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On the Universal Models of Commutative Systems of Linear Operators

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The universal models are constructed for a system of linear bounded non-selfadjoint operators $\{A_1, A_2\}$ acting in a Hilbert space H such that 1) $[A_1, A_2] = 0$, $[A_1^*, A_2] = 0$; 2) $\dfrac{A_k - A_k^*}{i} \geq 0$ $(k = 1, 2)$; 3) the function $A(\lambda) = A_1(\lambda_1) A_2(\lambda_2)$ $(A_k(\lambda_k) = A_k(I - \lambda_k A_k)^{-1}$, $k = 1, 2)$ is an entire function of the exponential type. It is proved that this class of linear operator systems is realized by the restriction on invariant subspaces of systems of operator of integration by independent variables in $L^2(\Omega) \otimes l^2$ where Ω is a rectangle in $\mathbb{R}^2$.

Key words: non-selfadjoint operators, universal models.

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Triangular models of linear non-selfadjoint operators first constructed by M.S. Livšic play an important role in problems of spectral analysis for this class of operators and have multiple applications [1–3]. An universal model for operators from the class $\Lambda^{\exp}$ was obtained in 1965 by G. Kisilevskii [4] ($A \in \Lambda^{\exp}$ if A is dissipative and its Fredholm resolvent $A(\lambda) = A(I - \lambda A)^{-1}$ is an entire function of the exponential type). A generalization of this result to the operators with infinite-dimensional imaginary part was obtained later with some restrictions by G. Kisilevskii and M.S. Brodskii [5], L.L. Vaksman [6] and in general case by L. Isayev in 1968 [7]. The theorem of Isayev, in which the universality of the Volterra integration operator in the space $L^2(0, l) \otimes l^2$ $(0 < l < \infty)$ is proved, is one of the profound results in the field. Namely, it is proved that every bounded linear operator A of the class $\Lambda^{\exp}$ is unitarily equivalent to restriction on the invariant subspace of the Volterra integration operator in $L^2(0,l)\otimes l^2$. The proof is based on the fact that the characteristic function of operators from the class $\Lambda^{\exp}$ is the proper divisor of the function $\exp\{i\lambda l\}\cdot I_{l^2}$ (l is the type of $A(\lambda)$). Another method of proving based on the Wiener–Paley theorem [8] was proposed by V.A. Zolotarev [3, 9]. The generalization of this method made it possible to prove the similar theorem for the case when $A(\lambda)$ has order $\rho > 1$ and type σ $(0<\sigma<\infty)$.

The aim of the paper is to prove the universality of the Volterra operator system of integrations by independent variables in the space $L^2_D\otimes l^2$, where D is a rectangle in $\mathbb{R}_+\times\mathbb{R}_+$. The proof is based on the generalization of the methods used in [3, 9] and on the Polya–Plancherel theorem which is an analogue of the Wiener–Paley theorem for entire functions of several complex variables.

I. First, following [3, 9], consider the general scheme of the construction of universal models for dissipative operators. Let the linear bounded A be given in a Hilbert space H such that a) A is dissipative, $A_I=\frac{1}{2i}(A-A^*)\geq 0$; b) $\operatorname{Ker}A=\operatorname{Ker}A^*=\{0\}$. Consider the Cauchy problem

$$\begin{cases} Ai\dfrac{d}{dt}f(t)=f(t)\\ f(0)=f, \quad t\in\mathbb{R}_+,\end{cases} \tag{1}$$

where $f(t)$ is a vector function from H and $f\in H$. This Cauchy problem generates a contractive semigroup [3, 9]

$$T_tf\overset{\text{def}}{=}f(t) \tag{2}$$

since

$$\langle (I-T_t^*T_t)\,f,f\rangle = 2\int_0^t \big\langle A_If'(\xi),f'(\xi)\big\rangle\,d\xi\geq 0. \tag{3}$$

Let $\varphi=\sqrt{2A_I}$, then (3) implies that [3]

$$\int_0^\infty \left\|\varphi T_t'f\right\|^2dt=\|Bf\|^2, \tag{4}$$

where $B^2=I-K$, $0\leq B\leq I$ and the operator K equals

$$K=s-\lim_{t\to\infty}T_t^*T_t, \tag{5}$$

besides, limit (5) always exists [3, 9]. Hereinafter the following statement plays an important role.

