Journal of Mathematical Physics, Analysis, Geometry
2013, vol. 9, No. 2, pp. 239–265

Spectrum of Two-Magnon non-Heisenberg Ferromagnetic Model of Arbitrary Spin with Impurity

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Received May 12, 2011, revised June 6, 2012

We consider a two-magnon system in the isotropic non-Heisenberg ferromagnetic model of an arbitrary spin s on a ν-dimensional lattice Z^ν. We establish that the essential spectrum of the system consists of the union of at most four intervals. We obtain lower and upper estimates for the number of three-particle bound states, i.e., for the number of points of discrete spectrum of the system.

Key words: non-Heisenberg ferromagnet, essential spectrum, discrete spectrum, three-particle discrete Schrödinger operator, compact operator, finite-dimensional operator, lattice, spin.

Mathematics Subject Classification 2010: 46L60, 47L90, 70H06. 70F05, 81Q10, 45B05, 45C05, 47B25, 47G10, 34L05.

We consider a two-magnon system in the isotropic non-Heisenberg ferromagnetic model of an arbitrary spin s with impurity on a ν-dimensional lattice Z^ν and study the discrete and essential spectra of the system. The system consists of three particles: two magnons and an impurity spin.

The Hamiltonian of the system has the form

$$H_{reg} = -\sum_{m,\tau}\sum_{n=1}^{2s} J_n(S^z_m S^z_{m+\tau} - s^2 + \frac{1}{2}(S^+_m S^-_{m+\tau} + S^-_m S^+_{m+\tau}))^n$$

$$-\sum_{\tau}\sum_{n=1}^{2s}(J^0_n - J_n)(S^z_0 S^z_\tau - s^2 + \frac{1}{2}(S^+_0 S^-_\tau + S^-_0 S^+_\tau))^n \qquad (1)$$

and acts on the symmetric Fock space $\mathscr{H}$. Here $J_n > 0$ are the parameters of the multipole exchange interaction between the nearest-neighbor atoms in the lattice Z^ν, $J^0_n \neq 0$ are the atom-impurity multipole exchange interaction parameters,

$\vec{S}_m = (S_m^x; S_m^y; S_m^z)$ is the atomic spin operator of spin s at the lattice site m, and $\tau = \pm e_j, j = 1, 2, \ldots, \nu$, where e_j are the unit coordinate vectors. Let φ_0 denote the vacuum vector uniquely defined by the conditions $S_m^+\varphi_0 = 0$ and $S_m^z\varphi_0 = s\varphi_0$, where $||\varphi_0|| = 1$. We set $S_m^{\pm} = S_m^x \pm iS_m^y$, where S_m^- and S_m^+ are the magnon creation and annihilation operators at the site m. The vector $S_m^- S_n^- \varphi_0$ describes the state of the system of two magnons located at the sites m and n with spin s. The vectors $\{\frac{1}{\sqrt{4s^2+(4s^2-4s)\delta_{m,n}}} S_m^- S_n^- \varphi_0\}$ form an orthonormal system. Let $\mathscr{H}_2$ be the Hilbert space spanned by these vectors. The space is called the two-magnon space of the operator H. We also denote the restriction of H to $\mathscr{H}_2$ by H_2.

Proposition 1. *The space $\mathscr{H}_2$ is an invariant subspace of H. The operator $H_2 = H/_{\mathscr{H}_2}$ is a bounded self-adjoint operator generating a bounded self-adjoint operator $\overline{H}_2$ whose kernel in the momentum representation, i.e., in $L_2(T^\nu)$, is given by the formula*

$$(\widetilde{H}_2 f)(x; y) = h(x; y)f(x; y) + \int_{T^\nu} h_1(x; y; t)f(t; x+y-t)dt + D\int_{T^\nu} h_2(x; s)f(s; y)ds$$

$$+E\int_{T^\nu} h_3(y; t)f(x; t)dt + \int_{T^\nu}\int_{T^\nu} h_4(x; y; s; t)f(s; t)dsdt, \qquad (2)$$

where

$$h(x; y) = 8sA\sum_{i=1}^{\nu}[1 - \cos\frac{x_k + y_k}{2}\cos\frac{x_k - y_k}{2}]$$

and

$$h_1(x; y; t) = -4s(2s-1)B$$

$$\times\sum_{i=1}^{\nu}\{1 + \cos(x_k + y_k) - 2\cos\frac{x_k + y_k}{2}\cos\frac{x_k - y_k}{2}\} - 4C\sum_{i=1}^{\nu}\{\cos\frac{x_k - y_k}{2}$$

$$-\cos\frac{x_k + y_k}{2}\}\cos(\frac{x_k + y_k}{2} - t_k), x, y, t \in T^\nu, h_2(x; s) = \sum_{i=1}^{\nu}\{1 + \cos(x_i - s_i)$$

$$-\cos s_i - \cos x_i\},\quad h_3(y; t) = \sum_{i=1}^{\nu}\{1 + \cos(y_i - t_i) - \cos t_i - \cos y_i\},$$

and

$$h_4(x; y; s; t) = F\sum_{i=1}^{\nu}[1 + \cos(x_i + y_i - s_i - t_i) + \cos(s_i + t_i) + \cos(x_i + y_i)$$

$$-\cos(x_i-s_i-t_i)-\cos(y_i-s_i-t_i)-\cos x_i-\cos y_i]+Q\sum_{i=1}^{\nu}[\cos(x_i-t_i)+\cos(y_i-s_i)]$$

$$+M\sum_{i=1}^{\nu}[\cos(x_i-s_i)+\cos(y_i-t_i)]+N\sum_{i=1}^{\nu}[\cos s_i+\cos t_i+\cos(x_i+y_i-s_i)$$
$$+\cos(x_i+y_i-t_i)],$$

here
$A = J_1-2sJ_2+(2s)^2J_3+\ldots+(-1)^{2s+1}J_{2s}$, $B = J_2-(6s-1)J_3+(28s^2-10s+1)J_4-(120s^3-68s^2+14s-1)J_5+\ldots$, $C = J_1+(4s^2-6s+1)J_2-(24s^3-32s^2+10s-1)J_3+(112s^4-160s^3+72s^2-14s+1)J_4-(480s^5-768s^4+448s^3-128s^2+18s-1)J_5+\ldots$, $D = -2\sum_{k=1}^{2s}(-2s)^k(J_k^0-J_k)$, $E = D$, $F = (2s-4s^2)(J_2^0-J_2)+(2s-16s^2+24s^3)(J_3^0-J_3)+\ldots+\ldots$, $Q = (-4s^2+2s)(J_2^0-J_2)+(-4s+20s^2-24s^3)(J_3^0-J_3)+\ldots+\ldots$, $M = 2[(J_1^0-J_1)-(1+5s+2s^2)(J_2^0-J_2)+(1-8s+22s^2-12s^3)(J_3^0-J_3)+\ldots+\ldots]$, $N = -(J_1^0-J_1)+(1-6s+4s^2)(J_2^0-J_2)-(1-10s+32s^2-24s^3)(J_3^0-J_3)+\ldots+\ldots.]$.

In the isotropic non-Heisenberg ferromagnetic model of an arbitrary spin s with impurity, the spectral properties of the above operator in the two-magnon case are closely related to those of its two-particle subsystems. The initial system is usually called a three-particle system, and the corresponding Hamiltonian is called a three-particle operator. We first study the spectrum and the corresponding eigenvectors, which we call the localized impurity states (LIS) of one-magnon impurity systems, and the spectrum and the corresponding eigenvectors, which we call the bound states (BS) of two-magnon systems.

1. One-Magnon Impurity States

The spectrum and the LIS in the one-magnon case of the isotropic non-Heisenberg ferromagnetic model of arbitrary spin with impurity were studied in [1].

The Hamiltonian of a one-magnon impurity system also has the form (1). The vector $S_m^-\varphi_0$ describes the one magnon state of spin s located at the site m. The vectors $\{\frac{1}{\sqrt{2s}}S_m^-\varphi_0\}$ form an orthonormal system. Let $\mathscr{H}_1$ be the Hilbert space spanned by these vectors. It is called the space of one-magnon states of the operator H. Denote by H_1 the restriction of the operator H to the space $\mathscr{H}_1$.

