Journal of Mathematical Physics, Analysis, Geometry
2013, vol. 9, No. 2, pp. 150–164

On Isomorphism Between Certain Group Algebras on the Heisenberg Group

M.E. Egwe and U.N. Bassey

Department of Mathematics, University of Ibadan
Ibadan, Nigeria

E-mail: me_egwe@yahoo.co.uk
unbassey@yahoo.com

Received July 6, 2009, revised March 27, 2012

Let $\mathbb{H}_n$ denote the $(2n+1)$-dimensional Heisenberg group and let K be a compact subgroup of $Aut(\mathbb{H}_n)$, the group of automorphisms of $\mathbb{H}_n$. We prove that the algebra of radial functions on $\mathbb{H}_n$ and the algebra of spherical functions arising from the Gelfand pairs of the form $(K, \mathbb{H}_n)$ are algebraically isomorphic.

Key words: Heisenberg group, spherical functions, radial functions, Heat kernel, algebra isomorphism.

Mathematics Subject Classification 2010: 43A80, 22E45, 33E99, 33C65.

1. Introduction

In [1], Krotz *et al.* studied the heat kernel transform for the Heisenberg group while Sikora and Zienkiewicz [2] described the analytic continuation of the heat kernel on the Heisenberg group. Earlier, Cowling *et al.* [3] derived a formula for the heat semigroup generated by a distinguished Laplacian on a large class of Iwasawa AN groups and proved that the maximal function constructed from the semigroup is of weak type $(1,1)$. Thangavelu [4] studied the spherical mean value operators L_r on the reduced Heisenberg group $\mathbb{H}_n/\Gamma$, where Γ is the subgroup $\{(0, 2\pi k) : k \in \mathbb{Z}\}$ of $\mathbb{H}_n$, and showed that all the eigenvalues of the operator L_r defined by $L_r f = \alpha f$ are of the form $\psi_\pi(r) = \int_G \phi_\pi(x) d\nu_r$.

In this paper, we show that the algebra of spherical functions generated by the Gelfand space $\Delta(K, \mathbb{H}_n)$, the space of bounded K-spherical functions on $\mathbb{H}_n$ modulo its center, equipped with compact-open topology associated to the Laplacian is algebraically isomorphic with the algebra of integrable radial functions on $\mathbb{H}_n$. This implies that these two algebras can be compared as sets considering their closed ideals as can be seen in [5]. Here $(K, \mathbb{H}_n)$ is a Gelfand pair with $K \subseteq U(n)$, the group of $Aut(\mathbb{H}_n)$.

1.1. The Heisenberg group. The $(2n+1)$-dimensional Heisenberg group, $\mathbb{H}_n$, is a noncommutative nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ with coordinates $(z,t) = (z_1, z_2, \ldots, z_n, t)$ and group law given by

$$(z,t)(z^{'},t^{'}) = (z + z^{'}, t + t^{'} + 2Imz.z^{'}),$$

where

$$z.z^{'} = \sum_{j=1}^{n} z_j \bar{z}_j, \quad z \in \mathbb{C}^n, \ \ t \in \mathbb{R}.$$

Setting $z_j = x_j + iy_j$, then $(x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ forms a real coordinate system for $\mathbb{H}_n$. In this coordinate system, we define the following vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \ \ Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \ \ T = \frac{\partial}{\partial t}.$$

It is clear from [6] that $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, T\}$ is a basis for the left invariant vector fields on $\mathbb{H}_n$ and the following commutation relations hold:

$$[Y_j, X_k] = 4\delta_{jk}T, \quad [Y_j, Y_k] = [X_j, T] = [X_j, X_k] = 0.$$

Similarly, we obtain the complex vector fields by setting

$$\begin{cases} Z_j = \dfrac{1}{2}(X_j - iY_j) = \dfrac{\partial}{\partial z_j} + i\bar{z}\dfrac{\partial}{\partial t} \\ \\ \bar{Z}_j = \dfrac{1}{2}(X_j + iY_j) = \dfrac{\partial}{\partial \bar{z}_j} - iz\dfrac{\partial}{\partial t} \end{cases}$$

and we have the commutation relations

$$[Z_j, \bar{Z}_k] = -2\delta_{jk}T, \ \ [Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] = [Z_j, T] = [\bar{Z}, T] = 0.$$

The Haar measure on $\mathbb{H}_n$ is the Lebesgue measure

$$dzd\bar{z}dt$$

on $\mathbb{C}^n \times \mathbb{R}$ [7]. In particular, for $n = 1$, we obtain the 3-dimensional Heisenberg group $\mathbb{H}_1 \cong \mathbb{R}^3$ (since $\mathbb{C}^n \cong \mathbb{R}^{2n}$).

Let us briefly recall the definition and properties of spherical functions which we shall need in the sequel.