Theorem 1. [3]. *Let a linear bounded dissipative completely non-selfadjoint operator A such that* $\operatorname{Ker} A = \operatorname{Ker} A^* = \{0\}$ *be given in a Hilbert space H. Then*

$$\int_0^\infty |F_t(f,g)|^2\, dt < \infty, \tag{6}$$

where the function $F_t(f,g)$ is given by $F_t(f,g) = \langle T_t f, g\rangle$ (or $F_t\langle f,g\rangle = \langle T_t' f, g\rangle$), besides, T_t is the contractive semigroup (2) and the vectors f and g belong to the following dense sets in H:

$$f \in AH; \quad g = \sum_{k=0}^{n} (A^*)^k A_I h_k, \quad \forall n \in \mathbb{Z}_+,\ \forall h_k \in H.$$

An entire operator function $F(\lambda)$: $\mathbb{C} \to [H,H]$ is said to be the function of the exponential type [2] if

$$\|F(\lambda)\| \le Ce^{a|\lambda|}, \quad \forall \lambda \in \mathbb{C},$$

where a, $C \in \mathbb{R}_+$. The exact lower bound of such a for which there exists a finite C such that this estimation takes place is said to be the type of the function $F(\lambda)$.

Class $\Lambda^{\exp}$. *A linear bounded operator A is said to belong to the class $\Lambda^{\exp}$* [2, 3] *if*

1) $A_I \ge 0$;

2) *the Fredholm resolvent $A(\lambda) = A(I - \lambda A)^{-1}$ of an operator A is an entire function of the exponential type.*

Denote the type of the resolvent $A(\lambda)$ by $l(A)$.

Consider the model example of the operators from the class $\Lambda^{\exp}$. Denote by $L_r^2(0,l)$ the tensor product of the Hilbert spaces $L^2(0,l) \otimes l_r^2$, $0 < l < \infty$, $1 \le r \le \infty$,

$$L_r^2(0,l) = \left\{ f(x) = \left(f^1(x), \ldots, f^r(x)\right) : \int_0^l \|f(x)\|_{l_r^2}^2 dx < \infty \right\}, \tag{7}$$

where $\|f(x)\|_{l_r^2}^2 = \sum_{k=1}^{r} \left\| f^k(x) \right\|^2$. Specify in $L_r^2(0,l)$ the operator $\tilde{A} = \left(i \int_x^l .\, dt \right) \otimes I_{l_r^2}$,

$$\left(\tilde{A} f\right)(x) = \left(i \int_x^l f^1(t) dt, \ldots, i \int_x^l f^r(t) dt \right), \tag{8}$$

where $f(x) = \big(f^1(x), \dots, f^r(x)\big) \in L^2_r(0,\infty)$. It is easy to see [3] that the operator $\tilde{A}$ (8) belongs to the class $\Lambda^{\exp}$ since $\left\|\tilde{A}(\lambda)\right\| \le l \cdot \exp\{l|\lambda|\}$. Moreover, the semigroup $\tilde{T}_t$ (2) generated by the operator $\tilde{A}$ (8) is nilpotent [3],

$$\left(\tilde{T}_t f\right)(x) = \chi_{(0,l)}(x) f(x+t), \tag{9}$$

where $f(x) \in L^2_r(0,l)$ and $\chi_{(0,l)}(x)$ is the characteristic function of the set $(0,l)$.

II. A classical result by Wiener and Paley establishing relation between the growth of the entire function and the support of its Fourier transform is as follows [3, 7].