Proposition 2. *The space $\mathscr{H}_1$ is an invariant subspace of the operator H. The operator $H_1 = H/_{\mathscr{H}_1}$ is a bounded self-adjoint operator generating a bounded self-adjoint operator $\overline{H}_1$ acting on the space $l_2(Z^\nu)$ according to the formula*

$$(\overline{H}_1 f)(p) = \sum_{k=1}^{\nu}(-1)^{k+1}J_k s^k\sum_{p,\tau}2^{k-1}[2f(p)-f(p+\tau)-f(p-\tau)]$$

$$+\sum_{k=1}^{\nu}(-1)^{k+1}(J_k^0-J_k)(2s)^k\sum_{p,\tau}(f(\tau)-f(0))(\delta_{p,\tau}-\delta_{p,0}), \tag{3}$$

where $\delta_{k,j}$ is the Kronecker symbol, and the summation over τ is over the nearest neighbors. The operator H_1 acts on the vector $\psi=(2s)^{-1/2}\sum_p f(p)S_p^-\varphi_0\in\mathscr{H}_1$ by the formula

$$H_1\psi=\sum_p(\overline{H}_1 f)(p)\frac{1}{\sqrt{2s}}S_p^-\varphi_0. \tag{4}$$

Proposition 2 is proved by using the well-known commutation relations for the operators S_m^+, S_p^-, and $S_q^z: [S_m^+,S_n^-]=2\delta_{m,n}S_m^z,\ \ [S_m^z,S_n^\pm]=\pm\delta_{m,n}S_m^\pm$.

Lemma 1. *The spectra of the operators H_1 and $\overline{H}_1$ coincide.*

P r o o f. Because H_1 and $\overline{H}_1$ are bounded self-adjoint operators, it follows that if $\lambda\in\sigma(H_1)$, then the Weyl criterion (see [2]) implies that there is a sequence $\{\psi_n\}_{n=1}^\infty$ such that $||\psi_n||=1$ and

$$\lim_{n\to\infty}||H_1\psi_n-\lambda\psi_n||=0. \tag{5}$$

We set $\psi_n=(2s)^{-1/2}\sum_p f_n(p)S_p^-\varphi_0$.

Then

$$||H_1\psi_n-\lambda\psi_n||^2=(H_1\psi_n-\lambda\psi_n,H_1\psi_n-\lambda\psi_n)$$

$$=\sum_p||(\overline{H}_1 f_n(p)-\lambda f_n(p)||^2(\frac{1}{\sqrt{2s}}S_p^-\varphi_0,\frac{1}{\sqrt{2s}}S_p^-\varphi_0)=||\overline{H}_1F_n-\lambda F_n||^2$$

$$\times(\frac{1}{2s}S_p^+S_p^-\varphi_0,\varphi_0)=||(\overline{H}_1-\lambda)F_n||^2(\frac{1}{2s}2s\varphi_0,\varphi_0)=||(\overline{H}_1-\lambda)F_n||^2\to 0,\ n\to\infty.$$

Here $F_n=(f_n(p))_{p\in Z^\nu}$ and $||F_n||^2=\sum_p|f_n(p)|^2=||\psi_n||^2=1$. It follows that $\lambda\in\sigma(\overline{H}_1)$. Consequently, $\sigma(H_1)\subset\sigma(\overline{H}_1)$. Conversely, let $\overline{\lambda}\in\sigma(\overline{H}_1)$. Then, by the Weyl criterion, there is a sequence $\{F_n\}_{n=1}^\infty$ such that

$$||F_n||=\sqrt{\sum_p|f_n(p)|^2}=1\quad\text{and}\quad||(\overline{H}_1F_n-\overline{\lambda}F_n||\to 0,\quad n\to\infty. \tag{6}$$

We conclude that $||\psi_n||=||F_n||=1$ and $||\overline{H}_1F_n-\overline{\lambda}F_n||=||H_1\psi_n-\overline{\lambda}\psi_n||$. Thus (6) and the Weyl criterion imply that $\overline{\lambda}\in\sigma(H_1)$ and hence $\sigma(\overline{H}_1)\subset\sigma(H_1)$. These two relations imply that $\sigma(\overline{H}_1)=\sigma(H_1)$.

The spectrum and the LIS of the operator H_1 can be easily studied in its quasimomentum representation. Denote by $\mathscr{F}$ the Fourier transformation

$$\mathscr{F}: l_2(Z^\nu)\to L_2(T^\nu).$$

Here T^ν is the $\nu-$ dimensional torus endowed with the normalized Lebesgue measure $d\lambda : \lambda(T^\nu) = 1$.

Proposition 3. *The operator $\widetilde{H}_1 = \mathscr{F}\overline{H}_1\mathscr{F}^{-1}$ acts on the space $L_2(T^\nu)$ by the formula*

$$(\widetilde{H}_1 f)(x) = p(s)h(x)f(x) + q(s)\int\limits_{T^\nu} h_1(x;t)f(t)dt, \tag{7}$$

where $h(x) = \nu - \sum_{i=1}^{\nu}\cos x_i$, $h_1(x;t) = \nu + \sum_{i=1}^{\nu}[\cos(x_i - t_i) - \cos x_i - \cos t_i]$, $p(s) = -2\sum_{k=1}^{2s}(-2s)^k J_k$, $q(s) = -2\sum_{k=1}^{2s}(-2s)^k(J_k^0 - J_k), t \in T^\nu$.

To prove Proposition 3, the Fourier transform of (3) should be considered directly.

It is clear that the continuous spectrum of the operator $\widetilde{H}_1$ is independent of $q(s)h_1(x;t)$ and it fills the whole closed interval $[m_\nu; M_\nu]$, where $m_\nu = \min_{x\in T^\nu} p(s)h(x)$, $M_\nu = \max_{x\in T^\nu} p(s)h(x)$.

Definition 1. *An eigenfunction $\varphi \in L_2(T^\nu)$ of the operator $\widetilde{H}_1$ corresponding to an eigenvalue $z \notin [m_\nu; M_\nu]$ is called the LIS of the operator $\widetilde{H}_1$, and z is called the energy of this state.*

We consider the operator $K_\nu(z)$ acting on the space $\widetilde{\mathscr{H}_1}$ according to the formula

$$(K_\nu(z)f)(x) = \int\limits_{T^\nu} \frac{h_1(x;t)}{p(s)h(t) - z} f(t)dt, x, t \in T^\nu.$$

It is a compact operator in the space $\widetilde{\mathscr{H}_1}$ for the values z lying outside the set $G_\nu = [m_\nu; M_\nu]$.

Set

$$\Delta_\nu(z) = (1 + q(s)\int\limits_{T^\nu} \frac{(1 - \cos t_1)(\nu - \sum_{i=1}^{\nu}\cos t_i)dt}{p(s)h(t) - z}) \times (1 + q(s)\int\limits_{T^\nu} \frac{\sin^2 t_1 dt}{p(s)h(t) - z})^\nu$$

$$\times(1 + \frac{q(s)}{2}\int\limits_{T^\nu} \frac{(\cos t_1 - \cos t_2)^2 dt}{p(s)h(t) - z})^{\nu-1}, \tag{8}$$

where $dt = dt_1 dt_2 \dots dt_\nu$.

Lemma 2. *A number $z_0 \notin [m_\nu; M_\nu]$ is an eigenvalue of the operator $\widetilde{H}_1$ if and only if it is a zero of the function $\Delta_\nu(z)$, i.e., $\Delta_\nu(z_0) = 0$.*

P r o o f. In the case under consideration, the equation for the eigenvalues is an integral equation with a degenerate kernel. Therefore it is equivalent to a homogeneous linear system of algebraical equations. A homogeneous linear system of algebraic equations has a nontrivial solution if and only if the determinant of the system is zero. Taking into account that the function $h(s_1; s_2; \dots; s_\nu)$ is symmetric and carrying out the corresponding transformations, we present the determinant of the system in the form $\Delta_\nu(z)$.

