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Dependence of Kolmogorov Widths on the Ambient Space

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We study the dependence of the Kolmogorov widths of a compact set on the ambient Banach space.

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Dedicated to the memory of Mikhail Iosifovich Kadets

Contents

1. Introduction

Let $\mathcal{Z}$ be a subset of a Banach space $\mathcal{X}$ and $x \in \mathcal{X}$. The *distance from x to $\mathcal{Z}$* is defined as

$$E(x, \mathcal{Z}) = \inf\{||x - z|| : \ z \in \mathcal{Z}\}.$$

Definition 1.1. Let K be a subset of a Banach space $\mathcal{X}$, $n \in \mathbb{N} \cup \{0\}$. The *Kolmogorov n-width* (or *n-th Kolmogorov number*) of K is given by

$$d_n(K, \mathcal{X}) = \inf_{\mathcal{X}_n} \sup_{x \in K} E(x, \mathcal{X}_n),$$

where the infimum is over all subspaces $\mathcal{X}_n \subset \mathcal{X}$, of dimension not exceeding n. We use the notation $d_n(K)$ if $\mathcal{X}$ is clear from context.

This notion was introduced by Kolmogorov [Kol36] in 1936. It has been a subject of an extensive study and has found many applications, both in Approximation Theory and in Functional Analysis, see [CS90], [LGM96], [Pie80], [Pin85], and [Tik60]. In [OS09] it was discovered that some general asymptotic properties of Kolmogorov widths are useful in the study of closures of sets of operators in the weak operator topology. More results on asymptotic properties of Kolmogorov widths were discovered in [Ost10]. The purpose of this paper is to continue analysis of asymptotic properties of widths.

Our emphasis in this paper is on dependence of asymptotic properties of widths on the ambient space. It is known for long time (see [Tik60, §7]) that if $\mathcal{Y}$ is a subspace of a Banach space $\mathcal{X}$ and $K \subset \mathcal{Y}$, then it can happen that $d_n(K, \mathcal{Y}) > d_n(K, \mathcal{X})$. Furthermore, the quotient $d_n(K, \mathcal{Y})/d_n(K, \mathcal{X})$ can be arbitrarily large. An example with in a certain sense optimal order of this quotient was found in [Ost10], where the following result was proved:

Theorem 1.2 ([Ost10]). *For each n the Banach space ℓ_1^{3n} contains a $2n$-dimensional subspace Y_{2n} and a compact $K_{2n} \subset \mathcal{Y}_{2n}$ such that $d_n(K_{2n}, \ell_1^{3n}) \leq 1$ but $d_n(K_{2n}, \mathcal{Y}_{2n}) \geq c\sqrt{n}$ for some absolute constant $c > 0$.*

R e m a r k 1.3. The order in Theorem 1.2 is optimal in the following sense: Proposition 2.7 implies that $d_n(K_{2n}, \mathcal{Y}_{2n}) \leq \sqrt{2n}\, d_n(K_{2n}, \ell_1^{3n})$.

The paper is structured as follows: in Section 2 we introduce the notion of the absolute width $d_n^a(K)$ (Definition 2.1), and collect the necessary basic facts. In general, $d_n^a(K) \leq d_n(K)$, but in some cases, we obtain the equality, or at least proportionality, of the two quantities. In Section 3 we study affine widths. This allows us to construct, in certain Banach spaces X, a compact convex set K so that $d_1(K) > d_1^a(K)$. In Section 4 we note some connections of Kolmogorov and absolute widths to other s-sequences (such as the sequence of Gelfand numbers). This provides us with some tools to be used later.

We then pass to the study of asymptotic behavior of Kolmogorov numbers. In Section 5 we exhibit a large class of Banach spaces which contain a sequence of compact subsets (K_n), so that $\lim_n d_{k_n}(K_n)/d_{k_n}^a(K_n) = \infty$, for some increasing sequence (k_n). In Section 6 we sharpen this result by showing that, if a space $\mathcal{X}$ satisfies certain conditions (for instance, if it is K-convex), then it contains a compact K with the property that $\limsup_n d_n(K)/d_n^a(K) = \infty$. If, furthermore, $\mathcal{X}$ contains ℓ_p $(1 < p < \infty)$ as a complemented subspace, then it contains a compact subset K so that $\liminf_n n^{-\sigma} d_n(K)/d_n^a(K) = \infty$, for some $\sigma > 0$. In Section 7 we examine compacts K for which $d_n(K) = d_n^a(K)$, for any ambient space. Finally, Section 8 is devoted to comparing the Kolmogorov widths of the sets K and $u(K)$, where u is compact operator.

Throughout the paper we pose some interesting geometric problems related to our study (Problems 2.5, 2.6, 5.12, 6.1, 6.4, 7.1, 8.1). Problem 5.12 could be of interest not only in the context of the theory of widths.

We use the basic Banach space theory and its standard notation. We denote by $\mathrm{B}(\mathcal{X})$ the closed unit ball of a space $\mathcal{X}$.

2. Absolute Widths

Dependence of the sequence $\{d_n(K)\}_{n=0}^{\infty}$ on the ambient Banach space leads to the introduction of the following definition.

Definition 2.1 ([Ism74]). Let K be a compact in a Banach space $\mathcal{Y}$ and $n \in \mathbb{N}$. The n-th *absolute width* (or *number*) $d_n^a(K)$ of K is defined by $d_n^a(K) = \inf_{\mathcal{X}} d_n(K, \mathcal{X})$, where the inf is over all Banach spaces $\mathcal{X}$ containing $\mathcal{Y}$ as a subspace.

Absolute widths were studied in [Ism74], [Koc90], [Oik95], and [Ost10]. Our main purpose in this paper is to study the asymptotic behavior of the quotients $d_n(K, \mathcal{Y})/d_n^a(K)$ under different assumptions. We start with the following natural open problem: characterize Banach spaces $\mathcal{Y}$ for which $d_n(K, \mathcal{Y}) = d_n^a(K)$ for all compacts $K \subset \mathcal{Y}$.

We present a class of Banach spaces having this property. The following definition goes back to [LP68]: Let $1 \le \lambda < \infty$. A Banach space $\mathcal{Y}$ is called an $\mathcal{L}_{\infty,\lambda}$-*space* if for every finite-dimensional subspace $S \subset \mathcal{Y}$ there is a finite-dimensional subspace $F \subset \mathcal{Y}$ such that $S \subset F$ and $d(F, \ell_\infty^m) \le \lambda$, where $m = \dim F$. A Banach space is called an $\mathcal{L}_{\infty,\lambda+}$-*space* if it is a $\mathcal{L}_{\infty,\nu}$-space for each $\nu > \lambda$. See [Bou81] and [LT73] for theory of $\mathcal{L}_p$-spaces.

More generally, a Banach space $\mathcal{X}$ is called an $\mathcal{N}_\lambda$*-space* if, for every finite dimensional subspace E of X, there exists a finite dimensional subspace F, satisfying $E \subset F \subset X$ and $\lambda(F) \le \lambda$. Here, following [Tom89], we define $\lambda(F)$ the (*absolute*) *projection constant* of F as follows: for a superspace $G \supset F$, define the *relative projection constant* $\lambda(F, G)$ as the infimum of $\|P\|$, where P is the projection from G onto F. Then $\lambda(F) = \sup \lambda(F, G)$, with the supremum taken over all superspaces G.

A Banach space X is called an $\mathcal{N}_{\lambda+}$*-space* if it is a $\mathcal{N}_\nu$-space for each $\nu > \lambda$, and an $\mathcal{N}$*-space* if it is a $\mathcal{N}_\lambda$-space for some $1 \le \lambda < \infty$.

It is easy to see that each $\mathcal{L}_{\infty,\lambda}$-space is an $\mathcal{N}_\lambda$-space. However, the converse is false, see e.g. [Sza90]. It is not known whether each $\mathcal{N}$-space is an $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda < \infty$. This problem is a version of the well-known P_λ-problem (see [LP68, Problem 7, p. 323]), which is still open. However, it is known [LL66] that, for a real Banach space $\mathcal{X}$, the following are equivalent: (i) $\mathcal{X}$ is a $\mathcal{N}_{1+}$-space; (i) $\mathcal{X}$ is a $\mathcal{L}_{\infty,1+}$-space; (iii) $\mathcal{X}^* = L_1(\mu)$, for some measure μ.

Proposition 2.2. *Let K be a compact in an $\mathcal{N}_{\infty,\lambda+}$-space $\mathcal{Y}$. Then $d_n(K, \mathcal{Y}) \le \lambda d_n^a(K)$ for all $n \in \mathbb{N}$.*

P r o o f. It suffices to show that for each $C > \lambda$ and $n \in \mathbb{N}$ we have $d_n(K, \mathcal{Y}) \le C d_n^a(K)$. Pick $\varepsilon > 0$ so that $(1 + 3\varepsilon + \varepsilon^2)\lambda < C$. By the definition of d_n^a there exists a Banach space $\mathcal{X} \supset \mathcal{Y}$ and an n-dimensional subspace $\mathcal{X}_n \subset \mathcal{X}$ such that $E(x, \mathcal{X}_n) \le (1+\varepsilon)d_n^a(K)$ for any $x \in K$. Let $\{k_i\} \subset K$ be an $\varepsilon\lambda d_n^a(K)$-net in K. Find a finite dimensional subspace $F \subset \mathcal{Y}$, containing $\{k_i\}$, so that there exists a projection $P : \mathcal{X} \to F$ satisfying $\|P\| \le \lambda(1 + \varepsilon)$. Let $\mathcal{Y}_n = P(\mathcal{X}_n)$. Then $E(k_i, \mathcal{Y}_n) = E(Pk_i, P\mathcal{X}_n) \le (1+\varepsilon)\lambda E(k_i, \mathcal{X}_n) \le (1+\varepsilon)^2\lambda d_n^a(K)$. Let $k \in K$ and k_i be such that $||k - k_i|| \le \varepsilon\lambda d_n^a(K)$, we have

$$E(k, \mathcal{Y}_n) \le ||k - k_i|| + E(k_i, \mathcal{Y}_n) \le ((1 + \varepsilon)^2 + \varepsilon)\lambda d_n^a(K) \le C d_n^a(K). \qquad \blacksquare$$

Corollary 2.3. *Let K be a compact in an $\mathcal{L}_{\infty,1+}$-space $\mathcal{Y}$. Then $d_n(K, \mathcal{Y}) = d_n^a(K)$ for all $n \in \mathbb{N}$.*

In this connection it is worth mentioning that all spaces of continuous functions on compacts with their sup-norms are $\mathcal{L}_{\infty,1+}$-spaces, see [LT73].