The Wiener–Paley Theorem 2. *The function $F(\lambda)$ can be represented as*

$$F(\lambda) = \int\limits_a^b e^{i\lambda t} f(t)dt, \quad -\infty < a < b < \infty,$$

where $f(t) \in L^2_{(a,b)}$, if and only if $F(\lambda)$ is an entire function of the exponential type and $F(\lambda) \in L^2_{\mathbb{R}}$ as $x \in \mathbb{R}$. The finite interval $[a,b]$ containing the support of $f(t)$ is defined by the formulas

$$a = -\overline{\lim_{y\to\infty}} \frac{\ln|F(iy)|}{y}; \quad b = \overline{\lim_{y\to\infty}} \frac{|F(-iy)|}{y}.$$

The main result on the universality of the integration operator $\tilde{A}$ (8) for the operators from the class $\Lambda^{\exp}$ is as follows [2, 3].

Theorem 3. *Each completely non-selfadjoint operator A from the class $\Lambda^{\exp}$ such that $\operatorname{Ker} A = \operatorname{Ker} A^* = \{0\}$ is unitarily equivalent to the restriction of the operator $\tilde{A}$ (8) on one of the invariant eigensubspaces in $L^2_r(0,l)$ (7), besides, $l = l(A)$.*

P r o o f. The proof of the theorem in general terms [3] is presented below. Equation (1) for T_t implies that

$$iA(\lambda) = \int\limits_0^\infty e^{i\lambda t} T_t dt, \quad \forall \lambda \in \mathbb{C}_+$$

since $\|T_t\| \le 1$. Therefore

$$i\langle A(\lambda)f, g\rangle = \int\limits_0^\infty e^{i\lambda t} \langle T_t f, g\rangle \, dt, \quad \forall \lambda \in \mathbb{C}_+, \tag{10}$$

where f and g belong to the dense sets in H (see Theorem 1). Since the function

$$F_t(f,g)=\begin{cases}\langle T_tf,g\rangle\,; & t\in\mathbb{R}_+;\\ 0; & t\in\mathbb{R}_-\end{cases}$$

belongs to $L^2_{\mathbb{R}}$ (Theorem 1), then its Fourier transform is also a function from $L^2_{\mathbb{R}}$ in view of the Plancherel theorem [8] and so $\langle A(x)f,g\rangle\in L^2_{\mathbb{R}}$ $(x\in\mathbb{R})$ (see (10)). Therefore $\langle T_tf,g\rangle=0$ as $t>l(A)$ in view of the Wiener – Paley Theorem 2. Taking into account the density of the set of vectors f and g in H (Theorem 1), we obtain that

$$iA(\lambda)=\int\limits_0^{l(A)}e^{i\lambda t}T_tdt.$$

The bilinear analogue of formula (4) in this case is given by

$$\int\limits_0^l\left\langle\varphi T'_tf,\varphi T'_tg\right\rangle dt=\langle f,g\rangle,\quad l=l(A).\tag{11}$$

Expanding the function $\varphi T'_tf$ in the series in terms of the orthonormal basis $\{e_k\}_1^r$ in $\overline{(A)_IH}$ $(1\le r\le\infty)$, we obtain that

$$\varphi T'_tf=\sum_{k=1}^r\phi_k(t,f)e_k,\tag{12}$$

where $\phi_k(t,f)=\langle\varphi T'_tf,e_k\rangle\in L^2_{(0,l)}$ (Theorem 1). (11) yields that

$$\langle f,g\rangle_H=\sum_{k=1}^r\langle\phi_k(t,f),\phi_k(t,g)\rangle_{L^2_{(0,l)}}.$$

Therefore the operator U: $H\to L^2_r(0,l)$,

$$Uf\overset{\text{def}}{=}\phi(t,f)=(\phi_1(t,f),\ldots,\phi_r(t,f))\,,$$

specified on the dense set Ah in H is an isometry. Let $f=iAh\in AH$, then

$$\varphi T'_tT_xf=\varphi T'_tiAT_xh=\varphi T_tT_xh=\varphi T_{t+x}h=\varphi\frac{d}{dt}T_{t+x}f$$

for all $x\in[0,l]$. By (12), we have

$$\langle T_xf,T_yg\rangle_H=\langle\phi(t+x,f),\phi(t+y,g)\rangle_{L^2_r(0,l)}$$

and so $UT_xf=\tilde{T}_xUf$ where $\tilde{T}_x$ is given by formula (9). Thus the semigroup T_x in H is realized in $L^2_r(0,l)$ (7) as a restriction of $\tilde{T}_x$ (9) on a subspace in $L^2_r(0,l)$ invariant under $\tilde{A}$ (8). ■