We denote a set of all pairs $\omega = (p(s); q(s))$ by Ω and introduce the following subsets in Ω for $\nu = 1$:

$$A_1 = \{\omega : p(s) > 0, -p(s) \le q(s) < 0\}, A_2 = \{\omega : p(s) > 0, q(s) < -p(s)\},$$

$$A_3 = \{\omega : p(s) < 0, q(s) < p(s)\}, A_4 = \{\omega : p(s) > 0, p(s) < q(s)\},$$

$$A_5 = \{\omega : p(s) > 0, 0 < q(s) \le p(s)\}, A_6 = \{\omega : p(s) < 0, q(s) \ge p(s)\},$$

$$A_7 = \{\omega : p(s) < 0, 0 < q(s) < -p(s)\}, A_8 = \{\omega : p(s) < 0, q(s) > -p(s)\}.$$

We write

$$z_1 = -\frac{[p(s)+q(s)][p(s)-3q(s)+\sqrt{D}]}{4q(s)},$$

$$z_2 = \frac{[p(s)+q(s)]^2}{2q(s)},$$

$$z_3 = -\frac{[p(s)+q(s)][p(s)-3q(s)-\sqrt{D}]}{4q(s)},$$

where $D = [p(s)+q(s)][p(s)+9q(s)]$.

The following theorem describes the variation of the energy spectrum of the operator $\widetilde{H}_1$ in the one-dimensional case.

Theorem 1. *(i) If $\omega \in A_2 \bigcup A_3$, $(\omega \in A_4 \bigcup A_8)$, then the operator $\widetilde{H}_1$ has exactly two LIS's, φ_1 and φ_2, with the respective energies z_1 and z_2 $(z_2$ and $z_3)$ satisfying the inequalities $z_1 < z_2$ $(z_2 < z_3)$ and $z_i < m_1$, $i = 1, 2$ $(z_j > M_1$, $j = 2, 3)$.*

(ii) If $\omega \in A_6$ $(\omega \in A_5)$, then the operator $\widetilde{H}_1$ has a single LIS φ with the energy $z = z_1$ $(z = z_3)$ satisfying the inequality $z_1 < m_1$ $(z_3 > M_1)$.

(iii) If $\omega \in A_1 \bigcup A_7$, then the operator $\widetilde{H}_1$ has no LIS.

We sketch the proof of Theorem 1. In the one-dimensional case, the equation $\Delta_1(z) = 0$ is equivalent to the system of two equations,

$$1 + q(s)\int\limits_T \frac{(1-\cos t)^2 dt}{p(s)h(t) - z} = 0, \tag{9}$$

and

$$1 + q(s)\int\limits_T \frac{\sin^2 t dt}{p(s)h(t) - z} = 0. \tag{10}$$

In the one-dimensional case, the integrals in equations (9) and (10) can be found explicitly for the values $z \notin G_1 = [m_1; M_1]$. We obtain:
(a) for $z < m_1$:

$$1 + \frac{q(s)}{p(s)} + \frac{zq(s)}{p^2(s)} + \frac{z^2 q(s)}{p^2(s)\sqrt{z[z-2p(s)]}} = 0, \tag{11}$$

and

$$1 + \frac{q(s)}{p(s)} - \frac{zq(s)}{p^2(s)} + \frac{zq(s)}{p(s)\sqrt{z[z-2p(s)]}} - \frac{zq(s)[z-2p(s)]}{p^2(s)\sqrt{z[z-2p(s)]}} = 0, \tag{12}$$

(b) for $z > M_1$:

$$1 + \frac{q(s)}{p(s)} + \frac{zq(s)}{p^2(s)} - \frac{z^2 q(s)}{p^2(s)\sqrt{z[z-2p(s)]}} = 0, \tag{13}$$

and

$$1 + \frac{q(s)}{p(s)} - \frac{zq(s)}{p^2(s)} - \frac{zq(s)}{p(s)\sqrt{z[z-2p(s)]}} + \frac{zq(s)[z-2p(s)]}{p^2(s)\sqrt{z[z-2p(s)]}} = 0. \tag{14}$$

In turn, these equations are equivalent to the next equations:
(a) for $z < m_1$:

$$\{p^2(s) + p(s)q(s) + zq(s)\}\sqrt{z[z-2p(s)]} + z^2 q(s) = 0, \tag{11$'$}$$

and

$$\{p^2(s) + p(s)q(s) - zq(s)\}\sqrt{z[z-2p(s)]} - zq(s)[z-2p(s)] = 0, \tag{12$'$}$$

(b) for $z > M_1$:

$$\{p^2(s) + p(s)q(s) + zq(s)\}\sqrt{z[z-2p(s)]} - z^2 q(s) = 0, \tag{13$'$}$$

and

$$\{p^2(s) + p(s)q(s) - zq(s)\}\sqrt{z[z-2p(s)]} + zq(s)[z-2p(s)] = 0. \tag{14$'$}$$

Solving equation (11′), we find the root $z = z_1$, and solving equation (12′), we find the root $z = z_2$. In turn, solving equation (13′), we find the root $z = z_3$, and solving equation (14′), we find the root $z = z_2$. Whence the proof of Theorem 1 immediately follows in view of the existence of conditions for these solutions.

In the case of the dimension $\nu = 2$, for the pairs ω, we introduce:
$B_1 = \{\omega : p(s) > 0, -p(s) \leq q(s) < 0\}$, $B_2 = \{\omega : p(s) < 0, 0 < q(s) \leq -p(s)\}$,

$B_3 = \{\omega : p(s) > 0, -\frac{25}{9}p(s) \le q(s) < -p(s)\}$, $B_4 = \{\omega : p(s) < 0, \frac{25}{9}p(s) \le q(s) < 0\}$, $B_5 = \{\omega : p(s) > 0,\ 0 < q(s) < \frac{25}{9}p(s)\}$, $B_6 = \{\omega : p(s) < 0, -p(s) \le q(s) < -\frac{25}{9}p(s)\}$, $B_7 = \{\omega : p(s) > 0, -\frac{100}{27}p(s) \le q(s) < -p(s)\}$, $B_8 = \{\omega : p(s) < 0, \frac{100}{27}p(s) \le q(s) < \frac{25}{9}p(s)\}$, $B_9 = \{\omega : p(s) > 0, \frac{25}{9}p(s) \le q(s) < \frac{100}{27}p(s)\}$, $B_{10} = \{\omega : p(s) < 0, -\frac{25}{9}p(s) \le q(s) < -\frac{100}{27}p(s)\}$, $B_{11} = \{\omega : p(s) > 0, q(s) \le -\frac{100}{27}p(s)\}$, $B_{12} = \{\omega : p(s) < 0, q(s) \le \frac{100}{27}p(s)\}$, $B_{13} = \{\omega : p(s) > 0, q(s) \ge \frac{100}{27}p(s)\}$, $B_{14} = \{\omega : p(s) < 0, q(s) > -\frac{100}{27}p(s)\}$.

The next theorem describes the variation of the energy spectrum of the operator $\widetilde{H}_1$ in the two-dimensional case.

Theorem 2. *(i) If $\omega \in B_1 \bigcup B_2$, then the operator $\widetilde{H}_1$ has no LIS.*

(ii) If $\omega \in B_3 \bigcup B_4$ ($\omega \in B_5 \bigcup B_6$), then the operator $\widetilde{H}_1$ has a single LIS φ with the energy z_1 (z_2), where $z_1 < m_2$ ($z_2 > M_2$). The energy level is of multiplicity one.

(iii) If $\omega \in B_7 \bigcup B_8$ ($\omega \in B_9 \bigcup B_{10}$) then the operator $\widetilde{H}_1$ has exactly two LIS's, φ_1 and φ_2, with the respective energies z_1 and z_2 (z_3 and z_4), where $z_i < m_2$, $i = 1, 2$ ($z_j > M_2$, $j = 3, 4$). The energy levels are of multiplicity one.