R e m a r k 2.4. Corollary 2.3 can be regarded as a generalization of the following result of Ismagilov [Ism74, Corollary of Theorem 2]: Let K be a compact in a Banach space $\mathcal{X}$ and $\mathcal{B}$ be the Banach space of all bounded functions on $\mathrm{B}(\mathcal{X}^*)$ (the unit ball of $\mathcal{X}^*$) with the sup-norm. Let i be the natural isometric embedding of $\mathcal{X}$ into $\mathcal{B}$. Then $d_n^a(K) = d(i(K), \mathcal{B})$. To get this result from Corollary 2.3 it suffices to combine the corollary with the well-known fact that $\mathcal{B}$ is an $\mathcal{L}_{\infty,1+}$-space (see [LT73]).

Do Proposition 2.2 and Corollary 2.3 characterize the $\mathcal{N}$ spaces and $\mathcal{L}_{\infty,1+}$ spaces, respectively?

Problem 2.5. *Let a Banach space $\mathcal{Y}$ be such that for some $1 \le \lambda < \infty$ the condition $d_n(K, \mathcal{Y}) \le \lambda d_n^a(K)$ holds for each compact $K \subset \mathcal{Y}$ and each $n \in \mathbb{N}$. Does it follow that $\mathcal{Y}$ is an $\mathcal{N}$-space?*

Problem 2.6. *Let a Banach space $\mathcal{Y}$ be such that $d_n^a(K) = d_n(K, \mathcal{Y})$ for each compact $K \subset \mathcal{Y}$ and each $n \in \mathbb{N}$. Does it follow that $\mathcal{Y}$ is an $\mathcal{L}_{\infty,1+}$-space?*

Approaches to these questions may rely on Zippin's solution [Zip81a, Zip81b, Zip84] to the close-to-isometric version of the P_λ-problem. (See [Tom89] for a presentation of this result of Zippin and [Zip00] for further results related to the P_λ-problem.)

Corollary 2.3 can be used to estimate from above the quotient $d_k(K)/d_k^a(K)$ for an n-dimensional compact K.

Proposition 2.7. *Let K be an n-dimensional compact in a Banach space $\mathcal{Y}$. Then $d_k(K, \mathcal{Y}) \le \sqrt{n} d_k^a(K)$ for all $k \in \mathbb{N}$.*

P r o o f. We may assume that $\mathcal{Y}$ is separable and so we may consider $\mathcal{Y}$ as a subspace of ℓ_∞. It is easy to see that ℓ_∞ is an $\mathcal{L}_{\infty,1+}$-space. By Corollary 2.3, $d_n^a(K) = d_n(K, \ell_\infty)$.

The inequality $d_k(K, \mathcal{Y}) \le \sqrt{n} d_k^a(K)$ is trivially true for $k \ge n$. So let $k \in \{0, \dots, n-1\}$. Consider an arbitrary $\varepsilon > 0$. Let $\mathcal{X}_k$ be a k-dimensional subspace of ℓ_∞ such that $E(x, \mathcal{X}_k) \le (1+\varepsilon) d_k^a(K)$ for all $x \in K$. Let $P : \ell_\infty \to \mathrm{span}[K]$ be a linear projection with norm $\le \sqrt{n}$, existing by the Kadets–Snobar theorem [KS71] and let $\mathcal{Y}_k = P\mathcal{X}_k$. Then for all $x \in K$ we have $E(x, \mathcal{Y}_k) = E(Px, P\mathcal{X}_k) \le ||P|| E(x, \mathcal{X}_k) \le \sqrt{n}(1+\varepsilon) d_k^a(K)$. ■

As we already mentioned in Remark 1.3, the estimate of Proposition 2.7 is optimal up to a multiplicative constant.

As a step towards the solution of Problems 2.6 and 2.5 we find a wide class of spaces X for which the quotients $d_n(K, X)/d_n^a(K)$ can be arbitrarily large. This is the subject of Sections 5 and 6.

3. Affine Widths, Geometry, and Injectivity

While dealing with arbitrary convex (not necessarily centrally symmetric) sets, it is convenient to use affine subspaces for approximation (see, e.g., [AO10]).

Definition 3.1. Let K be a compact in a Banach space Y and $n \in \mathbb{N} \cup \{0\}$. The n-th *affine width* $\tilde{d}_n(K)$ of K is set to be $\inf_Z \sup_{x \in K} E(x, Z)$, where the infimum runs over all affine subspaces of $Z \subset Y$ of dimension not exceeding n.

The n-th *absolute affine width* $\tilde{d}_n^a(K)$ of K is defined by $\tilde{d}_n^a(K) = \inf_X \tilde{d}_n(K, X)$, where the inf is over all Banach spaces X containing Y as a subspace.

It is clear that $\tilde{d}_n^a(K) \leq \tilde{d}_n(K, X)$, and the equality is attained if X is 1-injective. Moreover (see [AO10, Section 6.2]),

$$d_n(K) \geq \tilde{d}_n(K) \geq d_{n+1}(K \cup (-K)).$$

Furthermore, $d_n(K) = \tilde{d}_n(K)$ if K is centrally symmetric. The affine widths $\tilde{d}_0$ have been considered previously. To summarize the existing knowledge on them, recall a few definitions.

Definition 3.2. For a bounded subset K of a Banach space $\mathcal{Y}$, define its *diameter* $D(K)$ and *radius* $R(K)$ by setting

$$D(K) = \sup_{a,b \in K} \|a - b\|, \quad R(K) = \inf_{y \in \mathcal{Y}} \sup_{a \in K} \|a - y\|$$

(that is, $R(K)$ is the infimum of the radii of balls containing K). The *Jung constant* $J(\mathcal{Y})$ of a Banach space $\mathcal{Y}$ is defined as the supremum (over bounded sets $K \subset \mathcal{Y}$) of $2R(K)/D(K)$. Note that, in our notation, $R(K) = \tilde{d}_0(K)$.

Clearly, $2 \geq J(\mathcal{Y}) \geq 1$. The spaces $\mathcal{Y}$ with $J(\mathcal{Y}) = 1$ were described in [Dav77].

Theorem 3.3 ([Dav77]). *For a real Banach space $\mathcal{Y}$, the following are equivalent:*

1. *For any compact $K \subset \mathcal{Y}$, there exists $y \in \mathcal{Y}$ such that $K \subset \mathrm{B}(y, D(K)/2)$.*

2. *$\mathcal{Y}$ is 1-injective.*

3. $J(\mathcal{Y}) = 1$.

The equivalence (1) $\Leftrightarrow$ (2) in the above theorem precedes [Dav77] — it is due to [Nac50]. For certain Banach spaces, the Jung constant is known. For instance, [Bal87, Pic88] show that, for $1 \leq p < \infty$, $J(L_p(\mu)) = \max\{2^{1/p}, 2^{(p-1)/p}\}$. By [FS98], for any rearrangement invariant space $\mathcal{Y}$ which is not injective, $J(\mathcal{Y}) \geq \sqrt{2}$, and the equality holds iff $\mathcal{Y}$ is isometric to the Hilbert space. [AFS00] establishes the Jung constant for some classes of Banach lattices (such as Lorentz spaces). One is referred to the bibliography of the latter paper for additional information. In our notation, Theorem 3.3 implies that, for any bounded K in a 1-injective Banach space $\mathcal{Y}$, $\tilde{d}_0^a(K) = D(K)/2$. For any Banach space $\mathcal{Y}$, $J(\mathcal{Y}) = \sup_{K \subset \mathcal{Y} \, \mathrm{bounded}} \tilde{d}_0(K)/\tilde{d}_0^a(K)$. This leads to:

Proposition 3.4. *Suppose a real Banach space $\mathcal{X}$ is not 1-injective. Then $\tilde{\mathcal{X}} = \mathbb{R} \oplus_1 \mathcal{X}$ contains a bounded centrally symmetric subset K, such that $d_1^a(K) < d_1(K)$.*

P r o o f. By Theorem 3.3, $\mathcal{X}$ contains a bounded set A, such that $D(A) = 1/2$, while $R(A) = c \in (1/4, 1/2]$. By translation, we may assume that $\|x\| \leq 1/2$ for any $x \in A$. Consider the "skew cylinder"

$$K = \operatorname{conv}\big(1 \oplus A, (-1) \oplus (-A)\big) = \Big\{t \oplus \Big(\frac{1+t}{2} a_1 - \frac{1-t}{2} a_2\Big) : -1 \leq t \leq 1,\, a_1, a_2 \in A\Big\}.$$

We shall show that $d_1(K) \geq c$, while $d_1^a(K) \leq 1/4$ (in fact, equalities hold in both cases, but we do not need this for our purposes). We handle $d_1^a(K)$ first. Embed $\mathcal{X}$ into a 1-injective space $\tilde{\mathcal{X}}$. By the discussion above, there exists $\tilde{x} \in \tilde{\mathcal{X}}$ such that $\|\tilde{x} - a\| \leq 1/4$ for any $a \in A$. Consider the 1-dimensional space $F = \operatorname{span}[1 \oplus \tilde{x}] \subset \mathbb{R} \oplus_1 \tilde{\mathcal{X}}$, and show that, for any $y \in K$, $E(y, F) \leq 1/4$. Indeed, write $y = t \oplus a$, where $t \in [-1, 1]$, and

$$a = \frac{1+t}{2} a_1 - \frac{1-t}{2} a_2 \ (a_1, a_2 \in A).$$

Then $t \oplus t\tilde{x} \in F$, hence

$$E(y, F) \leq \|y - t \oplus t\tilde{x}\| = \|a - t\tilde{x}\| = \Big\|\frac{1+t}{2}(a_1 - \tilde{x}) - \frac{1-t}{2}(a_2 \tilde{x})\Big\|$$
$$\leq \frac{1}{4}\Big(\frac{1+t}{2} + \frac{1-t}{2}\Big) = \frac{1}{4}.$$

Turning to $d_1(K)$, we have to show that, for any 1-dimensional subspace F of $\mathbb{R} \oplus \mathcal{X}$, we have $\sup_{a \in A} E(1 \oplus a, F) \geq c$. If $F = \operatorname{span}[0 \oplus x] \subset \mathbb{R} \oplus_1 \mathcal{X}$, the previous inequality holds for every a. Now consider $F = \operatorname{span}[1 \oplus x] \subset \mathbb{R} \oplus_1 \mathcal{X}$. Note that, for $a \in A$, $E(1 \oplus a, F) = \inf_{t \in \mathbb{R}}(|1 - t| + \|tx - a\|)$. Consider the cases of $\|x\| \leq 1$ and $\|x\| > 1$ separately.

