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Isomorphically Polyhedral Banach Spaces

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We prove two theorems giving sufficient conditions for a Banach space to be isomorphically polyhedral.

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Devoted to the memory of M.I. Kadets

All Banach spaces we consider in this paper are infinite-dimensional and real. A Banach space is called polyhedral if the unit ball of each its finite-dimensional subspace is a polytope [7]. If a Banach space admits an equivalent norm in which it is polyhedral then we say that it is isomorphically polyhedral. In [8] is proved that polyhedral space cannot be isometric to a dual space.

In [9] is proved that if $\mathrm{ext}B_{X^*}$ is countable then X is not reflexive. In [6] this result is strengthen, namely it is proved that if $\mathrm{ext}B_{X^*}$ can be covered by a countable union of compact sets then X is not reflexive.

A subset $B \subset S_{X^*}$ of the unit sphere S_{X^*} of a Banach space X^* is called a boundary (for X) if for any $x \in X$ there is $f \in B$ such that $f(x) = ||x||$. An important example of a boundary is $\mathrm{ext}B_{X^*}$ (the Krein–Milman theorem).

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A subset B of the dual unit ball B_{X^*} has property (*) if, given any w^*-limit point f_0 of B (i.e. any w^*-neighborhood of f_0 contains infinitely many elements of B), we have $f_0(x) < 1$ whenever x is in the unit sphere S_X. Note that if B is a set such that $|||x||| = \sup\{f(x) : x \in B\}$ defines a norm, and B has (*) for this norm, then B is a boundary for the norm $|||.|||$.

Any separable polyhedral space has a countable boundary, and if a Banach space has a countable boundary then it is isomorphically polyhedral space [2]. Any (isomorphically) polyhedral space saturated by spaces isomorphic to c_0, that is any (infinite-dimensional) subspace of a polyhedral space contains an isomorphic copy of c_0 [4].

In this paper we prove two theorems giving sufficient conditions for a Banach space to be isomorphically polyhedral.

Theorem 1. *Let X be a Banach space. Then (a), (b) and (c) are equivalent and imply (d).*

(a) *There are a sequence of subsets of S_X, $\{S_k\}_{k=1}^{\infty}$, $S_X = \bigcup S_k$, and an increasing sequence of norm-compact subsets of S_{X^*}, $\{F_k\}_{k=1}^{\infty}$, $F_k = -F_k$, with the following properties:*

$$b_k := \inf_{x \in S_k} \max_{f \in F_k} \{f(x)\} > 0, k = 1, 2, \dots \quad \textit{and} \quad \lim_k b_k = 1.$$

(b) *There are a sequence $\{t_k\} \subset S_{X^*}$ and a sequence of positive numbers $\epsilon_k > 0$, $k = 1, 2, \dots$, such that*

(i) *the set $B = \{\pm(1 + \epsilon_k)t_k\}_k$ is a boundary with property (*) for the equivalent norm*

$$|||x||| = \sup_k |(1 + \epsilon_k)t_k(x)|, x \in X,$$

(ii) $B_{(X,|||.|||)} \subset \operatorname{int} B_X$.

(c) *There are a sequence $\{t_n\}_n \subset S_{X^*}$ and a sequence of positive numbers $\{\alpha_n\}$, $\lim_n \alpha_n = 0$, such that*

$$S_X \subset \bigcup_n S(t_n, \alpha_n),$$

where $S(t_n, \alpha_n)$ are the slices defined as $S(t_n, \alpha_n) = \{x \in B_X : t_n(x) \geq 1 - \alpha_n\}$.

(d) *X is a separable isomorphically polyhedral space.*

P r o o f.

(a) ⇒ (b). Fix a sequence $\delta_n > 0$ such that $0 < 3\delta_n < b_n$ and $\lim_n \delta_n = 0$. Choose γ_n defined by the equation $(1+\gamma_n) = (b_n - 3\delta_n)^{-1}$ and observe that $\gamma_n > 0$,

$$\lim_n \gamma_n = 0 \quad \text{and} \quad (1+\gamma_n)(b_n - 2\delta_n) > 1, n = 1, 2, \ldots$$

(notice that here we use that $1 \geq b_n > 0$, for any n and that $\lim_n b_n = 1$).

Since each set F_n is compact, for each n there is a δ_n-net $N_n = \{h_j^n : j = 1, 2, \ldots, p_n\} \subset F_n$ such that $\|h_i^n - h_j^n\| \geq \delta_n, i \neq j$ and for each $f \in F_n$ there is some h_j^n with $\|f - h_j^n\| < \delta_n$.

