\xi, \xi') = R(\xi, \xi') + \delta^2\langle\xi', J\xi\rangle R(\xi, J\xi)$, and R is the curvature operator of the base manifold M.

P r o o f. Using Lemma 1.2, find the derivative Γ'' and equalize it to zero

$$\begin{aligned} \Gamma'' = \tilde{\nabla}_{(x')^h+(\xi')^v}\big((x')^h + (\xi')^v\big) &= \Big(x'' + R(\xi, \xi')x' + \delta^2\langle\xi', J\xi\rangle R(\xi, J\xi)x'\Big)^h \\ &+ \Big(\xi'' + 2\delta^2\Big(\langle\xi', J\xi\rangle J\xi' - \frac{\delta^2}{1+\delta^2|\xi|^2}\langle\xi', \xi\rangle\langle\xi', J\xi\rangle\Big)J\xi\Big)^v \\ &= \Big(x'' + \Big(R(\xi, \xi') + \delta^2\langle\xi', J\xi\rangle R(\xi, J\xi)\Big)x'\Big)^h \\ &+ \Big(\xi'' + 2\delta^2\big\langle\xi', J\xi\big\rangle\Big(J\xi' - \frac{\delta^2}{1+\delta^2|\xi|^2}\big\langle\xi', \xi\big\rangle\Big)J\xi\Big)^v = 0. \end{aligned}$$

The lemma is proved. ∎

Consider now the unit tangent bundle T_1M.

Lemma 2.2. *Let (M^{2n}, g, J) be a Kählerian manifold and T_1M be its unit tangent bundle with the Berger-type deformed Sasaki metric. Set $c = |\xi'|$, $\mu = \big\langle\xi', J\xi\big\rangle$. The curve $\Gamma = \{x(\sigma), \xi(\sigma)\}$ is a geodesic on T_1M if and only if* (*a*) $c = const, \mu = const$;

(*b*) $x(\sigma)$ *and* $\xi(\sigma)$ *satisfy the equations*

$$\begin{aligned} & x'' + \mathcal{R}(\xi, \xi')x' = 0 \\ & \xi'' + c^2\xi + 2\delta^2\mu(J\xi' + \mu\xi) = 0, \end{aligned} \tag{4}$$

where $\mathcal{R}(\xi, \xi') = R(\xi, \xi') + \delta^2\mu R(\xi, J\xi)$ *and* R *is the curvature operator of the base manifold* M.

P r o o f. At each point $(q, \xi) \in T_1M$, the unit normal to T_1M is ξ^v. Indeed, with respect to metric (1), we have

$$\begin{aligned} &\big\langle\big\langle X^h, \xi^v \big\rangle\big\rangle = 0 \quad \text{for all } X \text{ tangent to } M, \\ &\big\langle\big\langle X^v, \xi^v \big\rangle\big\rangle = 0 \quad \text{for all } X \in \xi^\perp . \end{aligned}$$

As T_1M is a hypersurface in TM, the curve on T_1M is geodesic iff its second covariant derivative in TM is collinear to the unit normal, i.e. to ξ^v. That is why, to find the equations of geodesics on T_1M, it is sufficient to set $|\xi| = 1$ in (3) and to suppose the left-hand side of $(3)_2$ to be collinear ξ. Thus, we get

$$\begin{aligned} & x'' + \mathcal{R}(\xi, \xi')x' = 0 \\ & \xi'' + 2\delta^2\langle \xi', J\xi\rangle J\xi' = \rho\xi, \end{aligned} \tag{5}$$

where ρ is some function.

Put $c = |\xi'|$. Then $c = const$, since $\langle \xi'', \xi' \rangle = 0$ directly from $(5)_2$. Put $\mu = \langle \xi', J\xi \rangle$. Then $\mu = const$, since $\mu' = \langle \xi'', J\xi \rangle = 0$. Multiplying $(5)_2$ by ξ, we find

$$-\rho = c^2 + 2\delta^2\mu^2 = const.$$

After substituting it into (5), we get what was claimed. ∎

The difference in the description of solutions of (3) and (4) follows from different behavior of the operator $\mathcal{R}(\xi, \xi')$ along $\pi \circ \Gamma$.

Proposition 2.1. *Let* Γ *be a geodesic of the slashed or unit tangent bundle over Kählerian locally symmetric manifold* M *and* $\gamma = \pi \circ \Gamma$. *Then* $\mathcal{R}(\xi, \xi')$ *is parallel along* γ *for the case of* T_1M *and non-parallel for the case of* TM_0.