(i) If $\|x\| \leq 1$,

$$|1-t| + \|tx - a\| = |1-t| + \|(x-a) - (1-t)x\| \geq |1-t| + \|x-a\| - |1-t|\|x\| \geq \|x-a\|,$$

hence $\sup_{a \in A} E(1 \oplus a, F) \geq \sup_{a \in A} \|x - a\| \geq c$.

(ii) If $\|x\| > 1$,

$$|1-t| + \|tx - a\| \geq 1 - |t| + |t|\|x\| - \|a\| \geq 1 - \|a\| \geq \frac{1}{2}.$$

As $c \leq 1/2$, we are done. ■

We obtain a sharper result for $\mathcal{X} = L_1(\mu)$.

Proposition 3.5. *Suppose the real Banach space $L_1(\mu)$ (μ is a σ-finite measure) has dimension at least $n = 2^k + 1$, $k \geq 2$. Then $L_1(\mu)$ contains a closed finite dimensional centrally symmetric subset K, satisfying $d_1^a(K) \leq 1/4$, and $d_1(K) \geq (n-1)/(2n)$.*

This result is asymptotically optimal: by Proposition 4.3, $d_1(K) \leq 2d_1^a(K)$.

P r o o f. By assumption, $L_1(\mu)$ contains a contractively complemented copy of ℓ_1^n. Thus, it suffices to prove the existence of a set $K \subset \ell_1^n$ with desired properties. Write $\ell_1^n = \mathbb{R} \oplus_1 \ell_1^{n-1}$. By [Dol87], $J(\ell_1^{n-1}) = 2(n-1)/n$. By the compactness of the set of bounded compacts in a finite dimensional space (with respect to the Hausdorff distance), ℓ_1^{n-1} contains a set A with diameter 1/2, and radius $(n-1)/(2n)$. We construct K as in the proof of Proposition 3.4. ∎

R e m a r k 3.6. In fact, [Dol87] shows that $J(\ell_1^{n-1}) = 2(n-1)/n$ iff there exists a Hadamard matrix of order n. Walsh matrices are clearly Hadamard matrices of order 2^k. The existence of Hadamard matrices of order $4k$ for any $k \in \mathbb{N}$ is a long-standing conjecture.

4. Relations with Other Sequences of s-numbers

In this section, we consider the relations between Kolmogorov and absolute numbers of operators, on one hand, and other sequences of s-numbers, on the other hand. For general properties of s-numbers (or s-sequences), we refer to [Pie87]. We define the *Kolmogorov* and *absolute widths* (*numbers*) of an operator $T \in B(\mathcal{X}, \mathcal{Y})$ by setting $d_n(T) = d_n(\overline{T(\mathrm{B}(\mathcal{X}))})$, and $d_n^a(T) = d_n^a(\overline{T(\mathrm{B}(\mathcal{X}))})$. We also need to define the *approximation* and *Gelfand numbers* of T, denoted by c_n and a_n, respectively:

$$\begin{aligned} a_n(T) &= \inf\{\|T - S\| : S \in B(\mathcal{X}, \mathcal{Y}), \operatorname{rank} S \leq n\}, \\ c_n(T) &= \inf\{\|T|_E\| : E \subset \mathcal{X}, \operatorname{codim} E \leq n\}. \end{aligned}$$

Note that $d_n(T) \leq a_n(T)$, $c_n(T) \leq a_n(T)$, and $d_n(T) = \inf \|qT\|$, where the infimum runs over all quotient maps $q : \mathcal{Y} \to \mathcal{Y}/F$, with $\dim F \leq n$.

By [Pie87], s-numbers (such as $a_n(\cdot)$, $c_n(\cdot)$, and $d_n(\cdot)$) have an ideal property:

$$s_n(ATB) \leq \|A\| s_n(T) \|B\|$$

for any three operators A, B, and T.

The following lemma seems to be part of the Banach space lore.

Proposition 4.1. *Consider an operator $T \in B(\mathcal{X}, \mathcal{Y})$, and $n \in \mathbb{N}$.*

1. *If $\mathcal{Y}$ is λ-injective, then $a_n(T) \leq \lambda c_n(T)$.*

2. *If $\mathcal{X}$ is λ-projective, then $a_n(T) \leq \lambda d_n(T)$.*

P r o o f. We only prove (2). Suppose $d_n(T) < 1$, and show that there exists an operator $u : \mathcal{X} \to \mathcal{Y}$, of rank $\leq n$, with $\|T - u\| < \lambda$. To this end, pick a subspace $F \subset \mathcal{Y}$, such that $\dim F \leq n$, and $\|q_F T\| < 1$ (here, $q_F : \mathcal{Y} \to \mathcal{Y}/F$ is the quotient map). As $\mathcal{X}$ is λ-projective, qT admits a lifting $T_0 : \mathcal{X} \to \mathcal{Y}$, with $\|T_0\| < \lambda$ and $qT_0 = qT$. Let $u = T - T_0$. As $qu = 0$, the range of u must be contained in F, hence $\operatorname{rank} u \leq \dim F \leq n$. ■

In a similar fashion, one can show:

Proposition 4.2. *Consider $T \in B(\mathcal{X}, \mathcal{Y})$, and $n \in \mathbb{N}$.*

1. *If $\mathcal{X}$ is 1-projective, then $d_n^a(T) = c_n(T)$.*

2. *If $\mathcal{Y}$ is 1-injective, then $d_n^a(T) = d_n(T)$.*

P r o o f. Here, we prove (1). Let J be an embedding of $\mathcal{Y}$ into a 1-injective space $\mathcal{Y}_0$. By Proposition 4.1(2), $d_n^a(T) = d_n(JT) = a_n(JT) \geq c_n(JT) = c_n(T)$. Conversely, by Proposition 4.1(1), $a_n(JT) \leq c_n(JT)$. ■

Proposition 4.3. *For any $T \in B(\mathcal{X}, \mathcal{Y})$ and $k \in \mathbb{N}$, $d_k(T) \leq \sqrt{2(k+1)}\, d_k^a(T)$.*

P r o o f. Fix a quotient map $Q : X_0 \to X$, where X_0 is 1-projective. Clearly, $d_k(T) = d_k(TQ) \leq a_k(TQ)$, and $d_k^a(T) = d_k^a(TQ)$. By Proposition 4.2, $d_k^a(T) = c_k(TQ)$. By [CS90, Proposition 2.4.3], $a_k(TQ) \leq \sqrt{2(k+1)}\, c_k(TQ)$. ■

Lemma 4.4. *For any operator u, $c_n(u) \geq d_n^a(u)$.*

Some cases of equality are noted in Propositions 4.1 and 4.2.

P r o o f. For $u \in B(\mathcal{X}, \mathcal{Y})$, consider an isometric embedding j of $\mathcal{Y}$ into $\ell_\infty(I)$, for a sufficiently large index set I. Let $E \subset \mathcal{X}$ be a subspace of codimension n on which $\|u|_E\| < \lambda$. We need to show that $d_n^a(u(\mathrm{B}(\mathcal{X}))) < \lambda$. It suffices to show that $d_n(ju(\mathrm{B}(\mathcal{X}))) < \lambda$. Using the injectivity of $\ell_\infty(I)$, we obtain $\tilde{v} \in B(\mathcal{X}, \ell_\infty(I))$ so that $\tilde{v}|_E = ju|_E$, and $\|\tilde{v}\| = \|ju|_E\| < \lambda$. Let $w = \tilde{v} - ju$. Then $\|ju + w\| < \lambda$ and $\operatorname{rank} w \leq n$. This implies that $d_n(ju(\mathrm{B}(\mathcal{X}))) \leq E(ju(\mathrm{B}(\mathcal{X})), w(\mathcal{X}))) < \lambda$. ■

Finally, we state a well known result, to be used throughout the paper.

Lemma 4.5. *Suppose K is a subset of a Banach space $\mathcal{X}$, and $T \in B(\mathcal{X}, \mathcal{Y})$. Then, for any $n \in \mathbb{N}$, $d_n(T(K), \mathcal{Y}) \leq \|T\| d_n(K, \mathcal{X})$, and $d_n^a(T(K)) \leq \|T\| d_n(K)$.*

Sketch of the p r o o f. (i) For any $C > d_n(K, \mathcal{X})$, there exists $F \subset \mathcal{X}$, so that $\dim F \le n$, and $E(K, F) < C$. Then $d_n(T(K), \mathcal{Y}) \le E(T(K), T(F)) < C\|T\|$. Taking the infimum over all C's, we conclude that $d_n(T(K), \mathcal{Y}) \le \|T\| d_n(K, \mathcal{X})$.