Clearly, we can write the set $\pm \bigcup_n (1+\gamma_n) N_n$ in the form $B = \{\pm(1+\epsilon_k)t_k : k = 1, 2, \ldots\}$ for suitable t_k's and ϵ_k's, $\lim_k \epsilon_k = 0$. Define

$$|||x||| = \sup\{|(1+\epsilon_k)t_k(x)| : k = 1, 2, \ldots\}.$$

For $x \in S_X$ there is some n such that $x \in S_n$ and some $f \in F_n$ such that $f(x) > b_n - \delta_n$, and some $h_j^n \in N_n$ such that $\|f - h_j^n\| < \delta_n$. Then we have

$$|||x||| \geq (1+\gamma_n)h_j^n(x) \geq (1+\gamma_n)(f(x) - \delta_n) > (1+\gamma_n)(b_n - 2\delta_n) > 1 = \|x\|.$$

On the other hand we have $|||x||| \leq \max_k\{(1+\epsilon_k)\|x\|\}$. Therefore

$$\|x\| < |||x||| \leq \max_k\{(1+\epsilon_k)\}\|x\|, x \in X, x \neq 0.$$

This proves (ii) in (b).

Finally we prove that B is a boundary of $(X, |||.|||)$ with (*). Assume the contrary and choose f a w^* limit point of $(1+\gamma_n)h_j^n$ such that there is x with $|||x||| = 1$ and $f(x) = 1$. Since $\lim_n \gamma_n = 0$, we have that $\|f\| \leq 1$, and then $\|x\| \geq 1$ in contradiction with (ii) we have already proof.

(b) ⇒ (a). Put

$$A_k = \{u \in X : |||u||| = 1, (1+\epsilon_k)t_k(u) = 1\}, k = 1, 2, \ldots$$

Since B is boundary it follows that $S_{(X,|||.|||)} = \bigcup_k A_k$. Define $S_k = \{u/||u|| : u \in A_k\}$ and $F_k = \{\pm t_j : j = 1, 2, \ldots, k\}$. Clearly $S_{(X,\|\cdot\|)} = \bigcup_k S_k$ and

$$\begin{aligned} b_k &= \inf\{\max\{f(x) : f \in F_k\} : x \in S_k\} \\ &= \inf\{\max\{t_i(u/\|u\|) : i = 1, 2, \ldots, k\} : u \in A_k\} \\ &\geq \inf\{t_k(u/\|u\|) : u \in A_k\} = \frac{1}{(1+\epsilon_k)} \inf\{1/\|u\| : u \in A_k\} \geq \frac{1}{(1+\epsilon_k)} > 0. \end{aligned}$$

and clearly $\lim_k b_k = 1$.

(b) $\Rightarrow$ (c). Put $V = B_{(X,|||.|||)}$. Clearly $V = \bigcap_k \{x \in X : |(1+\epsilon_k)t_k(x)| \le 1\}$. From (ii) follows that $S_X \subset \bigcup_k \{x \in X : |(1+\epsilon_k)t_k(x)| > 1\}$. Put $\alpha_k = \frac{\epsilon_k}{1+\epsilon_k}$, $k = 1, 2 \ldots$ and finish the proof.

(c) $\Rightarrow$ (b). Put $\epsilon_k = \frac{2\alpha_k}{1-2\alpha_k}$, $k = 1, 2, \ldots$ and $V = \bigcap_k \{x \in X : |(1+\epsilon_k)t_k(x)| \le 1\}$. Since $S_X \subset \bigcup_k \{x \in X : |t_k(x)| \ge 1 - \alpha_k\}$, it follows that

$$\bigcap_k \{x \in X : |t_k(x)| < 1 - \alpha_k\} \subset int(B_X),$$

and an easy verification shows that $V \subset int(B_X)$. Now (i) easily follows from $\lim_k \epsilon_k = 0$.

(b) $\Rightarrow$ (d). Is clear, since X has an equivalent norm with a countable boundary with (*). ■

R e m a r k. In [3] is proved that if X satisfies (c) (in the given norm) then X is isomorphically polyhedral, and if X is isomorphically polyhedral then there is an equivalent norm on X in which it satisfies (c). However, the following question remains open: if X is isomorphically polyhedral then does it satisfy (c) in any equivalent norm? There is a partial answer changing $\alpha_k = \alpha > 0$ with α arbitrary [1]. Finally, let us mention that the implication $(a) \Rightarrow (d)$ is a particular case of one of the results in [5].

