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Universality at the Edge for Unitary Matrix Models

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Using the results on the $1/n$-expansion of the Verblunsky coefficients for a class of polynomials orthogonal on the unit circle with n varying weight, we prove that the local eigenvalue statistic for unitary matrix models is independent of the form of the potential, determining the matrix model. Our proof is applicable to the case of four times differentiable potentials and of supports, consisting of one interval.

Key words: unitary matrix models, local eigenvalue statistics, universality, polynomials orthogonal on the unit circle.

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1. Introduction

We study a class of random matrix ensembles known as unitary matrix models. These models are defined by the probability law

$$p_n(U)\, d\mu_n(U) = Z_{n,2}^{-1} \exp\left\{-n\mathrm{Tr}V\left(\frac{U+U^*}{2}\right)\right\} d\mu_n(U), \tag{1.1}$$

where $U = \{U_{jk}\}_{j,k=1}^{n}$ is an $n \times n$ unitary matrix, $\mu_n(U)$ is the Haar measure on the group $U(n)$, $Z_{n,2}$ is the normalization constant, and $V : [-1,1] \to \mathbb{R}$ is a continuous function called the potential of the model. Denote $e^{i\lambda_j}$ the eigenvalues of the unitary matrix U. The joint probability density of λ_j, corresponding to (1.1), is given by (see [1])

$$p_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{1\le j<k\le n} \left|e^{i\lambda_j} - e^{i\lambda_k}\right|^2 \exp\left\{-n\sum_{j=1}^{n} V(\cos\lambda_j)\right\}. \tag{1.2}$$

Normalized Counting Measure of eigenvalues (NCM) is given by

$$N_n(\Delta) = n^{-1}\sharp\left\{\lambda_l^{(n)} \in \Delta,\, l = 1,\dots,n\right\}, \quad \Delta \subset [-\pi,\pi].$$

The random matrix theory deals with several asymptotic regimes of the eigenvalue distribution. The global regime is centred around the weak convergence of NCM. It is well known (see e.g. [2]) that for some smooth conditions for the potential V there exists a measure $N \in \mathcal{M}_1([-\pi,\pi])$ with a compact support σ such that N_n converges to N in probability .

Let

$$p_l^{(n)}(\lambda_1,\dots,\lambda_l) = \int p_n(\lambda_1,\dots,\lambda_l,\lambda_{l+1},\dots,\lambda_n)\, d\lambda_{l+1}\dots d\lambda_n$$

be the l -th marginal density of p_n. The local regime of eigenvalue distribution describes the asymptotic behaviour of marginal densities when their arguments are on the distances of order of the typical distance between eigenvalues. The universality conjecture of marginal densities was suggested by Dyson (see [3]) in the early 60s. He supposed that their asymptotic behaviour depends only on the ensemble symmetry group and does not depend on other ensemble parameters. First rigorous proofs for the hermitian matrix models with non-quadratic V appeared only in the 90s. The case of general V which is locally C^3 function was studied in [4]. The case of real analytic V was studied in [5], where the asymptotic behaviour of orthogonal polynomials was obtained. For the unitary matrix models the bulk universality was proved for $V = 0$ (see [3]), and for the locally C^3 functions (see [6]). The edge universality was proved only in the case of the linear V (see [7]). In the present paper we prove the universality conjecture for UMM with a smooth potential V in the case of one-interval support σ of the limiting NCM.

It was proved in [2] that the limiting measure can be obtained as a unique minimizer of the functional

$$\mathcal{E}[m] = \int_{-\pi}^{\pi} V(\cos\lambda)m(d\lambda) - \int_{-\pi}^{\pi} \log\left|e^{i\lambda} - e^{i\mu}\right| m(d\lambda)m(d\mu)$$

in the class of unit measures on the interval $[-\pi,\pi]$ (see [8] for the existence and properties of the solution). It is well known, in particular, that for smooth V' the equilibrium measure has a density ρ which is uniquely defined by the condition that the function

$$u(\lambda) = V(\cos\lambda) - 2\int_{\sigma} \log\left|e^{i\lambda} - e^{i\mu}\right| \rho(\mu)\, d\mu \tag{1.3}$$

takes its minimum value if $\lambda \in \sigma = \operatorname{supp} \rho$. From this condition in the case of differentiable V one can obtain the following integral equation for the equilibrium density ρ:

$$(V(\cos\lambda))' = v.p. \int_\sigma \cot\frac{\lambda-\mu}{2}\rho(\mu)\,d\mu, \quad \text{for } \lambda \in \sigma. \tag{1.4}$$

We also use the weak convergence of the first marginal density $\rho_n(\lambda) = p_1^{(n)}$ proved in [2].

Proposition 1.1. *For any $\phi \in H^1(-\pi,\pi)$,*

$$\left|\int \phi(\lambda)\rho_n(\lambda)\,d\lambda - \int \phi(\lambda)\rho(\lambda)\,d\lambda\right| \le C\|\phi\|_2^{1/2}\|\phi'\|_2^{1/2} n^{-1/2}\ln^{1/2} n, \tag{1.5}$$

where $\|\cdot\|_2$ denotes L_2 norm on $[-\pi,\pi]$.

We consider here the case of one interval σ. Our main conditions on the potential V are

Condition C1. *The support σ of the equilibrium measure is a single symmetric subinterval of the interval $[-\pi,\pi]$, i.e.,*

$$\sigma = [-\theta,\theta], \text{ with } \quad \theta < \pi.$$

R e m a r k 1.2. In fact, there is one more possibility to have one-interval σ. Another case is some left symmetric arc of the circle, i.e., $[\pi-\theta,\pi+\theta]$. In this case we replace $V(\cos x)$ in (1.2) by $V(\cos(\pi-x))$. This replacement will rotate all eigenvalues on the angle π and we will have the support from condition C1.

Condition C2. *The equilibrium density ρ has no zeros in $(-\theta,\theta)$ and*

$$\rho(\lambda) \sim C|\lambda \mp \theta|^{1/2}, \text{ for } \lambda \to \pm\theta,$$

and the function $u(\lambda)$ of (1.3) attains its minimum if and only if λ belongs to σ.

R e m a r k 1.3. From this condition we obtain the necessary scaling for marginal densities at the edge of σ

$$\int_\Delta \rho(\lambda)\,d\lambda \sim n^{-1} \Rightarrow |\,\Delta\,| \sim n^{-2/3}, \tag{1.6}$$

hence the typical distance between eigenvalues is of order $n^{-2/3}$.

Condition C3. *$V(\cos\lambda)$ possesses four bounded derivatives on $\sigma_\varepsilon = [-\theta-\varepsilon,\theta+\varepsilon]$.*

The following simple representation of ρ plays an important role in our asymptotic analysis (see [9])

Proposition 1.4. *Under conditions C1-C3 the density ρ has the form*

$$\rho(\lambda) = \frac{1}{4\pi^2}\chi(\lambda)\, P(\lambda)\, \mathbf{1}_\sigma,$$

where

$$\chi(\lambda) = \sqrt{|\cos\lambda - \cos\theta|}, \quad P(\lambda) = \int\limits_{-\theta}^{\theta} \frac{(V(\cos\mu))' - (V(\cos\lambda))'}{\sin(\mu-\lambda)/2} \frac{d\mu}{\chi(\mu)}. \tag{1.7}$$

The main result of the paper is the following theorem

Theorem 1.5. *Consider the unitary matrix ensemble of the form* (1.1)*, satisfying conditions* C1–C3 *above. Then*

- *for the endpoints $\theta_\pm = \pm\theta$ and any positive integer l the rescaled marginal density*

$$\left(\gamma n^{2/3}\right)^{-l} \frac{n!}{(n-l)!} p_l^{(n)}\left(\theta_\pm \pm t_1/\gamma n^{2/3}, \ldots, \theta_\pm \pm t_l/\gamma n^{2/3}\right) \tag{1.8}$$

  with the sign $\pm$ corresponding to $\theta_\pm$ and

$$\gamma = \tan^{1/3}\theta/2 \left(\frac{P(\theta)}{4\pi}\right)^{2/3}$$

  converges weakly, as $n \to \infty$, to $\det\{Q_{Ai}(t_j, t_k)\}_{j,k=1}^l$ *, where $Q_{Ai}(x,y)$ is the Airy kernel*

$$Q_{Ai}(x,y) = \frac{Ai(x)\,Ai'(y) - Ai'(x)\,Ai(y)}{x-y}; \tag{1.9}$$

- *if $\Delta \subset \mathbb{R}$ is a finite union of disjoint bounded intervals and*

$$E_n(\Delta_n) = \mathbb{P}(\Delta_n \textit{does not contain eigenvalues of } U)$$

  is the hole probability for $\Delta_n = \theta_\pm \pm \Delta/\gamma n^{2/3}$, then

$$\lim_{n\to\infty} E_n(\Delta_n) = 1 + \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \int\limits_{\Delta} dt_1 \ldots dt_l \det\{\mathcal{K}(t_j, t_k)\}_{j,k=1}^l, \tag{1.10}$$

  i.e., the limit is the Fredholm determinant of the integral operator $\mathcal{K}_\Delta$ defined by the kernel $\mathcal{K}$ on the set Δ.