III. Produce an analogue of the above construction with two variables. Let a system of twice commuting operators $\{A_1, A_2\}$, $[A_1, A_2] = 0$, $[A_1^*, A_2] = 0$, be given, besides, every operator A_k is dissipative and $\operatorname{Ker} A_k = \operatorname{Ker} A_k^* = \{0\}$ ($k =$ 1, 2). Similarly to (9.1), (9.2), associate with every operator A_k the semigroup T_{t_k},

$$T_{t_k} f = f\,(t_k) : \begin{cases} iA_k \partial_k f\,(t_k) = f(t_k) \\ f(0) = f, \quad t_k \in \mathbb{R}_+, \end{cases} \tag{13}$$

where $f\,(t_k)$ is a vector function from H, $f \in H$, and $\partial_k = \partial/\partial t_k$, $k = 1, 2$. With the help of T_{t_k} (13), specify the two-parameter semigroup on the cone $\mathbb{R}_+^2$

$$T_t = T_{t_1} T_{t_2}, \tag{14}$$

where $t = (t_1, t_2) \in \mathbb{R}_+^2$. The correctness of the specification of semigroup T_t (14) follows from the commutativity of the Fredholm resolvents

$$A_k\,(\lambda_k) = A_k\,(I - \lambda_k A_k)^{-1}, \quad k = 1, 2, \tag{15}$$

of operators A_k. The Cauchy problems (13) imply

$$\left\langle \left(I - T_{t_k}^* T_{t_k}\right) f, f \right\rangle = 2 \int_0^{t_k} \langle (A_k)_I\, \partial_k f\,(\xi_k)\,, \partial_k f\,(\xi_k) \rangle \, d\xi_k, \quad k = 1, 2, \tag{16}$$

where $f\,(\xi_k)$ is the solution of (13) $k = 1, 2$; therefore every T_{t_k} is a semigroup of contractions in view of dissipativity of A_k, $(A_k)_I \geq 0$ ($k = 1$, 2). If $f \in A_1 A_2 H$ and $f(t) = T_t f$, where T_t is given by (14), then it is easy to see that

$$\begin{gathered} \left\langle \left(I - T_{t_1}^* T_{t_1} - T_{t_2}^* T_{t_2} + T_t^* T_t\right) f, f \right\rangle \\ = 4 \int_0^{t_1} \int_0^{t_2} \langle (A_1)_I\, (A_2)_I\, \partial_1 \partial_2 f(\xi), \partial_1 \partial_2 f(\xi) \rangle \, d\xi_1 d\xi_2 \geq 0, \end{gathered} \tag{17}$$

and thus

$$I - T_{t_1}^* T_{t_1} - T_{t_2}^* T_{t_2} + T_t^* T_t \geq 0, \quad \forall t \in \mathbb{R}_+^2. \tag{18}$$

Similarly to (5) (in view of (16), (17)), it is easy to show that the limits

$$K_s = s - \lim_{t_s \to \infty} T_{t_s}^* T_{t_s}, \quad s = 1, 2; \quad K = s - \lim_{t_1, t_2 \to \infty} T_t^* T_t \tag{19}$$

exist. It is obvious that the selfadjoint operators K_1, K_2 and K have the properties

$$0 \leq K_s \leq I, \; s = 1, 2; \quad 0 \leq K \leq I; \quad 0 \leq I - K_1 - K_2 + K.$$