1.2. Basic Definitions. Let G be a semisimple noncompact connected Lie group with finite center, and K be a maximal compact subgroup. Let

$C_c(K \setminus G/K)$ denote the space of continuous functions with compact support on G which satisfy $f(k_1 g k_2) = f(g)$ for all k_1, k_2 in K. Such functions are called spherical or K-bi-invariant. Then, $C_c(K \setminus G/K)$ forms a commutative Banach algebra under convolution [8]. An elementary spherical function ϕ is defined to be a K-bi-invariant continuous function which satisfies $\phi(e) = 1$ and such that $f \to f * \phi(e)$ defines an algebra homomorphism of $C_c(K \setminus G/K)$.

The elementary spherical functions are characterized by the following properties (see [9]):

(i) They are eigenfunctions of the convolution operator

$$f * \phi = \hat{\phi}(f)\phi,$$

where

$$\hat{\phi}(f) = \int_G f(x^{-1})\phi(x)dx.$$

(ii) They are eigenfunctions for a large class of left invariant differential operators on G.

(iii) They satisfy

$$\int_K \phi(xky)dk = \phi(x)\phi(y).$$

Now, on the Heisenberg group, we consider K, a compact group of subgroup of automorphisms of $\mathbb{H}_n$ such that the convolution algebra L^1_K of $K-$invariant functions is commutative. A bounded continuous $K-$invariant function φ such that $f \to \int f\varphi$ is an algebra homomorphism on L^1_K is called a $K-$spherical function. (For a complete characterization of the $K-$spherical functions and their properties (for various different K, see [10, 11].) In fact, when $K = U(n)$, the $K-$spherical functions include elementary spherical functions.

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be radial if there is a function ϕ defined on $[0, \infty)$ such that $f(x) = \phi(|x|)$ for almost every $x \in \mathbb{R}^n$.

Simple and classical examples of radial functions and their properties can be seen in, for example, [12, p. 464],[13, p. 266], [14, p. 134] and [15, p. 366].

Let ρ be a transformation on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then ρ is said to be orthogonal if it is a linear operator on $\mathbb{R}^n$ that preserves the inner product $\langle \rho x, \rho y\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$. If $det\rho = 1, \rho$ is called a rotation. Hence,

(1) We thus have that from the definition above a function f, defined on $\mathbb{R}^n$, is radial if and only if $f(\rho x) = f(x)$ for all orthogonal transformations ρ of $\mathbb{R}^n$.

(2) Also, f is radial if and only if $f(\rho x) = f(x)$ for all rotations ρ and all $x \in \mathbb{R}^n$ when $n > 1$.

(3) The basic property of Fourier with respect to orthogonal transformations is that the Fourier transformation F commutes with orthogonal transformations, i.e., if ρ is an orthogonal transformation. Let R_ρ be the mapping taking f on $\mathbb{R}^n$ into a function g whose values are $g(x) = (R_\rho f)(x) = f(\rho x)$ for $x \in \mathbb{R}^n$, then whenever $f \in L^1(\mathbb{R}^n)$,

$$\hat{g}(t) = (Fg)(t) = (FR_\rho f)(t) = (R_\rho Ff)(t) = (Ff)(\rho t) = \hat{f}(\rho t), \qquad (1.1)$$

i.e., the operators F and R_ρ commute: $FR_\rho = R_\rho F$ [21, p. 135].

To see this, we notice that the adjoint of ρ is also its inverse and the Jacobian in the change of variable $\omega = \rho x$ is one. Thus we have

$$\begin{aligned} \hat{g}(t) &= \int_{\mathbb{R}^n} e^{-2\pi i t.x} f(\rho x) dx = \int_{\mathbb{R}^n} e^{-2\pi i t.\rho^{-1}\omega} f(\omega) d\omega \\ &= \int_{\mathbb{R}^n} e^{-2\pi i \rho t \omega} f(\omega) d\omega = \hat{f}(\rho t). \end{aligned} \qquad (1.2)$$

Now, since whenever $|x_1| = |x_2|$ for two points of $\mathbb{R}^n$, there is an orthogonal transformation ρ such that $\rho x_1 = x_2$, we obtain the above mentioned property of the Fourier transform and thus we have that if f is a radial function in $L^1(\mathbb{R}^n)$, then $\widehat{f}$ is also radial [16].

2. Radial Functions On $\mathbb{H}_n$

Let $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$. We define an automorphism α_θ of $\mathbb{H}_n$ by

$$\alpha_\theta : (z, t) \mapsto (e^{i\theta} z, t) : \mathbb{H}_n \to Aut(\mathbb{H}_n),$$

where

$$e^{i\theta} z = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n).$$

We then have

$$\begin{cases} U^\lambda_{(e^{i\theta} z, t)} = A_\theta^{-1} U^\lambda_{(z,t)} A_\theta & \text{for } \lambda > 0, \\ U^\lambda_{(e^{i\theta} z, t)} = A_\theta U^\lambda_{(z,t)} A_\theta^{-1} & \text{for } \lambda < 0 \end{cases} \qquad (2.1)$$

where $A_\theta F(z) = F(e^{i\theta} z)$ and $U^\lambda_{(z,t)}$ denotes the irreducible unitary representation of $\mathbb{H}_n$. Also, for $\lambda = 0$, we obtain

$$\chi_\omega(e^{i\theta} z, t) = \chi_{e^{-i\theta}\omega}(z, t),$$

where $\chi_\omega(z, t) = e^{iRe\langle z, w\rangle}$ is the 1-dimensional representation of $\mathbb{H}_n$.