(ii) Embed $\mathcal{X}$ and $\mathcal{Y}$ isometrically into $\ell_\infty(I)$ and $\ell_\infty(J)$, respectively. Then T has an extension $S : \ell_\infty(I) \to \ell_\infty(J)$, with $\|T\| = \|S\|$. We know that $d_n^a(K) = d_n(K, \ell_\infty(I))$, and $d_n^a(T(K)) = d_n(S(K), \ell_\infty(J))$. By Part (i), $d_n(S(K), \ell_\infty(J)) \le \|S\| d_n(K, \ell_\infty(I))$. ■

5. A Class of Spaces for Which the Ratio Between Widths and Absolute Widths Can be Arbitrarily Large

Throughout this section, B_p^m stands for the unit ball of ℓ_p^m. We use $\mathrm{VR}(F)$ to denote the volume ratio of a finite-dimensional normed space F, that is $\mathrm{VR}(F) = \mathrm{vol}(\mathrm{B}(F))/\mathrm{vol}(\mathcal{E})$, where $\mathcal{E}$ is the maximum volume ellipsoid in $\mathrm{B}(F)$, see [ST80] or [Pis89] for basic facts about VR. The purpose of this section is to prove the following result.

Theorem 5.1. *Let $\mathcal{X}$ be a Banach space containing a sequence $\{\mathcal{X}_n\}$ of uniformly complemented subspaces with $\dim \mathcal{X}_n \to \infty$ and such that there exists $\gamma \in [0, 1/2)$ satisfying*

$$\liminf_{n\to\infty} \frac{\mathrm{VR}(\mathcal{X}_n)}{(\dim \mathcal{X}_n)^\gamma} = 0.$$

Then there exist a sequence of compacts $K_n \subset \mathcal{X}$ with

$$\lim_{n\to\infty} \frac{d_n^a(K_n)}{d_n(K_n, \mathcal{X})} = 0.$$

The proof relies on the following finite dimensional theorem.

Theorem 5.2. *Suppose $\gamma \in [0, 1/2)$ and $\sigma \in (\gamma, 1/2)$. Let $A \ge 5$ be a positive integer satisfying*

$$\frac{A-2}{2(A+1)} \ge \gamma \frac{A}{A-1} + (\sigma - \gamma).$$

Then there exists $N_0 \in \mathbb{N}$ with the following property: if $n \ge N_0$ is even, and X is a normed space of dimension An, with $\mathrm{VR}(X) \le n^\gamma$, then there exists a compact symmetric $K \subset X$, so that $d_n^a(K) \le C_1$, and $d_n(K, X) \ge n^{\sigma-\gamma}$, where C_1 is a constant which depends only on A.

Note that, for γ and σ as above, A satisfying the centered identity always exists. Indeed, as $A \to \infty$, the left hand side tends to $1/2$, and the right hand side – to $\sigma < 1/2$.

Tools which we use in this proof were invented by Gluskin [Glu81] and later developed by Szarek [Sza81] and [Sza86]. See [MT03] for a survey of related results. Throughout the proof we use Gaussian random variables. To describe them, denote an orthonormal basis in $\mathbb{R}^N$ by (e_i). We call a vector $\sum_{i=1}^N g_i e_i$ N-*standard Gaussian* if g_i are independent standard normal random variables (with $\mathbb{E}(|g_i|^2) = 1$). It is well known that the definition is actually independent of the choice of an orthonormal basis in $\mathbb{R}^N$. If P is an orthogonal projection on an M-dimensional subspace of $\mathbb{R}^N$, and $(\tilde{g}_j)_{j=1}^k$ are independent N-standard Gaussians, then $(P\tilde{g}_j)_{j=1}^k$ are independent M-standard Gaussians (see, e.g., [MT03, Fact 1]).

Proving Theorem 5.2 we identify X with $\mathbb{R}^{An}$, and naturally embed it into $\tilde{X} = \mathbb{R}^{(1+A)n}$, with the basis $(e_i)_{i=1}^{(1+A)n}$. We may and shall assume that the maximal volume ellipsoid, inscribed in $\mathrm{B}(X)$, is the Euclidean ball B_2^{An}. Let P_X be the orthogonal projection of $\tilde{X}$ onto X. Let $\tilde{g}_i = \tilde{g}_{i,\omega}$ $(1 \le i \le (1+A)n,\ \omega \in \Omega)$ be independent $(1+A)n$-standard Gaussian vectors in $\tilde{X}$. Then $g_i = g_{i,\omega} = P_X \tilde{g}_i$ are An-standard Gaussian vectors in X. We show that the set $K = K_\omega = \mathrm{absconv}(g_1, \ldots, g_{(1+A)n})$ has the desired properties with probability (relative to ω) of at least $1/2$, for sufficiently large n. We use the notation $\mathbb{G} = \mathbb{G}_\omega = (\tilde{g}_{i,\omega})_{i=1}^{(1+A)n}$. Let $\tilde{K} = \tilde{K}_\omega = \mathrm{absconv}(\tilde{g}_1, \ldots, \tilde{g}_{(1+A)n})$.

Lemma 5.3. *There exists a constant C_1, depending only on A, such that for each sufficiently large even number n*

$$\mathbb{P}_\omega(\mathcal{S}_1) \ge 1 - 3 \cdot \exp(-n/2),$$

where $\mathcal{S}_1$ is the set of those ω for which $\tilde{K}_\omega \cap X \subset C_1 \mathrm{B}_2^{An}$.

P r o o f. Let $\mathcal{U}$ be the group of unitary operators on $\mathbb{R}^{(1+A)n}$, with its normalized Haar measure. For $\mathbb{G} = (\tilde{g}_i)$, let $U\mathbb{G} = (U\tilde{g}_i)$. It is well known (see, e.g., [MP81, Proposition V.1.1]) that the distributions $(U\mathbb{G}_\omega)_{U\in\mathcal{U},\omega\in\Omega}$ and $(\mathbb{G}_\omega)_{\omega\in\Omega}$ are the same. Define the set $\mathcal{S}_1'$ of all pairs (U, ω) for which $\tilde{K}_\omega \cap U(X) \subset C_1 \mathrm{B}_2^{An}$. Then $\mathbb{P}_\omega(\mathcal{S}_1) = \mathbb{P}_{\omega,U}(\mathcal{S}_1')$. For any ω, let $\mathcal{S}_{1\omega}'$ be the set of all $U \in \mathcal{U}$ for which $(\omega, U) \in \mathcal{S}_1'$. It suffices to show that

$$\mathbb{P}_\omega\big(\mathbb{P}_U(\mathcal{S}_{1\omega}') \ge 1 - 2 \cdot \exp(-n/2)\big) \ge 1 - 2^{-n}. \tag{1}$$

Consider the set $\mathcal{F}$ of all ω for which there exists a subspace F of codimension $n/2$ in $\mathbb{R}^{(1+A)n}$, so that

$$F \cap \tilde{K}_\omega \subset F \cap C_1' \mathrm{B}_2^{(1+A)n},$$

where C_1' is a constant (depending only on A). By [LPT06, Theorem 2.4], if $\omega \in \mathcal{F}$, then $\mathbb{P}_U(\mathcal{S}_{1\omega}') \ge 1 - 2 \cdot \exp(-n/2)$ if $C_1 = C_1'(\kappa A)^{3/2}$, where κ is a universal constant. To prove (1), we need to show that $\mathbb{P}_\omega(\mathcal{F}) \ge 1 - 2^{-n}$.

To establish the last inequality, consider the (random) operator Γ_ω, mapping e_i $(1 \le i \le (1+A)n)$ to $\tilde{g}_{i,\omega}$. It is well known (see [Sza90, Lemma 2.8]) that there exists an absolute constant $\lambda > 0$ so that

$$\mathbb{P}_\omega(\|\Gamma_\omega\| \ge \lambda\sqrt{(1+A)n}) \le \exp(-(1+A)n)$$

for sufficiently large n (here we consider Γ_ω as an operator $\ell_2^{(1+A)n} \mapsto \ell_2^{(1+A)n}$).

On the other hand, by the well-known Kashin decomposition [Kas77] (see also [Sza78] and [Pis89, Theorem 6.1]), there exists a subspace $G \subset \mathbb{R}^{(1+A)n}$, of codimension $n/2$, so that

$$\sqrt{(1+A)n}\,\mathrm{B}_1^{(1+A)n} \cap G \subset 20^{2(1+A)}\mathrm{B}_2^{(1+A)n}.$$

In fact, most subspaces of given (proportional) codimension have this property, but one subspace is enough for us. If ω satisfies $\|\Gamma_\omega\| \le \lambda\sqrt{(A+1)n}$, we let $F = \Gamma_\omega(G)$. Note that Γ_ω maps $\mathrm{B}_1^{(1+A)n}$ onto $\tilde{K}_\omega$, hence $F \cap \tilde{K}_\omega \subset F \cap C_1'\mathrm{B}_2^{(1+A)n}$ for $C_1' = \lambda 20^{2(1+A)}$. ∎

Keeping the notation of Lemma 5.3, we obtain:

Corollary 5.4. *For any $\omega \in \mathcal{S}_1$, $d_n^a(K_\omega) \le C_1$, where C_1 is the constant from Lemma 5.3.*

P r o o f. Let $\tilde{X}$ be the normed space defined as $\mathbb{R}^{(1+A)n}$ with the norm whose unit ball is $\mathrm{B}(\tilde{X}) = \mathrm{conv}(C_1^{-1}\tilde{K}_\omega \cup \mathrm{B}(X))$. Clearly, $\mathrm{B}(\tilde{X}) \cap X = \mathrm{B}(X)$, hence the embedding of X into $\tilde{X}$ is isometric.