The paper is organized as follows. In Section 2 we give a brief outline of the orthogonal polynomials method. In Section 3 we prove the main Theorem 1.5 using some technical results. These results are proved in Section 4.

2. Orthogonal Polynomials

We prove Theorem 1.5, using the orthogonal polynomials technique. This method is based on a simple observation. Joint eigenvalue distribution (1.2) is expressed in terms of the Vandermonde determinant of powers of $e^{i\lambda_k}$, and therefore by the properties of determinants, can be written in terms of the determinant of any system of linearly independent trigonometric polynomials. We consider a system of polynomials orthogonal on the unit circle(OPUC) with a varying weight. Let

$$w_n(\lambda) = e^{-nV(\cos\lambda)}$$

be the weight function for the system of polynomials. Then the system can be obtained from $\left\{e^{ik\lambda}\right\}_{k=0}^{\infty}$ if we use the Gram-Schmidt procedure in $L^{(n)} := L_2\left([-\pi,\pi], w_n(\lambda)\right)$ with the inner product

$$\langle f,g\rangle_n = \int\limits_{-\pi}^{\pi} f(x)\,\overline{g(x)}w_n(x)\,dx.$$

Hence, for any n we get the system of trigonometric polynomials $\left\{P_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$ which are orthonormal in $L^{(n)}$. One can see from the Szegö's condition that the system $\left\{P_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$ is not complete in $L^{(n)}$. To construct the complete system one should also include polynomials with respect to $e^{-i\lambda}$. Thus, following [10], we introduce the Laurent polynomials

$$\begin{array}{lcl} \chi_{2k}^{(n)}(\lambda) & = & e^{ik\lambda}P_{2k}^{(n)}(-\lambda), \\ \chi_{2k+1}^{(n)}(\lambda) & = & e^{-ik\lambda}P_{2k+1}^{(n)}(\lambda). \end{array} \tag{2.1}$$

It is easy to check (see, e.g., [10, 11]) that the system $\left\{\chi_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$ is an orthonormal basis in $L^{(n)}$. Moreover, it was proved in [10] that the functions $\chi_k^{(n)}$ satisfy some five term recurrent relations. Let $\alpha_k^{(n)}$ and $\rho_k^{(n)}$ be the Verblunsky coefficients of the system $\left\{\chi_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$ (for the definition and properties see [9]). Denote by

$$\Theta_j^{(n)} = \begin{pmatrix} -\alpha_j^{(n)} & \rho_j^{(n)} \\ \rho_j^{(n)} & \alpha_j^{(n)} \end{pmatrix},$$

$$M^{(n)} = E_1 \oplus \Theta_2^{(n)} \oplus \Theta_4^{(n)} \oplus ..., \quad L^{(n)} = \Theta_1^{(n)} \oplus \Theta_3^{(n)} \oplus \Theta_5^{(n)} \oplus ...,$$
$$C^{(n)} = M^{(n)}L^{(n)}. \tag{2.2}$$

From the properties of the Verblunsky coefficients one can see that the semi-infinite matrices $M^{(n)}$ and $L^{(n)}$ are symmetric, three diagonal and unitary. $C^{(n)}$ is also a unitary five diagonal matrix. Finally, using the above notations, we can write the recurrence relations as

$$e^{i\lambda}\overrightarrow{\chi^{(n)}} = C^{(n)}\overrightarrow{\chi^{(n)}}.$$

Hence, $C^{(n)}$ is a matrix presentation of the multiplication operator by $e^{i\lambda}$ in the basis $\left\{\chi_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$.

The main advantage of the orthogonal polynomials technique is the determinant formulas which can be obtained in the same way as in [1],

$$\frac{n!}{(n-l)!}p_l^{(n)}(\lambda_1,\ldots,\lambda_l) = \det\left\{K_n^{(n)}(\lambda_j,\lambda_k)\right\}_{j,k=1}^{l}, \tag{2.3}$$

where

$$K_m^{(n)}(\lambda,\mu) = \sum_{k=0}^{m-1}\chi_k^{(n)}(\lambda)\overline{\chi_k^{(n)}(\mu)}w_n^{1/2}(\lambda)\,w_n^{1/2}(\mu) \tag{2.4}$$

is the reproducing kernel of the system $\left\{\chi_k^{(n)}(\lambda)\right\}_{k=0}^{\infty}$. Similarly to [12], the weak convergence of the kernel $K_n^{(n)}$ to $\mathcal{K}$ as $n\to\infty$ will prove Theorem 1.5.

3. Proof of Theorem 1.5

To prove the weak convergence of the reproducing kernel (2.4), we use the lemma (see [12])

Lemma 3.1. *Consider the sequence of functions $\mathcal{K}_n : \mathbb{R}\times\mathbb{R}\to\mathbb{C}$ and define for $\Im\zeta,\xi\neq 0$,*

$$\mathcal{F}_n(\zeta,\xi) = \iint \Im\frac{1}{x-\zeta}\Im\frac{1}{y-\xi}\left|\mathcal{K}_n(x,y)\right|^2 dxdy. \tag{3.1}$$

Assume that there exists $\mathcal{F}(\zeta,\xi)$ of the form

$$\mathcal{F}(\zeta,\xi) = \iint \Im\frac{1}{x-\zeta}\Im\frac{1}{y-\xi}\left|\mathcal{K}(x,y)\right|^2 dxdy, \tag{3.2}$$

with $\mathcal{K}$ bounded uniformly in each compact in $\mathbb{R}^2$ and such that for any fixed $A>0$ uniformly on the set

$$\Omega_A = \{\zeta,\xi : 1\le\Im\zeta,\Im\xi\le A, |\Re\zeta,\Re\xi|\le A\} \tag{3.3}$$

we have

$$\left|\mathcal{F}_n(\zeta,\xi)-\mathcal{F}(\zeta,\xi)\right|\le\varepsilon_n, \quad \varepsilon_n\to 0, \text{as } n\to\infty. \tag{3.4}$$

Then for any intervals $I_1, I_2 \subset \mathbb{R}$

$$\lim_{n\to\infty} \int_{I_1} dx \int_{I_2} dy \left|\mathcal{K}_n(x,y)\right|^2 = \int_{I_1} dx \int_{I_2} dy \left|\mathcal{K}(x,y)\right|^2.$$

The lemma helps to prove the convergence of $|\mathcal{K}_n|^2$ to $|\mathcal{K}|^2$. Similarly, we can check the convergence of $\mathcal{K}_n(t_1,t_2)\,\mathcal{K}_n(t_2,t_3)\ldots\mathcal{K}_n(t_l,t_1)$ for any $l \in \mathbb{N}$. To prove the second part of Theorem 1.5, we use another proposition from [12].

Proposition 3.2. *Let* $\Delta \subset \mathbb{R}$ *be a system of disjoint intervals as in Theorem 1.5 and let* $\mathcal{K}_n : L_2(\Delta) \to L_2(\Delta)$ *be a sequence of positive definite integral operators with kernels* $\mathcal{K}_n(x,y)$ *and* $\mathcal{K} : L_2(\Delta) \to L_2(\Delta)$ *a positive definite integral operator with kernel* $\mathcal{K}(x,y)$*, such that for any* $l \in \mathbb{N}$, $\det\{\mathcal{K}_n(t_j,t_k)\}_{j,k=1}^{l} \to \det\{\mathcal{K}(t_j,t_k)\}_{j,k=1}^{l}$ *weakly as* $n \to \infty$*. Assume also that for any* Δ *there exists* C_Δ *such that*

$$\int_{\Delta} \mathcal{K}_n(s,s)\,ds \le C_\Delta. \tag{3.5}$$

Then, for the Fredholm determinants of $\mathcal{K}_n$ *and* $\mathcal{K}$ *we have*

$$\lim_{n\to\infty} \det(1-\mathcal{K}_n) = \det(1-\mathcal{K}).$$

We are going to use Lemma 3.1 for the scaled reproducing kernel of the system of OPUC. Let

$$\mathcal{K}_n(x,y) = n^{-2/3} K_n^{(n)}\left(\theta + x n^{-2/3}, \theta + y n^{-2/3}\right) 1_{|x,y| \le c_\theta n^{2/3}} \tag{3.6}$$

for some small enough θ-dependent constant c_θ. This will be sufficient in view of the following lemma (the analogue of Theorem 11.1.4, [13])

Lemma 3.3. *Let the model* (1.1) *satisfy conditions C1-C3. Then, for any* n*-independent* $\varepsilon > 0$*, there exists a constant* $d_\varepsilon > 0$ *such that*

$$\int_{\sigma_\varepsilon^c} K_n^{(n)}(\lambda,\lambda)\,d\lambda \le C e^{-n d_\varepsilon}.$$

Since the polynomials $\chi_k^{(n)}$ are functions of $e^{i\lambda}$, it is more convenient to define a little bit different from (3.1) transformation and estimate the difference between it and (3.1). Hence, we consider the following transformation:

$$F_n(z,w) = n^{-4/3} \iint_{[-\pi,\pi]} G(z-\lambda)\,G(w-\mu) \left|K_n^{(n)}(\lambda,\mu)\right|^2 d\lambda d\mu, \tag{3.7}$$

with

$$G(z) = \Re g(z), \text{and}\, g(z) = \frac{1+e^{iz}}{1-e^{iz}} \tag{3.8}$$

being the analogues of the Poisson and the Herglotz transformations.

