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Inverse Problems for Jacobi Operators II: Mass Perturbations of Semi-Infinite Mass-Spring Systems

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We consider an inverse spectral problem for infinite linear mass-spring systems with different configurations obtained by changing the first mass. We give results on the reconstruction of the system from the spectra of two configurations. Necessary and sufficient conditions for two real sequences to be the spectra of two modified systems are provided.

Key words: infinite mass-spring system, Jacobi matrices, two-spectra inverse problem.

Mathematics Subject Classification 2010: 34K29, 47A75, 47B36, 70F17.

1. Introduction

In this work, we treat the two spectra inverse problem for Jacobi operators in $l_2(\mathbb{N})$. The Jacobi operators considered here are obtained from each other by a particular kind of rank-two perturbation. The special form of the perturbation has a physical motivation; it is the extension to the semi-infinite case of an inverse problem for the finite mass-spring systems studied in [7] and [20].

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The Jacobi operator J in the Hilbert space $l_2(\mathbb{N})$ is the operator whose matrix representation with respect to the canonical basis in $l_2(\mathbb{N})$ is a semi-infinite Jacobi matrix of the form

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

where $q_n \in \mathbb{R}$ and $b_n > 0$ for any $n \in \mathbb{N}$ (see the definition of the matrix representation of an unbounded symmetric operator in [2]). J is closed by definition and it may be self-adjoint or have deficiency indices (1,1). In this work, we deal with self-adjoint operators, so, if $J \neq J^*$, we consider its self-adjoint extensions denoted $J^{(g)}$, where $g \in \mathbb{R} \cup \{\infty\}$ (see Definition 1 a)). If $J = J^*$, we assume $J^{(g)} = J$ for all $g \in \mathbb{R} \cup \{\infty\}$ (see Definition 1 b)).

The two spectra inverse problem for Jacobi operators $J^{(g)}$ takes as input data the spectra of two operators in an operator family obtained by perturbing $J^{(g)}$ in a certain way. The solution of the problem is the finding of the matrix (1.1) and the "boundary condition at infinity" g if necessary. The case of the operator family consisting of rank-one perturbations of a self-adjoint Jacobi operator has been amply studied in [8, 12, 13] and, in the more general setting of the rank-one perturbations of $J^{(g)}$, in [22, 26]. The rank-one perturbations can be viewed as a change of the "boundary condition at the origin" for the corresponding difference equation (see [22, Appendix]). We remark that the case of finite Jacobi matrices has also been thoroughly studied (see [5, 6, 9, 11, 14]).

It is known that the dynamics of a finite mass-spring system is characterized by the spectral properties of a finite Jacobi matrix [11]. Accordingly, in solving the inverse problem for mass-spring systems mentioned above, [20] provides necessary and sufficient conditions for two point sets to be the spectra of two finite Jacobi matrices corresponding to two mass-spring systems, one of which has a mass and a spring modified. The results of [20] are related to the study of microcantilevers [24, 25], which are modeled by a spring-mass system whose masses and springs constants correspond to the mechanical parameters of the system. The inverse problem treated in [20] could be used as a theoretical framework for the problem of measuring micromasses with a help of microcantilevers [24, 25].

Let us consider a semi-infinite spring-mass system with masses $\{m_j\}_{j=1}^{\infty}$ and spring constants $\{k_j\}_{j=1}^{\infty}$ as in Fig. 1. By a standard reasoning (see [11, 17, 18]), one verifies that the infinite system of Fig. 1 is modeled by the spectral properties

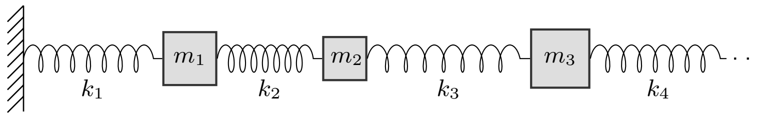

Fig. 1. Semi-infinite mass-spring system

of the Jacobi operator J with

$$q_j = -\frac{k_{j+1}+k_j}{m_j}\,, \qquad b_j = \frac{k_{j+1}}{\sqrt{m_j m_{j+1}}}\,, \qquad j \in \mathbb{N}\,. \tag{1.2}$$

We remark that in [11, 17, 18] the obtained matrix corresponds to $-J$. An alternative physical interpretation is provided by a one-dimensional harmonic crystal [27, Sec. 1.5].

In this work, we consider the spectrum of $J^{(g)}$ to be discrete (if $J \neq J^*$ this is always the case). Below, in Remarks 3 and 4 we comment on matrices of the form (1.1) whose corresponding operator $J^{(g)}$ has discrete spectrum.

The discreteness of $\sigma(J^{(g)})$ implies that the movement of our mechanical system is a superposition of harmonic oscillations whose frequencies are the square roots of the modules of the eigenvalues.

Along with the self-adjoint operator $J^{(g)}$, we consider the family of operators $J^{(g)}(\theta)$ $(\theta > 0)$ being self-adjoint extensions of the Jacobi operator whose matrix representation with respect to the canonical basis in $l_2(\mathbb{N})$ is

$$\begin{pmatrix} \theta^2 q_1 & \theta b_1 & 0 & 0 & \cdots \\ \theta b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}. \tag{1.3}$$

Here $J^{(g)}(\theta)$ $(\theta > 0)$ is the family of perturbed Jacobi operators. Note that the operators of the family are not obtained from each other by a rank-one perturbation (see (2.4) below).

Going from $J^{(g)}$ to $J^{(g)}(\theta)$ corresponds to changing the first mass by $\Delta m = m_1(\theta^{-2} - 1)$. In other words, θ^2 is the ratio of the original mass m_1 to the new mass $m_1 + \Delta m$. This is illustrated in Fig. 2. It is worth mentioning that we also consider here the cases when $\Delta m < 0$, equivalently, $\theta > 1$, although physical applications correspond to $\theta < 1$ [24, 25].

The problem of reconstructing the initial and the perturbed matrices by their spectra can be then interpreted from the physical point of view as the problem of

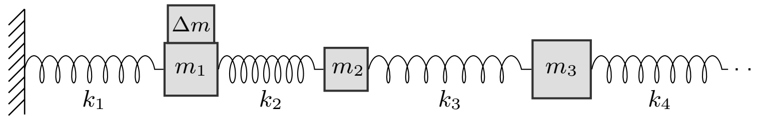

Fig. 2. Perturbed semi-infinite mass-spring system

finding the mechanical parameters of the spring-mass system from the frequencies of its oscillations before and after the modification.

We emphasize that, although the operators and the particular kind of perturbation considered here were motivated by a physical system, the general mathematical setting is considered throughout the work. Thus, the entries in (1.1) have no restriction other than J being a Jacobi operator ($q_n \in \mathbb{R}$, $b_n > 0$) and $J^{(g)}$ having discrete spectrum (see Remarks 3, 4). Note that J is then not necessarily semibounded though it actually is when J corresponds to a mass-spring system.

This work is organized as follows. In Section 2. we lay down the notation, introduce the Jacobi operators and their perturbations, and present some preparatory facts related with the inverse spectral problems of these operators. Section 3. gives an account of the spectral properties of the family of perturbed Jacobi operators $J^{(g)}(\theta)$. The problem of reconstruction is treated in Section 4. This section gives some necessary conditions for the spectra of $J^{(g)}(\theta)$, provides an algorithm for reconstruction of the matrix and establishes uniqueness of the reconstruction. Finally, Section 5 gives necessary and sufficient conditions for two sequences of real numbers to be the spectra of $J^{(g)}$ and its perturbation $J^{(g)}(\theta)$ ($\theta \neq 1$).

2. Preliminaries

Let Υ be a second order symmetric difference expression such that for any sequence $f = \{f_k\}_{k=1}^{\infty}$

$$(\Upsilon f)_1 := q_1 f_1 + b_1 f_2 \,, \tag{2.1}$$

$$(\Upsilon f)_k := b_{k-1} f_{k-1} + q_k f_k + b_k f_{k+1} \,, \quad k \in \mathbb{N} \setminus \{1\}, \tag{2.2}$$

where, for $n \in \mathbb{N}$, b_n is positive and q_n is real. Let $l_{\text{fin}}(\mathbb{N})$ be the linear space of complex sequences with a finite number of non-zero elements. In the Hilbert space $l_2(\mathbb{N})$, let us consider the operator whose domain is $l_{\text{fin}}(\mathbb{N})$ and acts as the expression Υ. This operator is symmetric since it is densely defined and Hermitian, and thus it is closable. Now, let J be the closure of this operator.