P r o o f. First, consider the case of T_1M. Using (4), we get

$$\begin{aligned} \mathcal{R}'(\xi, \xi') &= R(\xi, \xi'') + \delta^2\mu R(\xi', J\xi) + \delta^2\mu R(\xi, J\xi') \\ &= -2\delta^2\mu R(\xi, J\xi') - \delta^2\mu R(J\xi', \xi) + \delta^2\mu R(\xi, J\xi') = 0. \end{aligned}$$

Here we also used the fact that $R(JX,JY)=R(X,Y)$.
A similar but longer calculation shows that for the of case of TM_0

$$\mathcal{R}'(\xi,\xi')=\frac{2\delta^6\langle\xi',J\xi\rangle\langle\xi',\xi\rangle(1-|\xi|^2)}{1+\delta^2|\xi|^2}R(\xi,J\xi),$$

which completes the proof. ■

Theorem 2.1. *Let Γ be a geodesic of the unit tangent bundle with the Berger-type deformed Sasaki metric over the Kählerian locally symmetric manifold M, and $\gamma=\pi\circ\Gamma$. Then all geodesic curvatures of γ are constant.*

P r o o f. For the case of T_1M, Proposition 2.1 implies that if Γ is a geodesic on T_1M, then along each curve $\gamma=\pi\circ\Gamma$

$$x^{(p+1)}(\sigma)=-\mathcal{R}(\xi,\xi')\,x^{(p)}(\sigma)\quad p\ge 1. \tag{6}$$

On the other hand, it is rather evident that

$$\langle\mathcal{R}(\xi,\xi')X,Y\rangle=-\langle\mathcal{R}(\xi,\xi')Y,X\rangle.$$

This fact and (6) imply

$$|x^{(p)}(\sigma)|=const\quad\text{for all } p\ge 1. \tag{7}$$

Indeed,

$$\tfrac{d}{d\sigma}\,|x^{(p)}(\sigma)|^{\,2}=2\,\langle x^{(p+1)}(\sigma),x^{(p)}(\sigma)\rangle=-2\,\langle\mathcal{R}(\xi,\xi')\,x^{(p)}(\sigma),x^{(p)}(\sigma)\rangle=0.$$

Denote by s an arc length parameter on γ. Then $x'_\sigma=x'_s\frac{ds}{d\sigma}$, and therefore

$$1=\|\Gamma'\|^2=\left|\frac{ds}{d\sigma}\right|^2+|\xi'|^2+\delta^2\langle\xi',J\xi\rangle^2=\left|\frac{ds}{d\sigma}\right|^2+c^2+\delta^2\mu^2.$$

Hence

$$\frac{ds}{d\sigma}=\sqrt{1-c^2-\delta^2\mu^2}=\sqrt{1-\lambda^2}, \tag{8}$$

where $\lambda^2=c^2+\delta^2\mu^2=const$.

Denote by $\nu_1,\dots,\nu_{2n-1}$ the Frenet frame along γ and by $k_1,\dots,k_{2n-1}$ the geodesic curvatures of γ. Then, keeping in mind (8), we have

$$x'=\sqrt{1-\lambda^2}\,\nu_1,$$
$$x''=(1-\lambda^2)k_1\nu_2.$$

Now (7) implies $k_1=const$. Next,

$$x^{(3)}=(1-\lambda^2)^{3/2}\,k_1(-k_1\nu_1+k_2\nu_3),$$

and again (7) implies $k_2=const$. By continuing the process, we finish the proof. ■

As proven in [10], for the case of T_1CP^n and TCP^n with the Sasaki metric, the curvatures of $\gamma = \pi\circ\Gamma$ are zeroes starting from k_6. It is rather remarkable that this property is still valid for the case of the Berger-deformed Sasaki metric on the unit tangent bundle over the Kählerian manifold of constant holomorphic curvature. It is well known that the complete simply connected Kählerian manifold of the constant holomorphic sectional curvature k is isometric to: the complex projective space CP^n for $k > 0$; the open ball $D^n \subset C^n$ for $k < 0$; C^n for $k = 0$.

Theorem 2.2. *Let Γ be a geodesic of the unit tangent bundle with the Berger-type deformed Sasaki metric over Kählerian manifold M^{2n} ($n \geq 3$) of the constant holomorphic curvature. Then the geodesic curvatures of $\gamma = \pi\circ\Gamma$ are all constant, and $k_6 = \dots = k_{2n-1} = 0$.*

P r o o f. For the case of the Kählerian manifold of constant holomorphic curvature k we have

$$R(X,Y)Z = \frac{k}{4}\Big(\langle Y,Z\rangle X - \langle X,Z\rangle Y + \big\langle JY,Z\big\rangle JX - \big\langle JX,Z\big\rangle JY + 2\big\langle X,JY\big\rangle JZ\Big).$$

Rewrite equations (4) as follows:

$$x'' = -\frac{k}{4}\Big(\langle \xi',x'\rangle\xi - \langle\xi,x'\rangle\xi' + \big\langle J\xi',x'\big\rangle J\xi - \big\langle J\xi,x'\big\rangle J\xi' + 2\big\langle\xi,J\xi'\big\rangle Jx'\Big) - \frac{1}{2}\delta^2\mu\Big(\big\langle J\xi,x'\big\rangle\xi - \big\langle\xi,x'\big\rangle J\xi - Jx'\Big) \qquad (9)$$

$$\xi'' = -(c^2+2\delta^2\mu^2)\xi - 2\delta^2\mu J\xi'. \qquad (10)$$

Equation (9) shows that x'' is a linear combination of at most $\xi,\xi',J\xi,J\xi'$ and Jx'. Therefore, Jx'' is a linear combination of at most $J\xi,J\xi',\xi,\xi',x'$. Equation (10) shows that ξ'' is a linear combination of at most ξ and $J\xi'$. Therefore, $J\xi''$ is a linear combination of at most $J\xi$ and ξ'.