As $t_1 \to \infty$, $t_2 \to \infty$ in formulas (16) and (17), we have

$$\begin{gathered} 2\int\limits_0^\infty \langle (A_s)_I\, \partial_s f\,(t_s)\,, \partial_s f\,(t_s)\rangle\, dt_s = \|B_s f\|^2\,, \quad s = 1,2; \\ 4\int\limits_0^\infty \int\limits_0^\infty \langle (A_1)_I\,(A_2)_I\, \partial_1\partial_2 f(t), \partial_1\partial_2 f(t)\rangle\, dt_1 dt_2 = \|Bf\|^2, \end{gathered} \tag{20}$$

where $B_s^2 = I - K_s$, $0 \le B_s \le I$ ($s = 1, 2$); and $B^2 = I - K_1 - K_2 + K$, $0 \le B \le I$. The inequality $B \le I$ follows from

$$0 \le T_{t_2}^* \left(I - T_{t_1}^* T_{t_1}\right) T_{t_2} + T_{t_1}^* T_{t_1} \le I$$

(see (18)) after proceeding to the limit.

Formulate an analogue of Theorem 2 with two variables.

Theorem 4. *Let in a Hilbert space H the system of twice commuting linear bounded operators $\{A_1, A_2\}$ be given such that a) every operator A_k is completely non-selfadjoint; b) A_k is dissipative, $(A_k)_I \ge 0$; c)* $\operatorname{Ker} A_k = \operatorname{Ker} A_k^* = \{0\}$, *$k = 1, 2$. Then*

$$\int\limits_{\mathbb{R}_+^2} |F_t(f,g)|^2\, dt < \infty, \tag{21}$$

where the function $F_t(f,g)$ equals $F_t(f,g) = \langle T_t f, g\rangle$ (or $F_t(f,g) = \langle \partial_1 T_t f, g\rangle$, $F_t(f,g) = \langle \partial_2 T_t f, g\rangle$, $F_t(f,g) = \langle \partial_1\partial_2 T_t f, g\rangle$), besides, T_t is given by (14) *and the vectors f and g belong to the following dense in H sets:*

$$f \in A_1 A_2 H; \quad g = \sum_{k=0}^{n}\sum_{s=0}^{m} (A_1^*)^k\,(A_2^*)^s\,(A_1)_I\,(A_2)_I\, h_{k,s}$$

for all n, $m \in \mathbb{Z}_+^2$ and all $h_{k,s} \in H$.

P r o o f. Restrict oneself to the proof of statement of the theorem for $F_t\langle f, g\rangle = \langle T_t f, g\rangle$ (for other $F_t(f,g)$ the proof is similar). First of all, note that the density of the manifold g in H is proved in [5]. Let $f \in A_1 A_2 H$ and $g = (A_1^*)^n\,(A_2^*)^m\,(A_1)_I\,(A_1)_I\,(A_2)_I\, h$, n, $m \in \mathbb{Z}_+$, $h \in H$, then

$$\int\limits_{\mathbb{R}_+^2} |F_t(f,g)|^2\, dt = \int\limits_{\mathbb{R}_+^2} |\langle (A_1)_I\,(A_2)_I\, \varphi_t(f), h\rangle|^2\, dt,$$

where $\varphi_t(f) = \partial_1\partial_2 T_t A_1^{n+1} A_2^{m+1} f$. Since $(A_1)_I\,(A_2)_I \ge 0$, then

$$|\langle (A_1)_I\,(A_2)_I\, h, \varphi\rangle|^2 \le \langle (A_1)_I\,(A_2)_I\, h, h\rangle \cdot \langle (A_1)_I\,(A_2)_I\, \varphi, \varphi\rangle$$

for all h, $\varphi \in H$. Hence,

$$\int\limits_{\mathbb{R}_+^2} |F_t(f,g)|^2 \, dt \le \int\limits_{\mathbb{R}_+^2} \langle (A_1)_I \, (A_2)_I \, \varphi_t(f), \varphi_t(f) \rangle \, \langle (A_1)_I \, (A_2)_I \, h, h \rangle \, dt$$

$$= \langle (A_1)_I \, (A_2)_I \, h, h \rangle \cdot \frac{1}{4} \left\| B A_1^{n+1} A_2^{m+1} f \right\|^2 < \infty$$