Proposition 3.4. *It follows from the definition of $g(z)$ that*

$$g(z) = i\cot\frac{z}{2}, \quad g(z-\lambda) = \frac{e^{i\lambda}+e^{iz}}{e^{i\lambda}-e^{iz}}.$$

For $z = x+iy$ we have $g(x+iy) = \dfrac{i\sin x + \sinh y}{\cosh y - \cos x}$, hence $\overline{g(z)} = -g(\overline{z})$. And for $G(z)$ we get

$$G(x+iy) = \frac{\sinh y}{\cosh y - \cos x}, \quad G(z-\lambda) = \Im\cot\frac{\lambda - z}{2}.$$

Moreover, $G(z)$ is a Nevanlinna function and

$$|g(z)|^2 = -1 + 2\coth\Im z \cdot G(z). \tag{3.9}$$

The difference between the new transformation and the old one can be estimated in the following way:

Proposition 3.5. *Let $z = \theta + \zeta n^{-2/3}$ and $w = \theta + \xi n^{-2/3}$ with $|\zeta|, |\xi| \le c_\theta n^{-2/3}$ and $\Im\zeta, \Im\xi \ge 1$. Then,*

$$|F_n(z,w) - 4\mathcal{F}_n(\zeta,\xi)| \le Cn^{-1/6}\left(|F_n(z,w)| + 1\right). \tag{3.10}$$

The next step is to prove the convergence of $F_n(z,w)$ to the transformation $\mathcal{F}$ (3.2) of the Airy kernel Q_{Ai} (1.9). $\mathcal{F}$ can be calculated in terms of the Airy functions, thus we are concentrated on the calculations of F_n. First, using the properties of CMV matrices, we present $F_n(z,w)$ in terms of the "resolvent" of $C^{(n)}$. After that we use the asymptotic behaviour of the Verblunsky coefficients, obtained in [9], to get an approximation of the "resolvent". The approximation will be given in terms of the Airy functions. Then we will estimate the error of the "resolvent" approximation and prove the uniform bound (3.4).

We start with a simple corollary from the spectral theorem and Proposition 3.4.

Proposition 3.6. *Let*

$$g^{(n)}(z) = \left(C^{(n)} + e^{iz}\right)\left(C^{(n)} - e^{iz}\right)^{-1},$$

be the "resolvent" of the CMV matrix $C^{(n)}$. *Then,*

$$\left(g^{(n)}(z)\right)^{\dagger}=-g^{(n)}(\overline{z}),\quad G^{(n)}(z):=\frac{1}{2}\left(g^{(n)}(z)-g^{(n)}(\overline{z})\right),$$

$$g^{(n)}(z)\left(g^{(n)}(z)\right)^{\dagger}=-\mathrm{I}+2\cot\Im z\cdot G^{(n)}(z)$$

and

$$F_n(z,w)=n^{-4/3}\sum_{j,k=0}^{n-1}G_{j,k}^{(n)}(z)\,G_{k,j}^{(n)}(w). \tag{3.11}$$

First of all, we would like to restrict the summation above by $j,k\le M=\left[Cn^{1/2}\log n\right]$ with some constant C.

Lemma 3.7. *There exists V-depended constants C such that under the conditions of Theorem 1.5 uniformly in Ω_A of* (3.3) *we have*

$$n^{-2/3}\sum_{j=M+1}^{n}G_{n-j,n-j}^{(n)}(z)\le Cn^{-1/12}\log n.$$

Now we present the approximation for the matrix elements $G_{n-j,n-k}^{(n)}$. Using the three-diagonal matrices expansion (2.2) of the $C^{(n)}$, we can write the matrix $g^{(n)}$ as

$$g^{(n)}(z)=\left(M^{(n)}e^{-iz/2}+L^{(n)}e^{iz/2}\right)\left(M^{(n)}e^{-iz/2}-L^{(n)}e^{iz/2}\right)^{-1}.$$

From the definitions of $M^{(n)}$ and $L^{(n)}$ one can find their matrix elements

$$\begin{array}{ll}M_{n+k,n+k-1}^{(n)}=d_{n+k}\rho_{n+k}^{(n)}, & M_{n+k,n+k}^{(n)}=d_{n+k}\alpha_{n+k}^{(n)}-d_{n+k+1}\alpha_{n+k+1}^{(n)},\\ L_{n+k,n+k-1}^{(n)}=d_{n+k+1}\rho_{n+k}^{(n)}, & L_{n+k,n+k}^{(n)}=d_{n+k+1}\alpha_{n+k}^{(n)}-d_{n+k}\alpha_{n+k+1}^{(n)},\end{array}$$

where $d_k=(1+s_k)/2$ and $s_k=(-1)^k$. Denote

$$C_{\pm}^{(n)}(z)=M^{(n)}e^{-iz/2}\pm L^{(n)}e^{iz/2}.$$

At the first step we derive the representation for the matrix elements of the inverse matrix of $C_{-}^{(n)}(z)$. Note that $C_{r-}^{(n)}$ is three-diagonal and symmetric, and its entries are

$$\begin{array}{rl}C_{-_{n+k,n+k-1}}^{(n)}(z)= & s_{n+k}\rho_{n+k}^{(n)}e_{n+k}(z),\\ C_{-_{n+k,n+k}}^{(n)}(z)= & s_{n+k}\alpha_{n+k}^{(n)}e_{n+k}(z)+s_{n+k}\alpha_{n+k+1}^{(n)}e_{n+k+1}(z)\end{array}$$

with

$$e_k(z)=\cos\frac{z}{2}-is_k\sin\frac{z}{2}.$$

For the Verblunsky coefficients we use the result of [9].

Lemma 3.8. *Consider the system of orthogonal polynomials and the Verblunsky coefficients defined above. Let the potential V satisfy conditions* C1–C3 *above. Then, for any k,*

$$\begin{aligned}\alpha_{n+k}^{(n)} &= (-1)^k s^{(n)} \left(\cos\frac{\theta}{2} - p_\theta x_k^{(n)} n^{-2/3}\right) + \underline{O}\left(\varepsilon_{n,k}\right),\\ \rho_{n+k}^{(n)} &= \sin\frac{\theta}{2} + \cot\frac{\theta}{2} p_\theta x_k^{(n)} n^{-2/3} + \underline{O}\left(\varepsilon_{n,k}\right),\end{aligned}$$

where $s^{(n)} = 1$ or $s^{(n)} = -1$ and

$$x_k^{(n)} = kn^{-1/3}, \quad \varepsilon_{n,k} = n^{-4/3}\log^{11} n \left(1 + \left(x_k^{(n)}\right)^2\right) 1_{|k|<n} + 1_{|k|\geq n},$$

with $p_\theta = \dfrac{\pi\sqrt{2}}{P(\theta)}$ and P defined in (1.7).

To introduce the approximation for the resolvent, we define two "rotation" matrices which help to present the matrix $C_{r_-}^{(n)}$ in the form, similar to the discrete Laplacian matrix. Let $U^{(n)}$ and $V^{(n)}$ be two semi-infinite matrices with the entries

$$U_{n+j,n+k}^{(n)} = \left(is^{(n)}\right)^{2nk-k-1}\delta_{jk}, \quad V_{n+j,n+k}^{(n)} = \left(is^{(n)}\right)^{2nk-k}\delta_{jk}$$

and

$$C_{r_\pm}^{(n)}(z) = U^{(n)} C_\pm^{(n)} V^{(n)}, \quad R^{(n)}(\zeta) = \left(C_{r_-}^{(n)}(z)\right)^{-1}, \text{ where } z = \theta + \zeta n^{-2/3}.$$

Then the entries of the new matrix are

$$\begin{aligned}\left(C_{r_-}^{(n)}\right)_{n+k,n+k-1}(z) &= \rho_{n+k}^{(n)} e_{n+k}(z),\\ \left(C_{r_-}^{(n)}\right)_{n+k,n+k}(z) &= -is^{(n)} s_n \left(\alpha_{n+k}^{(n)} e_{n+k}(z) + \alpha_{n+k+1}^{(n)} e_{n+k+1}(z)\right).\end{aligned}$$

Using the above definitions, we write

$$g^{(n)}(z) = \mathrm{I} + 2L^{(n)} V^{(n)} R^{(n)}(\zeta)\, U^{(n)} e^{iz/2}. \tag{3.12}$$

Now we prove that the matrix elements of $R^{(n)}(\zeta)$ can be expressed in terms of the Airy functions. For this aim we present an approximation matrix $R^\star$ and find the difference between $R^\star$ and $R^{(n)}$. Note that

$$\begin{aligned}e^{iz/2} &= e^{i\theta/2} + ie^{i\theta/2}\zeta n^{-2/3} + \underline{O}\left(|\zeta|^2 n^{-4/3}\right),\\ e_{n+k}(z) &= e_{n+k}(\theta) - is_{n+k} e_{n+k}(\theta)\,\zeta n^{-2/3} + \underline{O}\left(|\zeta|^2 n^{-4/3}\right),\end{aligned}$$