We have defined the operator J such that the semi-infinite Jacobi matrix (1.1) is its matrix representation with respect to the canonical basis $\{\delta_n\}_{n=1}^{\infty}$ in $l_2(\mathbb{N})$ (see [2, Sec. 47] for the definition of the matrix representation of an unbounded symmetric operator). Indeed, J is the minimal closed symmetric operator satisfying

$$\begin{array}{l}\langle\delta_n, J\delta_n\rangle = q_n\,, \quad \langle\delta_{n+1}, J\delta_n\rangle = \langle\delta_n, J\delta_{n+1}\rangle = b_n\,, \\ \langle J\delta_n, \delta_{n+k}\rangle = \langle\delta_n, J\delta_{n+k}\rangle = 0\,,\end{array} \qquad n\in\mathbb{N}\,,\ k\in\mathbb{N}\setminus\{1\}\,.$$

We shall refer to J as the *Jacobi operator* and to (1.1) as its associated matrix.

The operator J^* turns out to be given by

$$\operatorname{dom}(J^*) = \{f\in l_2(\mathbb{N}) : \Upsilon f\in l_2(\mathbb{N})\}, \qquad J^* f = \Upsilon f\,,$$

which follows directly from the definition of J [1, Chap. 4 Sec. 1.1], [23, Thm. 2.7].

If one gives the complex number f_1, then the solution of the difference equation

$$\Upsilon f = \zeta f\,, \qquad \zeta\in\mathbb{C}\,,$$

is uniquely determined from (2.1) and (2.2) by recurrence. For the elements of this solution when $f_1 = 1$, the following notation is standard [1, Chap. 1, Sec. 2.1]:

$$P_{k-1}(\zeta) := f_k\,, \qquad k\in\mathbb{N}\,,$$

where the polynomial $P_k(\zeta)$ (of degree k) is referred to as the k-th orthogonal polynomial of the first kind associated with the matrix (1.1). Now, let us solve the difference equation

$$(\Upsilon f)_k = \zeta f_k, \qquad k\in\mathbb{N}\setminus\{1\},$$

under the assumption that $f_1 = 0$ and $f_2 = b_1^{-1}$, and define

$$Q_{k-1}(\zeta) := f_k\,, \qquad k\in\mathbb{N}\,.$$

$Q_k(\zeta)$ is a polynomial of degree $k-1$ and it is called the k-th orthogonal polynomial of the second kind associated with the matrix (1.1).

The sequence $P(\zeta) := \{P_{k-1}(\zeta)\}_{k=1}^{\infty}$ is not in $l_{\text{fin}}(\mathbb{N})$, but it may happen that

$$\sum_{k=0}^{\infty} |P_k(\zeta)|^2 < \infty\,, \tag{2.3}$$

in which case $P(\zeta)\in\ker(J^* - \zeta I)$. Since J is symmetric, if the series in (2.3) is convergent for one ζ in the upper half-plane $\mathbb{C}_+$ (the lower half-plane $\mathbb{C}_-$), then it is convergent in all $\mathbb{C}_+$ ($\mathbb{C}_-$). Actually, because of the reality of the coefficients of $P_{k-1}(\zeta)$ for all $k\in\mathbb{N}$, the series in (2.3) is then convergent in all $\mathbb{C}\setminus\mathbb{R}$ and J has deficiency indices $(1,1)$. When the series in (2.3) is divergent for one ζ in $\mathbb{C}\setminus\mathbb{R}$, J has deficiency indices $(0,0)$ and the operator is self-adjoint since J is closed. There are known conditions on the matrix (1.1) which guarantee that J is self-adjoint [1, Addenda 1], [3, Chap. 7, Thms. 1.2–1.4].

We now introduce the operators that will be at the center of our considerations in this work.

Definition 1. *Let the operator $J^{(g)}$ be defined as follows:*

a) In case $J \neq J^*$, define the sequence $v(g) = \{v_k(g)\}_{k=1}^{\infty}$ such that $\forall k \in \mathbb{N}$

$$v_k(g) := P_{k-1}(0) + gQ_{k-1}(0)\,, \quad g \in \mathbb{R},$$

and

$$v_k(\infty) := Q_{k-1}(0)\,.$$

Let $J^{(g)}$ be the restriction of J^* to the set

$$\left\{ f = \{f_k\}_{k\in\mathbb{N}} \in \operatorname{dom}(J^*) : \lim_{k\to\infty} b_k(v_k(g) f_{k+1} - f_k v_{k+1}(g)) = 0 \right\}.$$

When $g \in \mathbb{R}\cup\{\infty\}$, $J^{(g)}$ runs over all self-adjoint extensions of J. Moreover, different values of g imply different self-adjoint extensions [27, *Lemma* 2.20].

b) In case $J = J^*$, define $J^{(g)} := J$ for all $g \in \mathbb{R}\cup\{\infty\}$.

Alongside the operator $J^{(g)}$, we consider the operators $J_n^{(g)}$ ($n \in \mathbb{N}$) in the Hilbert space $l_2(\mathbb{N}) \ominus \operatorname{span}\{\delta_1, \dots, \delta_n\}$ defined by restricting $J^{(g)}$ to $l_2(\mathbb{N}) \ominus \operatorname{span}\{\delta_1, \dots, \delta_n\}$. Thus, $J_n^{(g)}$ is a self-adjoint extension of the Jacobi operator whose associated matrix is (1.1) with the first n columns and n rows removed.

Finally we introduce the perturbed operators $J^{(g)}(\theta)$. They are defined as follows. Consider $J^{(g)}$ with fixed $g \in \mathbb{R}\cup\{\infty\}$ and take any $\theta > 0$. Then

$$J^{(g)}(\theta) := J^{(g)} + q_1(\theta^2 - 1)\langle \delta_1, \cdot\rangle \delta_1 + b_1(\theta - 1)(\langle \delta_1, \cdot\rangle \delta_2 + \langle \delta_2, \cdot\rangle \delta_1)\,, \tag{2.4}$$

where we take the inner product to be antilinear in its first argument. By this definition, $J^{(g)}(\theta)$ is a self-adjoint extension of the Jacobi operator whose associated matrix is (1.3). Note that $J^{(g)}(\theta)$ is a finite-rank perturbation of $J^{(g)}$ and thus $\operatorname{dom}(J^{(g)}) = \operatorname{dom}(J^{(g)}(\theta))$.

Fix $g \in \mathbb{R}\cup\{\infty\}$ and take the resolution of the identity $E^{(g)}(t)$ of $J^{(g)}$, so

$$J^{(g)} = \int_{\mathbb{R}} t dE^{(g)}(t)\,.$$

Since $J^{(g)}$ is simple [1, Sec. 2.2, Chap. 4], it is particularly useful to consider the function

$$\rho^{(g)}(t) := \left\langle \delta_1, E^{(g)}(t)\delta_1 \right\rangle, \quad t \in \mathbb{R}\,. \tag{2.5}$$

It turns out that all the moments of the measure generated by $\rho^{(g)}$ are finite [1, Thm. 4.1.3], that is,

$$s_k = \int_{\mathbb{R}} t^k d\rho^{(g)}(t) < \infty \qquad \forall k \in \mathbb{N} \cup \{0\}\,, \tag{2.6}$$

and the polynomials are dense in $L_2(\mathbb{R}, d\rho^{(g)})$ [1, Thms. 2.3.2, 4.1.4], [23, Prop. 4.15].