Definition 2.1. *A function f, defined on $\mathbb{H}_n$, is said to be radial if*

$$f(z,t) = f(e^{i\theta}z,t) \text{ for all } \theta. \tag{2.2}$$

Thus, by 2.0(1), if f is radial, then the operators U_f^λ, $\lambda \neq 0$, and A_θ commute and, since

$$A_\theta \phi_n^\lambda = e^{i\langle\theta,n\rangle}\phi_n^\lambda, \tag{2.3}$$

we have

$$U_f^\lambda \phi_n^\lambda = \hat{f}(\lambda,n)\phi_n^\lambda, \text{ where } \hat{f}(\lambda,n) \in \mathbb{C}. \tag{2.4}$$

Also, for $\lambda = 0$, we write

$$\hat{f}(0,\rho) = \int_{\mathbb{H}_n} f(z,t)\chi_\omega(z,t)dzdt, \tag{2.5}$$

where $\rho = (|\omega_1|,\cdots,|\omega_n|)$.

In what follows, let $\mathcal{A}$ denote the space of radial functions in $L^1(\mathbb{H}_n)$. Now, since α_θ are automorphisms of $\mathbb{H}_n$, $\mathcal{A}$ is a closed *-subalgebra of $L^1(\mathbb{H}_n)$. And it follows from (2.4) that the algebra $\mathcal{A}$ is commutative. Since $L^1(\mathbb{H}_n)$ is symmetric [17], the *-subalgebra $\mathcal{A}$ is also symmetric. The following results are well known.

Proposition 2.2. [18] *All non-zero multiplicative functionals on $\mathcal{A}$ are either of the form*

(a) $f \longrightarrow \hat{f}(\lambda,n)$ (*as in* (2.4))

or of the form

(b) $f \longrightarrow \hat{f}(0,\rho)$ (*as in* (2.5)).

P r o o f. Let ψ be a non-zero multiplicative linear functional on $\mathcal{A}$. Since $\mathcal{A}$ is a symmetric *-subalgebra of $L^2(\mathbb{H}_n)$, there exists an irreducible *-representation π of $L^1(\mathbb{H}_n)$ and a unit vector ξ in the Hilbert space $\mathfrak{H}_\pi$ such that $\pi_f\xi = \psi(f)\xi$ for f in $\mathcal{A}$. If $\mathfrak{H}_\pi$ is one-dimensional, then ψ has the form (b). Otherwise, $\pi = U^\lambda$ for some $\lambda \neq 0$ and $\mathfrak{H}_\pi = \mathfrak{H}_\lambda$. Since $\{U_f^\lambda : f \in \mathcal{A}\}$ is a *-algebra of operators which are diagonal on the basis ϕ_n^λ, we have $\xi = \phi_n^\lambda$ for some n and (a) follows. ■

Proposition 2.3. [18] *If $f \in \mathcal{A}$, then*

$$\hat{f}(\lambda,n) = \int_{\mathbb{H}_n} f(z,t)e^{-i\lambda t}e^{-|\lambda||z|^2}\prod_{j=1}^{r} L_{n_j}(2|\lambda||z_j|^2)dzdt, \tag{2.6}$$

where L_k is the Laguerre polynomial of degree k, that is,

$$L_k(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{(-x)^j}{j!}.$$

For $r > 0$, recall that a dilation of $\mathbb{H}_n$ is defined by

$$\delta_r(z,t) = (r^{-1/2}z, r^{-1}t).$$

δ_r is an automorphism of $\mathbb{H}_n$ and so $\delta_r(f)(z,t) = r^{-n-1}f(\delta_r(z,t))$ defines an automorphism of $L^1(\mathbb{H}_n)$ which preserves $\mathcal{A}$. For a functional ψ on $\mathcal{A}$, let $\langle f, \delta_r^*\psi\rangle = \langle \delta_r f, \psi\rangle$. δ_r^* maps the Gelfand space $\mathcal{M}(\mathcal{A})$ of non-zero multiplicative functionals on $\mathcal{A}$ homeomorphically onto itself. On the other hand, if $f \in L^1(\mathbb{H}_n)$ and $\int_{\mathbb{H}_n} f(z,t)dzdt = 1$, $\{\delta_r f\}$ is an approximate identity in $L^1(\mathbb{H}_n)$ as $r \longrightarrow 0$.

Proposition 2.4. *$\mathcal{A}$ is a (commutative) regular algebra and the set of functions f in $\mathcal{A}$ whose Gelfand transform $\hat{f}$ has support in $\mathcal{M}(\mathcal{A})$ is dense in $\mathcal{A}$.*

We give some examples of radial functions on $\mathbb{H}_n$.