Let $y_k^{(n)} = x_k^{(n)} - n^{-1/3}/2$ and $r_{k,\zeta}^{(n)} = n^{-4/3}\varepsilon_{n,k} + |\zeta|^2$. Then

$$\begin{aligned}\left(C_{r_-}^{(n)}\right)_{n-k,n-k-1}(\zeta) &= \sin\frac{\theta}{2}e_{n+k}(\theta) - \cot\frac{\theta}{2}e_{n+k}(\theta)\,p_\theta y_k^{(n)}n^{-2/3}\\ &\quad -is_{n+k}\sin\frac{\theta}{2}e_{n+k}(\theta)\,\zeta n^{-2/3} - \frac{1}{2}\cot\frac{\theta}{2}e_{n+k}(\theta)\,p_\theta n^{-1}\\ &\quad +n^{-4/3}\underline{O}\left(r_{k,\zeta}^{(n)}\right),\end{aligned} \tag{3.13}$$

$$\begin{aligned}\left(C_{r_-}^{(n)}\right)_{n-k,n-k}(\zeta) &= -\sin\theta - 2\sin\frac{\theta}{2}p_\theta y_k^{(n)}n^{-2/3}\\ &- 2\cos^2\frac{\theta}{2}\zeta n^{-2/3} - is_{n+k}p_\theta\cos\frac{\theta}{2}n^{-1} + n^{-4/3}\underline{O}\left(r_{k,\zeta}^{(n)}\right).\end{aligned} \tag{3.14}$$

The matrix elements of $C_{r_-}^{(n)}$ are similar to the matrix elements of the discrete Laplace operator with some potential in the $n^{-1/3}$ scale, but off-diagonal elements contain alternating terms $is_{n+k}\sin^2\dfrac{\theta}{2}$. Hence, we define the approximate resolvent in terms of the Airy function with some shift. Set

$$\delta_k^{(n)} = is_{n+k+1}\delta, \quad \delta = \frac{1}{2}\tan\frac{\theta}{2}, \quad h = n^{-1/3}$$

and

$$R^\star_{n-k,n-j}(\zeta) = h^{-1}\mathcal{R}_\zeta\left(y_k^{(n)} + \delta_k^{(n)}h, y_j^{(n)} + \delta_j^{(n)}h\right), \tag{3.15}$$

where $\mathcal{R}_\zeta(z,w)$, defined by

$$\mathcal{R}_\zeta(z,w) = ab^{-1}\pi\begin{cases}\psi_-(z,\zeta)\,\psi_+(w,\zeta), & \Re z \le \Re w,\\ \psi_+(z,\zeta)\,\psi_-(w,\zeta), & \Re z \ge \Re w\end{cases} \tag{3.16}$$

with $\psi_\pm$ defined in the Appendix, is the extension of the resolvent of the operator $\mathcal{L}$

$$\mathcal{L}[f](x) = a^3 f''(x) - b^3 x f(x) \tag{3.17}$$

to the complex plane, where $a^3 = \sin\theta$ and $b^3 = 2p_\theta\sin^{-1}(\theta/2)$. For the properties, asymptotic behaviour, and the integral representation of $\mathcal{R}_\zeta$ see Appendix. Denote by $D^{(n)}$ the error of the approximation

$$D^{(n)}(\zeta) = C_{r_-}^{(n)}(\zeta)\,R^\star(\zeta) - I. \tag{3.18}$$

To present the bounds for $D_{n-k,n-j}^{(n)}$, we introduce the notations

$$d_{n-k,n-j}^{(p)} = \sup_{|s|\le\delta+1}\left|\frac{\partial^p}{\partial z^p}\mathcal{R}_\zeta\left(y_k^{(n)} + sh, y_j^{(n)} + \delta_j^{(n)}h\right)\right|.$$

One can see from the definition of $\mathcal{R}_\zeta$ that $\dfrac{\partial}{\partial z}\mathcal{R}_\zeta$ is not defined for $z = w$. In this case, by $\dfrac{\partial}{\partial z}$ we denote the half of the sum of the left and the right derivatives $\dfrac{1}{2}\left(\dfrac{\partial_+}{\partial z} + \dfrac{\partial_-}{\partial z}\right)$. Then $D^{(n)}$ satisfies the following bound.

Lemma 3.9. *There exists constants C_1, C_2 such that uniformly in k, j and $\zeta \in \Omega_A$*

$$D^{(n)}_{n-k,n-j}(\zeta) \le C_1 h^2 \log^{C_2} n$$
$$\left(\left(1 + h^2 \left|y_k^{(n)}\right|^2\right) d^{(0)}_{n-k,n-j} + \left(\left|y_k^{(n)}\right| + |\zeta|\right) d^{(1)}_{n-k,n-j}\right). \tag{3.19}$$

Now we are ready to analyse the r.h.s of (3.11). From (3.15), (3.12), and Lemma 3.9 one can see that $G^{(n)}_{n-k,n-j} \approx h^{-1}\Im\mathcal{R}_\zeta\left(y_k^{(n)}, y_j^{(n)}\right)$, and if we could neglect the remainder, then

$$F_n(\zeta,\xi) \approx h^2 \sum \Im\mathcal{R}_\zeta\left(y_k^{(n)}, y_j^{(n)}\right)\Im\mathcal{R}_\xi\left(y_j^{(n)}, y_k^{(n)}\right).$$

On the other hand, changing a double sum by the double integral and using (5.4), we obtain $\mathcal{F}[Q_{Ai}]$. Hence, our main goal now is to estimate the remainder that appears after replacement of the "resolvent" of $C^{(n)}_{r_-}$ by the resolvent of the differential operator. We will do these calculations in several steps.

We start from the proof of the bound for

$$\Sigma_M = n^{-2/3}\sum_{j=0}^{M} G^{(n)}_{n-j,n-j}(z) \tag{3.20}$$

with $M = \left[C_0 n^{1/2}\log n\right]$. It follows from (3.12) and the definition of $G^{(n)}$ that

$$G^{(n)}(z) = L^{(n)}V^{(n)}\left(R^{(n)}(\zeta)e^{iz/2} - R^{(n)}\left(\overline{\zeta}\right)e^{i\overline{z}/2}\right)U^{(n)}.$$

Using the definition of $D^{(n)}$, we can write $R^{(n)}$ as

$$R^{(n)}(\zeta) = R^\star - R^{(n)}(\zeta)D^{(n)}(\zeta).$$

Then,

$$\Sigma_M = n^{-2/3}\sum_{j=0}^{M}\left(L^{(n)}V^{(n)}\left(R_e^\star(\zeta) - R_e^{(n)}D^{(n)}(\zeta)\right)U^{(n)}\right)_{n-j,n-j} = \Sigma_M^* - \Sigma_M^{D^{(n)}},$$

where $R_e^\star(\zeta) = R^\star(\zeta)\, e^{iz/2} - R^\star\left(\overline{\zeta}\right) e^{i\overline{z}/2}$ and the same with $R^{(n)}$ and $R_e^{(n)}$. Here $\Sigma_M^\star$ can be estimated immediately by using Proposition 5.5, and $\Sigma_M^{D^{(n)}}$ can be estimated by multiplying $\Sigma_M^{1/2}$ by some small factor which we get using the Cauchy inequality and the bounds (3.19) for $D_{n-k,n-j}^{(n)}$. Thus we obtain the quadratic inequality (3.23). Solving this inequality, we will obtain (3.20). Indeed,

$$\begin{aligned} |\Sigma_M^*| \quad &\leq \quad C\sum_{j=0}^{M}\sum_{|k-j|\leq 1} h\left|\Im\mathcal{R}_\zeta\left(y_k^{(n)}+\delta_k^{(n)}h, y_j^{(n)}+\delta_j^{(n)}h\right)\right| \qquad (3.21)\\ &\quad +h^3\left|\mathcal{R}_\zeta\left(y_k^{(n)}+\delta_k^{(n)}h, y_j^{(n)}+\delta_j^{(n)}h\right)\right|. \qquad (3.22)\end{aligned}$$

Using Proposition 5.5, we can estimate $\Sigma_M^\star$ as follows:

$$|\Sigma_M^*| \leq C.$$

To estimate $\Sigma_M^{D^{(n)}}$, we start with the relation

$$\begin{aligned} L^{(n)}V^{(n)}R_e^{(n)}D^{(n)}U^{(n)} &= L^{(n)}V^{(n)}R_e^{(n)}U^{(n)}\left(U^{(n)}\right)^{-1}D^{(n)}U^{(n)}\\ &= \left(g^{(n)}(z) - g^{(n)}(\overline{z})\right)\widehat{D^{(n)}},\end{aligned}$$

where $\widehat{D^{(n)}}$ entries have the same bounds as $D^{(n)}$, and we will write below $D^{(n)}$ to simplify notations. Note that

$$\begin{aligned} &\left(g^{(n)}D^{(n)}\right)_{n-j,n-j} = \left\langle g^{(n)}D^{(n)}e_{n-j}, e_{n-j}\right\rangle = \left\langle D^{(n)}e_{n-j}, \left(g^{(n)}\right)^\dagger e_{n-j}\right\rangle\\ &\leq \left\|D^{(n)}e_{n-j}\right\|\left\|\left(g^{(n)}\right)^\dagger e_{n-j}\right\| = \left(\left(D^{(n)}\right)^\dagger D^{(n)}\right)_{n-j,n-j}^{1/2}\left(\left(g^{(n)}\right)^\dagger g^{(n)}\right)_{n-j,n-j}^{1/2},\end{aligned}$$

and by the Cauchy inequality and (3.9),

$$\begin{aligned} \left|\Sigma_M^{D^{(n)}}\right| \quad &\leq \quad Cn^{-2/3}\left(\sum_{j=0}^{M}\left(\left(D^{(n)}\right)^\dagger D^{(n)}\right)_{n-j,n-j}\right)^{1/2}\\ &\quad \times\left(M + 2\coth\Im z\sum_{j=M_1+1}^{M_2} G_{n-j,n-j}^{(n)}\right)^{1/2}\\ &= \quad S_{D^{(n)}}^{1/2}\left(\underline{O}\left(n^{-5/6}\log n\right) + 2n^{-2/3}\coth\left(\Im\zeta n^{-2/3}\right)\Sigma_M\right).\end{aligned}$$