In this work we also make use of the so-called Weyl m-function

$$m^{(g)}(\zeta) := \left\langle \delta_1, (J^{(g)} - \zeta I)^{-1}\delta_1 \right\rangle\,, \quad \zeta \notin \sigma(J^{(g)})\,. \tag{2.7}$$

The functions (2.5) and (2.7) are related by the Borel transform, viz.,

$$m^{(g)}(\zeta) = \int_{\mathbb{R}} \frac{d\rho^{(g)}(t)}{t-\zeta}\,,$$

so $m^{(g)}$ is a Herglotz function, i. e.,

$$\frac{\operatorname{Im} m^{(g)}(\zeta)}{\operatorname{Im}\zeta} > 0\,, \qquad \operatorname{Im}\zeta > 0\,.$$

Using the von Neumann expansion for the resolvent (cf. [27, Chap. 6, Sec. 6.1])

$$(J^{(g)} - \zeta I)^{-1} = -\sum_{k=0}^{N-1} \frac{(J^{(g)})^k}{\zeta^{k+1}} + \frac{(J^{(g)})^N}{\zeta^N}(J^{(g)} - \zeta I)^{-1}\,,$$

where $\zeta \in \mathbb{C} \setminus \sigma(J^{(g)})$, one can easily obtain the asymptotic formula

$$m^{(g)}(\zeta) = -\frac{1}{\zeta} - \frac{q_1}{\zeta^2} - \frac{b_1^2 + q_1^2}{\zeta^3} + O(\zeta^{-4})\,, \tag{2.8}$$

as $\zeta \to \infty$ ($\operatorname{Im}\zeta \geq \epsilon$, $\epsilon > 0$).

The inverse Stieltjes transform allows to recover the spectral function (2.5) from its corresponding Weyl m-function (2.7). So they are in one-to-one correspondence. Furthermore, either (2.5) or (2.7) uniquely determines the Jacobi operator $J^{(g)}$, i. e., the matrix (1.1) and the parameter g in the non-self-adjoint case. Indeed, there are two general methods for recovering the matrix (1.1) that work without any assumption on the spectrum. One method, developed in [9] (see also [26]), makes use of the asymptotic behavior of the Weyl m-function and the Riccati equation [9, Eq. 2.15], [26, Eq. 2.23],

$$b_n^2 m_n^{(g)}(\zeta) = q_n - \zeta - \frac{1}{m_{n-1}^{(g)}(\zeta)}\,, \quad n \in \mathbb{N}\,, \tag{2.9}$$

where $m_n^{(g)}(\zeta)$ is the Weyl m-function of the Jacobi operator $J_n^{(g)}$ ($m_0 = m$).

The other method of reconstruction (see [3, Chap. 7, Sec. 1.5] and, particularly, [3, Chap. 7, Thm. 1.11]) has its starting point in the sequence $\{t^k\}_{k=0}^{\infty}$, $t \in \mathbb{R}$. From (2.6) all the elements of the sequence $\{t^k\}_{k=0}^{\infty}$ are in $L_2(\mathbb{R}, d\rho^{(g)})$ and one can apply, in this Hilbert space, the Gram-Schmidt procedure of orthonormalization to the sequence $\{t^k\}_{k=0}^{\infty}$. One, thus, obtains a sequence of polynomials $\{P_k(t)\}_{k=0}^{\infty}$ normalized and orthogonal in $L_2(\mathbb{R}, d\rho^{(g)})$. These polynomials satisfy a three-term recurrence equation [3, Chap. 7, Sec. 1.5], [23, Sec. 1]

$$tP_{k-1}(t) = b_{k-1}P_{k-2}(t) + q_k P_{k-1}(t) + b_k P_k(t)\,, \quad k \in \mathbb{N} \setminus \{1\}\,, \tag{2.10}$$

$$tP_0(t) = q_1 P_0(t) + b_1 P_1(t)\,, \tag{2.11}$$

where all the coefficients b_k ($k \in \mathbb{N}$) turn out to be positive and q_k ($k \in \mathbb{N}$) are real numbers. The system (2.10) and (2.11) defines a Jacobi matrix which is the matrix representation of either $J^{(g)}$ or a restriction of $J^{(g)}$ depending on whether $J = J^*$ or not.

The function (2.7), equivalently (2.5), determines the parameter g which defines the self-adjoint extension when the reconstructed matrix turns out to be the matrix representation of a non-self-adjoint operator. Indeed, consider a pole γ of $m^{(g)}$ (see Remark 1 bellow) and evaluate $P_k(\gamma)$, $k \in \mathbb{N}$. Then either

$$\lim_{k\to\infty} b_k(Q_{k-1}(0)P_k(\gamma) - P_{k-1}(\gamma)Q_{k-1}(0)) = 0,$$

which means that $g = \infty$, or

$$g = \frac{\lim_{k\to\infty} b_k(P_{k-1}(0)P_k(\gamma) - P_{k-1}(\gamma)P_{k-1}(0))}{\lim_{k\to\infty} b_k(Q_{k-1}(0)P_k(\gamma) - P_{k-1}(\gamma)Q_{k-1}(0))}\,.$$

The details of this technique are explained, for instance, in [22, Sec. 2].

Since any simple self-adjoint operator in an infinite dimensional Hilbert space is unitarily equivalent to some operator $J = J^*$ [1, Thm. 4.2.3], [2, Sec. 69], in the case $J = J^*$, $\sigma(J^{(g)})$ may be any non-empty closed infinite set in $\mathbb{R}$. In particular, $J^{(g)}$ may have discrete spectrum, that is, $\sigma_{ess}(J^{(g)}) = \emptyset$. When $J \neq J^*$, this is always the case, that is, all self-adjoint extensions $J^{(g)}$ of the non-self-adjoint operator J have a discrete spectrum [27, Lem. 2.19].

Assume that J has discrete spectrum (this always happens if $J \neq J^*$), so the spectrum is a sequence of real numbers, $\{\lambda_k\}_k$, without finite points of accumulation. The simplicity of $J^{(g)}$ implies that all eigenvalues are of multiplicity one. In this case the function $\rho^{(g)}(t)$, defined by (2.5), can be written as follows:

$$\rho^{(g)}(t) = \sum_{\lambda_k < t} \frac{1}{\alpha_k}\,, \tag{2.12}$$

where the coefficients $\{\alpha_k\}_k$ are called the normalizing constants and according to [3, Chap. 7, Thm. 1.17] are given by

$$\alpha_n = \sum_{k=0}^{\infty} |P_k(\lambda_n)|^2 \, . \tag{2.13}$$

Thus, from (2.12) and (2.7) one has that

$$m^{(g)}(\zeta) = \sum_k \frac{1}{\alpha_k(\lambda_k - \zeta)} \, . \tag{2.14}$$

R e m a r k 1. In the case of discrete spectrum, the set of poles of the meromorphic Weyl m-function coincides with $\sigma(J^{(g)})$. By (2.9), the set of zeros coincides with $\sigma(J_1^{(g)})$. The zeros and poles of the Weyl m-function are simple and interlace as occurred to any nonconstant meromorphic Herglotz function. Interlacing means that between two contiguous poles there is exactly one zero and between two contiguous zeros there is exactly one pole (see the proof of [16, Chap. 7, Thm. 1]).

R e m a r k 2. By elementary perturbation theory (Weyl theorem), $J^{(g)}$ has discrete spectrum if and only if $J^{(g)}(\theta)$ has a discrete spectrum. Note that $J^{(g)}(\theta)$ has simple spectrum since it is a self-adjoint extensions of a Jacobi operator.

R e m a r k 3. Let us comment briefly on the criteria for discreteness of $\sigma(J^{(g)})$ on the basis of the matrix entries in (1.1) when $J = J^*$. Consider a matrix whose main diagonal is a sequence $\{q_k\}_{k=1}^{\infty}$ of pairwise distinct real numbers without finite accumulation points and the sequence defining the off-diagonals $\{b_k\}_{k=1}^{\infty}$ is such that $b_k = o(q_k)$ as $k \to \infty$. Then it can be shown that J is the sum of the operator D whose matrix representation is $\operatorname{diag}\{q_k\}_{k=1}^{\infty}$ and a perturbation relatively compact with respect to D. By perturbation theory, J is thus self-adjoint and has discrete spectrum. Of course, there are other examples of self-adjoint Jacobi operators having discrete spectrum and whose matrix representation diagonals do not satisfy the conditions just given (see, for instance, [19, 21])

R e m a r k 4. There are conditions on the entries of (1.1) which guarantee that $J \neq J^*$ (see, for instance, [1, Addenda 1] and [3, Thm. 7.1.5]). Thus, for (1.1) satisfying those conditions, $J^{(g)}$ has discrete spectrum [27, Lem. 2.19].

R e m a r k 5. Consider the mass-spring system of the Introduction. On the basis of Remarks 3, 4, and by means of the recurrence equations given below in Remark 11, one can construct a mass-spring system whose corresponding operator $J^{(g)}$ has a discrete spectrum.