(iv) If $\omega \in B_{11} \bigcup B_{12}$ ($\omega \in B_{13} \bigcup B_{14}$), then the operator $\widetilde{H}_1$ has three LIS's, φ_1, φ_2 and φ_3, with the respective energies z_1, z_2 and z_3 (z_4, z_5 and z_6), where $z_i < m_2$, $i = 1, 2, 3$ ($z_j > M_2$, $j = 4, 5, 6$). The energy levels z_1 and z_3 (z_4 and z_6) are of multiplicity one, while z_2 (z_5) is of multiplicity two.

P r o o f. The functions

$$\varphi(z) = \int_{T^2} \frac{(1 - \cos t_1)(2 - \cos t_1 - \cos t_2)dt}{p(s)h(t) - z}, \quad \psi(z) = \int_{T^2} \frac{\sin^2 t_1 dt}{p(s)h(t) - z},$$

$$\theta(z) = \int_{T^2} \frac{(\cos t_1 - \cos t_2)^2 dt}{p(s)h(t) - z}$$

are the monotone increasing functions of z for $z \notin [m_2; M_2]$. Their values can be exactly calculated at the points $z = m_2$ and $z = M_2$. For $z < m_2$ and $p(s) > 0$, the function $\varphi(z)$ increases from 0 to $(p(s))^{-1}$, the function $\psi(z)$ increases from 0 to $9(25p(s))^{-1}$, and the function $\theta(z)$ increases from 0 to $27(50p(s))^{-1}$. For $z > M_2$ and $p(s) > 0$, these functions increase from $-\infty$ to 0, from $-9(25p(s))^{-1}$ to 0, and from $-27(50p(s))^{-1}$ to 0, respectively. If $p(s) < 0$ and $z < m_2$, then they increase from 0 to ∞, from 0 to $-9(25p(s))^{-1}$, and from 0 to $-27(50p(s))^{-1}$, respectively. For $p(s) < 0$ and $z > M_2$, the functions $\varphi(z)$, $\psi(z)$, and $\theta(z)$ increase from $(p(s))^{-1}$

to 0, from $9(25p(s))^{-1}$ to 0, and from $27(50p(s))^{-1}$ to 0. Investigating the equation $\Delta_2(z) = 0$ outside the domain of the continuous spectrum, we immediately prove the assertion of Theorem 2.

In the case $\nu = 3$, we introduce the notation:

$$a = \int_{T^3} \frac{\sin^2 s_1\, ds_1 ds_2 ds_3}{3 - \cos s_1 - \cos s_2 - \cos s_3} = \int_{T^3} \frac{\sin^2 s_1\, ds_1 ds_2 ds_3}{3 + \cos s_1 + \cos s_2 + \cos s_3},$$

$$b = \int_{T^3} \frac{(\cos s_1 - \cos s_2)^2\, ds_1 ds_2 ds_3}{3 - \cos s_1 - \cos s_2 - \cos s_3} = \int_{T^3} \frac{(\cos s_1 - \cos s_2)^2\, ds_1 ds_2 ds_3}{3 + \cos s_1 + \cos s_2 + \cos s_3}.$$

As it is seen, we have $0 < a < b < 1$ and $2a < b$. We now consider the following subsets in Ω for the case $\nu = 3$:

$$Q_1 = \{\omega : p(s) > 0,\ -p(s) < q(s) < 0\}, \quad Q_2 = \{\omega : p(s) > 0, \quad 0 < q(s) < \frac{p(s)}{3}\},$$

$$Q_3 = \{\omega : p(s) < 0,\ \frac{p(s)}{3} < q(s) < 0\}, \quad Q_4 = \{\omega : p(s) < 0,\ 0 < q(s) < -p(s)\},$$

$$Q_5 = \{\omega : p(s) > 0,\ -\frac{2p(s)}{b} < q(s) \le -p(s), \quad Q_6 = \{\omega : p(s) < 0,$$

$$\frac{2p(s)}{b} < q(s) \le \frac{p(s)}{3}\}, \quad Q_7 = \{\omega : p(s) > 0,\ \frac{p(s)}{3} < q(s) \le \frac{2p(s)}{b}\},$$

$$Q_8 = \{\omega : p(s) < 0,\ -p(s) < q(s) \le -\frac{2p(s)}{b}\}, \quad Q_9 = \{\omega : p(s) > 0,$$

$$-\frac{p(s)}{a} \le q(s) < -\frac{2p(s)}{b}\}, \quad Q_{10} = \{\omega : p(s) < 0, \frac{p(s)}{a} < q(s) \le \frac{2p(s)}{b}\},$$

$$Q_{11} = \{\omega : p(s) > 0, \frac{2p(s)}{b} \le q(s) < \frac{p(s)}{a}\}, \quad Q_{12} = \{\omega : p(s) < 0,$$

$$-\frac{2p(s)}{b} \le q(s) < -\frac{p(s)}{a}\}, \quad Q_{13} = \{\omega : p(s) > 0,\ q(s) \le -\frac{p(s)}{a}\},$$

$$Q_{14} = \{\omega : p(s) < 0,\ q(s) \le \frac{p(s)}{a}\}, \quad Q_{15} = \{\omega : p(s) > 0,\ \frac{p(s)}{a} \le q(s)\},$$

$$Q_{16} = \{\omega : p(s) < 0,\ -\frac{p(s)}{a} \le q(s)\}.$$

Theorem 3. *(i) If $\omega \in Q_1 \bigcup Q_2 \bigcup Q_3 \bigcup Q_4$, then the operator $\widetilde{H}_1$ has no LIS.*

(ii) If $\omega \in Q_5 \bigcup Q_6$ $(\omega \in Q_7 \bigcup Q_8)$, then the operator $\widetilde{H}_1$ has a single LIS φ with the energy $z < m_3$ $(z > M_3)$. The energy level is of multiplicity one.

(iii) If $\omega \in Q_9 \bigcup Q_{10}$ $(\omega \in Q_{11} \bigcup Q_{12})$, *then the operator* $\widetilde{H}_1$ *has two LIS's,* φ_1 *and* φ_2, *with the energy levels* z_1 *and* z_2 *(*z_3 *and* z_4*), where* $z_i < m_3$, $i = 1, 2$ $(z_j > M_3$, $j = 3, 4)$. *Furthermore, the energy level* z_1 *(*z_3*) is of multiplicity one, while* z_2 *(*z_4*) is of multiplicity two.*

(iv) If $\omega \in Q_{13} \bigcup Q_{14}$ $(\omega \in Q_{15} \bigcup Q_{16})$, *then the operator* $\widetilde{H}_1$ *has exactly three LIS's,* φ_1, φ_2 *and* φ_3, *with the energies* z_1, z_2 *and* z_3 *(*z_4, z_5 *and* z_6*) satisfying the inequalities* $z_i < m_3$, $i = 1, 2, 3$ $(z_j > M_3$, $j = 4, 5, 6)$. *Moreover, the energy level* z_1 *(*z_4*) is of multiplicity one,* z_2 *(*z_5*) is of multiplicity two, and* z_3 *(*z_6*) is of multiplicity three.*

Theorem 3 is proved basing on the monotonicity of the functions

$$\varphi(z) = \int\limits_{T^3} \frac{(1-\cos t_1)(3-\cos t_1-\cos t_2-\cos t_3)dt}{p(s)h(t)-z},\ \psi(z) = \int\limits_{T^3} \frac{\sin^2 t_1 dt}{p(s)h(t)-z},$$

$$\theta(z) = \int\limits_{T^3} \frac{(\cos t_1-\cos t_2)^2 dt}{p(s)h(t)-z}$$

for $z \notin [m_3; M_3]$. Further we will use the values of the Watson integral [3]. It should be taken into account that the measure is normalized in the case under consideration.

It can be similarly proved that in the $\nu-$ dimensional lattice, the system has at most three types of LIS's (not counting the degeneracy multiplicities of their energy levels) with the energies $z_i \notin [m_\nu; M_\nu]$. Furthermore, for $i = 1, 2, 3$, the corresponding energy levels are of multiplicity one, of multiplicity ν and of multiplicity $(\nu - 1)$. The domains of these LIS's can also be found.