Using Lemma 3.9, the Cauchy inequality, and Proposition 5.4, we estimate $S_{D^{(n)}}$ as follows:

$$\begin{aligned} S_{D^{(n)}} &= \sum_{j=0}^{M} \left(\left(D^{(n)} \right)^{\dagger} D^{(n)} \right)_{n-j,n-j} \\ &\le C_1 n^{-4/3} \log^{C_2} n \sum_{j=0}^{M} \sum_{k=0}^{\infty} \left(\left| y_k^{(n)} \right|^2 + |\zeta|^2 \right) \left| d_{n-k,n-j}^{(1)} \right|^2 + \left| d_{n-k,n-j}^{(0)} \right|^2 \\ &+ h^4 \left(\left| y_k^{(n)} \right|^4 + |\zeta|^4 \right) \left| d_{n-k,n-j}^{(0)} \right|^2 \\ &\le C_1 n^{-1} \log^{C_2} n \sum_{j=0}^{M} \left(1 + \left| y_j^{(n)} \right| \right)^{3/2} + h^4 \left(1 + \left| y_j^{(n)} \right| \right)^{5/2} \\ &\le C_1 n^{-2/3} \log^{C_2} n \left(M n^{-1/3} \right)^{5/2} \le C_1 n^{-1/4} \log^{C_2} n. \end{aligned}$$

Combining this inequality with the above estimate of $\Sigma_M^{D^{(n)}}$, we obtain the inequality for Σ_M

$$|\Sigma_M| \le C_1 + C_2 n^{-1/8} \log^{C_3} n \left(\underline{O} \left(n^{-5/6} \log n \right) + |\Sigma_M| \right)^{1/2} \tag{3.23}$$

which gives (3.20).

Now we are ready to find the limit of the r.h.s. of (3.11). Combining Lemma 3.7 with (3.21), we get

$$n^{-2/3} \sum_{j=0}^{n} G_{n-j,n-j}^{(n)}(z) \le C. \tag{3.24}$$

Using the definition of $G^{(n)}$, the sum in (3.11) can be splitted into four parts with different products of $g^{(n)}$ and $\overline{g^{(n)}}$. For each sum, the Cauchy inequality yields

$$\begin{aligned} & n^{-4/3} \left| \sum_{j,k} g_{n-j,n-k}^{(n)}(z)\, g_{n-k,n-j}^{(n)}(w) \right| \\ \le & \left(n^{-4/3} \sum_{j} \left(g^{(n)} \left(g^{(n)} \right)^{\dagger} \right)_{n-j,n-j} (z) \right)^{1/2} \\ \times & \left(n^{-4/3} \sum_{j} \left(g^{(n)} \left(g^{(n)} \right)^{\dagger} \right)_{n-j,n-j} (w) \right)^{1/2}, \end{aligned}$$

where each of the brackets is bounded because of (3.9) and (3.24). Changing the summation limits in the previous bound to $j \in [M, n]$ and using Lemma 3.7, we obtain that under the conditions of Lemma 3.1

$$F_n(z,w) = n^{-4/3} \sum_{j,k=0}^{M} G^{(n)}_{n-k,n-j}(z)\, G^{(n)}_{n-j,n-k}(w) + \underline{O}\left(n^{-1/24}\log n\right).$$

Now we use once more the identity

$$G^{(n)} = G^{\star} - G^{(n)}\widehat{D^{(n)}}.$$

Repeating the above arguments, we obtain

$$F_n(z,w) = F_n^{\star}(z,w) + F_{D^{(n)}}(z,w),$$

and

$$F_{D^{(n)}}(z,w) \le C_1 n^{-1/8}\log^{C_2} n.$$

Since $G^{\star} = L^{(n)}V^{(n)}R_e^{\star}U^{(n)}$ with $R_e^{\star}$ defined above, we have

$$G^{\star}_{n-k,n-j} = n^{1/3}\Im\mathcal{R}_{\zeta}\left(y_k^{(n)}, y_j^{(n)}\right) + r_{k,j}^{G^{\star}},$$

where $r_{k,j}^{G^{\star}}$ contains terms with some derivatives of the $\mathcal{R}_{\zeta}$ multiplied by h in some non-negative power. Thus, from the boundness of the corresponded integrals (see proof of Proposition 5.4 for the arguments)

$$h^{p+q}\int_0^{Mn^{-1/3}}\int_0^{Mn^{-1/3}} \left|\frac{\partial^{p+q}}{\partial x^p \partial y^q}\mathcal{R}_{\zeta}(x,y)\right|^2 dxdy \le C_{p,q,r,s},$$

we obtain that we can neglect terms from $r_{k,j}^{G^{*}}$ and

$$F_n^{\star}(z,w) = \int_0^{Mn^{-1/3}}\int_0^{Mn^{-1/3}} \Im\mathcal{R}_{\zeta}(x,y)\,\Im\mathcal{R}_{\xi}(y,x)\,dxdy + \underline{O}\left(h^{1/2}\right).$$

Finally we note that by (5.7) and (5.8),

$$\int_{Mn^{-1/3}}^{\infty} dx \int dy\, |\mathcal{R}_{\zeta}(x,y)|^2 \le \int_{Mn^{-1/3}}^{\infty} \Im\mathcal{R}_{\zeta}(x,x)\,dx \le Cn^{-1/12}\log n,$$

and

$$\int_0^{\infty}\int_0^{\infty} \Im\mathcal{R}_{\zeta}(x,y)\,\Im\mathcal{R}_{\xi}(y,x)\,dxdy \le C.$$

Hence,

$$F_n(z,w) = \int_0^\infty \int_0^\infty \Im \mathcal{R}_\zeta(x,y) \Im \mathcal{R}_\xi(y,x)\, dxdy + \underline{O}\left(Cn^{-1/24} \log^C n\right). \tag{3.25}$$

Estimate (3.25), integral representation (5.4), and the following relation (see [14])

$$Q_{Ai}(x,y) = \int_0^\infty Ai(x+t)\, Ai(y+t)\, dt$$

imply (3.4) with

$$\mathcal{K}(x,y) = a^{-2} b^{-4} Q_{Ai}\left(a^{-1}b^{-2}x, a^{-1}b^{-2}y\right).$$

Proposition 3.2 implies that it is sufficient to check (3.5) to finish the proof of Theorem 1.5. We use an evident relation

$$G(t+i\varepsilon - s) = \frac{d}{dt} 2 \arctan\left(\tan\left(\frac{t-s}{2}\right)\cot\frac{\varepsilon}{2}\right)$$

that implies the inequality valid for any $s \in [a,b] \subset \mathbb{R}$

$$\int_{a-1}^{b+1} G\left((t+i-s)\, n^{-2/3}\right) dt \geq Cn^{2/3},$$

with some absolute constant C. The last inequality, the positiveness of $\mathcal{K}_n$ and G, and definition of $G^{(n)}$ imply

$$\begin{aligned} \int_a^b \mathcal{K}_n(s,s)\, ds \quad & \leq Cn^{-2/3} \int_a^b ds \int_{a-1}^{b+1} dt \mathcal{K}_n(s,s)\, G\left((t+i-s)\, n^{-2/3}\right) \\ & \leq C \int_{a-1}^{b+1} \sum_{j=1}^{n} G^{(n)}_{n-j,n-j}\left(\theta + (t+i)\, n^{-2/3}\right) dt. \end{aligned}$$

Hence, by (3.24) for any finite $\Delta \subset [-A+1, A-1]$ we obtain (3.5).