3. Direct Spectral Analysis of $J^{(g)}$ and $J^{(g)}(\theta)$

We begin this section by noting that

$$J_1^{(g)} = J_1^{(g)}(\theta)\,, \qquad \forall\theta > 0\,.$$

Fix $g \in \mathbb{R} \cup \{\infty\}$ and consider the Weyl m-functions $m^{(g)}$, $m^{(g,\theta)}$ of the operators $J^{(g)}$ and $J^{(g)}(\theta)$. Therefore, taking into account that $m_1^{(g)}$ and $m_1^{(g,\theta)}$ coincide, (2.9) implies that

$$\theta^2\left(\zeta + \frac{1}{m^{(g)}(\zeta)}\right) = \zeta + \frac{1}{m^{(g,\theta)}(\zeta)}\,. \tag{3.1}$$

Let us now consider the function

$$\mathfrak{m}(\zeta) := \frac{m^{(g)}(\zeta)}{m^{(g,\theta)}(\zeta)}\,. \tag{3.2}$$

R e m a r k 6. In view of Remark 2, if $J^{(g)}$ has discrete spectrum, then the function $\mathfrak{m}$ is meromorphic by (3.2). Since the zeros of $m^{(g)}$ and $m^{(g,\theta)}$ are the same (see Remark 1), it follows that for all $\theta > 0$ the set of poles of $\mathfrak{m}$ is a subset of $\sigma(J^{(g)})$, while $\sigma(J^{(g)}(\theta))$ contains all the zeros of $\mathfrak{m}$. Observe also that, from (3.1), $0 \in \sigma(J^{(g)})$ if and only if $0 \in \sigma(J^{(g)}(\theta))$. Moreover, whenever $\theta \neq 1$, (3.1) implies that the sets $\sigma(J^{(g)})$ and $\sigma(J^{(g)}(\theta))$ can intersect only at 0.

R e m a r k 7. By [15, Chap. 7, Thm. 3.9], the zeros of $\mathfrak{m}$ are analytic functions of the parameter θ. The same is true for the eigenvectors of $J^{(g)}(\theta)$.

Proposition 3.1. *Let $J^{(g)}$ have a discrete spectrum and let $\{\lambda_k(\theta)\}_k$ be the set of eigenvalues of $J^{(g)}(\theta)$ ($\theta > 0$). For a fixed k the following holds:*

$$\frac{d}{d\theta}\lambda_k(\theta) = \frac{2\lambda_k(\theta)}{\theta\alpha_k(\theta)}\,,$$

where $\alpha_k(\theta)$ is the normalizing constant corresponding to $\lambda_k(\theta)$.

P r o o f. Let us denote by $f(\theta)$ the eigenvector of $J^{(g)}(\theta)$ corresponding to $\lambda_k(\theta)$. We assume that $f(\theta)$ is normalized in such a way that

$$\langle\delta_1, f(\theta)\rangle = 1\,. \tag{3.3}$$

Pick any small real τ (it suffices that $|\tau| < \theta$). Then, taking into account that $\operatorname{dom}(J^{(g)}) = \operatorname{dom}(J^{(g)}(\theta))$ and the self-adjointness of $J^{(g)}(\theta)$ for any $\theta > 0$, we have that

$$\begin{aligned}(\lambda_k(\theta+\tau)-\lambda_k(\theta))\left\langle f(\theta), f(\theta+\tau)\right\rangle &= \left\langle f(\theta), J^{(g)}(\theta+\tau) f(\theta+\tau)\right\rangle \\ &- \left\langle J^{(g)}(\theta) f(\theta), f(\theta+\tau)\right\rangle \\ &= \left\langle f(\theta), (J^{(g)}(\theta+\tau)-J^{(g)}(\theta)+J^{(g)}(\theta)) f(\theta+\tau)\right\rangle \\ &- \left\langle J^{(g)}(\theta) f(\theta), f(\theta+\tau)\right\rangle \\ &= \left\langle f(\theta), (J^{(g)}(\theta+\tau)-J^{(g)}(\theta)) f(\theta+\tau)\right\rangle .\end{aligned}$$

From (3.3) it follows that the entries $f(\theta+\tau)$ and $f(\theta)$ are the polynomials of the first kind associated to the matrix of $J^{(g)}(\theta+\tau)$ and $J^{(g)}(\theta)$, so

$$f_2(\theta+\tau)=\frac{\lambda_k(\theta+\tau)-(\theta+\tau)^2 q_1}{(\theta+\tau) b_1}, \qquad f_2(\theta)=\frac{\lambda_k(\theta)-\theta^2 q_1}{\theta b_1}.$$

Now, taking into account these last equalities and (3.3), together with

$$J^{(g)}(\theta+\tau)-J^{(g)}(\theta)=\begin{pmatrix}(2\theta\tau+\tau^2) q_1 & \tau b_1 & 0 & 0 & \cdots \\ \tau b_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & & \ddots & \ddots\end{pmatrix},$$

one obtains that

$$(\lambda_k(\theta+\tau)-\lambda_k(\theta))\left\langle f(\theta), f(\theta+\tau)\right\rangle=\tau\left(\frac{\lambda_k(\theta+\tau)}{\theta+\tau}+\frac{\lambda_k(\theta)}{\theta}\right).$$

Therefore, on the basis of Remark 7, one has

$$\lim_{\tau\to 0}\frac{\lambda_k(\theta+\tau)-\lambda_k(\theta)}{\tau}=\lim_{\tau\to 0}\frac{1}{\left\langle f(\theta), f(\theta+\tau)\right\rangle}\left(\frac{\lambda_k(\theta+\tau)}{\theta+\tau}+\frac{\lambda_k(\theta)}{\theta}\right)=\frac{2\lambda_k(\theta)}{\theta\alpha_k(\theta)}.$$

■

The proposition below can be proven by means of Remark 6, 7, and Proposition 3.1. However, we present an alternative proof based on the expression

$$\mathfrak{m}(\zeta)=\zeta(\theta^2-1) m^{(g)}(\zeta)+\theta^2, \tag{3.4}$$

which follows from (3.1) and (3.2).

Proposition 3.2. *Fix $g \in \mathbb{R} \cup \{\infty\}$ and let $J^{(g)}$ have a discrete spectrum. The spectra $\sigma(J^{(g)})$, $\sigma(J^{(g)}(\theta))$ interlace in $\mathbb{R}_+$ and $\mathbb{R}_-$. Moreover, $\sigma(J^{(g)}(\theta))$ in $\mathbb{R}_+$ ($\mathbb{R}_-$) is shifted with respect to $\sigma(J^{(g)})$ to the left (right) if $\theta < 1$, and to the right (left) if $\theta > 1$.*

P r o o f. In view of Remark 6, one only needs to verify that between two positive and contiguous eigenvalues of $J^{(g)}$ there is only one eigenvalue of $J^{(g)}(\theta)$ and vice versa. Take two positive and contiguous eigenvalues of $\sigma(J^{(g)})$, $\lambda < \widetilde{\lambda}$. Due to (2.14), one has

$$\lim_{\substack{t \to \widetilde{\lambda}^- \\ t \in \mathbb{R}}} m^{(g)}(t) = +\infty\,, \qquad \lim_{\substack{t \to \lambda^+ \\ t \in \mathbb{R}}} m^{(g)}(t) = -\infty\,. \tag{3.5}$$

Now, in (3.4) assume that $\theta > 1$. Thus, because of the positivity of $\lambda, \widetilde{\lambda}$, (3.4) and (3.5) imply that

$$\lim_{\substack{t \to \widetilde{\lambda}^- \\ t \in \mathbb{R}}} \mathfrak{m}(t) = +\infty\,, \qquad \lim_{\substack{t \to \lambda^+ \\ t \in \mathbb{R}}} \mathfrak{m}(t) = -\infty\,.$$

Since $\mathfrak{m}$ is analytic on the interval $(\lambda, \widetilde{\lambda})$, it should cross the 0-axis an odd number of times. If it crosses this axis three or more times as in Fig. 3 (a), then, by Remarks 1 and 6, there are at least two elements of $\sigma(J_1^{(g)})$ in $(\lambda, \widetilde{\lambda})$. But, because of Remark 1, this would contradict the fact that $\lambda, \widetilde{\lambda}$ are contiguous.