On the other hand, $d_n(C_1^{-1}K_\omega, \tilde{X}) \le 1$. In fact, the space $X^\perp = \ker P_X$ (the orthogonal complement of X in $\tilde{X}$) is n-dimensional. In addition, for any $x \in C_1^{-1}K_\omega$ there exists $\tilde{x} \in C_1^{-1}\tilde{K}_\omega \cap P_X^{-1}(x)$. Therefore, $x - \tilde{x} \in X^\perp$, and $\|\tilde{x}\|_{\tilde{X}} \le 1$. Thus, $d_n(C_1^{-1}K_\omega, \tilde{X}) \le 1$. ∎

Thus, with overwhelming probability, $d_n^a(K_\omega) \le C_1$. We shall show that, with overwhelming probability, $d_n(K_\omega, X) \ge 4n^{\sigma-\gamma}$.

The following easy observation provides a useful tool for us. If E is a subspace of X, denote by P_E the orthogonal projection from X (or $\tilde{X}$) onto E. We shall view E as equipped with the norm whose unit ball $\mathrm{B}(E) = P_E(\mathrm{B}(X))$.

Lemma 5.5. *Suppose S is a subset of X. Then $d_m(S, X) \ge c$ if and only if for every $E \subset X$ with $\mathrm{codim}\, E = m$, we have $P_E(S) \not\subseteq c\mathrm{B}(E)$.*

P r o o f. The proof can be viewed as a standard exercise: the orthogonal complement of E satisfying $P_E(S) \subseteq c\mathrm{B}(E)$ is a subspace witnessing $d_m(S, X) \le c$. ∎

We have to show that, with high probability, $P_E(\tilde{K}_\omega) \not\subseteq C_2 n^{\sigma-\gamma} \mathrm{B}(E)$ holds for any E of dimension $(A-1)n$ and some C_2, when n is large enough. Note that $P_E(\tilde{K}_\omega)$ is the absolute convex hull of the vectors $g_{E,i} := P_E g_i = P_E \tilde{g}_i$ $(1 \le i \le (1+A)n)$, which are independent $(A-1)n$-standard Gaussians.

Our next auxiliary result is well known. For the sake of brevity, set $\mathcal{V} = \mathrm{VR}(X)$.

Lemma 5.6. *For any $t \in (0,1]$, $\mathrm{B}(X)$ contains a set $(x_i)_{i=1}^N$, with $N \le ((1+2t^{-1})\mathcal{V})^{An}$, so that, for every $x \in \mathrm{B}(X)$, there exists i satisfying $\|x-x_i\|_2 \le t$.*

P r o o f. Suppose $(x_i)_{i=1}^N$ is a maximal subset of $\mathrm{B}(X)$ with the property that $\|x_i - x_j\|_2 > t$ whenever $i \ne j$. Consider $S = \cup_i \{x_i + t/2\mathrm{B}_2^{An}\}$ (a disjoint union of N balls). Then $S \subset \mathrm{B}(X) + t/2\mathrm{B}_2^{An} \subset (1+t/2)\mathrm{B}(X)$, hence

$$N(t/2)^{An}\mathrm{vol}(\mathrm{B}_2^{An}) = \mathrm{vol}(S) \le (1+t/2)^{An}\mathrm{vol}(\mathrm{B}(X)) \le (1+t/2)^{An}\mathcal{V}^{An}\mathrm{vol}(\mathrm{B}_2^{An}),$$

yielding the desired inequality. ∎

Corollary 5.7. *If E is a subspace of X of dimension $(A-1)n$, then $\mathrm{vol}(\mathrm{B}(E)) \le 3^{An}\mathcal{V}^{An}\mathrm{vol}(\mathrm{B}_2^{(A-1)n})$.*

P r o o f. Suppose $(x_i)_{i=1}^N$ is as in the statement of Lemma 5.6, with $t = 1$ (hence $N \le 3^{An}\mathcal{V}^{An}$). Then $\mathrm{B}(X) \subset \cup_{i=1}^N \{x_i + \mathrm{B}_2^{An}\}$, hence

$$\mathrm{B}(E) = P_E(\mathrm{B}(X)) \subset \cup_{i=1}^N \{P_E x_i + \mathrm{B}_2^{(A-1)n}\}.$$

Therefore, $\mathrm{vol}(\mathrm{B}(E)) \le N\mathrm{vol}(\mathrm{B}_2^{(A-1)n})$. ∎

Lemma 5.8. *For any $\lambda > 0$, we have: for any $E \subset X$ of dimension $(A-1)n$,*

$$\mathbb{P}\big(P_E(K_\omega) \subset \lambda\mathrm{B}(E)\big) \le \Big(\frac{\mathcal{V}'}{\sqrt{(A-1)n}}\lambda\Big)^{(A-1)(A+1)n^2},$$

where $\mathcal{V}' = (3\mathcal{V})^{A/(A-1)}\sqrt{e}$.

P r o o f. Recall that $P_E(K_\omega)$ is the absolute convex hull of $(1+A)n$ independent $(A-1)n$-standard Gaussian vectors $g_{E,i}$. Thus,

$$\mathbb{P}\big(P_E(K_\omega) \subset \lambda\mathrm{B}(E)\big) = \Big(\mathbb{P}\big(g \in \lambda\mathrm{B}(E)\big)\Big)^{(1+A)n},$$

where g is a $(A-1)n$-standard Gaussian vector. By [MT03, Fact 1],

$$\begin{aligned}\mathbb{P}(g \in \lambda\mathrm{B}(E)) &\le e^{(A-1)n/2}\mathrm{vol}\big(((A-1)n)^{-1/2}\lambda\mathrm{B}(E)\big)/\mathrm{vol}(\mathrm{B}_2^{(A-1)n}) \\ &\le \Big(\frac{e}{(A-1)n}\Big)^{(A-1)n/2}(3\mathcal{V})^{An}\lambda^{(A-1)n}.\end{aligned}$$

Therefore,

$$\mathbb{P}\big(P_E(K_\omega) \subset \lambda \mathrm{B}(E)\big) \le \Big(\frac{\mathcal{V}'}{\sqrt{(A-1)n}}\lambda\Big)^{(A-1)(A+1)n^2}. \qquad \blacksquare$$

Denote by $\mathcal{E}$ the set of all subspaces of X of dimension $(A-1)n$, equipped with the distance $\mathrm{dist}(E,F) = \|P_E - P_F\|_2$. Here, for an operator T on X, we denote by $\|\cdot\|_2$ its operator norm on ℓ_2^{An}.

Lemma 5.9. *For any $E, F \in \mathcal{E}$, and $x \in X$,*

$$\|P_F x\|_F \le \|P_E x\|_E + (\|P_E x\|_E \sqrt{An} + \|x\|_2)\|P_E - P_F\|_2.$$

P r o o f. For simplicity, let $a = \|P_E x\|_E$, and $b = \|x\|_2$. By the definition of the norm on E, we can write $x = x_1 + x_2$, with $x_1 \in a\mathrm{B}(X)$, and $x_2 \in E^\perp$. Recall that B_2^{An} is the maximal volume ellipsoid contained in $\mathrm{B}(X)$, hence, by the well known theorem of F. John (see [MS86, p. 10]), $\mathrm{B}(X) \subset \sqrt{An}\,\mathrm{B}_2^{An}$. Therefore, $\|x_2\|_2 \le \|x_1\|_2 + \|x\|_2 \le a\sqrt{An} + b$. We have

$$P_F x = P_F x_1 + P_F x_2 = P_F x_1 + (P_F - P_E)x_2.$$

Thus,

$$\|P_F x\|_F \le \|P_F x_1\|_F + \|(P_F - P_E)x_2\|_2$$
$$\le a + \|P_F - P_E\|_2 \|x_2\|_2 \le a + \|P_F - P_E\|_2 (a\sqrt{An} + b). \qquad \blacksquare$$

Corollary 5.10. *Suppose $E \in \mathcal{E}$ and ω are such that*

$$P_E(K_\omega) \subset a\mathrm{B}(E),$$

and

$$\max_{1 \le i \le (A+1)n} \|\tilde{g}_i\|_2 \le b\sqrt{An}.$$

Then, for any $F \in \mathcal{E}$,

$$P_F(K_\omega) \subset \big(a + \|P_F - P_E\|_2 (a+b)\sqrt{An}\big)\mathrm{B}(F).$$

P r o o f of Theorem 5.2. Consider the set $\mathcal{S}_2$ of all ω for which $\|g_i\|_2 \le 4\sqrt{(A-1)n}$ for every i. By [MT03, Fact 1], if g is an An-standard Gaussian, then

$$\mathbb{P}\big(\|g\|_2 > 4\sqrt{(A-1)n}\big) \le \big(\sqrt{2}e^{-4(A-1)/A}\big)^{An},$$

hence

$$\mathbb{P}(\mathcal{S}_2) \geq 1 - (A+1)n\big(\sqrt{2}e^{-4(A-1)/A}\big)^{An} \geq 1 - e^{-2(A+1)n} \tag{2}$$

for n large enough (recall that $A \geq 5$).

We shall prove that, for n large enough, there exists $\omega \in \mathcal{S}_1 \cap \mathcal{S}_2$, with the property that $P_E(K_\omega) \not\subseteq C\mathrm{B}(E)$ for any $E \in \mathcal{E}$, where $C = 4n^{\sigma-\gamma}$ ($\mathcal{S}_1$ is defined as in Lemma 5.3).

Let $t = (An)^{-1/2}$. By [Sza81] (see also [Paj99, Proposition 6]), $\mathcal{E}$ has a t-net $\mathcal{E}^\dagger$, of cardinality not exceeding $(C_3/t)^{(A-1)n^2}$, where C_3 is a universal constant. Suppose $P_E(K_\omega) \subset C\mathrm{B}(E)$, for some E. Find $F \in \mathcal{E}^\dagger$ so that $\|P_E - P_F\|_2 \leq t$. By Corollary 5.10, $P_F(K_\omega) \subset (2C+4)\mathrm{B}(F)$.