in virtue of (20). ■

IV. An entire operator function $F(\lambda)$: $\mathbb{C}^2 \to [H,H]$ is said to be the function of exponential type [10] if

$$\|F(\lambda)\| \le C \exp\{a_1 |\lambda_1| + a_2 |\lambda_2|\}$$

for all $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, where C, a_1, $a_2 \in \mathbb{R}_+$. An exact lower bound of such a_1 and a_2 for which the finite $C \in \mathbb{R}_+$ exists and this estimation is true is said to be [8] the system of conjugate types $\{a_1, a_2\}$ of a function $F(\lambda)$.

Class $\Lambda_0^{\exp}$. *A system of the linear bounded operators $\{A_1, A_2\}$ acting in a Hilbert space H is said to belong to the class $\Lambda_0^{\exp}$ if*

1) $[A_1, A_2] = 0$, $[A_2, A_1^*] = 0$;

2) $(A_k)_I \ge 0$, $k = 1, 2$;

3) *the function $A(\lambda) = A_1(\lambda_1) \, A_2(\lambda_2)$ is an entire function of the exponential type, where $A_k(\lambda_k)$ are given by* (15), $k = 1, 2$.

Denote by $l(A) = \{l_1, l_2\}$ the system of the conjugate types of a function $A(\lambda)$, besides, it is obvious that $l_k = l(A_k)$, where $l(A_k)$ is the type of the Fredholm resolvent $A_k(\lambda_k)$ (15), $k = 1, 2$.

Give the model example of an operator system $\{A_1, A_2\}$ of the class $\Lambda_0^{\exp}$. Let $\Omega = [0, l_1] \times [0, l_2]$ be a rectangle in $\mathbb{R}_+^2$, $0 < l_k < \infty$, $k = 1, 2$. Similarly to (7), consider the Hilbert space $L_r^2(\Omega) = L^2(\Omega) \otimes l_r^2$, $1 \le r \le \infty$,

$$L_r^2(\Omega) = \left\{ f(x) = \left(f^1(x), \dots, f^r(x)\right) : \int\limits_{\Omega} \|f(x)\|_{l_r^2}^2 dx < \infty \right\}, \tag{22}$$

where $x = (x_1, x_2) \in \Omega$, $\|f(x)\|_{l_r^2}^2 = \sum\limits_{k=1}^{r} \left| f^k(x) \right|^2$ and $dx = dx_1 dx_2$. In $L_r^2(\Omega)$ define a system of the twice commuting operators

$$\left(\tilde{A}_1 f\right)(x) = i \int\limits_{x_1}^{l_1} f(\xi_1, x_2) \, d\xi_1; \quad \left(\tilde{A}_2 f\right)(x) = i \int\limits_{x_2}^{l_2} f(x_1, \xi_2) \, d\xi_2, \tag{23}$$

where $f(x) \in L_r^2(\Omega)$. Since $\left\|\tilde{A}_k(\lambda_k)\right\| \leq l_k \cdot \exp\{l_k |\lambda_k|\}$ $(k = 1, 2)$, then $\left\|\tilde{A}(\lambda)\right\| \leq l_1 l_2 \exp\{l_1 |\lambda_1| + l_2 |\lambda_2|\}$, and thus the operator system $\left\{\tilde{A}_1, \tilde{A}_2\right\}$ (23) belongs to the class $\Lambda_0^{\exp}$. It is obvious that the semigroup $\tilde{T}_t$ (14) corresponding to $\left\{\tilde{A}_1, \tilde{A}_2\right\}$ (23) is nilpotent,

$$\left(\tilde{T}_t f\right)(x) = \chi_\Omega(x) f(x + t), \tag{24}$$

where $f(x) \in L_r^2(\Omega)$, and $\chi_\Omega(x)$ is the characteristic function of the set Ω.