E x a m p l e 2.5. Let

$$D_n = \{(z, z_0) \in \mathbb{C}^n \times \mathbb{C} : \; Im z_0 > |z|^2\}$$

on which the Heisenberg group $\mathbb{H}_n$ acts by translations [6]

$$(\omega, u)(z, z_0) \longrightarrow (\omega, u)\cdot(z, z_0) = (\omega+z, z_0+u+i|\omega|^2+2i\langle z,\omega\rangle : \mathbb{H}_n \times D_n \longrightarrow D_n.$$

Introducing new coordinates t, ϵ, z

$$\begin{aligned} z_0 &= t + i(\epsilon + |z|^2), \\ z &= z, \end{aligned}$$

$D_n \cong \mathbb{H}_n \times \mathbb{R}^+$ and the level surfaces for the variable ϵ are the orbits of $\mathbb{H}_n$ in D_n. Also, $\mathbb{H}_n$ is identified with the boundary ∂D_n of D_n.

Let Δ be the Laplace–Beltrami operator for the Bargman metric on D_n. The bounded harmonic functions u on D_n, i.e., $\Delta u = 0$, have boundary values a.e. on ∂D_n, i.e.,

$$\lim_{\epsilon \longrightarrow 0} u(z,t,\epsilon) = \varphi(z,t) \;\; a.e.,$$

where $\varphi \in L^\infty(\mathbb{H}_n)$. Moreover,

$$u(z,t,\epsilon) = (\varphi * P_\epsilon)(z,t),$$

where

$$P_\epsilon(z,t) = c_n \epsilon^{n+1}((|z|^2+\epsilon)^2+t^2)^{-n-1},$$

$C_n = \dfrac{2^{r-1}n!}{\pi^{n+1}}$ and the convolution is on $I\!H_n$.
We notice that $P_\epsilon \in L^1(I\!H_n)$ and is radial. P_ϵ can be expressed as

$$P_\epsilon = c_n^{-1}\epsilon^{n+1}|S_\epsilon|^2,$$

where

$$S_\epsilon(z,t) = c_n(\epsilon + |z|^2 - it)^{-n-1}$$

is the Szego Kernel for D_n, which determines the orthogonal projection of $L^2(I\!H_n)$ on the Hardy space $H^2(D_n)$ and precisely the spherical harmonics earlier obtained. It has been shown in [18] that: For every $\epsilon > 0$. $\hat{P}_\epsilon$ does not varnish at any point in the Gelfand space $\mathcal{M}(\mathcal{A}_r)$.
We give next the group Fourier transform of radial functions on the Heisenberg group. Recall that the group Fourier transform of an integrable function g on $I\!H_n$ is, for each $\lambda \neq 0$, an operator-valued function on the Hilbert space $L^2(I\!R^n)$ given by

$$\hat{g}(\lambda)\varphi(\xi) = W_\lambda(g^\lambda)\varphi(\xi),$$

where

$$W_\lambda(f^\lambda)\varphi(\xi) = \int\limits_{\mathbb{C}^n} g^\lambda(z)\pi_\lambda(z)\varphi(\xi) \quad \text{and} \quad g^\lambda(z) = \int\limits_{I\!R} g(z,t)e^{i\lambda t}dt.$$

Now, if g is also radial on $I\!H_n$, which means that it depends only on $|z|$ and t, then it follows that the operators $\hat{g}(\lambda)$ are diagonal on the Hermite basis for $L^2(I\!R^n)$.

The following functions are required in Theorem 2.6 below. For $\delta > -1$, the Laguerre functions of type δ are given by

$$\Lambda_k^\delta(x) = \left(\frac{k!}{(k+\delta)!}\right)^{1/2} L_k^\delta(x)e^{-\frac{1}{2}x}x^{\frac{\delta}{2}}.$$

Also, for each $\lambda > 0$,

$$\ell_k^\lambda(r) = (|\lambda|r^2)^{\frac{1-n}{2}}\Lambda_k^{n-1}(\frac{1}{2}|\lambda|r^2),\ r \in I\!R^+.$$

Theorem 2.6. *If $g \in L^1(\mathbb{H}_n)$ and $g(z,t) = g_0(|z|,t)$, then*

$$\hat{g}(\lambda)h_\alpha^\lambda(x) = C_n \mu(|\alpha|,\lambda)h_\alpha^\lambda(x),$$

where

$$\mu(k,\lambda) = \left(\frac{k!}{(k+n-1)!}\right)^{1/2} \int_0^\infty g_0^\lambda(s)(\frac{1}{2}|\lambda|s^2)^{\frac{1-n}{2}}\Lambda_k^{n-1}(\frac{1}{2}|\lambda|s^2)s^{2n-1}ds,$$

and C_n is a constant which depends only on n.