For the sake of brevity, denote a point-wise linear combination of the corresponding vectors by $l.c.(\cdot,\cdot,\dots)$. Then we can write

$$x'' = l.c.(\xi,\xi',J\xi,J\xi',Jx'), \quad Jx'' = l.c.(J\xi,J\xi',\xi,\xi',x'),$$
$$\xi'' = l.c.(\xi,J\xi'), \qquad J\xi'' = l.c.(J\xi,\xi').$$

Hence

$$x''' = l.c.(\xi',\xi'',J\xi',J\xi'',Jx'') = l.c.\Big(\xi',l.c.(\xi,J\xi'),J\xi',l.c.(J\xi,\xi'),l.c.(J\xi,J\xi',\xi,\xi',x')\Big) = l.c.(\xi,\xi',J\xi,J\xi',x').$$

In a similar way,
$$x^{(4)} = l.c.(\xi, \xi', J\xi, J\xi', Jx').$$
Continuing the process, we conclude that
$$x^{(p)} = l.c.(\xi, \xi', x', J\xi, J\xi', Jx')$$
for all p. This means that no more than the first six derivatives could be linearly independent, and hence at least
$$x^{(7)} = l.c.(x', x'', \dots, x^{(6)}). \tag{11}$$
On the other hand, the Frenet formulas yield
$$x' = \sqrt{1-\lambda^2}\,\nu_1 \quad x'' = (1-\lambda^2)k_1\nu_2, \quad x''' = (1-\lambda^2)^{3/2}(-k_1^2\nu_1 + k_1k_2\nu_3)$$
and, in general,
$$x^{(2m)} = l.c.(\nu_2, \dots, \nu_{2m-2}) + (1-\lambda^2)^m\, k_1 \dots k_{2m-1}\, \nu_{2m},$$
$$x^{(2m+1)} = l.c.(\nu_1, \dots, \nu_{2m-1}) + (1-\lambda^2)^{m+1/2}\, k_1 \dots k_{2m}\, \nu_{2m+1},$$
where λ is given by (8). For $m = 3$, we have
$$x^{(7)} = l.c.(\nu_1, \dots, \nu_5) + (1-\lambda^2)^{7/2}\, k_1 \dots k_6\, \nu_7.$$
Notice that for all $p \le 6$,
$$x^{(p)} = l.c.(\nu_1, \dots, \nu_6).$$
From (11) it follows that
$$(1-\lambda^2)^{7/2}\, k_1 \dots k_6\, \nu_7 = l.c.(\nu_1, \dots, \nu_6),$$
and therefore, at least $k_6 = 0$. ■

Acknowledgement. The author thanks Professor P.T. Nagy for the idea of considering the deformation proposed and a hospitality during the visit to the University of Debrecen (Hungary).

References

[1] *S. Sasaki*, Geodesics on the Tangent Sphere Bundle over Space Forms. — *J. Reine Angew. Math.* **288** (1976), 106–120.

[2] *K. Sato*, Geodesics on the Tangent Bundle over Space Forms. — *Tensor* **32** (1978), 5–10.

[3] *P.T. Nagy*, Geodesics on the Tangent Sphere Bundle of a Riemannian Manifold. — *Geom. Dedic.* **7** (1978), No. 2, 233–244.

[4] *J. Cheeger and D. Gromoll*, On the Structure of Complete Manifolds of Nonnegative Curvature. — *Ann. Math.* **96** (1972), 413–443.

[5] *M. Sekizawa*, Curvatures of Tangent Bundles with Cheeger–Gromoll Metric. — *Tokyo J. Math.* **14** (1991), No. 2, 407–417.

[6] *M.T.K. Abbassi and M. Sarih*, On Riemannian g-Natural Metrics of the Form $ag^s+bg^h+cg^v$ on the Tangent Bundle of a Riemannian Manifold (M,g). — *Mediter. J. Math.* **2** (2005), 19–43.

[7] *V. Oproiu*, A Kähler Einstein Structure on the Tangent Bundle of a Space Form. — *Int. J. Math. Math. Sci.* **25** (2001), 183–195.

[8] *M. Munteanu*, Cheeger Gromoll Type Metrics on the Tangent Bundle. arXiv:math.DG/0610028 (30 Sept. 2006).

[9] *O. Kowalski*, Curvature of the Induced Riemannian Metric on the Tangent Bundle of a Riemannian Manifold. — *J. Reine Angew. Math.* **250** (1971), 124–129.

[10] *A. Yampolsky*, On Characterisation of Projections of Geodesics of Sasaki Metric TCP^n and T_1CP^n. — *Ukr. Geom. Sbornik* **34** (1991), 121–126. (Russian) (Engl. transl.: *J. Math. Sci.* **69** (1994), No. 1, 916–920.)