Theorem 2. *Let X be a Banach space. Assume that $S_X = \bigcup_k S_k$ such that each S_k has an ε-approximative countable boundary, for any $\varepsilon > 0$, in the following sense*
(P) for any $k \in \mathbb{N}$ and $\varepsilon > 0$ there is a sequence $\{h_i^k\} \subset (1+\varepsilon)B_{X^*}$ such that if

$$V_k^* = w^* - cl\,co\{B_{X^*}, \{\pm h_i^k\}_{i\in\mathbb{N}}\}, \qquad V_k = \{x : \max x(V_k^*) \le 1\},$$

then
$(i)_k$ $V_k \bigcap S_k = \emptyset$
$(ii)_k$ for any $x \in \partial V_k \setminus S_X$ there is h_i^k with $h_i^k(x) = 1$.
Then X is isomorphically polyhedral.

P r o o f. Fix $\epsilon \in (0,1)$ and a sequence $\{\epsilon_k\}, \epsilon_k \in (0,\epsilon), \lim_k \epsilon_k = 0$. Next by using property (P) for every $k \in \mathbb{N}$ find $\{h_i^k\}_{i,k=1,2,\ldots} \subset (1+\epsilon_k)B_{X^*}, V_k^*, V_k$ such that $(i)_k$ and $(ii)_k$ are satisfied. Put

$$V^* = w^* - clco\{\pm h_i^k\}_{i,k\in\mathbb{N}}, V = \{x \in X : \max x(V^*) \le 1\},$$

and introduce in X a new norm $|||.|||$ with the unit ball V. By using $S_X = \cup_k S_k$ and $(i)_k, k = 1, 2, \ldots$, we get

$$V \subset intB_X. \tag{1}$$

Indeed, let $x \in X$ with $\|x\| \geq 1$ and take $z = x/\|x\| \in S_X$, then $z \in S_k$ for some k and $z \notin V_k$. By the definition of V_k it is possible to find $x_1^* \in B_{X^*}$, $x_2^* \in co\{\pm h_i^k\}_{i\in\mathbb{N}}$ and $\lambda \in [0,1)$ such that $\lambda x_1^*(z)+(1-\lambda)x_2^*(z) > 1$, $(1-\lambda)x_2^*(z) > 1-\lambda$ and $x_2^*(x) > \|x\| \geq 1$. Thus we have that $x \notin V$. Note that by (1) and its definition we have that $V = \bigcap_k V_k$.

From $\epsilon_k \in (0,\epsilon), k = 1,2,\ldots$, it follows that $V \supset (1+\epsilon)^{-1}B_X$. So the norm $|||.|||$ is $(1+\epsilon)$-equivalent to the original one. We show that the set $B = \{\pm h_i^k\}_{i,k\in\mathbb{N}}$ is a (countable) boundary for the space $(X, |||.|||)$, and then by [2] we conclude that $(X, |||.|||)$ is isomorphically polyhedral. Fix $x_0 \in \partial V$. From (1) follows that $x_0 \in intB_X$. Assume to the contrary that for any $h \in B$ we have $h(x_0) < 1$. Since $x_0 \in \partial V$ there is a sequence $\{h_{i_n}^{k_n}\}_{n=1}^{\infty}$ with $\lim_n h_{i_n}^{k_n}(x_0) = 1$. We consider two cases.

Case 1. $\limsup_n k_n = \infty$. WLOG we can assume that $\lim_n k_n = \infty$. Let $h_0 \in X^*$ be a w^*-limit point of $\{h_{i_n}^{k_n}\}_{n=1}^{\infty}$. Since $h_{i_n}^{k_n} \in (1+\epsilon_{k_n})B_{X^*}$ and $\lim_k \epsilon_k = 0$, it follows that $h_0 \in B_X$. However $h_0(x_0) = 1$, and hence $||x_0|| \geq 1$, contradicting $x_0 \in V \subset intB_X$.

Case 2. $\sup_n k_n < \infty$. WLOG we can assume that $k_n = l$, for any $n \in \mathbb{N}$. Since $x_0 \in V_l$ (recall that $x_0 \in V = \cap_k V_k$), and $\lim_n h_{i_n}^l(x_0) = 1$, it follows that $x_0 \in \partial V_l$. By $(ii)_l$ there is h_i^l with $h_i^l(x_0) = 1$, which proves that B is a (countable) boundary for $(X, |||.|||)$.

The proof of the theorem is complete.

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