P r o o f. It is clear that $g^\lambda(z) = g_0^\lambda(|z|)$ for some function g_0^λ. We can therefore write

$$g_0^\lambda(r) = \sum_{k=0}^\infty \left(\int_0^\infty g_0^\lambda(s)\ell_k^\lambda(s)|\lambda|^n s^{2n-1}ds\right)\ell_k^\lambda(r).$$

From this we see that we formally have

$$g^\lambda(z) = C_n \sum_{k=0}^\infty \mu(k,\lambda)\varphi_k^\lambda(z),$$

where $C_n = (2\pi)^n 2^{1-n}$. It now follows that

$$g^\lambda *_\lambda \varphi_k^\lambda(z) = C_n\mu(k,\lambda)\varphi_k^\lambda(z),$$

and hence from [7] we have that this formal Laguerre expansion in fact agrees with the special Hermite expansion

$$g^\lambda(z) = \sum_{k=0}^\infty g^\lambda *_\lambda \varphi_k^\lambda(z) = C_n \sum_{k=0}^\infty \mu(k,\lambda)\varphi_k^\lambda(z). \tag{2.7}$$

Now, since $\hat{g}(\lambda) = W_\lambda(g^\lambda)$, the theorem follows immediately from the last equation and Lemma 10 of [19]. ■

We now consider a comparison of the algebras of radial and spherical functions in what follows.

Let $\mathfrak{h}_n$ denote the $(2n+1)$-dimensional Heisenberg algebra with generators

$$X_1, \ldots, X_n, U_1, \ldots, U_n, Z$$

satisfying the commutation relations $[Z_j, U_j] = Z_j$. We identify $\mathfrak{h}_n$ with $I\!R^{2n+1} := I\!R^n \times I\!R^n \times I\!R$. For this, let $x = (x_1, \ldots, x_n)$ and $u = (u_1, u_2, \ldots, u_n)$ denote the canonical coordinates on $I\!R^{2n+1}$. The map

$$p : I\!R^{2n+1} \longrightarrow \mathfrak{h}_n : (x, u, \xi) \mapsto \sum_{j=1}^{\infty} x_j X_j + \sum_{j=1}^{n} u_j U_j \xi Z$$

is a linear isomorphism providing suitable coordinates for $\mathfrak{h}_n$, using the Mackev basis.

We identify $I\!H_n$ with $\mathfrak{h}_n$ through the exponential map

$$\exp : \mathfrak{h}_n \longrightarrow I\!H_n$$

with the usual group law and Haar measure dh in such a way that it coincides with the product of Lebesgue measures, i.e.,

$$\int\limits_{I\!H_n} f(h) dh = \int\limits_{I\!R^{2n+1}} f(x, u, \xi) dx du d\xi.$$

Here, for $(x, u, \xi) \in I\!H_n$, we have

$$(x, u, \xi)^{-1} = (-x, -u, -\xi).$$

The automorphisms are the dilations

$$\delta_r(z, \xi) := (rz, r^2 \xi), \quad z = (x, u).$$

For $(x, u, \xi) \in I\!H_n$, define the Koranyi-norm by

$$|(x, u, \xi)| := (|x + iu|^4 + 16\xi^2)^{1/4} = ||x + iu|^2 \pm 14i\xi|^{1/2}.$$

This norm has the following properties:

(i) $|\delta_r g| = r|g| \;\; \forall \;\; g \in I\!H_n, \;\; r > 0,$

(ii) $|g| = 0 \Leftrightarrow g = 0,$

(iii) $|g^{-1}| = |g|,$

(iv) $|g_1 g_2| \leq |g_1| + |g_2| \;\; g_1, g_2 \in I\!H_n.$

In particular, $|\cdot|$ is a homogeneous norm and $d_K(g_1, g_2) := |g_1^{-1} g_2|$ defines a left-invariant metric on $I\!H_n$.

R e m a r k 2.7. $I\!H_n$, endowed with the Koranyi metric d_k and the Haar measure, forms a space of homogeneous type in the sense of Coifman and Weiss [20].

In fact, denote by

$$B_r(g) := \{h \in I\!H_n : |g^{-1}h| < r\}$$

the ball of radius $r > 0$ centred at $g \in I\!H_n$. Then, by left-invariance and (i) above, we have

$$|B_r(g)| = |B_r(0)| = |\delta_r(B_1(0))| = r^Q|B_1(0)|,$$

where $Q = 2n + 2$
is the homogeneous dimension of $I\!H_n$.

Next, let $\mathcal{U}(\mathfrak{h}_n)$ denote the universal enveloping algebra of $\mathfrak{h}_n$ and let the Laplace element in $\mathcal{U}(\mathfrak{h}_n)$ be given by

$$\mathcal{L} := \sum_{j=1}^{n} X_j^2 + \sum_{j=1}^{n} U_j^2 + Z^2.$$

For $X \in \mathfrak{h}_n$, we shall write $\tilde{X}$ for the left-invariant vector field on $I\!H_n$, i.e.,

$$(\tilde{X}f)((h) = \frac{d}{dt}\bigg|_{t=0} f(h\exp(tX))$$

for f a function on $I\!H_n$ which is differentiable at $h \in I\!H_n$. Let ρ be the right regular representation of $I\!H_n$ on $L^2(I\!H_n)$, i.e.,

$$(\rho(h)f)(x) = f(xh)$$

for $x, h \in I\!H_n$ and $f \in L^2(I\!H_n)$. If $d\rho$ is the derived representation, then we have $d\rho(X) = \tilde{X}$ for all $X \in \mathfrak{h}_n$. In particular, if