We now consider the case $p(s) \equiv 0$. If $p(s) \equiv 0$ and $J_n \neq 0, n = 1, 2, \ldots, 2s$, then the function $\Delta_\nu(z) = 0$ takes the form $\Delta_\nu(z) = detA \times detB$, where $A =$

$$\begin{pmatrix} a_1 & b_1 & b_1 & \cdots & b_1 \\ a_2 & b_2 & 0 & \cdots & 0 \\ a_2 & 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_2 & 0 & 0 & \cdots & b_2 \end{pmatrix} \text{is a } (\nu+1)\times(\nu+1) \text{ matrix, } B = \begin{pmatrix} b_2 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_2 \end{pmatrix}$$

is a diagonal $\nu \times \nu$ matrix. Here

$$a_1 = 1 - \frac{\nu q(s)}{2z},\ a_2 = \frac{q(s)}{2z},\ b_1 = \frac{q(s)}{z},\ b_2 = 1 - \frac{q(s)}{2z}.$$

Theorem 4. *If* $p(s) \equiv 0$, *and* $J_n \neq 0, n = 1, 2, \ldots, 2s$, *then the operator* $\widetilde{H}_1$ *has exactly two LIS's (not counting the multiplicities of degeneration of their* *energy levels),* φ_1 *and* φ_2*, with the energies* $z_1 = \frac{q(s)}{2}$ *and* $z_2 = \frac{2\nu+1}{2}q(s)$*. The energy* z_1 *is of multiplicity* $(2\nu-1)-$*, while* z_2 *is of multiplicity one. Moreover,* $z_i < m_\nu$*,* $i = 1, 2$*,* $(z_i > M_\nu$*,* $i = 1, 2)$*, if* $q(s) < 0$ $(q(s) > 0)$*.*

P r o o f. The equation $\Delta_\nu(z) = 0$ is equivalent to the system of two equations,

$$b_2^{2\nu-1} = 0 \tag{15}$$

and

$$a_1 b_2 - \nu a_2 b_1 = 0. \tag{16}$$

Equation (15) has a root equal to $z = \frac{q(s)}{2}$, and it is clear that its multiplicity is $2\nu-1$, while equation (16) has a solution $z = z_2$. Consequently, for the arbitrary values of ν, the system has at most three types of LIS's.

2. Two-Magnon States

The Hamiltonian of a two-magnon system has the form

$$H' = -\sum_{m,\tau}\sum_{n=1}^{2s} J_n(\vec{S}_m\vec{S}_{m+\tau})^n, \tag{17}$$

where $J_n > 0$ are the parameters of the multipole exchange interaction between the nearest-neighbor atoms in the lattice. Hamiltonian (17) acts on the symmetric Fock space $\mathscr{H}$. The vector $S_m^- S_n^- \varphi_0$ describes the state of a system of two magnons with spin s located at the sites m and n. The vectors $\{\frac{1}{\sqrt{4s^2+(4s^2-4s)\delta_{m,n}}} S_m^- S_n^- \varphi_0\}$ form an orthonormal system. Denote the Hilbert space spanned by these vectors by $\mathscr{H}_2$. It is called the space of two-magnon states of the operator H'. By H_2', we denote the restriction of the operator H' to $\mathscr{H}_2 : H_2' = H'/_{\mathscr{H}_2}$.

We find the action of operator (17) on the space $l_2(Z^\nu \times Z^\nu)$, i.e., the coordinate representation for the spin values $s = 1, s = 3/2, s = 2, s = 5/2$, and obtain the momentum representation of these operators in the space $L_2(T^\nu \times T^\nu)$. Finally, we generalize these formulas for the arbitrary values of s. The operator $\tilde{H}_2'$ in the momentum representation acts on the space $\tilde{\mathscr{H}}_2$ according to the formula

$$(\tilde{H}_2' f)(x; y) = h(x; y)f(x; y) + \int_{T^\nu} h_1(x; y; t)f(t; x + y - t)dt, \tag{18}$$

where

$$h(x; y) = A\sum_{i=1}^{\nu}[1 - cos\frac{x_i + y_i}{2}cos\frac{x_i - y_i}{2}]$$

and

$$h_1(x;y;t) = B\sum_{i=1}^{\nu}[1 - 2\cos\frac{x_i+y_i}{2}\cos\frac{x_i-y_i}{2} + \cos(x_i+y_i)]$$

$$-C\sum_{i=1}^{\nu}[\cos\frac{x_i-y_i}{2} - \cos\frac{x_i+y_i}{2}]\cos(\frac{x_i+y_i}{2} - t_i), \quad x,y,t \in T^{\nu}.$$

Here

$$A = \begin{cases} 8(J_1 - 2J_2), & \text{if} \quad s=1, \\ 12(J_1 - 3J_2 + 9J_3), & \text{if} \quad s=3/2, \\ 16(J_1 - 4J_2 + 16J_3 - 64J_4), & \text{if} \quad s=2, \\ 20(J_1 - 5J_2 + 25J_3 - 125J_4 + 625J_5), & \text{if} \quad s=5/2, \end{cases}$$

$$B = \begin{cases} -4J_2, & \text{if} \quad s=1, \\ -12(J_2 - 8J_3), & \text{if} \quad s=3/2, \\ -24(J_2 - 11J_3 + 93J_4), & \text{if} \quad s=2, \\ -40(J_2 - 15J_3 + 151J_4 - 1484J_5), & \text{if} \quad s=5/2, \end{cases}$$

$$C = \begin{cases} -4(J_1 - J_2), & \text{if} \quad s=1, \\ -4(J_1 + J_2 - 23J_3), & \text{if} \quad s=3/2, \\ -4(J_1 + 5J_2 - 83J_3 + 773J_4), & \text{if} \quad s=2, \\ -4(J_1 + 11J_2 - 199J_3 + 2291J_4 - 23119J_5), & \text{if} \quad s=5/2. \end{cases}$$

Proposition 4. *The space $\mathscr{H}_2$ is invariant with respect to the operator H'. The operator $H'_2 = H'/_{\mathscr{H}_2}$ is a bounded self-adjoint operator generating a bounded self-adjoint operator $\overline{H'}_2$ acting on the space $l_2(Z^{\nu} \times Z^{\nu})$. The operator H'_2 in the momentum representation in the space $L_2(T^{\nu} \times T^{\nu})$ acts according to the formula*

$$(\widetilde{H'_2}f)(x;y) = h(x;y)f(x;y) + \int_{T^{\nu}} h_1(x;y;s)f(s;x+y-s)ds, \tag{19}$$

where

$$h(x;y) = 8sA\sum_{k=1}^{\nu}[1 - \cos\frac{x_k+y_k}{2}\cos\frac{x_k-y_k}{2}],$$

$$h_1(x;y;t) = -4s(2s-1)B\sum_{k=1}^{\nu}\{1 + \cos(x_k+y_k) - 2\cos\frac{x_k+y_k}{2}\cos\frac{x_k-y_k}{2}\}$$

$$-4C\sum_{k=1}^{\nu}\{\cos\frac{x_k-y_k}{2} - \cos\frac{x_k+y_k}{2}\}\cos(\frac{x_k+y_k}{2} - t_k), \quad x,y,t \in T^{\nu},$$

here $A = J_1 - 2sJ_2 + (2s)^2J_3 + \ldots + (-1)^{2s+1}J_{2s}$, $B = J_2 - (6s-1)J_3 + (28s^2 - 10s + 1)J_4 - (120s^3 - 68s^2 + 14s - 1)J_5 + \ldots$, $C = J_1 + (4s^2 - 6s + 1)J_2 - (24s^3 -$

$32s^2+10s-1)J_3+(112s^4-160s^3+72s^2-14s+1)J_4-(480s^5-768s^4+448s^3-128s^2+18s-1)J_5+\ldots.$

The spectra and bound states of the energy operator of two-magnon systems in the isotropic non-Heisenberg ferromagnetic model of arbitrary spin s with impurity were studied in [4]. We consider the manifolds $\Gamma_\Lambda=\{(x;y): x+y=\Lambda\}$.

The following fact is important for further studying of the spectrum of the operator $\widetilde{H'}_2$.