4. Auxiliary Results

P r o o f of Proposition 3.5. Using Lemma 3.3 with $\varepsilon = 2c_\theta$ and inequality

$$\left| K^{(n)}_n(\lambda,\mu) \right|^2 \leq K^{(n)}_n(\lambda,\lambda)\, K^{(n)}_n(\mu,\mu), \tag{4.1}$$

we obtain

$$\int\limits_{\lambda\in\sigma_\varepsilon^c} G(z-\lambda)\left|K_n^{(n)}(\lambda,\mu)\right|^2 d\lambda \le Ce^{-nd(\varepsilon)} \sup_{\lambda\in\sigma_\varepsilon^c} G(z-\lambda)\, K_n^{(n)}(\mu,\mu)\,.$$

Due to the restrictions on λ and z we get $G(z-\lambda) \le C'$ when $\lambda \in \sigma_\varepsilon^c$. Thus,

$$\iint\limits_{\sigma_\varepsilon^c} G(z-\lambda)\, G(w-\mu)\left|K_n^{(n)}(\lambda,\mu)\right|^2 d\lambda d\mu = e^{-cn}\underline{O}\left(\Im^{-1}z + \Im^{-1}w\right).$$

Changing the variables by the scaled ones in (3.7), we get

$$F_n(z,w) = n^{-4/3}\iint \Im\cot\frac{\zeta-x}{2n^{2/3}}\Im\cot\frac{\xi-y}{2n^{2/3}}\left|\mathcal{K}_n(x,y)\right|^2 dxdy + \underline{O}\left(e^{-cn}\right).$$

Finally we estimate the difference between F_n and $4\mathcal{F}_n$

$$4\mathcal{F}_n(\zeta,\xi) - F_n(z,w) = n^{-4/3}\left(I_1(\zeta,\xi) + I_2(\zeta,\xi) + I_2(\xi,\zeta)\right) + \underline{O}\left(e^{-cn}\right)$$

with I_1 and I_2 of (4.2) and (4.3). It is easy to see that

$$\begin{aligned}|I_1(\zeta,\xi)| &= \left|\iint \Im\left(\frac{2n^{2/3}}{\zeta-x} - \cot\frac{\zeta-x}{2n^{2/3}}\right)\Im\left(\frac{2n^{2/3}}{\xi-y} - \cot\frac{\xi-y}{2n^{2/3}}\right)\left|\mathcal{K}_n(x,y)\right|^2 dxdy\right| \\ &\le C\iint \left|\mathcal{K}_n(x,y)\right|^2 dxdy \le Cn, \qquad (4.2)\end{aligned}$$

where we have used that for $0 < |z| \le 2c_\theta$

$$\left|\cot z - \frac{1}{z}\right| \le C.$$

In addition, since the kernel $\left|K_n^{(n)}(\lambda,\mu)\right|^2$ is positive definite, we can use the Cauchy inequality to get

$$\begin{aligned}|I_2(\zeta,\xi)| &= \left|\iint \Im\left(\frac{2n^{2/3}}{\zeta-x} - \cot\frac{\zeta-x}{2n^{2/3}}\right)\Im\cot\frac{\xi-y}{2n^{2/3}}\left|\mathcal{K}_n(x,y)\right|^2 dxdy\right| \\ &\le |I_1(\zeta,\xi)|^{1/2}\left|n^{4/3}F_n(z,w)\right|^{1/2} \le Cn^{7/6}\left|F_n(z,w)\right|^{1/2}. \qquad (4.3)\end{aligned}$$

Finally, collecting the above bounds, we obtain

$$|F_n(z,w) - \mathcal{F}_n(\zeta,\xi)| \le Cn^{-1/6}\left|F_n(z,w)\right|^{1/2} + C'n^{-1/3},$$

and using the Cauchy inequality, we get (3.10). ■

P r o o f of Lemma 3.9. The proof is based on the direct calculations of the matrix elements $D^{(n)}_{n-j,n-k}$. We start with the case $j \neq k$. Then all derivatives of $\mathcal{R}_\zeta$ are well defined and the points $y^{(n)}_{j-1}, y^{(n)}_j, y^{(n)}_{j+1}$ are laying on the same side of $y^{(n)}_k$. Now we are going to calculate $D^{(n)}_{n-j,n-k}$ using the Taylor expansion and definition of the $C^{(n)}_{r_-}$. These calculations are a little bit involved, so we present them in several steps. First, we calculate $R^\star_{n-k\mp1,n-j}$,

$$R^\star_{n-k\mp1,n-j} = h^{-1}\mathcal{R}_\zeta\left(y^{(n)}_k \pm h - \delta^{(n)}_k h, y^{(n)}_j + \delta^{(n)}_j h\right)$$
$$= h^{-1}\mathcal{R}_\zeta\left(y^{(n)}_k, y^{(n)}_j + \delta^{(n)}_j h\right) + \left(\pm1 - \delta^{(n)}_k\right)\frac{\partial}{\partial z}\mathcal{R}_\zeta\left(y^{(n)}_k, y^{(n)}_j + \delta^{(n)}_j h\right)$$
$$+ \left(\pm1 - \delta^{(n)}_k\right)^2 h\frac{\partial^2}{\partial z^2}\mathcal{R}_\zeta\left(y^{(n)}_k, y^{(n)}_j + \delta^{(n)}_j h\right) + h^2\underline{O}\left(r^\star_{n-k,n-j}(\delta+1)\right)$$

with the remainder

$$r^\star_{n-k,n-j}(d) = \sup_{|s|<d}\left|\frac{\partial^3}{\partial z^3}\mathcal{R}_\zeta\left(y^{(n)}_k + s, y^{(n)}_j + \delta^{(n)}_j h\right)\right|,$$

where the last bound follows from differential equation (5.1) valid for the functions $\psi_\pm$. To simplify calculations for $C^{(n)}_{r_-}$, we use the following notations:

$$S_k := \left(C^{(n)}_{r_-}\right)_{n-k,n-k-1} + \left(C^{(n)}_{r_-}\right)_{n-k,n-k+1},$$
$$D_k := \left(C^{(n)}_{r_-}\right)_{n-k,n-k-1} - \left(C^{(n)}_{r_-}\right)_{n-k,n-k+1}.$$

Then, combining the above expansion with (3.13)–(3.14), we obtain

$$\begin{aligned} D^{(n)}_{n-k,n-j} &= h^{-1}\mathcal{R}_\zeta\left(y^{(n)}_k, y^{(n)}_j + \delta^{(n)}_j h\right)\left(S_k + \left(C^{(n)}_{r_-}\right)_{n-k,n-k}\right) \\ &+ \frac{\partial}{\partial z}\mathcal{R}_\zeta\left(y^{(n)}_k, y^{(n)}_j + \delta^{(n)}_j h\right)\left(D_k - \delta^{(n)}_k S_k + \delta^{(n)}_k\left(C^{(n)}_{r_-}\right)_{n-k,n-k}\right) \\ &+ h\frac{\partial^2}{\partial z^2}\mathcal{R}_\zeta\left(y^{(n)}_k, y^{(n)}_j + \delta^{(n)}_j h\right)\left(\frac{1}{2}S_k - \delta^{(n)}_k D_k - \frac{\delta^2}{2}\left(S_k + \left(C^{(n)}_{r_-}\right)_{n-k,n-k}\right)\right) \\ &+ \underline{O}\left(r^\star_{n-k,n-j}(\delta+1)\right), \end{aligned} \tag{4.4}$$

where for the last term we have used the uniform bound for elements $\left(C^{(n)}_{r_-}\right)_{n-j,n-k}$.

Now it is sufficient to calculate every expression in the brackets. We start with S_k and D_k,

$$S_k = \sin\theta - 2\cos\frac{\theta}{2}\cot\frac{\theta}{2}p_\theta y^{(n)}_k h^2 - 2\sin^2\frac{\theta}{2}\zeta h^2 + is_{n+k}p_\theta\cos\frac{\theta}{2}h^3 + h^4\underline{O}\left(r^{(n)}_{k,\zeta}\right),$$

$$D_k = -2is_{n+k}\sin^2\frac{\theta}{2} + 2is_{n+k}\cos\frac{\theta}{2}p_\theta y_k^{(n)}h^2 - is_{n+k}\sin\theta\zeta h^2$$
$$-\cos\frac{\theta}{2}\cot\frac{\theta}{2}p_\theta h^3 + h^4\underline{O}\left(r_{k,\zeta}^{(n)}\right).$$

Therefore, with an error of order $h^4\underline{O}\left(r_{k,\zeta}^{(n)}\right)$ we can write

$$S_k + \left(C_{r_-}^{(n)}\right)_{n-k,n-k} \approx -2h^2\left(p_\theta\sin^{-1}(\theta/2)\,y_k^{(n)} + \zeta\right),$$

$$D_k - \delta_k^{(n)}S_k + \delta_k^{(n)}\left(C_{r_-}^{(n)}\right)_{n-k,n-k} \approx -2\delta_k^{(n)}h^2\Big(p_\theta\sin^{-1}(\theta/2)\,y_k^{(n)} - \zeta + is_{n+k}p_\theta\cos(\theta/2)\sin^{-2}(\theta/2)\,h\Big).$$

Finally, combining the above relations and the equation for $\mathcal{R}_\zeta$ in the form

$$\sin\theta\frac{\partial^2}{\partial z^2}\mathcal{R}_\zeta\left(y_k^{(n)}, y_j^{(n)} + \delta_j^{(n)}h\right)$$
$$-\left(2p_\theta\sin^{-1}\theta/2y_k^{(n)} + \zeta\right)\mathcal{R}_\zeta\left(y_k^{(n)}, y_j^{(n)} + \delta_j^{(n)}h\right) = 0,$$

we obtain the remainder in (4.4) with all terms of order less than h^2. Gathering all these remainders and the remainder $h^4\underline{O}\left(r_{k,\zeta}^{(n)}\right)$, we get (3.19). For $j = k$, the calculations can be performed similarly if we take into account jump condition (5.2). ■