Denote by $\mathcal{S}_{3,F}$ the set of all $\omega \in \mathcal{S}_2$ for which $P_F(K_\omega) \subset (2C+4)\mathrm{B}(F)$, and let $\mathcal{S}_3 = \cup_{F \in \mathcal{E}^\dagger} \mathcal{S}_{3,F}$. For a given F, Lemma 5.8 yields

$$\mathbb{P}(\mathcal{S}_{3,F}) \leq \Big(\frac{\mathcal{V}'}{\sqrt{(A-1)n}}(2C+4)\Big)^{(A-1)(A+1)n^2} \leq \Big(\frac{\mathcal{V}'}{\sqrt{An}}3C\Big)^{(A-1)(A+1)n^2}.$$

Thus,

$$\begin{aligned}
\mathbb{P}(\mathcal{S}_3) &\leq |\mathcal{E}^\dagger|\Big(\frac{\mathcal{V}'}{\sqrt{An}}3C\Big)^{(A-1)(A+1)n^2} \\
&\leq \big(C_3\sqrt{An}\big)^{(A-1)n^2}\Big(\frac{\mathcal{V}'}{\sqrt{An}}3C\Big)^{(A-1)(A+1)n^2} \\
&= \Big(C_3(An)^{-A/2}\big(3\mathcal{V}'C\big)^{A+1}\Big)^{(A-1)n^2}.
\end{aligned}$$

Note that $C^{A+1} = 4^{A+1}n^{(\sigma-\gamma)(A+1)}$, and $\mathcal{V}'^{(A+1)} \leq n^{\gamma A(A+1)/(A-1)}$. By our choice of A,

$$\frac{A}{2} > (\sigma-\gamma)(A+1) + \gamma\frac{A(A+1)}{A-1},$$

and therefore, $\mathbb{P}(\mathcal{S}_3) \leq (C_4 n)^{-C_5 n^2}$, where C_4 and C_5 are positive constants.

On the other hand, combining Lemma 5.3 with (2), we obtain, for n large enough,

$$\mathbb{P}(\mathcal{S}_1 \cap \mathcal{S}_2) \geq 1 - 3e^{-n/2} - e^{-2(A+1)n}.$$

Thus, for large n, $\mathbb{P}(\mathcal{S}_3) < \mathbb{P}(\mathcal{S}_1 \cap \mathcal{S}_2)$. Thus, there exists $\omega \in \mathcal{S}_1 \cap \mathcal{S}_2$, so that $P_E(K_\omega) \not\subseteq C\mathrm{B}(E)$, for any E. By Lemma 5.5, we are done. ■

To prove Theorem 5.1, we need also the following lemma.

Lemma 5.11. *Suppose X is an m-dimensional space. Then, for any $k \leq m$, there exists a k-dimensional subspace Y, so that* $\dim Y = k$, *and* $\mathrm{VR}(Y) \leq \mathrm{VR}(X)$.

P r o o f. Denote the norm of X by $\|\cdot\|$. Without loss of generality, the maximal volume ellipsoid inscribed into $\mathrm{B}(X)$ is the Euclidean ball. By, e.g., [Pis89, Section 6],

$$\mathrm{VR}(X) = \int_{\mathbf{S}^{m-1}} \|x\|^{-m}\, d\sigma_{m-1},$$

where σ_{m-1} is the uniform probability measure on the unit sphere $\mathbf{S}^{m-1}$. As explained in, e.g., [MS86, 1.6], we can write

$$\mathrm{VR}(X) = \int_{\mathbf{G}} \int_{\mathbf{S}^{k-1}(Y)} \|x\|^{-m}\, d\sigma_{k-1}\, d\mu,$$

where μ is the rotation invariant probability measure on the Grassman manifold $\mathbf{G}$ of k-dimensional subspaces $Y \subset X$, and σ_{k-1} is the probability measure on the unit sphere of Y. Clearly, for some $Y \in \mathbf{G}$,

$$\int_{\mathbf{S}^{k-1}(Y)} \|x\|^{-m}\, d\sigma_{k-1} \leq \mathrm{VR}(X).$$

Then

$$\mathrm{VR}(Y) = \int_{\mathbf{S}^{k-1}(Y)} \|x\|^{-k}\, d\sigma_{k-1} \leq \int_{\mathbf{S}^{k-1}(Y)} \|x\|^{-m}\, d\sigma_{k-1} \leq \mathrm{VR}(X). \qquad \blacksquare$$

P r o o f of Theorem 5.1. Pick $\sigma \in (\gamma, 1/2)$. As in Theorem 5.2, find a positive integer $A \geq 5$, so that

$$\frac{A-2}{2(A+1)} \geq \gamma \frac{A}{A-1} + (\sigma - \gamma).$$

Now we use Lemma 5.11 to obtain a sequence $\{X_n\}_{n=1}^{\infty}$ of uniformly complemented subspaces so that $\dim X_n = Ak_n$, where k_n is even, $\lim_{n\to\infty} k_n = \infty$), and $\mathrm{VR}(X_n) \leq k_n^{\gamma}$. Theorem 5.2 yields, for n large enough, compact sets $K_n \subset X_n$, so that $\sup_n d^a_{k_n}(K_n) < \infty$, and $\lim_n d_{k_n}(K_n, X) = \infty$. $\blacksquare$

Can we use the techniques of Theorem 5.1 for other spaces? Below, we outline a possible approach. As in Section 2, we use the notation $\lambda(F)$ and $\lambda(F, G)$ for absolute and relative projection constants. On the first step, find (when possible) a sequence of uniformly complemented subspaces $X_n \subset X$ such that $\lambda(X_n) \to \infty$. The second step consists of picking a sequence $\{Y_n\}$ of superspaces $Y_n \supset X_n$ such that $\lim_n \lambda(X_n, Y_n) = \infty$, and $k_n = \dim(Y_n/X_n) = \dim X_n/2$ (or more generally, $\lim_n \big(\dim(Y_n/X_n)/\dim X_n\big) = \alpha \in (0,1)$). The third step proceeds as in the proof Theorem 1.2 — namely, by selecting projections $P_n : Y_n \to X_n$ so that $\lim_n d_{k_n}(P_n(\mathrm{B}(Y_n)), X_n) = \infty$. Then we would also have $\lim_n d_{k_n}(P_n(\mathrm{B}(Y_n)), X) = \infty$ (due to the uniform complementability of X_n's), and $d^a_{k_n}(K_n) \leq 1$. We believe that the possibility of implementing the second step of this program is an interesting problem, which can find other applications as well:

Problem 5.12. *Suppose that finite-dimensional spaces X_n are such that $\lambda(X_n) \to \infty$. Does this imply that there exist $Y_n \supset X_n$ such that*

$$\dim (Y_n/X_n) \leq \dim X_n/2 \quad \text{and} \quad \lambda(X_n, Y_n) \to \infty?$$

The problem is of interest if we replace 2 by any positive constant.

Problem 5.12 can be considered as a problem on possibility to generalize the isometric, one-codimensional result of Davis [Dav77].

The possibility of making the third step is still a problem (even if we assume that Problem 5.12 has a positive answer): Can Y_n and P_n be chosen in such a way that $P_n(\mathrm{B}(Y_n))$ has large k-width in X_n, where $k = \dim (Y_n/X_n)$?

R e m a r k 5.13. There exist non-$\mathcal{L}_\infty$-spaces for which the scheme above cannot be realized because they do not contain uniformly complemented finite-dimensional spaces with growing dimensions. One example of this type was constructed by Pisier [Pis83] (see [Pis86] for a simpler version of the construction).

6. Ratios of Widths to Absolute Widths

In this section, we modify Problem 2.5.

Problem 6.1. (1) *Describe the Banach spaces $\mathcal{Y}$ which contain compact subsets K so that* $\limsup_n d_n(K)/d^a_n(K) = \infty$.

(2) *What can be said about the Banach spaces $\mathcal{Y}$ satisfying a stronger property: they contain compact subsets K so that* $\liminf_n d_n(K)/d^a_n(K) = \infty$.

To answer Part (1) of this question, we state:

Proposition 6.2. *Suppose a Banach space $\mathcal{Y}$ is such that there exist $\gamma > 0$ and $\sigma \in [0, 1/2)$ so that, for infinitely many positive integers n, there exist operators $A_n : \ell_2^n \to \mathcal{Y}$ and $B_n : \mathcal{Y} \to \ell_2^n$, so that $B_n A_n = I_{\ell_2^n}$, and $\|A_n\|\|B_n\| \leq \gamma n^\sigma$. Then $\mathcal{Y}$ contains a compact subset K, so that*

$$\limsup d_n(K)/d^a_n(K) = \infty.$$

If $\mathcal{Y}$ is K-convex, then there exists a sequence of projections P_n from $\mathcal{Y}$ onto subspaces F_n, where $\sup_n \|P_n\| < \infty$, and $d(F_n, \ell_2^n) < 2$ (see [Pis82] or [DJT95, Theorem 19.3]). Thus, K-convex spaces $\mathcal{Y}$ satisfy the conditions of this proposition. By [FLM77, Example 3.5], Proposition 6.2 is also applicable to $\mathcal{Y} = (\oplus_n \ell_1^n)_{c_0}$, $(\oplus_n \ell_1^n)_\infty$, $c_0(\ell_1)$, or $\ell_\infty(\ell_1)$.

P r o o f. Find a sequence $4 < n(1) < n(2) < \dots$ so that, for any $j \in \mathbb{N}$, $n(j+1) > 4n(j)$, and there exist operators $U_j : \ell_2^{n(j)} \to \mathcal{Y}$ and $V_j : \mathcal{Y} \to \ell_2^{n(j)}$, so that $\|U_j\| \le 1$, and $\|V_j\| \le \gamma n(j)^\sigma$. Define $m(j) = \lceil n(j)/2 \rceil$ and $k(j) = m(j) - \sum_{i=1}^{j-1} m(i)$ (note that $k(j) \ge 3m(j)/5$). Furthermore, set $\alpha_1 = 1$, and $\alpha_{j+1} = \alpha_j / \sqrt{n(j)}$.