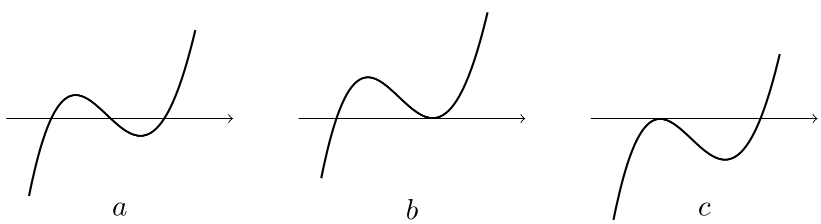

Fig. 3. Impossible crossings of the 0-axis by $\mathfrak{m}$

Observe that one should discard the possibility of one crossing of the 0-axis and a tangential touch of it as in Fig. 3 (b) and (c). But again the impossibility of this follows from the fact that the poles of $m^{(g,\theta)}$ are simple (see Remark 1). Analogously, between two contiguous eigenvalues of $J^{(g)}(\theta)$, the function $\frac{1}{\mathfrak{m}} \restriction_{\mathbb{R}}$ crosses the 0-axis exactly once. Thus, the interlacing in $\mathbb{R}_+$ has been established. By the same token, the spectra interlace in $\mathbb{R}_-$. The case $\theta < 1$ is treated in a similar way. The second assertion follows directly from Proposition 3.1. ■

R e m a r k 8. We note that $\sigma(J^{(g)}) \cap \mathbb{R}_+$, $\sigma(J^{(g)}) \cap \mathbb{R}_-$, may be finite or empty.

4. Inverse Spectral Analysis for $J^{(g)}$ and $J^{(g)}(\theta)$

In this section we find some necessary conditions for the spectra of $J^{(g)}(\theta)$ ($\theta > 0$). Also we provide a reconstruction algorithm of the Jacobi matrix and establish uniqueness of the reconstruction. Some of the formulae obtained in this section (see, for instance, Corollary 4.1) have their analogous in the finite case [7], [20].

A central part of our approach is the Weyl m-function and its properties. We begin our discussion by setting out a convention for enumerating the elements of the spectra.

Convention. For a given countable set of real numbers S without finite points of accumulation, let M be an infinite subset of consecutive integers such that there is a one-to-one increasing function $h : M \to S$ with the property that $h^{-1}(0) = \{0\}$ when 0 is in S. Thus, M is semi-bounded from above (below) if and only if the same holds for S. We write $S = \{\lambda_k\}_{k\in M}$, where $\lambda_k = h(k)$. Note that in the sequence $\{\lambda_k\}_{k\in M}$ only λ_0 is allowed to be zero. Thus, if $-1, 1 \in M$, then

$$\lambda_{-1} < 0 < \lambda_1 \,.$$

In the sequel, the spectra of all operators will be enumerated according to this convention.

When $\{\lambda_k\}_{k\in M}$ is considered together with a sequence interlacing with it, we use the same set M for enumerating both sequences. For instance, if $\{\lambda_k\}_{k\in M}$ and $\{\mu_k\}_{k\in M}$ are interlacing and not semi-bounded, then one can assume that

$$\lambda_k < \mu_k < \lambda_{k+1}\,, \quad \forall k \in M.$$

The following auxiliary result can be found in [22, Sec. 4]. We sketch the proof here for the reader's convenience.

Lemma 4.1. *Let $J^{(g)}$ have a discrete spectrum and assume that $\sigma(J^{(g)}) = \{\lambda_k\}_{k\in M}$, and $\sigma(J_1^{(g)}) = \{\eta_k\}_{k\in M}$. Then, the following formula holds for the Weyl m-function of $J^{(g)}$:*

$$m^{(g)}(\zeta) = C\frac{\zeta-\eta_0}{\zeta-\lambda_0}\prod_{\substack{k\in M\\ k\neq 0}}\left(1-\frac{\zeta}{\eta_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1}. \tag{4.1}$$

Moreover, $C < 0$ and

$$\eta_k < \lambda_k < \eta_{k+1} \quad \forall k \in M \tag{4.2}$$

if $\sigma(J^{(g)})$ is semi-bounded from above, while, $C > 0$ and

$$\lambda_k < \eta_k < \lambda_{k+1} \quad \forall k \in M \tag{4.3}$$

otherwise.

P r o o f. Assume first that $\sigma(J^{(g)})$ is semi-bounded from below. Since the greatest lower bound of J does not exceed the greatest lower bound of $J_1^{(g)}$, the smallest element of $\{\lambda_k\}_{k\in M}$ is less than the smallest one of $\{\eta_k\}_{k\in M}$ (see [4, Chap. 6, Sec. 1.3]). Thus one can enumerate the sequences $\{\lambda_k\}_{k\in M}$ and $\{\eta_k\}_{k\in M}$ so that they obey our convention and (4.3). According to [16, Chap. 7, Thm. 1], (4.1) holds with $C > 0$.

Clearly, when $\sigma(J^{(g)})$ is not semi-bounded, the sequences can be arranged to obey (4.3), and then (4.1) holds with $C > 0$.

Now suppose that $\sigma(J^{(g)})$ is semi-bounded from above. Then $\sigma(-J^{(g)})$ is semi-bounded from below and, consequently, the greatest element of $\{\eta_k\}_{k\in M}$ is less than the greatest one of $\{\lambda_k\}_{k\in M}$. Thus $\{\lambda_k\}_{k\in M}$, and $\{\eta_k\}_{k\in M}$ cannot be arranged according to (4.3). However, we are still able to use (4.3) for arranging the zeros and poles of the meromorphic Herglotz function $-\frac{1}{m^{(g)}}$, that is, we use (4.2). Therefore [16, Chap. 7, Thm. 1] gives

$$-\frac{1}{m^{(g)}(\zeta)} = \widetilde{C}\frac{\zeta-\lambda_0}{\zeta-\eta_0}\prod_{\substack{k\in M\\ k\neq 0}}\left(1-\frac{\zeta}{\lambda_k}\right)\left(1-\frac{\zeta}{\eta_k}\right)^{-1}, \qquad \widetilde{C} > 0\,.$$

For completing the proof it only remains to note that the last equation can be rewritten as asserted in the lemma. The infinite product in (4.1) is convergent because of (4.2) (see the proof of [16, Chap. 7, Thm. 1]). ■

Another auxiliary simple result to be used later is the following lemma.

Lemma 4.2. *Let $J^{(g)}$ have a discrete spectrum and $\{\lambda_k(\theta)\}_k$ be the set of eigenvalues of $J^{(g)}(\theta)$. Then, the series*

$$\sum_{k\in M}\frac{\lambda_k(\theta)}{\alpha_k(\theta)} \tag{4.4}$$

converges uniformly in $[\theta_1,\theta_2]\subset\mathbb{R}_+$ to $s_1(\theta)$ (see (2.6)).

P r o o f. From (2.6) and (2.12), it follows that the series converges pointwise to $s_1(\theta)$. The series

$$\sum_{k\in M}\frac{\lambda_k^2(\theta)}{\alpha_k(\theta)} \tag{4.5}$$

converges also pointwise to the function $s_2(\theta)$. Since this function is continuous in $[\theta_1,\theta_2]$, then (4.5) is uniformly convergent in that interval (see [28, Sec. 1.31]). Now, for any $\theta\in[\theta_1,\theta_2]$ and $|\lambda_k| > 1$, one has

$$|\lambda_k| < \lambda_k^2\,,$$

so (4.4) is uniformly convergent in $[\theta_1,\theta_2]$. ■

R e m a r k 9. Proposition 3.2 tells that the interlacing of the sequences $\sigma(J^{(g)}) = \{\lambda_k\}_k$ and $\sigma(J(\theta)) = \{\mu_k\}_k$ is different in $\mathbb{R}_+$ and $\mathbb{R}_-$. So let us agree to enumerate the sequences according to our convention (the subscripts of the sequences run over M and only the eigenvalues with subscript equal zero are allowed to be zero) and obeying

$$\lambda_k < \mu_k < \lambda_{k+1} \quad \text{in } \mathbb{R}_+ , \qquad \mu_k < \lambda_k < \mu_{k+1} \quad \text{in } \mathbb{R}_- ,$$

when $\theta > 1$, and

$$\mu_k < \lambda_k < \mu_{k+1} \quad \text{in } \mathbb{R}_+ , \qquad \lambda_k < \mu_k < \lambda_{k+1} \quad \text{in } \mathbb{R}_- ,$$

if $\theta < 1$.