P r o o f of Lemma 3.7. We start with estimate of

$$X_n(\zeta) = n^{-2/3}\int\mathcal{K}_n(x,x)\,G\left((\zeta - x)\,n^{-2/3}\right)dx,$$

where $\mathcal{K}_n$ is defined as in (3.6) but without any restriction. Let $\zeta = s + i\varepsilon$. Changing variables to $z = \theta + \zeta n^{-2/3}$ and using (3.6) with (3.8), we obtain

$$X_n(\zeta) = n^{1/3}\Re h_n(z),$$

where

$$h_n(z) = \int_{-\pi}^{\pi} g(z-\lambda)\;\rho_n(\lambda)\;d\lambda.$$

For further estimates we use the "quadratic" equation obtained in [6],

$$h_n^2(z) - 2iV'(\Re z)\,h_n(z) - 2iQ_n(z) - 1 = -\frac{2}{n^2}\delta_n(z),$$

with

$$Q_n(z) = \int_{-\pi}^{\pi} g(z-\lambda)\left(V'(\lambda) - V'(\Re z)\right) \rho_n(\lambda)\, d\lambda,$$

$$\delta_n(z) = \iint_{-\pi}^{\pi} \left|K_n^{(n)}(\lambda,\mu)\right|^2 (g(z-\lambda) - g(z-\mu))^2\, d\lambda d\mu.$$

Solving the "quadratic" equation, we get

$$X_n(\zeta) = n^{1/3} \Re \sqrt{f_n(s,\varepsilon) - 2n^{-2}\delta_n(z)},$$

where the function

$$f_n(s,\varepsilon) = -V'^2\left(\theta + sn^{-2/3}\right) + 2iQ_n\left(\theta + (s+i\varepsilon)\, n^{-2/3}\right) + 1$$

is twice differentiable in both variables. Using the symmetry of the kernel $K_n^{(n)}$ and (4.1), we can estimate $\delta_n(z)$ as

$$\left|n^{-2}\delta_n(z)\right| \le 4n^{-2} \int_{-\pi}^{\pi} K_n^{(n)}(\lambda,\lambda)\, |g(z-\lambda)|^2\, d\lambda.$$

Then the identity (3.9) yields

$$\left|n^{-2}\delta_n(z)\right| \le 4n^{-1} + 2n^{-4/3} \coth\left(\varepsilon n^{-2/3}\right) \cdot X_n(\zeta) \le Cn^{-2/3}\left(n^{-1/3} + \varepsilon^{-1} X_n(\zeta)\right),$$

as $\varepsilon = \underline{O}(1)$. Now we continue the estimation of $Q_n(z)$. For the density ρ_n, we use the bound (see [6])

$$\left|\rho_n'(\lambda)\right| \le C\left(\left|\psi_{n-1}^{(n)}\right|^2 + \left|\psi_n^{(n)}\right|^2 + 1\right),$$

where $\psi_k^{(n)} = P_k^{(n)} w_n^{1/2}$ are orthonormal functions. Hence, the density ρ_n is uniformly bounded and therefore, similarly to (2.17) of [6], we have

$$|Q_n(z) - Q_n(\Re z)| \le C \Im z \left|\log \Im z\right|.$$

The weak convergence (1.5) with

$$\phi(\lambda) = \left(V'(\lambda) - V'\left(\theta + s/\gamma n^{2/3}\right)\right) \cot \frac{\lambda - \theta - s/\gamma n^{2/3}}{2}$$

implies

$$\left|Q_n\left(\theta + s/\gamma n^{2/3}\right) - Q\left(\theta + s/\gamma n^{2/3}\right)\right| \le Cn^{-1/2} \log^{1/2} n$$

if $|s| \le c_\theta n^{2/3}$. Hence, combining the above relations, we obtain

$$|f_n(s,\varepsilon) - f(s)| \le Cn^{-2/3}\log n \left(|\log \varepsilon| + n^{1/6}\right),$$

with $f(s) := f(s,0)$. The properties of the Herglotz transformation yield (see [6])

$$\rho(\lambda) = \frac{1}{2\pi}\lim_{\varepsilon\to+0}\Re h(\lambda + i\varepsilon).$$

Therefore, at the edge point θ we obtain $f(0) = 0$ and $f'(0) < 0$. Hence, by the differentiability of $f(s)$, we obtain

$$X(\zeta) = \Re\sqrt{\underline{O}\left(s + \varepsilon^{-1}X(\zeta) + n^{1/6}\log n\right)}. \tag{4.5}$$

Solving the quadratic inequality, we estimate $X(\zeta)$ as follows:

$$X(\zeta) \le C\left(\varepsilon^{-1} + s^{1/2} + n^{1/12}\log^{1/2} n\right).$$

Now we write (4.5) more precisely

$$X(\zeta) = \Re\sqrt{-Cs + \varepsilon^{-2}\underline{O}\left(1 + \varepsilon s^{1/2} + \varepsilon n^{1/12}\log^{1/2} n\right)}.$$

Below we need the estimate of $X(\zeta)$ for $s > Cn^{1/6}\log n$ and $\varepsilon = \underline{O}(1)$. Hence we obtain

$$X(\zeta) \le C_1\left|s - C_2 n^{1/6}\log n\right|^{-1/2}. \tag{4.6}$$

Note that all constants in the above estimates depend only on V and can be bounded by some combination of $\sup|V|$, $\sup|V''|$ and $\sup|V'''|$. Now we return to the estimate of the sum in Lemma 3.7. By the spectral theorem,

$$I(M) = n^{-2/3}\sum_{j=M+1}^{n} G^{(n)}_{n-j,n-j}(z) = n^{-2/3}\sum_{j=0}^{n-M-1}\int G(\lambda - z)\left|\chi_j^{(n)}(\lambda)\right|^2 w_n(\lambda)\,d\lambda.$$

Let us consider the analogue of the joint eigenvalue distribution of model (1.1) in the form

$$p^{(n-M)}_{n-M}(\lambda_1,\dots,\lambda_{n-M}) = \frac{1}{Z_n^{(n-M)}}\prod_{1\le j<k\le n-M}\left|e^{i\lambda_j} - e^{i\lambda_k}\right|^2 \exp\left\{-n\sum_{j=1}^{n-M}V(\cos\lambda_j)\right\}.$$

Then, by the same argument as above for model (1.1), we define the first marginal density

$$\rho^{(n-M)}_{n-M}(\lambda) = \frac{1}{n-M}\sum_{j=0}^{n-M-1}\left|\chi_j^{(n)}(\lambda)\right|^2 w_n(\lambda).$$

On the other hand, this density can be considered as the first marginal density for model (1.1) with the potential $\widetilde{V} = \dfrac{n}{n-M} V$. Hence,

$$I(M) = n^{-2/3} \int G(\lambda - z) \, K_{n-M}^{(n-M,\widetilde{V})}(\lambda, \lambda) \, d\lambda = X_{n-M}^{\widetilde{V}}(\zeta).$$

But it follows from the result of [15] that the support of the equilibrium density for $\widetilde{V}$ is $[\theta_M, \theta_M]$ with $\theta_M = \theta - c_V \left(Mn^{-1}\right) + \overline{o}\left(Mn^{-1}\right)$ with some $c_V > 0$. Hence, by (4.6),

$$X_{n-M}^{\widetilde{V}} \leq C n^{-1/12},$$

and Lemma 3.7 is proved. ■

5. Appendix

In this section we present the properties and the asymptotic analysis of the resolvent of the Airy operator. Denote by $\mathcal{L}$ the second order differential operator on the set of twice continuously differentiable functions on $\mathbb{R}$,

$$\mathcal{L}[f](x) = a^3 f''(x) - b^3 x f(x).$$

Let $\mathcal{R}_\zeta(x, y)$ be the kernel of the resolvent $(\mathcal{L} - \zeta I)^{-1}$ for $\Im\zeta \neq 0$. By the general principles (for example see [16], Section 72)

Proposition 5.1. *Let $Ai(z)$ and $Bi(z)$ be the standard Airy functions. Denote by $\psi_\pm$ the following functions:*

$$\psi_-(x,\zeta) = Ci(X_{x,\zeta}), \quad \psi_+(x,\zeta) = Ai(X_{x,\zeta}),$$

with

$$Ci(X) = iAi(X) - Bi(X) \quad \text{and} \quad X_{x,\zeta} = a^{-1}bx + a^{-1}b^{-2}\zeta.$$

Then these functions are the unique solutions of the differential equation

$$a^3 \frac{\partial^2}{\partial x^2} \psi_\pm(x,\zeta) - \left(b^3 x + \zeta\right) \psi_\pm(x,\zeta) = 0, \tag{5.1}$$

that are square integrable on the right (left) half axis and fixed by jump condition

$$\psi_-(x,\zeta) \frac{d}{dx}\psi_+(x,\zeta) - \psi_+(x,\zeta) \frac{d}{dx}\psi_-(x,\zeta) = a^{-1} b \pi^{-1}. \tag{5.2}$$

And the resolvent $\mathcal{R}_\zeta$ has two representations

$$\mathcal{R}_\zeta(x,y) = ab^{-1}\pi \begin{cases} \psi_-(x,\zeta)\,\psi_+(y,\zeta), & x \leq y, \\ \psi_+(x,\zeta)\,\psi_-(y,\zeta), & x \geq y, \end{cases} \tag{5.3}$$

$$\mathcal{R}_\zeta(x,y) = a^{-2}b^{-1} \int \frac{1}{t-\zeta} Ai\left(a^{-1}bx + a^{-1}b^{-2}t\right) Ai\left(a^{-1}by + a^{-1}b^{-2}t\right) dt. \tag{5.4}$$

The following asymptotic behaviour of the Airy functions can be found in [17].