Let $id_{12}^{(j)}$ be the formal identity map from $\ell_1^{n(j)}$ to $\ell_2^{n(j)}$, and set $\tilde{K}_j = id_{12}^{(j)} \mathrm{B}(\ell_1^{n(j)})$. By [GG84],

$$d^a_{k(j)}(\tilde{K}_j) \le c_{k(j)}(id_{12}^{(j)}) < C_1 n(j)^{-1/2}$$

($C_1 > 0$ is an absolute constant). On the other hand, by [Pin85, Theorem VI.2.7], $d_{m(j)}(\tilde{K}_j) > 1/2$.

Let $K_j = \alpha_j A_j(\tilde{K}_j)$. Then the set $K = \mathrm{conv}(K_1, K_2, \dots)$ is compact and convex. We claim that, for any j, $d_{m(j)}(K) \ge \alpha_j \gamma^{-1} n(j)^{-\sigma}/2$, while $d^a_{m(j)}(K) \le C_1 \alpha_j n(j)^{-1/2}$.

To estimate $d_{m(j)}(K)$ from below, note that $V_j(K) \supset \alpha_j^{-1} \tilde{K}_j$. By Lemma 4.5,

$$\frac{1}{2} < d_{m(j)}(\tilde{K}_j) \le \alpha_j^{-1} \|V_j\| d_{m(j)}(K).$$

As $\|V_j\| \le \gamma n(j)^\sigma$, we obtain $d_{m(j)}(K) \ge \alpha_j \gamma^{-1} n(j)^{-\sigma}/2$.

Next obtain an upper estimate for $d^a_{m(j)}(K)$. Embed $\mathcal{Y}$ isometrically into a 1-injective Banach space $\mathcal{Y}'$ (we can take, for instance, $\mathcal{Y}' = \ell_\infty(I)$). Find $F \subset \mathcal{Y}'$ so that $\dim F \le k(j)$, and $E(K_j, F) \le C_1 \alpha_j n(j)^{-1/2}$. Now let $G = \mathrm{span}[F, \mathrm{ran}\, V_1, \dots, \mathrm{ran}\, V_{j-1}]$. Clearly, $\dim G \le k(j) + \sum_{i=1}^{j-1} n(i) \le m(j)$. We show that $E(K, G) \le C_1 \alpha_j n(j)^{-1/2}$. By convexity, it suffices to establish the inequality $E(x, G) \le C_1 \alpha_j n(j)^{-1/2}$ for $x \in K_s$, for $s \in \mathbb{N}$. For $s < j$, we have $x \in G$, hence $E(x, G) = 0$. For $s = j$, $E(x, G) \le E(x, F) < C_1 \alpha_j n(j)^{-1/2}$, by our choice of F. For $s > j$,

$$E(x, G) \le \|x\| \le \alpha_s \le \alpha_{j+1} = \alpha_j n(j)^{-1/2}.$$

Taken together, the results above yield $d_{m(j)}(K)/d^a_{m(j)}(K) \ge \beta m(j)^{1/2-\sigma}$, where β is a constant. ∎

In [Ost10], a special case of the previous proposition was established: it was proved that ℓ_2 contains an infinite dimensional compact K for which

$\limsup_{n\to\infty} d_n(K)/d_n^a(K) = \infty$. This result leads to the following question [Ost10, Problem 4.2]: Does there exist an infinite-dimensional compact K in some Banach space $\mathcal{Y}$ such that

$$\lim_{n\to\infty} d_n(K)/d_n^a(K) = \infty?$$

Below, we provide a positive answer.

Proposition 6.3. 1. *Suppose $1 < p \le 2$, and $\alpha \in (0, 1/q)$, where $1/p + 1/q = 1$. Then there exists an operator $u_p : \ell_1 \to \ell_p$, so that, for every n,*

$$d_n^a(u_p) \le c_n(u_p) \le \beta_{p\alpha}(1 + \log n)n^{-1/q} \text{ and } d_n(u_p) \ge \gamma_{p\alpha} n^{-\alpha}.$$

2. *Suppose $2 < p < \infty$, and $\alpha \in (0, 1/p)$. Then there exists an operator $u_p : \ell_1 \to \ell_p$, so that, for every n,*

$$d_n^a(u_p) \le c_n(u_p) \le \beta_{p\alpha}(1 + \log n)n^{-1/2} \text{ and } d_n(u_p) \ge \gamma_{p\alpha} n^{1/p-1/2-\alpha}.$$

Here $\beta_{p\alpha}$ and $\gamma_{p\alpha}$ depend on p and α only.

P r o o f. By Lemma 4.4, $d_n^a(u) \le c_n(u)$ for any n, and any operator u.

Throughout the proof, we denote by $(e_j^{(p)})_{j\in\mathbb{N}}$ the canonical basis in ℓ_p. The projection onto the first N elements of this basis is denoted by $P_N^{(p)}$. For $p \le q$, id_{pq} (id_{pq}^N) stands for the formal identity from ℓ_p to ℓ_q (resp. from ℓ_p^N to ℓ_q^N). We identify the range of $P_N^{(p)}$ with ℓ_p^N.

In both (1) and (2), we consider a diagonal operator u_p, taking $e_j^{(1)}$ to $j^{-\alpha} e_j^{(p)}$. We make repeated use of the following formula: if $v = \operatorname{diag}(a_j)_{j=1}^\infty$ is a diagonal operator from ℓ_1 to ℓ_2, then, by [Pin85, Theorem VI.2.7 on p. 207],

$$d_n(v) = \sup_{r>n} \sqrt{\frac{r-n}{\sum_{j=1}^r a_j^{-2}}}. \tag{3}$$

(1) $1 < p \le 2$. To estimate $d_n(u_p)$, note that $id_{p2}u_p = u_2$, hence $d_n(u_p) \ge d_n(u_2)$. By (3), $d_n(u_2) \ge \gamma_\alpha n^{-\alpha}$. Now let $N = \lceil n^{1/(\alpha q)} \rceil$. By [GG84],

$$c_n(id_{1p}^N) \le \frac{c_p}{\alpha q}\big(1 + \log n\big)^{1/q} n^{-1/q},$$

for some universal constant $c_p > 1$. Thus, there exists a subspace $F \subset \operatorname{span}[e_j^{(1)} : 1 \le j \le N]$, so that

$$\|id_{1p}|_F\| \le \frac{c_p}{\alpha q}\big(1 + \log n\big)^{1/q} n^{-1/q}.$$

Denote by v_p the diagonal operator on ℓ_p^N, mapping $e_j^{(p)}$ to $j^{-\alpha}e_j^{(p)}$, and note that $u_p = v_p id_{1p}$. Therefore,

$$\|u_p|_F\| \leq \frac{c_p}{\alpha q}\big(1+\log n\big)^{1/q} n^{-1/q}.$$

Now let $G = \mathrm{span}[F, e_{N+1}^{(1)}, e_{N+2}^{(1)}, \ldots]$. Then $\dim \ell_1/G \leq n$, and, by our choice of N,

$$c_n(u_p) \leq \|u_p|_G\| \leq \frac{c_p}{\alpha q}\big(1+\log n\big)^{1/q} n^{-1/q}.$$

As $d_n^a(u_p) \leq c_n(u_p)$, we are done.

(2) $2 \leq p < \infty$. Note that $u_p = id_{2p}u_2$, and id_{2p} is contractive. Using the estimates for $c_n(u_2)$ obtained in Part (1), we get:

$$c_n(u_p) \leq \|id_{2p}\| c_n(u_2) \leq \beta_{2\alpha}\big(1+\log n\big)^{1/2} n^{-1/2}.$$

On the other hand, $d_n(u_p) \geq d_n(u_p P_{2n}^{(1)})$. By (3), $d_n(u_2 P_{2n}^{(1)}) \geq 2\gamma_\alpha n^{-1/\alpha}$, for some constant γ_α. Furthermore, $(id_{2p}^{2n})^{-1} u_p P_{2n}^{(1)} = u_2 P_{2n}^{(1)}$, hence

$$d_n(u_p P_{2n}^{(1)}) \geq \|(id_{2p}^{2n})^{-1}\|^{-1} d_n(u_2 P_{2n}^{(1)}) \geq (2n)^{-(1/2-1/p)} \cdot 2\gamma_\alpha n^{-1/\alpha} \geq \gamma_\alpha n^{1/p-1/2-\alpha}.$$

■

Problem 6.4. *Which Banach spaces $\mathcal{Y}$ contain a compact K with the property that*

$$\lim \frac{d_n(K)}{d_n^a(K)} = \infty?$$

By Proposition 6.3, the answer is affirmative if $\mathcal{Y}$ contains a complemented copy of ℓ_p, for some $p \in (1,\infty)$. This occurs, for instance, for $\mathcal{Y} = L_p(\mu)$. Large classes of rearrangement invariant function spaces contain complemented copies of ℓ_2, see e.g. [LT79, Theorem 2.b.4].

7. Restricted Widths

The following problem was raised in [Ost10].

Problem 7.1 ([Ost10]). *Characterize compacts K for which the absolute widths do not differ much from their widths in $\overline{\mathrm{span}}[K]$.*

The importance of this problem is illustrated by Lemma 8.2 below.

It is worth mentioning that any Banach space $\mathcal{Y}$ contains a compact K whose widths in $\overline{\mathrm{span}}[K]$ are the same as the absolute widths. To construct an example, we use a technique of Tikhomirov [Tik60]. Let $\{Z_n\}$ be a family of subspaces in a Banach space $\mathcal{Y}$ satisfying $\dim Z_n = n$ and $Z_n \subset Z_{n+1}$, let B_n be their unit balls and let $\{t_n\}$ be a decreasing sequence of positive numbers with $\lim_{n\to\infty} t_n = 0$. Consider the compact

$$K = \overline{\mathrm{conv}\,(\cup_{n=1}^{\infty} t_n \mathrm{B}_n)}.$$

Then $d_n(K, \mathcal{X}) = t_{n+1}$ for each $n \in \mathbb{N}$ and each Banach space $\mathcal{X}$ containing $\overline{\mathrm{span}[K]}$ as a subspace. The reasons: (1) Estimate from above: $K \subset Z_n + t_{n+1}\mathrm{B}(\mathcal{X})$. (2) Estimate from below: $K \supset t_{n+1}\mathrm{B}_{n+1}$ and the result of [KKM48] saying that the maximal distance from a unit ball of an $(n+1)$-dimensional subspace to an n-dimensional subspace is equal to 1.