Proposition 4.1. *Fix $g \in \mathbb{R} \cup \{\infty\}$ and $0 < \theta_1 < \theta_2$. Let $J^{(g)}$ have a discrete spectrum and assume that $\sigma(J^{(g)}(\theta_1)) = \{\lambda_k\}_{k\in M}$ and $\sigma(J^{(g)}(\theta_2)) = \{\mu_k\}_{k\in M}$, where the sequences have been arranged according to Remark 9. Then,*

$$\sum_{k\in M} (\mu_k - \lambda_k) = q_1(\theta_2^2 - \theta_1^2) .$$

P r o o f. Observe that from Proposition 3.1 it follows that

$$\mu_k - \lambda_k = 2 \int_{\theta_1}^{\theta_2} \frac{\lambda_k(\theta) d\theta}{\theta \alpha_k(\theta)} .$$

Consider a sequence $\{M_n\}_{n=1}^{\infty}$ of the subsets of M such that $M_n \subset M_{n+1}$ and $\cup_n M_n = M$. Thus

$$\sum_{k\in M} (\mu_k - \lambda_k) = 2 \lim_{n\to\infty} \int_{\theta_1}^{\theta_2} \left(\sum_{k\in M_n} \frac{\lambda_k(\theta)}{\alpha_k(\theta)} \right) \frac{d\theta}{\theta}.$$

By Lemma 4.2 and the fact that

$$s_1(\theta) = \left\langle \delta_1, J^{(g)}(\theta)\delta_1 \right\rangle = q_1\theta^2 ,$$

one obtains

$$\sum_{k\in M} (\mu_k - \lambda_k) = 2q_1 \int_{\theta_1}^{\theta_2} \theta d\theta = q_1(\theta_2^2 - \theta_1^2).$$

■

Proposition 4.2. *Fix $g \in \mathbb{R} \cup \{\infty\}$ and $0 < \theta \neq 1$. Let $J^{(g)}$ have a discrete spectrum and assume that $\sigma(J^{(g)}) = \{\lambda_k\}_{k\in M}$ and $\sigma(J^{(g)}(\theta)) = \{\mu_k\}_{k\in M}$, where the sequences have been arranged according to Remark 9. Then,*

$$\mathfrak{m}(\zeta) = \prod_{k\in M} \frac{\zeta - \mu_k}{\zeta - \lambda_k}.$$

P r o o f. Consider a sequence $\{M_n\}_{n=1}^{\infty}$ of subsets of M such that $M_n \subset M_{n+1}$ and $\cup_n M_n = M$. From (4.1) and (3.2) it follows that

$$\begin{aligned}\mathfrak{m}(\zeta) &= C\frac{\zeta-\mu_0}{\zeta-\lambda_0}\lim_{n\to\infty}\frac{\prod\limits_{\substack{k\in M_n\\k\neq 0}}\left(1-\frac{\zeta}{\eta_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1}}{\prod\limits_{\substack{k\in M_n\\k\neq 0}}\left(1-\frac{\zeta}{\eta_k}\right)\left(1-\frac{\zeta}{\mu_k}\right)^{-1}}\\ &= C\frac{\zeta-\mu_0}{\zeta-\lambda_0}\prod_{\substack{k\in M\\k\neq 0}}\left(1-\frac{\zeta}{\mu_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1}. \qquad (4.6)\end{aligned}$$

On the other hand, by Proposition 4.1, it holds true that

$$\prod_{\substack{k\in M\\k\neq 0}}\left(1-\frac{\zeta}{\mu_k}\right)\left(1-\frac{\zeta}{\lambda_k}\right)^{-1} = \prod_{\substack{k\in M\\k\neq 0}}\frac{\lambda_k}{\mu_k}\prod_{\substack{k\in M\\k\neq 0}}\frac{\zeta-\mu_k}{\zeta-\lambda_k}. \qquad (4.7)$$

From (2.8) and (3.4) it follows that

$$\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geq\epsilon>0}} \mathfrak{m}(\zeta) = 1. \qquad (4.8)$$

Also, on the basis that the second product on the r. h. s of (4.7) converges uniformly, one has

$$\lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geq\epsilon}}\prod_{k\in M}\frac{\zeta-\mu_k}{\zeta-\lambda_k} = \lim_{\substack{\zeta\to\infty\\ \operatorname{Im}\zeta\geq\epsilon}}\prod_{k\in M}\left(1+\frac{\mu_k-\lambda_k}{\lambda_k-\zeta}\right) = 1. \qquad (4.9)$$

Thus, (4.6), (4.7), (4.8), and (4.9) imply that

$$C = \prod_{\substack{k\in M\\k\neq 0}}\frac{\mu_k}{\lambda_k}$$

and the proposition is proven. ∎

Corollary 4.1. *Fix $g \in \mathbb{R} \cup \{\infty\}$ and $\theta > 0$. Let $J^{(g)}$ have discrete spectrum and assume that $\sigma(J^{(g)}) = \{\lambda_k\}_k$ and $\sigma(J^{(g)}(\theta)) = \{\mu_k\}_k$, where the sequences have been arranged according to Remark 9. Then,*

$$\theta^2 = \prod_{k \in M} \frac{\eta - \mu_k}{\eta - \lambda_k},$$

where η is any element of $\sigma(J_1^{(g)})$. Moreover, when $0 \notin \sigma(J^{(g)})$,

$$\theta^2 = \prod_{k \in M} \frac{\mu_k}{\lambda_k} \tag{4.10}$$

and, if $0 \in \sigma(J^{(g)})$,

$$\theta^2 = \frac{1}{\alpha_0 - 1} \left\{ \alpha_0 \prod_{\substack{k \in M \\ k \neq 0}} \frac{\mu_k}{\lambda_k} - 1 \right\}, \tag{4.11}$$

where α_0 is given in (2.13).

P r o o f. The first two identities for θ^2 are a straightforward consequence of Proposition 4.2 and (3.4). As regards to (4.11), note that from (2.14) one has

$$\alpha_k^{-1} = - \operatorname*{Res}_{\zeta = \lambda_k} m(\zeta). \tag{4.12}$$

Thus, according to (3.4),

$$\theta^2 - \alpha_0^{-1}(\theta^2 - 1) = \mathfrak{m}(0) = \prod_{\substack{k \in M \\ k \neq 0}} \frac{\mu_k}{\lambda_k}. \tag{4.13}$$

■

R e m a r k 10. *Due to (4.13) and the properties of the normalizing constants, when $0 \in \sigma(J^{(g)})$, one of the following inequalities holds depending on the value of $\theta \neq 1$:*

$$\theta^2 < \mathfrak{m}(0) = \prod_{\substack{k \in M \\ k \neq 0}} \frac{\mu_k}{\lambda_k} < 1, \qquad 1 < \mathfrak{m}(0) = \prod_{\substack{k \in M \\ k \neq 0}} \frac{\mu_k}{\lambda_k} < \theta^2.$$

Theorem 4.1. *Fix $g \in \mathbb{R} \cup \{\infty\}$ and $\theta > 0$. Let $J^{(g)}$ have discrete spectrum and assume that $0 \notin \sigma(J^{(g)})$. The spectra $\sigma(J^{(g)})$, $\sigma(J^{(g)}(\theta))$ ($\theta \neq 1$) uniquely determine the Jacobi matrix (1.1), that is the operator J, the parameter θ defining the perturbation, and the parameter g specifying the self-adjoint extension when $J \neq J^*$.*

P r o o f. Given the sequences $\sigma(J^{(g)})$ and $\sigma(J^{(g)}(\theta))$, one finds the parameter θ from (4.10). Proposition 4.2 yields the function $\mathfrak{m}$ and equation (3.4), the Weyl function $m^{(g)}$. According to the Preliminaries, this function allows to recover the matrix associated to the Jacobi operator and the parameter g which determines the self-adjoint extension when $J \neq J^*$. ■

Theorem 4.2. *Fix $g \in \mathbb{R} \cup \{\infty\}$ and $\theta > 0$. Let $J^{(g)}$ have a discrete spectrum and assume that $0 \in \sigma(J^{(g)})$. The spectra $\sigma(J^{(g)})$, $\sigma(J^{(g)}(\theta))$ ($\theta \neq 1$), together with either q_1 or α_0, uniquely determine the matrix associated to J, the parameter θ, and the parameter g when $J \neq J^*$. Alternatively, the spectra $\sigma(J^{(g)})$, $\sigma(J^{(g)}(\theta))$ and the parameter $\theta \neq 1$ uniquely determine the matrix corresponding to J and the parameter g when J turns out to be nonself-adjoint.*

P r o o f. This follows immediately from the proof of the previous theorem, taking into account (4.11). Note that θ can be determined either by Proposition 4.1 or by the asymptotic formula

$$\mathfrak{m}(\zeta) = 1 + \frac{q_1(1-\theta^2)}{\zeta} + O(\zeta^{-2}),$$

as $\zeta \to \infty$ ($\mathrm{Im}\zeta \geq \epsilon$, $\epsilon > 0$), obtained by combining (2.8) and (3.4). ■

R e m a r k 11. Theorems 4.1 and 4.2 solve the problem of reconstructing the matrix from spectral data. However, in order to solve the inverse problem for the mass-spring system, one should also recover the masses and spring constants from the matrix entries. This is actually not difficult as it is shown below (cf. [17, Chap. 8]).