Let the total quasi-momentum of the system $x+y=\Lambda$ be fixed. By $L_2(\Gamma_\Lambda)$, we denote the space of functions that are square integrable over the manifold $\Gamma_\Lambda=\{(x;y): x+y=\Lambda\}$. It is known [5] that the operators $\widetilde{H'}_2$ and the space $\tilde{\mathscr{H}}_2$ can be decomposed into the direct integrals $\widetilde{H'}_2=\bigoplus\int_{T^\nu}\widetilde{H'}_{2\Lambda}d\Lambda, \widetilde{\mathscr{H}_2}=\bigoplus\int_{T^\nu}\widetilde{\mathscr{H}_{2\Lambda}}d\Lambda$ of the operators $\widetilde{H'}_{2\Lambda}$ and the space $\tilde{\mathscr{H}}_{2\Lambda}$ such that the spaces $\tilde{\mathscr{H}}_{2\Lambda}$ are invariant under $\widetilde{H'}_{2\Lambda}$, and the operator $\widetilde{H'}_{2\Lambda}$ acts on the space $\tilde{\mathscr{H}}_{2\Lambda}$ as

$$(\widetilde{H'}_{2\Lambda}f_\Lambda)(x)=h_\Lambda(x)f_\Lambda(x)-\int\limits_{T^\nu}h_{1\Lambda}(x;t)f_\Lambda(t)dt,$$

where $h_\Lambda(x)=h(x;\Lambda-x), h_{1\Lambda}(x;t)=h_1(x;\Lambda-x;t)$ and $f_\Lambda(x)=f(x;\Lambda-x)$.

It is known that the continuous spectrum of the operator $\widetilde{H'}_2$ is independent of the functions $h_{1\Lambda}(x;t)$ and it consists of the intervals $G_\Lambda=[m_\Lambda;M_\Lambda]$, where $m_\Lambda=min_x h_\Lambda(x), M_\Lambda=max_x h_\Lambda(x)$.

The eigenfunction $\varphi_\Lambda\in L_2(T^\nu)$ of the operator $\widetilde{H'}_2$ corresponding to an eigenvalue $z_\Lambda\notin G_\Lambda$ is called the bound state of the operator $\widetilde{H'}_2$, and z_Λ is called the energy of this BS.

Denote the $2s$–th $(J_1;J_2;\ldots;J_{2s})$ by P and introduce the following subsets of the $2s$–th P for $\nu=1$:

$$Q_1=\{P: A<0, B<0, C<0\},\quad Q_2=\{P: A>0, B>0, C>0\},$$

$$Q_3=\{P: A>0, B>0, C<0\},\quad Q_4=\{P: A<0, B<0, C>0\},$$

$$Q_5=\{P: A<0, B>0, C<0\},\quad Q_6=\{P: A>0, B<0, C>0\},$$

$$Q_7=\{P: B=0, A=C>0\},\quad Q_8=\{P: B=0, A=C<0\}.$$

Let $\Delta^\nu_\Lambda(z)=detD$, where

$$D=\begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \cdots & d_{1,\nu+1}\\ d_{2,1} & d_{2,2} & d_{2,3} & \cdots & d_{2,\nu+1}\\ d_{3,1} & d_{3,2} & d_{3,3} & \cdots & d_{3,\nu+1}\\ \vdots & \vdots & \vdots & \vdots & \\ d_{\nu,1} & d_{\nu,2} & d_{\nu,3} & \cdots & d_{\nu,\nu+1}\\ d_{\nu+1,1} & d_{\nu+1,2} & d_{\nu+1,3} & \cdots & d_{\nu+1,\nu+1}\end{pmatrix},$$

and

$$d_{1,1} = 1 - 4s(2s-1)B \int\limits_{T^\nu} \frac{g_\Lambda(s)ds}{h_\Lambda(s)-z},$$

$$d_{1,k+1} = -4C \int\limits_{T^\nu} \frac{f_{\Lambda_k}(s_k)ds}{h_\Lambda(s)-z}, \quad k=\overline{1,\nu},$$

$$d_{k+1,1} = -4s(2s-1)B \int\limits_{T\nu} \frac{\eta_{\Lambda_k}(s_k)g_\Lambda(s)ds}{h_\Lambda(s)-z}, \quad k=\overline{1,\nu},$$

$$d_{k+1,k+1} = 1 - 4C \int\limits_{T^\nu} \frac{\eta_{\Lambda_k}(s_k)f_{\Lambda_k}(s_k)ds}{h_\Lambda(s)-z}, \quad k=\overline{1,\nu},$$

$$d_{k+1,i+1} = -4C \int\limits_{T^\nu} \frac{\eta_{\Lambda_k}(s_k)f_{\Lambda_i}(s_i)ds}{h_\Lambda(s)-z}, \quad k=\overline{1,\nu}, \quad i=\overline{1,\nu}, \quad k\neq i.$$

In these formulas

$$g_\Lambda(s) = \sum_{k=1}^{\nu}[1+\cos\Lambda_k - 2\cos\frac{\Lambda_k}{2}\cos(\frac{\Lambda_k}{2}-s_k)],$$

$$f_{\Lambda_k}(s_k) = \cos(\frac{\Lambda_k}{2}-s_k) - \cos\frac{\Lambda_k}{2},\ k=\overline{1,\nu}, \quad \eta_{\Lambda_k}(s_k) = \cos(\frac{\Lambda_k}{2}-s_k),\ k=\overline{1,\nu}.$$

Lemma 3. *A number $z=z_0 \notin G_\Lambda$ is an eigenvalue of the operator $\tilde{H}'_{2\Lambda}$ if and only if it is a zero of the function $\Delta_\Lambda^\nu(z)$, i.e., $\Delta_\Lambda^\nu(z_0)=0$.*

The proof of Lemma 3 is similar to that of Lemma 2.

In the case when $\nu=1$, the change of the energy spectrum is described by the theorems below.

Theorem 5. 1. *Let $P\in Q_1$ and $\Lambda \in]0;\pi[\quad (\Lambda\in]\pi;2\pi[)$.*

a) If $C\neq 2s(2s-1)B$, then the operator $\tilde{H}'_2$ has two BS's, φ_1 and φ_2, with the energy levels $z_1<m_\Lambda$ and $z_2>M_\Lambda$.

b) If $C=2s(2s-1)B$, then the operator $\tilde{H}'_2$ has only one BS φ with the energy level $z<m_\Lambda$.

2. *Let $P\in Q_2$ and $\Lambda\in]0;\pi[$ $(\Lambda\in]\pi;2\pi[)$.*

a) If $2sA<C<2s(2s-1)B$, $\cos\frac{\Lambda}{2}>\frac{C}{2s(2s-1)B}$, $(C>2s(2s-1)B$, $A<(2s-1)B)$, then the operator $\tilde{H}'_2$ has three BS's, $\varphi_i, i=1,2,3$; with the energy values $z_k<m_\Lambda$, $k=1,2$; and $z_3>M_\Lambda$.

b) If $C<2sA<2s(2s-1)B$, $\cos\frac{\Lambda}{2}>\frac{C}{2s(2s-1)B}$, $(C>2s(2s-1)B$, $A=(2s-1)B)$, then the operator $\tilde{H}'_2$ has two BS's, $\varphi_i, i=1,2$, corresponding to the *energy values* $z_1 < m_\Lambda$ *and* $z_2 > M_\Lambda$. *In this case the third BS vanishes because it is absorbed by the continuous spectrum.*

c) If $C < 2s(2s-1)B < (2s-1)A$, $\cos\frac{\Lambda}{2} > \frac{C}{2s(2s-1)B}$, $(C > 2s(2s-1)B$, $A > (2s-1)B)$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z > M_\Lambda$.

d) If $C = 2s(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z < m_\Lambda$.

e) If $C > 2s(2s-1)B$ $(C < 2s(2s-1)B)$, *then the operator* $\tilde{H}_2'$ *has two BS's,* φ_1, φ_2, *corresponding to the energy values* $z_1 < m_\Lambda$, $z_2 > M_\Lambda$.