There are other classes of K's for which $d_n(K) = d_n^a(K)$ holds. Suppose $1 \le q \le p \le \infty$. In [Oik95] it was shown that the natural image of $\mathrm{B}(\ell_p^m)$ in ℓ_q^m satisfies this. Furthermore [Koc90], $d_n(u) = d_n^a(u)$ if $u : \ell_p^m \to \ell_q^m$ is a diagonal map. Another example of a set K with $d_n(K) = d_n^a(K)$ is provided below.

Proposition 7.2. *Suppose F is an m-dimensional space with a 1-unconditional basis $(f_i)_{i=1}^m$, and $id : \ell_\infty^m \to F$ is the formal identity map, taking δ_i to f_i for every i (here, $(\delta_i)_{i=1}^m$ denotes the canonical basis for ℓ_∞^m). Then $d_n(id) = d_n^a(id)$ for any n.*

P r o o f. If $n \ge m$, we have $d_n(id) = d_n^a(id) = 0$. Now consider $n \in \{1, \ldots, m-1\}$. Relabeling if necessary, we can assume that $C = \|\sum_{i=1}^{m-n} f_i\|_F \le \|\sum_{i\in\mathcal{F}} f_i\|_F$ whenever $|\mathcal{F}| = m - n$. We claim that $d_n(id) = d_n^a(id) = C$. First take $G = \mathrm{span}[f_i : m - n < i \le m]$, and let $q_G : F \to F/G$ be the quotient map. By the 1-unconditionality of (f_i), $d_n(id) \le \|q_G \circ id\| = C$. For the opposite inequality, we apply [Oik95, Lemma 4] in the situation where V is the unit cube. A direct calculation shows that $d_n^a(id) \ge C$. ■

8. Widths of Images of Compacts Under Compact Operators

The purpose of this section is to make some comments on the following intriguing problem

Problem 8.1. *Let K be a compact in a Banach space $\mathcal{X}$ and $T : \mathcal{X} \to \mathcal{Y}$ be a compact operator. Does it follow that $d_n(TK) = o(d_n(K))$?*

Set $\hat{d}_n(K) = d_n(K, \overline{\mathrm{span}[K]})$. [OS09, Lemma 6.1] states:

Lemma 8.2 ([OS09]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, K be a compact set in $\mathcal{X}$ and $T : \mathcal{X} \to \mathcal{Y}$ be a compact operator. Then $\hat{d}_n(TK)/\hat{d}_n(K) \to 0$ as $n \to \infty$.*

For Hilbert spaces $\hat{d}_n(K) = d_n(K)$ and so the result of Lemma 8.2 remains true if we replace $\hat{d}_n$ by d_n. Problem 8.1 asks whether one can generalize this result to the Banach space case. Of course, Problem 8.1 would be solved if one would prove that $\hat{d}_n(K) \le Cd_n(K)$ for some absolute constant C. However, as we know, for example, from Theorem 1.2 this turned out not to be the case.

If a compact K is such that $\{d_n(K)\}$ decreases more slowly than any geometric progression, then $d_n(TK) = o(d_n(K))$. More precisely:

Proposition 8.3. *Suppose a compact $K \subset \mathcal{X}$ and $C \in (1, \infty)$ have the following property: for any $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $d_n(K)/d_{n+k}(K) < C$ for each $n \ge N$. Then $d_n(TK) = o(d_n(K))$ for each compact operator $T : \mathcal{X} \to \mathcal{Y}$.*

P r o o f. It suffices to show that for each $\delta > 0$ there exists $M \in \mathbb{N}$ such that $d_m(TK) \le C\delta d_m(K)$ for each $m \ge M$. To show this we observe that for each $\delta > 0$ there exists $k \in \mathbb{N}$ and a k-dimensional subspace $\mathcal{Y}_k \subset \mathcal{Y}$ such that

$$T\mathrm{B}(\mathcal{X}) \subset \mathcal{Y}_k + \delta \mathrm{B}(\mathcal{Y}). \tag{4}$$

By the assumption there exists N such that $d_n(K) < Cd_{n+k}(K)$ for each $n \ge N$. Let $M \ge N + k$ and $m \ge M$. Then $d_{m-k}(K) < Cd_m(K)$ and therefore there is an $(m-k)$-dimensional subspace $\mathcal{X}_{m-k} \subset \mathcal{X}$ such that

$$K \subset \mathcal{X}_{m-k} + Cd_m(K)\mathrm{B}(\mathcal{X}).$$

Combining with (4) we get

$$TK \subset T\mathcal{X}_{m-k} + Cd_m(K)T\mathrm{B}(\mathcal{X}) \subset T\mathcal{X}_{m-k} + \mathcal{Y}_k + C\delta d_m(K)\mathrm{B}(\mathcal{Y}).$$

The subspace $T\mathcal{X}_{m-k} + \mathcal{Y}_k$ is at most m-dimensional, therefore $d_m(TK) \le C\delta d_m(K)$. ∎

Proposition 8.4. *Let K be a compact subset of a Banach space X, and $T : X \to Y$ be a compact operator. Let $\phi : \mathbb{N} \to \mathbb{N}$ be a function, satisfying $\lim_n(\phi(n) - n) = +\infty$. Then $d_{\phi(n)}(TK) = o(d_n(K))$.*

Lemma 8.5. *Suppose K is a compact subset of a Banach space $\mathcal{X}$, and (δ_n) is a sequence of positive numbers. Then $\mathcal{X}$ contains a separable subspace $\tilde{\mathcal{X}}$ such that, for every $n \in \mathbb{N}$, $d_n(K, \tilde{\mathcal{X}}) \le (1 + \delta_n)d_n(K, \mathcal{X})$.*

P r o o f. For each $n \in \mathbb{N}$ find an n-dimensional subspace $Z_n \subset \mathcal{X}$ such that $E(K, Z_n) \le (1 + \delta_n)d_n(K, \mathcal{X})$. We can take $\tilde{\mathcal{X}}$ to be the closure of $\mathrm{span}[K, Z_1, Z_2, \ldots]$ in $\mathcal{X}$. ∎

P r o o f of Proposition 8.4. By Lemma 8.5, we can assume that $\mathcal{X}$ is separable. Furthermore, we assume that $d_n(K) > 0$ for every n (otherwise, the conclusion of the proposition is immediate). Let $(x_i)_{i=1}^\infty$ be a countable dense subset of the unit sphere of X. For $n \in \mathbb{N}$, let $\psi(n)$ be the smallest positive integer m with the property that $\phi(k) - k \geq n$ for any $k \geq m$. Let $\tilde{K}$ be the closed convex hull of the union of K and the sequence $(d_{\psi(i)}(K)x_i)$. Then $d_{\phi(n)}(\tilde{K}) \leq d_n(K)$. Indeed, fix $c > 1$, and find an n-dimensional subspace Z in $\mathcal{X}$, such that $E(K, Z) < cd_n(K)$. Let $\tilde{Z}$ be the linear span of Z, and of $x_1, \ldots, x_{\phi(n)-n}$. Then $\dim \tilde{Z} \leq \phi(n)$, and $E(\tilde{K}, \tilde{Z}) \leq cd_n(K)$. As $c > 1$ is arbitrary, we conclude that $d_{\phi(n)}(\tilde{K}) \leq d_n(K)$. We conclude the proof by applying Lemma 8.2 to $\tilde{K}$. ∎

It may be tempting to approach Problem 8.1 by fixing $C_1 > C > 1$, finding subspaces $Z_n \hookrightarrow \mathcal{X}$ such that $E(K, Z_n) \leq Cd_n(K)$ and $\dim Z_n = n$, and then considering $\tilde{K} = \cap_n (Z_n + C_1 d_n(K)\mathrm{B}(\mathcal{X}))$ as a subset of $\tilde{\mathcal{X}} = \overline{\mathrm{span}[Z_n : n \in \mathbb{N}]} \subset \mathcal{X}$. Then $K \subset \tilde{K}$, and $d_n(\tilde{K}, \tilde{\mathcal{X}}) \leq C_1 d_n(K, \mathcal{X})$. If we had $\tilde{\mathcal{X}} = \overline{\mathrm{span}[\tilde{K}]}$, we would then use Lemma 8.2 to conclude that

$$\frac{d_n(T\tilde{K})}{\hat{d}_n(K)} \leq \frac{d_n(TK)}{\hat{d}_n(K)} \underset{n\to\infty}{\longrightarrow} 0$$

However, the above construction may lead to $\overline{\mathrm{span}[\tilde{K}]}$ being a strict subset of $\tilde{\mathcal{X}}$, as the following example shows. Let $\mathcal{X} = \ell_2$, and take K to be the set of all $(x_i) \in \ell_2$ s.t. $x_1 = 0$, and $|x_2|^2 + \sum_{i=3}^\infty 4^{3-i}|x_i|^2 \leq 1$. By [Pie87], $d_1(K) = 1$, and $d_n(K) = 2^{2-n}$ for $n \geq 2$. Take $Z_1 = \mathrm{span}[e_1]$, and $Z_n = \mathrm{span}[e_3, \ldots, e_{n+1}]$ for $n \geq 2$. Then $E(K, Z_n) = d_n(K)$ for any n. However, $Z_1 \cap \mathrm{span}[\tilde{K}] = \{0\}$. Indeed, denote by P the orthogonal projection onto $\mathrm{span}[e_1]$. Then, for $n \geq 2$ and $x \in Z_n + C_1 d_n(K)\mathrm{B}(\mathcal{X})$, $\|Px\| \leq 2^{n-2}C_1$. Consequently, for $x \in \tilde{K}$, we have $Px = 0$. In other words, $\tilde{K} \subset Z_1^\perp$.

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