On the basis of (1.2), one finds the equations

$$k_{j+1} = -(k_j + q_j m_j),$$
$$m_{j+1} = \frac{k_{j+1}^2}{m_j b_j^2},$$

which allow to find recursively all spring constants and masses of the system from the first spring constant and mass. Note that, when the parameters k_1 and m_1 are given, only the quotient $\frac{k_1}{m_1}$ does not depend on the choice of mass unit. This quotient has a concrete physical meaning: it equals the squared natural frequency of the mass m_1 attached with the spring k_1 to a fixed support. Thus, it is physically convenient to find a way of expressing k_j/m_j in terms of k_1/m_1. This is achieved by means of the following continued fraction:

$$\frac{k_{j+1}}{m_{j+1}} = \cfrac{-b_j^2}{q_j - \cfrac{b_{j-1}^2}{\cdots q_2 - \cfrac{b_1^2}{q_1 + \frac{k_1}{m_1}}}},$$

which is constructed from $\frac{k_1}{m_1}$ upwards (cf. [17 p. 76]). We remark that, unlike the finite matrix case, here one cannot apply without substantial changes, the method developed in [17, Chap. 8] for determining the set of admissible values for the quotient $\frac{k_1}{m_1}$. Admissible values of $\frac{k_1}{m_1}$ are those for which $\frac{k_{j+1}}{m_{j+1}}$ is a positive real number for any $j \in \mathbb{N}$.

5. Necessary and Sufficient Conditions for the Spectra of $J^{(g)}$ and $J^{(g)}(\theta)$

The following statement gives an if-and-only-if criterion for two sequences to be the spectra of $J^{(g)}$ and $J^{(g)}(\theta)$. In the finite case the interlacing condition given in a) (see below) is necessary and sufficient [7],[20].

Theorem 5.1. *Given two infinite real sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ without finite points of accumulation, such that none of them contains the zero, there is a unique positive θ, a unique operator J, and a unique $g \in \mathbb{R} \cup \{\infty\}$ if $J \neq J^*$, such that $\{\mu_k\}_k$ is the spectrum of $J^{(g)}(\theta)$ and $\{\lambda_k\}_k$ is the spectrum of $J^{(g)}$ if and only if the following conditions are satisfied.*

a) $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ interlace in $\mathbb{R}_+$, $\mathbb{R}_-$ with one sequence shifted to the right (left) in $\mathbb{R}_+$, ($\mathbb{R}_-$) with respect to the other one. Thus, the sequences can be ordered according to Remark 9.

b) The following series converges

$$\sum_{k \in M} (\mu_k - \lambda_k).$$

By condition b) the products $\prod_{\substack{k \in M \\ k \neq n}} \frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}$, $\prod_{k \in M} \frac{\mu_k}{\lambda_k}$ are convergent, so define

$$\tau_n := \frac{(\mu_n - \lambda_n) \prod_{\substack{k \in M \\ k \neq n}} \frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}}{\lambda_n \left(\prod_{k \in M} \frac{\mu_k}{\lambda_k} - 1 \right)}, \quad \forall n \in M . \tag{5.1}$$

c) The sequence $\{\tau_n\}_{n \in M}$ is such that, for $m = 0, 1, 2, \ldots$, the series

$$\sum_{k \in M} \lambda_k^{2m} \tau_k \quad \textit{converges.}$$

d) If a sequence of complex numbers $\{\beta_k\}_{k\in M}$ is such that the series

$$\sum_{k\in M} |\beta_k|^2 \tau_k \quad \textit{converges}$$

and, for $m = 0, 1, 2, \dots,$

$$\sum_{k\in M} \beta_k \lambda_k^m \tau_k = 0\,,$$

then $\beta_k = 0$ for all $k \in M$.

P r o o f. In view of Propositions 3.2 and 4.1, for proving the necessity of the conditions, it only remains to show that for all $n \in M$, $\tau_n = \alpha_n^{-1}$. Indeed c) and d) will follow from the fact that all moments of the spectral measure (2.12) exist and that the polynomials are dense in $L_2(\mathbb{R}, \rho^{(g)})$.

From (3.4), (4.12), and Proposition 4.2 , it follows that

$$\begin{aligned}\alpha_n^{-1} &= \frac{1}{\theta^2 - 1} \lim_{\zeta\to\lambda_n} \frac{\lambda_n - \zeta}{\zeta}\mathfrak{m}(\zeta)\\ &= \frac{\mu_n - \lambda_n}{\lambda_n(\theta^2 - 1)} \prod_{\substack{k\in M\\ k\neq n}} \frac{\lambda_n - \mu_k}{\lambda_n - \lambda_k}\,.\end{aligned}$$

Hence, by Corollary 4.1, one verifies that $\tau_n = \alpha_n^{-1}$.

We now prove that conditions a), b), c) and d) are sufficient.

The condition a) implies that

$$\frac{\lambda_n - \mu_k}{\lambda_n - \lambda_k} > 0, \qquad \forall k \in M\,,\ k \neq n.$$

On the other hand, by b) one can define the number

$$\kappa = \prod_{k\in M} \frac{\mu_k}{\lambda_k}, \tag{5.2}$$

which is clearly positive and also $\kappa > 1$ if $|\mu_k| > |\lambda_k|$ for all $k \in M$ and $\kappa < 1$ if $|\mu_k| < |\lambda_k|$ for all $k \in M$. Thus,

$$\frac{\mu_n - \lambda_n}{\lambda_n(\kappa - 1)} > 0 \qquad \forall n \in M.$$

Hence, for all $n \in M$, $\tau_n > 0$, so define the function

$$\rho(t) := \sum_{\lambda_k < t} \tau_k\,. \tag{5.3}$$

It follows from c) that the moments of the measure corresponding to ρ are finite.

Now, on the basis of a) and b), define the meromorphic functions

$$\widetilde{\mathfrak{m}}(\zeta) := \prod_{k\in M} \frac{\zeta - \mu_k}{\zeta - \lambda_k}$$

and

$$\widetilde{m}(\zeta) := \frac{\widetilde{\mathfrak{m}}(\zeta) - \prod_{k\in M} \frac{\mu_k}{\lambda_k}}{\zeta \left(\prod_{k\in M} \frac{\mu_k}{\lambda_k} - 1\right)} . \tag{5.4}$$

Thus, taking into account (5.1), one has

$$\operatorname*{Res}_{\zeta=\lambda_n} \widetilde{m}(\zeta) = \left(\prod_{k\in M} \frac{\mu_k}{\lambda_k} - 1\right)^{-1} \lim_{\zeta\to\lambda_n} \frac{\zeta - \lambda_n}{\zeta} \widetilde{\mathfrak{m}}(\zeta) = -\tau_n . \tag{5.5}$$

In view of what was done earlier,

$$\lim_{\substack{\zeta\to\infty \\ \mathrm{Im}\zeta\ge\epsilon>0}} \widetilde{\mathfrak{m}}(\zeta) = 1 . \tag{5.6}$$

Therefore,

$$\lim_{\substack{\zeta\to\infty \\ \mathrm{Im}\zeta\ge\epsilon>0}} \widetilde{m}(\zeta) = \left(\prod_{k\in M} \frac{\mu_k}{\lambda_k} - 1\right)^{-1} \lim_{\substack{\zeta\to\infty \\ \mathrm{Im}\zeta\ge\epsilon>0}} \frac{\widetilde{\mathfrak{m}}(\zeta)}{\zeta} = 0. \tag{5.7}$$