$$\Delta_{I\!H_n} := \sum_{i=1}^{n} \tilde{X}_i^2 + \sum_{i=1}^{n} \tilde{U}_i^2 + \tilde{Z}$$

denotes the Laplacian on $I\!H_n$, then $d\rho(\mathcal{L}) = \Delta_{I\!H_n}$. Now set $I\!R^+ = (0, \infty)$. We have already seen that $\Delta_{I\!H_n}$ is not globally solvable. We now turn to the Heisenberg heat equation defined on $I\!H_n \times I\!R^+$ by

$$\partial_t U(h,t) = \Delta U(h,t),$$

$U(h,t) \in I\!H_n \times I\!R^+$. The fundamental solution of this equation is given by the heat kernel $K_t(h)$ which is obtained explicitly in [1] as

$$K_t(x,u,\xi) = c_n \int\limits_{I\!R} e^{-i\lambda E} e^{-t\lambda^2} \left(\frac{\lambda}{\sin h\lambda t}\right)^n e^{-\frac{1}{4}\lambda(\cot ht\lambda)(x\cdot x + u\cdot u)} d\lambda,$$

where $c_n = (4\pi)^{-n}, \;\; \lambda \in I\!R^* = I\!R \setminus \{0\}$.

Let φ_λ^k be the K-spherical function on $I\!H_n$. That is, the distinguished spherical function restricted to $L^1(K\backslash G/K)$ where (K, G) is a Gelfand pair, K a compact subgroup of $Aut(I\!H_n)$. In this case, G may be taken as a semi-direct product of K and $I\!H_n$ (i.e., $G := K \ltimes I\!H_n$) [10]. Thus φ_λ^k is a unique radial function since it is a radial eigenfunction of $\Delta_{I\!H_n}$ [13, p. 38]. (In fact, elementary spherical functions are radial functions [15]), i.e.,

$$\varphi_\lambda^k(u) = \psi(|u|).$$

Now rewriting the heat kernel, we have

$$\begin{aligned} K_t(h) &= c_n \int\limits_{I\!R^n} e^{-\lambda\xi} e^{-t\lambda^2} \varphi^n(\lambda t) e^{-\frac{1}{4}|h|^2 \phi(\lambda t)} d\lambda \\ &= c_n \int\limits_{I\!R^n}^{\lambda(\xi+\lambda^2)} \varphi^n(\lambda t) e^{-\frac{1}{4}|h|^2\phi(\lambda t)} d\lambda \\ &= c_n \psi_\lambda(|h|, t) \end{aligned}$$

which gives a radial function for $K := U(n)$ and

$$K_t(h) = c_n \psi_\lambda(e^{-i\theta}|h|, t)$$

which gives a polyradial function for $K := \mathbb{T}^n$. Applying dilations to the radial function, we obtain

$$\begin{aligned} K_t(h) &= \delta_r(c_n \psi_\lambda(|h|, t) \\ &= c_n \psi_\lambda(r|h|, r^2 t) \\ &= c_n t^{-n/2} \varphi^n(h) \delta_r^{-2}(h) e^{|h|^2/4t}. \end{aligned}$$

Let $\mathcal{A}$ be the subalgebra $L^1(I\!H_n)$ (with respect to the right invariant Haar measure) generated by $K_t, t > 0$. We wish to state a lemma (Tauberian theorem) which gives conditions, in terms of non-vanishing of transforms, for a closed ideal I in $L^1(I\!H_n)$ to be all the space $L^1(I\!H_n)$.

First, we consider the spherical transform of any $f \in L^1(I\!H_n)$. The Gelfand spherical transform is defined for the commutative Banach algebra $\mathcal{A}$ as the mapping from $\mathcal{A}$ to the continuous functions on its maximal ideal space $\mathcal{M}(\mathcal{A})$. The maximal ideal space consists of all the non-zero continuous homomorphisms from $\mathcal{A}$ to the complex numbers $\mathbb{C}$. As $L^1(K\backslash G/K)$ is a commutative Banach algebra, the spherical transform can be defined. Now the maximal ideal space $\mathcal{M}(L^1(K\backslash G/K))$ may also be expressed using the bounded spherical functions.

The set of bounded spherical functions consists of a Laguerre part and a Bessel part. They are the following [14, 18]:

$$\varphi_k^\lambda(z,t) = e^{2\pi i\lambda t} e^{-2\pi|\lambda||z|^2} \prod_{j=1}^{n} L_{k_j}^{(0)}(4\pi|\lambda||z_j|^2), \quad \lambda \in \mathbb{R}^*, k \in (\mathbb{Z}_+)^n,$$

$$\mathcal{J}_0^\rho = \prod_{j=1}^{n} J_0(\rho_j|z_j|), \quad \rho \in (\mathbb{R}_+)^n,$$

respectively. Here $L_k^{(0)}$ is the Laguerre polynomial of degree k and J_0 is the Bessel function (of the first kind) of index 0. The spherical transform of a function is then given by

$$\tilde{f}(\lambda;k) = \int\limits_{\mathbb{H}_n} f(z,t)\overline{\varphi_k^\lambda(z,t)}\, dzdt,$$

$$\tilde{f}(0;\rho) = \int\limits_{\mathbb{H}_n} f(z,t)\overline{\mathcal{J}_0^\rho(z)}\, dzdt.$$