3. *Let* $P \in Q_3$ *and* $\Lambda \in]0;\pi[$ $\quad (\Lambda \in]\pi; 2\pi[)$.

a) If $C \geq -2s(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has two BS's,* φ_1 *and* φ_2, *with the energy values* z_1 *and* z_2, *where* $z_1 < m_\Lambda$, *and* $z_2 > M_\Lambda$.

b) If $C < 2s(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z < m_\Lambda$.

4. *Let* $P \in Q_4$ *and* $\Lambda \in]0;\pi[$ $(\Lambda \in]\pi; 2\pi[)$.

a) If $2sA - 2s(2s-1)B - C > 0$, $\cos\frac{\Lambda}{2} > \frac{C}{2sA-2s(2s-1)B-C}$ $(\cos\frac{\Lambda}{2} \neq \frac{C}{2s(2s-1)B})$, *then the operator* $\tilde{H}_2'$ *has three (two) BS's,* φ_i, $i = 1,2,3$ $(\varphi_j, j = 1,2)$ *corresponding to the energy values* $z_k < m_\Lambda, k = 1,2; z_3 > M_\Lambda (z_1 < m_\Lambda,\ z_2 > M_\Lambda)$.

b) If $2sA - 2s(2s-1)B - C > 0$, $-\frac{C}{2s(2s-1)B} < \cos\frac{\Lambda}{2} < \frac{C}{2sA-2s(2s-1)B-C}$ *or* $2sA - 2s(2s-1)B - C < 0$ $(\cos\frac{\Lambda}{2} = \frac{C}{2s(2s-1)B})$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z > M_\Lambda$.

5. *Let* $P \in Q_5$ *and* $\Lambda \in]0;\pi[$ $(\Lambda \in]\pi; 2\pi[)$.

a) If $\cos\frac{\Lambda}{2} > -\frac{C}{2s(2s-1)B}$, $C \geq 2sA$ $(\cos\frac{\Lambda}{2} < \frac{C}{2s(2s-1)B}$, $C \geq 2sA)$, *then the operator* $\tilde{H}_2'$ *has three BS's,* φ_1, φ_2 *and* φ_3, *corresponding to the energy values* $z_i < m_\Lambda$, $i = 1,2$; *and* $z_3 > M_\Lambda$.

b) If $C < 2sA, 2sA - 2s(2s-1)B - C < 0, \cos\frac{\Lambda}{2} > \frac{C}{2sA-2s(2s-1)B-C}$ $(C < 2sA, 2sA - 2s(2s-1)B - C < 0$, $\cos\frac{\Lambda}{2} < -\frac{C}{2sA-2s(2s-1)B-C})$, *then the operator* $\tilde{H}_2'$ *has three BS's,* φ_1, φ_2 *and* φ_3, *corresponding to the energy values* $z_i < m_\Lambda, i = 1,2$; *and* $z_3 > M_\Lambda$.

c) If $C < 2sA, 2sA - 2s(2s-1)B - C < 0$, $-\frac{C}{2s(2s-1)B} < cos\frac{\Lambda}{2} < \frac{C}{2s(2s-1)B}$ $(C < 2sA,\ 2sA - 2s(2s-1)B - C < 0,\ -\frac{C}{2sA-2s(2s-1)B-C} \leq \cos\frac{\Lambda}{2} < \frac{C}{2s(2s-1)B})$ *or* $C < 2sA,\ 2sA - 2s(2s-1)B - C \geq 0$ $(C > 2sA,\ 2sA - 2s(2s-1)B - C \geq 0)$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z > M_\Lambda$.

d) If $\cos\frac{\Lambda}{2} = -\frac{C}{2s(2s-1)B}$, $C \geq 2sA$ $(cos\frac{\Lambda}{2} = \frac{C}{2s(2s-1)B}$, $C \geq 2sA)$, *then the operator* $\tilde{H}_2'$ *has two BS's,* φ_1 *and* φ_2, *with the energy values* $z_1 < m_\Lambda$ *and* $z_2 > M_\Lambda$.

e) If $\cos\frac{\Lambda}{2} < -\frac{C}{2s(2s-1)B}$ $(\cos\frac{\Lambda}{2} > \frac{C}{2s(2s-1)B})$, *then the operator* $\tilde{H}_2'$ *has two BS's,* φ_1 *and* φ_2, *with the energy values* $z_1 < m_\Lambda$ *and* $z_2 > M_\Lambda$.

f) If $\cos\frac{\Lambda}{2}=-\frac{C}{2s(2s-1)B}$, $C<2sA$ ($\cos\frac{\Lambda}{2}>\frac{C}{2s(2s-1)B}$, $C<2sA$), *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z>M_\Lambda$.

6. *Let* $P\in Q_6$ *and* $\Lambda\in]0;\pi[$ ($\Lambda\in]\pi;2\pi[$).

a) If $\cos\frac{\Lambda}{2}<-\frac{C}{2s(2s-1)B}$ ($\cos\frac{\Lambda}{2}>\frac{C}{2s(2s-1)B}$), *then the operator* $\tilde{H}_2'$ *has two BS's,* φ_1 *and* φ_2, *with the energy values* $z_1<m_\Lambda$, *and* $z_2>M_\Lambda$.

b) If $\cos\frac{\Lambda}{2}\geq-\frac{C}{2s(2s-1)B}$ ($\cos\frac{\Lambda}{2}\leq\frac{C}{2s(2s-1)B}$), *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z<m_\Lambda$.

7. *Let* $P\in Q_7\bigcup Q_8$ *and* $\Lambda\neq 0$.

Then the operator $\tilde{H}_2'$ *has two BS's,* φ_1 *and* φ_2, *with the energy values* $z_1<m_\Lambda$, *and* $z_2>M_\Lambda$.

In the case where $\nu=1$ and $\Lambda=0$, the change of the energy spectrum is described by the following theorems.

Theorem 6. *Let* $\Lambda=0$. *a) If* $P\in Q_1$, $C>2s(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has two BS's,* φ_1 *and* φ_2, *with the energy values* $z_1<m_\Lambda$, *and* $z_2>M_\Lambda$.

b) If $P\in Q_1$, $C\leq 2s(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z<m_\Lambda$.

2.*a) If* $P\in Q_2$, $2sA<C<2s(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has three BS's,* $\varphi_i, i=1,2,3$; *with the energy values* $z_j<m_\Lambda$, $j=1,2$; *and* $z_3>M_\Lambda$.

b) If $P\in Q_2$, $C\leq 2sA$, $C<2s(2s-1)B$ *or* $P\in Q_2$, $2sA<2s(2s-1)B<C$, *then the operator* $\tilde{H}_2'$ *has two BS's,* φ_i, $i=1,2$ *with the energy values* $z_1<m_\Lambda$ *and* $z_2>M_\Lambda$.

c) If $P\in Q_2$, $C=2s(2s-1)B>2sA$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z<m_\Lambda$.

d) If $P\in Q_2$, $C=2sA\geq 2s(2s-1)B$ *or* $P\in Q_2$, $2s(2s-1)B<2sA<C$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z>M_\Lambda$.

e) If $P\in Q_2$, $C=2s(2s-1)B<2sA$ *or* $P\in Q_2$, $2s(2s-1)B<2sA<C$, *then the operator* $\tilde{H}_2'$ *has no BS.*

3.*a) If* $P\in Q_3$, $C<-2s(2s-1)B$, $A\geq(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has two BS's,* φ_i, $i=1,2$, *with the energy values* $z_1<m_\Lambda$ *and* $z_2>M_\Lambda$.

b) If $P\in Q_3$, $A<(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z>M_\Lambda$.

c) If $P\in Q_3$, $C\geq-2s(2s-1)B$, $A\geq(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z<m_\Lambda$.