By (5.5) and (5.7), [16, Chap. 7, Thm. 2] implies that

$$\widetilde{m}(\zeta) = \sum_{k\in M} \frac{\tau_k}{\lambda_k - \zeta} . \tag{5.8}$$

On the other hand, using (5.6), one obtains

$$\lim_{\substack{\zeta\to\infty \\ \mathrm{Im}\zeta\ge\epsilon>0}} \zeta\widetilde{m}(\zeta) = \left(\prod_{k\in M} \frac{\mu_k}{\lambda_k} - 1\right)^{-1} \lim_{\substack{\zeta\to\infty \\ \mathrm{Im}\zeta\ge\epsilon>0}} \left(\widetilde{\mathfrak{m}}(\zeta) - \prod_{k\in M} \frac{\mu_k}{\lambda_k}\right) = -1 .$$

But

$$\lim_{\substack{\zeta\to\infty \\ \mathrm{Im}\zeta\ge\epsilon>0}} \zeta\widetilde{m}(\zeta) = -\sum_{k\in M} \tau_k ,$$

so it has been proven that, for the function given in (5.3),

$$\int_{\mathbb{R}} d\rho(t) = 1\,.$$

Thus the measure corresponding to ρ is appropriately normalized and all the moments exist, so in $L_2(\mathbb{R}, \rho)$ apply the Gram–Schmidt procedure of orthonormalization to the sequence $\{t_k\}_{k=0}^{\infty}$ to obtain a Jacobi matrix as was explained in the Preliminaries. Denote by J the operator whose matrix representation is the obtained matrix (cf. [2, Sec. 47]). Now, depending on the sequence of moments, J is self-adjoint or not. If $J = J^*$, the function ρ is the resolution of the identity of J, while if $J \neq J^*$, ρ corresponds to the resolution of the identity of a self-adjoint extension of J. This is a consequence of condition d) since it means that the polynomials are dense in $L_2(\mathbb{R}, \rho)$ [23, Prop. 4.15].

Finally, denote by $J^{(g)}$ the self-adjoint extension of J corresponding to ρ and consider the operator $J^{(g)}(\theta)$ obtained from $J^{(g)}$ as indicated in the Preliminaries with θ given by (4.10). By the construction, the sequence $\{\lambda_k\}_{k\in M}$ is the spectrum of $J^{(g)}$. For the proof to be complete it only remains to show that $\{\mu_k\}_{k\in M}$ is the spectrum of $J^{(g)}(\theta)$. For the function given in (3.2), taking into account (3.4) and (2.14), one has

$$\mathfrak{m}(\zeta) = \theta^2 + \zeta\left(\theta^2 - 1\right) \sum_{k\in M} \frac{1}{\alpha_k(\lambda_k - \zeta)}\,.$$

On the other hand, from (5.4) and (5.8), it follows that

$$\widetilde{\mathfrak{m}}(\zeta) = \theta^2 + \zeta\left(\theta^2 - 1\right) \sum_{k\in M} \frac{\tau_k}{\lambda_k - \zeta}\,.$$

But we have already proven that $\alpha_k^{-1} = \tau_k$ for $k \in M$. Thus $\mathfrak{m} = \widetilde{\mathfrak{m}}$, meaning that the zeros of $\mathfrak{m}$ are given by the sequence $\{\mu_k\}_{k\in M}$. ∎

Theorem 5.2. *Let $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ be two infinite real sequences without finite points of accumulation such that each of them contains exactly one element equal zero, and consider any positive real number $\theta \neq 1$. There exists a unique operator J, and a unique $g \in \mathbb{R} \cup \{\infty\}$ if $J \neq J^*$, such that $\{\mu_k\}_k$ is the spectrum of $J^{(g)}(\theta)$ and $\{\lambda_k\}_k$ is the spectrum of $J^{(g)}$ if and only if the conditions a), b),* *c), and d) hold with*

$$\tau_n := \frac{\mu_n - \lambda_n}{\lambda_n\,(\theta^2 - 1)} \prod_{\substack{k\in M\\ k\neq n}} \frac{\mu_k - \lambda_n}{\lambda_k - \lambda_n}\,, \quad n \in M\,, n \neq 0\,,$$

$$\tau_0 := (\theta^2 - 1)^{-1} \left(\theta^2 - \prod_{\substack{k\in M\\ k\neq 0}} \frac{\mu_k}{\lambda_k}\right),$$

where

$$\theta^2 \begin{cases} < \prod_{\substack{k\in M\\ k\neq 0}} \frac{\mu_k}{\lambda_k} & \text{if } \{\mu_k\}_k \text{ is shifted to the left in } \mathbb{R}_+ \text{ w.r.t. } \{\lambda_k\}_k\,, \\ > \prod_{\substack{k\in M\\ k\neq 0}} \frac{\mu_k}{\lambda_k} & \text{otherwise.} \end{cases} \tag{5.9}$$

P r o o f. The proof is analogous to the proof of Theorem 5.1. Recall that by our convention for enumerating the sequences $\lambda_0 = \mu_0 = 0$. Thus, for proving the necessity of the conditions a)–d), one only should verify that $\tau_0 = a_0^{-1}$ and (5.9) holds. This is immediate in view of (4.13) and Remark 10. The sufficiency of the conditions is established as in the proof of Theorem 5.1. Here, one substitutes (5.2) by

$$\kappa = \prod_{\substack{k\in M\\ k\neq 0}} \frac{\mu_k}{\lambda_k}$$

and (5.4) by

$$\widetilde{m}(\zeta) := \frac{\widetilde{\mathfrak{m}}(\zeta) - \theta^2}{\zeta\,(\theta^2 - 1)}\,, \qquad \zeta \neq 0\,.$$

Then, one verifies that $\operatorname{Res}_{\zeta=\lambda_n} \widetilde{m}(\zeta) = -\tau_n$ for all $n \in M$ and $\sum_{k\in M} \tau_k = 1$. Note that (5.9) guarantees that $\tau_n > 0$ for all $n \in M$. The rest of the proof repeats that of Theorem 5.1 taking into account that now the zeros of $\mathfrak{m}$ are given by $\{\mu_k\}_{k\in M} \setminus \{0\}$. ∎

Theorem 5.3. *Given two infinite real sequences $\{\lambda_k\}_k$ and $\{\mu_k\}_k$ without finite points of accumulation such that none of them contains the zero, there is a unique positive θ and a unique operator $J = J^*$ such that $\{\mu_k\}_k$ is the spectrum of $J^{(g)}(\theta)$ and $\{\lambda_k\}_k$ is the spectrum of J if and only if conditions a), b), c), together* *with*

$$d')\qquad \lim_{n\to\infty}\frac{\det\begin{pmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ s_n & s_{n+1} & \cdots & s_{2n}\end{pmatrix}}{\det\begin{pmatrix} s_4 & s_5 & \cdots & s_{n+2} \\ s_5 & s_6 & \cdots & s_{n+3} \\ \cdots & \cdots & \cdots & \cdots \\ s_{n+2} & s_{n+3} & \cdots & s_{2n}\end{pmatrix}} = 0\,,$$

where $s_n := \sum_{k\in M}\lambda_k^n\tau_k$ *for* n *in* $\mathbb{N}\cup\{0\}$ *are fulfilled. Note that by our convention on the notation* $J^{(g)}(\theta)$ *is a non-singular finite-rank perturbation of* J *which does not depend on* g.

P r o o f. We again repeat the reasoning of the proof of Theorem 5.1. Clearly, s_n ($n \in \mathbb{N}\cup\{0\}$) are the numbers given in (2.6). Thus, on the basis of Hamburger's criterion (see [1, Addenda 2, Sec. 9]), *d')* holds when $J = J^*$. For the sufficiency, note that due to [1, Addenda 2, Sec. 9], *d')* implies that the measure corresponding to the function given in (5.3) is the unique solution of the moment problem, so $J = J^*$ and *d)* is not needed. ■

R e m a r k 12. Admittedly, *d')* is not easy to check, however it allows to give the necessary and sufficient conditions in the self-adjoint case. Note that one can also give the analogous self-adjoint version of Theorem 5.2 by substituting condition *d)* for *d')*.

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