V. To formulate an analogue of the Wiener–Paley Theorem 2 with many parameters, list the necessary information from the theory of functions of multiple complex variables [10]. Associate every entire function $f(\lambda)$: $\mathbb{C}^n \to \mathbb{C}$ of the exponential type with

$$h_f(x, y) \stackrel{\text{def}}{=} \lim_{R \to \infty} \frac{1}{R} \ln |f(x + iRy)|,$$

where $\lambda = x + iy$ and x, $y \in \mathbb{R}^n$. The function

$$h_f(y) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} h_f(x, y) \tag{25}$$

is said to be the P-indicator (the Polya–Plancherel indicator) [10] of an entire function of the exponential type $f(\lambda)$.

The function

$$H_M(y) \stackrel{\text{def}}{=} \sup_{x \in M} \left(\sum_{k=1}^n x_k y_k\right) \tag{26}$$

is said to be [10] the support function $H_M(y) \leq \infty$, $y \in \mathbb{R}^n$, of the convex set $M \in \mathbb{R}^n$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are points from $\mathbb{R}^n$. (26) implies that the bounded set M is contained in the semispace $\left\{x \in \mathbb{R}^n : \sum_{k=1}^n x_k y_k \leq H_M(y)\right\}$, $H_M(y) < \infty$, and is not contained in the non-semispace $\left\{x \in \mathbb{R}^n : \sum_{k=1}^n x_k y_k \leq H_M(y) - \epsilon\right\}$, $H_M(y) < \infty$ and $\epsilon > 0$. The hyperplane $\left\{x \in \mathbb{R}^n : \sum_{k=1}^n x_k y_k = H_M(y)\right\}$, which is said to be the support hyperplane of the set M, is the boundary of the semispace $\left\{x \in \mathbb{R}^n : \sum_{k=1}^n x_k y_k < H_M(y)\right\}$. The support function $H_M(y)$ (26) has the homogeneity and semiadditivity properties [10]

$$H_M(ty) = tH_M(y), \quad \forall t \in \mathbb{R}_+;$$
$$H_M(x + y) \leq H_M(x) + H_M(y), \quad \forall x, y \in \mathbb{R}^n.$$

The Polya–Plancherel Theorem 5. *For a function $F(\lambda)$ to be given by*

$$F(\lambda) = \int\limits_{\mathbb{R}^n} f(t) \exp\left\{ i \sum_{k=1}^{n} t_k \lambda_k \right\} dt,$$

where $f(t) \in L^2_{\mathbb{R}^n}$ and equals zero outside a compact set in $\mathbb{R}^n$, it is necessary and sufficient that $F(\lambda)$ be a function of the exponential type in $\mathbb{C}^n$ and $F(x) \in L^2_{\mathbb{R}^n}$, $x \in \mathbb{R}^n$. Besides, the P-indicator $h_F(y)$ (25) of the function $F(\lambda)$ has to coincide with $H_f(y)$, where $H_f(y)$ is the support function (26) of the minimal convex domain outside of which the function $f(t)$ is zero.

The proof of this theorem is given in [10].

VI. The statement on the universality of the system of the integration operators $\left\{\tilde{A}_1; \tilde{A}_2\right\}$ (23) for operator systems of the class $\Lambda_0^{\exp}$ is as follows.

Theorem 6. *Suppose that a system of the linear bounded operators $\{A_1, A_2\}$ from the class $\Lambda_0^{\exp}$ is such that a) every operator A_k is completely non-selfadjoit; b) $\operatorname{Ker} A_k = \operatorname{Ker} A_k^* = \{0\}$, $k = 1, 2$. Then the operator system $\{A_1, A_2\}$ is unitarily equivalent to the restriction of the system $\left\{\tilde{A}_1, \tilde{A}_2\right\}$ (23) on the general invariant subspace with respect to $\tilde{A}_1$ and $\tilde{A}_2$ in $L^2_r(\Omega)$ (22), where $l(A) = \{l_1, l_2\}$ is the system of conjugate types of the entire function $A(\lambda)$.*