Proposition 5.2. *For any $\delta > 0$, the following asymptotics are uniform in the corresponding domains:*

$$\begin{aligned}
Ai(z) &= \pi^{-1/2} z^{-1/4} e^{-\frac{2}{3} z^{3/2}} \left(1 + \underline{O}\left(z^{-3/2}\right)\right), && |\arg z| < \pi - \delta, \\
Ai(-z) &= \pi^{-1/2} z^{-1/4} \sin\left(\frac{2}{3} z^{3/2} + \frac{\pi}{4}\right) \left(1 + \underline{O}\left(z^{-3/2}\right)\right), && |\arg z| < \frac{2}{3}\pi - \delta, \\
Ci(z) &= \pi^{-1/2} z^{-1/4} e^{\frac{2}{3} z^{3/2}} \left(1 + \underline{O}\left(z^{-3/2}\right)\right), && |\arg z| < \frac{1}{3}\pi - \delta, \\
Ci(-z) &= \pi^{-1/2} z^{-1/4} e^{i\frac{2}{3} z^{3/2} + i\frac{\pi}{4}} \left(1 + \underline{O}\left(z^{-3/2}\right)\right), && |\arg z| < \frac{2}{3}\pi - \delta.
\end{aligned}$$

The main term for the derivatives can be obtained by direct differentiation of the asymptotics. The last proposition and the definition of the functions $\psi_\pm$ yield the asymptotic behaviour of them

Proposition 5.3. *The functions $\psi_\pm$ are entire in x and ζ and have the following asymptotic behaviour in x for $\Im\zeta = \varepsilon > 0$:*

$$|\psi_+(x,\zeta)| =$$

$$\pi^{-1/2} |X_{x,\zeta}|^{-1/4} \left(1 + \underline{O}\left(|X_{x,\zeta}|^{-3/2}\right)\right) \begin{cases} \exp\left\{-\frac{2}{3} |\Re X_{x,\zeta}|^{3/2}\right\}, & x \to \infty \\ \exp\left\{a^{-1} b^{-2} \varepsilon |\Re X_{x,\zeta}|^{1/2}\right\}, & x \to -\infty \end{cases}$$

$$|\psi_-(x,\zeta)| =$$

$$(4\pi)^{-1/2} |X_{x,\zeta}|^{-1/4} \left(1 + \underline{O}\left(|X_{x,\zeta}|^{-3/2}\right)\right) \begin{cases} \exp\left\{\frac{2}{3} |\Re X_{x,\zeta}|^{3/2}\right\}, & x \to \infty \\ \exp\left\{-a^{-1} b^{-2} \varepsilon |\Re X_{x,\zeta}|^{1/2}\right\}, & x \to -\infty \end{cases}$$

Proposition 5.4. *For any non-negative integers s, q and any $A \in \mathbb{R}_+$ there exists a constant $C_{A,s,q}$ such that for any $x \geq -A$ and $\zeta \in \Omega_A$*

$$I(s;q) = \int_{-\infty}^{\infty} |y|^s \left| \frac{\partial^q}{\partial y^q} \mathcal{R}_\zeta(x,y) \right|^2 dy \leq C_{A,s,q} (1 + |x|)^{s+q-3/2}. \tag{5.5}$$

P r o o f of Proposition 5.4. In view of equation (5.1), two extra derivatives in (5.4) give the extra factor of order $|y|^2 + |\zeta|^2$ to the integrand. Therefore, we start with $I(s;0)$. Since $|\mathcal{R}_\zeta(x,y)| \le C_A e^{-c_A|x-y|^{1/2}}$ for $x \ge -A$ and $\zeta \in \Omega_A$, we split the integral from (5.5) into two parts

$$I(s;0) = \int\limits_{|y-x|<2|x|} + \int\limits_{|y-x|>2|x|} \le C_s\left(x^s + |\zeta|^s\right) \int \left|\mathcal{R}_\zeta(x,y)\right|^2 dy$$
$$+ C_A \int\limits_{t>2|x|} (t+x)^s e^{-c_A t^{1/2}} dt. \tag{5.6}$$

For the first integral we note that by the spectral theorem and the resolvent identity,

$$\int\limits_{-\infty}^{\infty} \left|\mathcal{R}_\zeta(x,y)\right|^2 dy = \frac{\Im \mathcal{R}_\zeta(x,x)}{\Im \zeta}. \tag{5.7}$$

The asymptotic behaviour of $\psi_\pm$ from Proposition 5.3 implies

$$|\mathcal{R}_\zeta(x,x)| \le C_A (1+|x|)^{-1/2}, \quad \text{and} \quad |\Im \mathcal{R}_\zeta(x,x)| \le C_A(1+|x|)^{-3/2}. \tag{5.8}$$

Combining (5.6) with (5.7) and (5.8), we obtain (5.5) with $q=0$. In view of equation (5.1), it is sufficient to prove (5.4) only for $q=0,1$. If $q=1$, similarly to the above argument, we split the integral into two parts. In the first term, integrating by parts, we have

$$\int\limits_{-\infty}^{\infty} \left|\frac{\partial}{\partial y}\mathcal{R}_\zeta(x,y)\right|^2 dy = \int\limits_{-\infty}^{\infty} (c_1 y + c_2\zeta) \left|\mathcal{R}_\zeta(x,y)\right|^2 dy.$$

The r.h.s satisfies the necessary bound for $q=1$, hence the proposition is proved. ■

Proposition 5.5. *Let $h = n^{-1/3}$, $M = \left[C_0 n^{1/2}\log n\right]$. Also, denote by $x_j = jh$ the equidistant set and $z_j^{(1,2)} = x_j + \delta_j^{(1,2)} h$ two shifted sets, with complex shifts $\left|\delta_j^{(1,2)}\right| \le C$ for some absolute constant C. Then,*

$$h\sum_{j=0}^{M} \left|\Im \mathcal{R}_\zeta\left(z_j^{(1)}, z_j^{(2)}\right)\right| \le C, \tag{5.9}$$

$$h\sum_{j=0}^{M} \left|\mathcal{R}_\zeta\left(z_j^{(1)}, z_j^{(2)}\right)\right| \le C(Mh)^{1/2}, \tag{5.10}$$

and for any non-negative integer p, $d=0$ or 1 and $k \le M$

$$h\sum_{j=0}^{\infty} |x_j|^p \left|\frac{\partial^d}{\partial z^d}\mathcal{R}_\zeta\left(z_j^{(1)}, z_k^{(2)}\right)\right|^2 \le C(1+|x_k|)^{p+d-3/2}. \tag{5.11}$$

P r o o f of Proposition 5.5. Since $\left|z_j^{(1,2)} - x_j\right| = \underline{O}(h)$, $|\Im\mathcal{R}_\zeta(x,x)| \leq C(1+|x|)^{-3/2}$ and derivatives of $\mathcal{R}_\zeta$ are bounded near the real line, we obtain that

$$\left|\Im\mathcal{R}_\zeta\left(z_j^{(1)}, z_j^{(2)}\right)\right| \leq 2C(1+|x_j|)^{-3/2}$$

for $n > n_0$ with some integer n_0. Hence,

$$h\sum_{j=0}^{M}\left|\Im\mathcal{R}_\zeta\left(z_j^{(1)}, z_j^{(2)}\right)\right| \leq Ch\sum_{j=0}^{M}(1+|x_j|)^{-3/2} \leq C.$$

The second statement can be checked in a similar way. The proof of the third statement consists of several steps. First, we change z_j by x_j in (5.11). The error of this change is a combination of sums of higher derivatives with extra factors h. These sums are small, because for z_j far from z_k these derivatives admit the exponential bound, and for $z_j \sim z_k$, in view of equation (5.1) and restriction $|z_k| \leq Cn^{1/6}\log n$, every two extra derivatives will give us the sum as in (5.11) with the factor of order $n^{-1/2}\log n$. After the change of z_j by x_j, we obtain the sum which can be estimated by the integral

$$C\int_0^\infty x^p\left|\frac{\partial^d}{\partial z^d}\mathcal{R}_\zeta\left(x, z_k^{(2)}\right)\right|^2 dx,$$

because of the smoothness and exponential decreasing of $\mathcal{R}_\zeta$. And finally, the identity (5.7) and Proposition 5.4 yield (5.11). We used the identity (5.7) which is valid for real x, but it remains valid for complex x because the l.h.s and r.h.s of the (5.7) are entire functions equal at the real line. ∎

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