Definition 2.8. *Let A be an algebra. (Here,* an Ideal of A *is always a two-sided ideal.) The primitive ideal space of A, denoted by $Prim(A)$, is the space of all ideals I of A of the form $I = Ker(T)$, where $T(V)$ denotes an algebraically irreducible representation of A on a vector space V. We provide $Prim(A)$ with the Jacobson topology. In this topology, a subset C of $Prim(A)$ is closed if it is the hull $H(I)$ of some ideal I of A, i.e., if*

$$C = H(I) = \{J \in Prim(A) : J \supset I\}.$$

For a subset $C \subset Prim(A)$, let

$$Ker(C) = \bigcap_{j\in C} J \subset A \text{ and } I(C) = \bigcap_{H(I)=C} I.$$

The hull of $I(C)$ contains C.

For certain algebras A, we have $H(I(C)) = C$, i.e., there exists a minimal ideal $j(C)$ with hull C. That means there exists an ideal $j(C)$ of A such that the hull of $j(C)$ is equal to C and $j(C) \subset I$ for every ideal I of A whose hull is contained in C.

R e m a r k 2.9. It was shown in [5] that $j(C)$ exists for every closed subset C in the primitive ideal space for the Schwartz algebra of a nilpotent Lie group.

Lemma 2.10. *Let $I\!I \subset L^1(I\!H_n)$ be a closed ideal such that*

(i) for each $(\lambda, k) \in I\!R^* \times (Z\!\!Z_+)^n$, there exists $f \in I\!I$ such that

$$\tilde{f}(\lambda; k) \neq 0,$$

(ii) for each $\rho \in (I\!R_+)^n$, there exists $f \in I\!I$ such that

$$\tilde{f}(0, \rho) \neq 0.$$

Then $I\!I = L^1(I\!H_n)$.

P r o o f. Assume without loss of generality that $f \in \mathcal{S}(I\!H_n)$. This is possible since $\mathcal{S}(I\!H_n)$ is dense in $L^1(I\!H_n)$. Now, by hypothesis, $I\!I$ is closed and therefore must be the hull of some ideal, say, $\mathcal{J}$ of $L^1(I\!H_n)$. This makes $I\!I$ a subset of $Prim(L^1(I\!H_n))$ since for any f spherical, $\varphi_k^\lambda(0) = \tilde{f}(\lambda, k) \neq 0$, and $\tilde{f}(0, \rho) \neq 0$. Thus $I\!I = H(I) = \{M \in Prim(L^1(I\!H_n) : M \supset J\}$.
Now, since $I\!H_n$ is a nilpotent Lie group, it follows from Remark 2.9 that

$$H(I(I\!I)) = I\!I,$$

$$\Longrightarrow L^1(I\!H_n) = I\!I$$

since $I\!I$ is a closed ideal. ■

Theorem 2.11. *Let $A_r(I\!H_n)$ and $\mathcal{S}_p(I\!H_n)$ denote the algebras of radial and spherical functions on $I\!H_n$, respectively. Define an operator $T : \mathcal{S}_p(I\!H_n) \longrightarrow A_r(I\!H_n)$ by*

$$T(\varphi) = C_n \varphi_\lambda^k(u) \delta_r(u) e^{|u|^2/4} e^{-i\lambda t}, \ \ u \in I\!H_n$$

$$= C_n \varphi_\lambda^k(|u|, t).$$

Then T is an algebraic isomorphism of $A_r(I\!H_n)$ and $\mathcal{S}_p(I\!H_n)$.

P r o o f. First recall that the heat equation on $I\!R^n$ is given by

$$\begin{aligned} u_t(t, x) &= \Delta u(t, x), \\ u(0, x) &= \delta(x). \end{aligned}$$

Now the calculation of the Gaussian integral

$$u(\epsilon, x) = 2\pi^{-n} \int\limits_{I\!R^n} e^{ix\cdot\xi - \epsilon|\xi|^2} d\xi$$

gives explicitly the fundamental solution of the heat equation as [20, p. 289]

$$e^{t\Delta}\delta(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}, \quad t > 0, x \in \mathbb{R}^n.$$

Thus we have the heat kernel $q_t(x)$ on $\mathbb{R}^n$ as given above. Now, let V denote the vector space of all linear combinations of $q_t, t > 0$. By the formula in the theorem, T restricted to V is an isomorphism of algebras. Moreover, for all $f \in V$, we have

$$\int\limits_{S_p(\mathbb{H}_n)} T\varphi = \int\limits_{A_r(\mathbb{H}_n)} \varphi.$$