P r o o f. The contractiveness of the semigroup T_t (14) yields

$$-A(\lambda) = \int\limits_{\mathbb{R}^2_+} e^{i(t_1\lambda_1 + t_2\lambda_2)} T_t dt, \quad \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2_+,$$

therefore (similarly to (10))

$$-\langle A(\lambda) f, g\rangle = \int\limits_{\mathbb{R}^2_+} e^{i(t_1\lambda_1 + t_2\lambda_2)} \langle T_t f, g\rangle . \tag{27}$$

In view of the Plancherel theorem [8], the Fourier transform of the function

$$F_t(f, g) = \begin{cases} \langle T_t f, g\rangle, & t \in \mathbb{R}^2_+, \\ 0, & t \notin \mathbb{R}^2_+, \end{cases}$$

from $L^2_{\mathbb{R}^2}$ (Theorem 4) is the function from $L^2_{\mathbb{R}^2}$ and so $\langle A(\lambda) f, g\rangle \in L^2_{\mathbb{R}^2}$ when $\lambda = x \in \mathbb{R}^2$. Now, using the Polya–Plancherel Theorem 5 for the entire function of the exponential type $\langle A(\lambda) f, g\rangle$, we conclude from (27) that $\langle T_t f, g\rangle = 0$ as

$t_1 > l_1$ and $t_2 > l_2$, where $\{l_1, l_2\} = l(A)$ is the system of conjugate types of the function $A(\lambda)$. The fact that the support of the function $\langle T_t f, g\rangle$ is contained in the rectangle $\Omega = [0, l_1] \times [0, l_2]$ follows from the structure of the function $A(\lambda)$ representing the product of the Fredholm resolvents $A_1(\lambda_1) A_2(\lambda_2)$. Taking into account the density of the manifolds of vectors f and g in H (Theorem 4), we obtain that

$$-A(\lambda) = \int_\Omega e^{i(t_1\lambda_1 + t_2\lambda_2)} T_t dt \quad (\lambda \in \mathbb{C}^2_+).$$

The bilinear analogue of formula (20) is given by

$$\int_\Omega \langle \varphi\partial_1\partial_2 T_t f, \varphi\partial_1\partial_2 T_t g\rangle \, dt = \langle f, g\rangle, \tag{28}$$

where $\varphi = 2\sqrt{(A_1)_I (A_2)_I} H$. Expand $\varphi\partial_1\partial_2 T_t f$ in terms of the orthonormal basis $\{e_k\}_1^r$ in $\overline{(A_1)_I (A_2)_I H}$,

$$\varphi\partial_1\partial_2 T_t f = \sum_{k=1}^r \phi_k(t, f) e_k,$$

where $\phi_k(t, f) = \langle \varphi\partial_1\partial_2 T_t f, e_k\rangle \in L^2_r(\Omega)$ (Theorem 4), $1 \le k \le r$. (28) implies that

$$\langle f, g\rangle = \sum_{k=1}^r \langle \phi_k(t, f), \phi_k(t, g)\rangle_{L^2_{(\Omega)}}.$$

Thus, the operator U: $H \to L^2_r(\Omega)$,

$$Uf \stackrel{\text{def}}{=} \phi(t, f) = (\phi_1(t, f), ..., \phi_r(t, f)),$$

given on the dense set $A_1 A_2 H$ in H is an isometry. Expand U on the whole H by continuity, then for $f = A_1 A_2 h$ $(h \in H)$ we obtain

$$\varphi\partial_1\partial_2 T_t (T_x f) = \varphi\partial_1\partial_2 T_{t+x} f.$$

Therefore $UT_x = \tilde{T}_x U f$, which gives the realization in question of the semigroup T_x (14) in $L^2_r(\Omega)$ (22) via restriction of the shift semigroup $\tilde{T}_x$ (24) on the subspace in $L^2_r(\Omega)$ invariant under $\tilde{A}_1$, $\tilde{A}_2$ (23). ∎

O b s e r v a t i o n 1. *It is easy to see that this statement is easily extending to the case of the system of n twice commuting operators $\{A_k\}_1^n$ for every finite $n \in \mathbb{N}$.*

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