On the other hand, if we denote E the space $L^1_{rad}(\mathbb{R}^n, e^{C|x|}dx)$ for sufficiently large C, then T is continuous from E to $\mathcal{S}_p(\mathbb{H}_n) \subset L^1(\mathbb{H}_n)$. Since V is dense in E, it follows that $T(E) \subset L^1(K\backslash \mathbb{H}_n/K)$ and for all $\varphi \in E$, we have

$$\int\limits_{L^1(K\backslash \mathbb{H}_n/K)} T\varphi = \int\limits_{\mathbb{R}^n} \varphi.$$

From [15], any $f \in E$ can be decomposed into its positive and negative parts with each component belonging to E. Thus decomposing φ yields

$$\|\varphi\|_{L^1_{rad}} = \int\limits_{\mathbb{R}^n} \varphi_+ + \int\limits_{\mathbb{R}^n} \varphi_- = \int\limits_{\mathcal{S}_p(\mathbb{H}_n)} T\varphi_+ + \int\limits_{\mathcal{S}_p(\mathbb{H}_n)} T\varphi_- = \|T\varphi\|_{L^1(\mathbb{H}_n)}$$

showing that the closure of $T|_V$ is an isometry of $L^1_{rad}(\mathbb{R}^n)$ with $S_p(\mathbb{H}_n)$ and this closure is equal to T. Hence the proof follows by Lemma 2.10. ■

Acknowledgement. The authors quite appreciate the complete and constructive suggestions of the referee on the center of the Heisenberg group which was initially assumed intrinsic.

References

[1] *B. Krotz, S. Thangavelu, and Y. Xu*, The Heat Kernel Transform for the Heisenberg Group. arXiv:math.CA/0401243v2, 2005.

[2] *A. Sikora and J. Zienkiewicz*, A Note on Heat Kernel on the Heisenberg Group. — *Bull. Austral Math. Soc.* **65** (2002), No. 1, 116–120.

[3] *M. Cowling, S. Giulini, and G. Mauceri*, Weak (1, 1) Estimates for Heat Kernel Maximal Functions on Lie Groups. — *Trans. Amer. Math. Soc.* **323** (1991), 637–649.

[4] *S. Thangavelu*, Spherical Means on the Heisenberg Group and a Restriction Theorem for the Symplectic Fourier Transform. — *Revista Math. Iberoamericana* **7** (1991), No. 2, 135–165.

[5] *J. Ludwig*, Hull-Minimal Ideals in the Schwartz Algebra of the Heisenberg Group. — *Stud. Math.* **130** (1998), No. 1, 77–98.

[6] *G.B. Folland and E.M. Stein*, Estimates for the $\bar{\partial}_b$ Complex and Analysis on the Heisenberg Group. — *Commun. Pure and Appl. Math.* Vol. XXVII (1974) 429–522.

[7] *G. Mauceri*, Zonal Multipliers On the Heisenberg Group. — *Pacific J. Math.* **95** (1981), 143–169.

[8] *S. Lang*, $SL(2, \mathbb{R})$. Addison–Wesley, Reading, MA, 1975.

[9] *R. Gangolli*, Spherical Functions on Semisimple Lie Groups. In: *Symmetric Spaces*, W. Boothy and G. Weiss (Eds.). Marcel Dekker, Inc. New York, 1972.

[10] *B. Astengo, D.B. Blasio, and F. Ricci*, Gelfand Pairs on the Heisenberg Group and Schwartz Functions. — *J. Funct. Anal.* **256** (2009), No. 5, 1565–1587.

[11] *C. Benson, J. Jenkins, and G. Ratcliff*, Bounded K-Spherical Functions on the Heisenberg Groups. — *J. Funct. Anal.* **105** (1992), 409–443.

[12] *G.E. Andrews, R. Askey, and R. Roy*, Special Functions. Encyclopedia of Math. and its Appl., Cambridge Univ. Press. UK, Cambridge, 2000.

[13] *S. Helgason*, Groups and Geometric Analysis: Integral Geometry, Differential Operators and Spherical Functions. Acad. Press Inc., New York, 1984.

[14] *E.M. Stein and G. Weiss*, Introduction to Fourier Analysis on Euclidean Spaces. Princeton, Univ. Press, London, 1971.

[15] *G. Szego*, Orthogonal Polynomials. AMS Colloq. Publ., Vol. XXIII, AMS Providence, RI, 1939.

[16] *S. Bochner and K. Chandrasekharan*, Fourier Transforms. Princeton Univ. Press, London, 1949.

[17] *N. Leptin*, On Group Algebras of Nilpotent Groups. — *Stud. Math.* **47** (1973), 37–49.

[18] *A. Hulanicki and F. Ricci*, A Tauberian Theorem and Tangential Convergence of Bounded Harmonic Functions on Balls in $\mathbb{C}^n$. — *Inv. Math.* **62** (1980), 325–331.

[19] *N. Lyall*, The Heisenberg Group Fourier Transform. — *Trans. Amer. Math. Soc.* **359** (2007), 4467–4488.

[20] *M.E. Taylor*, Noncommutative Harmonic Analysis. AMS Providence, RI, USA, 1986.