4.*a) If* $P\in Q_4$, $C>-2s(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has two BS's,* φ_1 *and* φ_2, *with the energy values* $z_i<m_\Lambda$, $i=1,2$.

b) If $P\in Q_4$, $C<-2s(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has only one BS* φ *with the energy value* $z<m_\Lambda$.

c) If $P\in Q_4$, $C=-2s(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has no BS.*

5.*a) If* $P\in Q_5$, $-2s(2s-1)B<C<2sA$, $C>sA-s(2s-1)B$, *then the operator* $\tilde{H}_2'$ *has two BS's,* φ_1 *and* φ_2, *with the energy values* $z_i<m_\Lambda$, $i=1,2$.

b) If $P \in Q_5$, $-2s(2s-1)B < C < 2sA$, $C \leq sA - s(2s-1)B$ or $P \in Q_5$, $C = -2s(2s-1)B < 2sA$, then the operator $\tilde{H}_2'$ has no BS.

c) If $P \in Q_5$, $C = -2s(2s-1)B \geq 2sA$ or $P \in Q_5$, $C < -2s(2s-1)B$, then the operator $\tilde{H}_2'$ has only one BS φ with the energy value $z < m_\Lambda$.

6.a) If $P \in Q_6$, $2sA \leq C < -2s(2s-1)B$, then the operator $\tilde{H}_2'$ has two BS's, φ_1 and φ_2, with the energy values $z_i > M_\Lambda$, $i = 1, 2$.

b) If $P \in Q_6$, $C = 2sA > -2s(2s-1)B$ or $P \in Q_6$, $C < -2s(2s-1)B$, $C < 2sA$, then the operator $\tilde{H}_2'$ has no BS.

c) If $P \in Q_6$, $C = -2s(2s-1)B < 2sA$ or $P \in Q_6$, $C > -2s(2s-1)B$, $C \neq 2sA$, then the operator $\tilde{H}_2'$ has only one BS φ with the energy value $z > M_\Lambda$.

7. If $P \in Q_7$ ($P \in Q_8$), then the operator $\tilde{H}_2'$ has only one BS φ with the energy value $z > M_\Lambda$ ($z < m_\Lambda$).

A sketch of the proofs of Theorems 5, 6 is given below. In the case under consideration, the equation for eigenvalues is an integral equation with a degenerate kernel. It is therefore equivalent to a system of the linear homogeneous algebraic equations. The system is known to have a nontrivial solution if and only if its determinant is equal to zero. In this case, the equation $\Delta_\Lambda^\nu(z) = 0$ is therefore equivalent to the equation stating that the determinant of the system is zero. In the case where $\nu = 1$, the determinant has the form

$$\Delta_\Lambda^1(z) = \det D,$$

where

$$D = \begin{pmatrix} d_{1,1} & d_{1,2} \\ d_{2,1} & d_{2,2} \end{pmatrix}.$$

Here

$$d_{1,1} = 1 - 4s(2s-1)B \int_T \frac{g_\Lambda(s)ds}{h_\Lambda(s) - z}, \quad d_{1,2} = -4C \int_T \frac{f_\Lambda(s)ds}{h_\Lambda(s) - z},$$

$$d_{2,1} = -4s(2s-1) \int_T \frac{\eta_\Lambda(s) g_\Lambda(s)ds}{h_\Lambda(s) - z}, \quad d_{2,2} = 1 - 4C \int_T \frac{\eta_\Lambda(s) f_\Lambda(s)ds}{h_\Lambda(s) - z},$$

$$g_\Lambda(s) = 1 + \cos\Lambda - 2\cos\frac{\Lambda}{2}\cos(\frac{\Lambda}{2} - s), \quad f_\Lambda(s) = \cos(\frac{\Lambda}{2} - s) - \cos\frac{\Lambda}{2},$$

$$\eta_\Lambda(s) = \cos(\frac{\Lambda}{2} - s).$$

Expressing all integrals in the equation $\Delta_\Lambda^1(z) = 0$ via the integral

$$J^\star(z) = \int_T \frac{dt}{h_\Lambda(t) - z},$$

we can see that the equation $\Delta^1_\Lambda(z)=0$ is equivalent to the equation

$$\{C(z-8sA)^2+8sA[2s(2s-1)B+C]cos^2\frac{\Lambda}{2}(z-8sA)+128s^3(2s-1)A^2B\cos^4\frac{\Lambda}{2}\}$$

$$\times J^\star(z)=-C(z-8sA)+8sA[2sA-C-2s(2s-1)B]\cos^2\frac{\Lambda}{2}. \qquad (20)$$

Because $\frac{1}{h_\Lambda(t)-z}$ is a continuous function for $z\notin[m_\Lambda;M_\Lambda]$ and

$$[J^\star(z)]'=\int\limits_T\frac{1}{[h_\Lambda(t)-z]^2}>0,$$

the function $J^\star(z)$ is an increasing function of z for $z\notin[m_\Lambda;M_\Lambda]$. Moreover, $J^\star(z)\to 0$ as $z\to-\infty, J^\star(z)\to+\infty$ as $z\to m_\Lambda-0, J^\star(z)\to-\infty$ as $z\to M_\Lambda+0$, and $J^\star(z)\to 0$ as $z\to+\infty$. Analyzing equation (20) outside the set $G_\Lambda=[m_\Lambda;M_\Lambda]$, we get the proof of Theorems 5, 6.

The energy spectrum of the operator $\tilde{H}_2'$ in the case where $\nu=2$ for the total quasi-momentum of the form $\Lambda=(\Lambda_1;\Lambda_2)=(\Lambda_0;\Lambda_0)$ is described below. It is easy to see that if the parameters $J_n, n=\overline{1,2s}$ and Λ_0 satisfy the conditions of Theorems 5, 6, then the statements of the theorems are true. Only one additional BS $\tilde{\varphi}$ appears, whose energy value is $\tilde{z}$, because $\tilde{z}<m_\Lambda$ $(\tilde{z}>M_\Lambda)$ if $C>0$ $(C<0)$. If $C=0$, the operator $\tilde{H}_2'$ does not have an additional BS.

The proof of this statement is based on the fact that if $\nu=2$ and $\Lambda=(\Lambda_0;\Lambda_0)$, then the function $\Delta^\nu_\Lambda(z)$ has the form

$$\Delta^\nu_\Lambda(z)=[1-2C\int\limits_{T^2}\frac{[\cos(\frac{\Lambda_0}{2}-t_1)-\cos(\frac{\Lambda_0}{2}-t_2)]^2dt_1dt_2}{h_\Lambda(t_1;t_2)-z}]\Psi_\Lambda(z), \qquad (21)$$

where

$$\Psi_\Lambda(z)=\{1-4s(2s-1)B\int\limits_{T^2}\frac{g_\Lambda(t)}{h_\Lambda(t_1;t_2)-z}dt_1dt_2\}[1-4C$$

$$\times\int\limits_{T^2}\frac{f_\Lambda(t_1)\eta_\Lambda(t_1;t_2)}{h_\Lambda(t_1;t-2)-z}dt_1dt_2]-32s(2s-1)BC\int\limits_{T^2}\frac{\xi_\Lambda(t_1)}{h_\Lambda(t_1;t_2)-z}dt_1dt_2$$

$$\times\int\limits_{T^2}\frac{f_\Lambda(t_1)g_\Lambda(t)}{h_\Lambda(t_1;t_2)-z}dt_1dt_2, t\in T^2, \Lambda\in T^\nu.$$

Here $g_\Lambda(t)=2+2\cos\Lambda_0-2\cos\frac{\Lambda_0}{2}[\cos(\frac{\Lambda_0}{2}-t_1)+\cos(\frac{\Lambda_0}{2}-t_2)]$, $f_\Lambda(t_1)=\cos(\frac{\Lambda_0}{2}-t_1)$, $\eta_\Lambda(t_1;t_2)=\cos(\frac{\Lambda_0}{2}-t_1)+\cos(\frac{\Lambda_0}{2}-t_2)-2\cos\frac{\Lambda_0}{2}$, $\xi_\Lambda(t_1)=\cos(\frac{\Lambda_0}{2}-t_1)-\cos\frac{\Lambda_0}{2}$.

Therefore the equation $\Delta^{\nu}_{\Lambda}(z) = 0$ holds if either the equation

$$1 - 2C \int\limits_{T^2} \frac{[\cos(\frac{\Lambda_0}{2} - t_1) - \cos(\frac{\Lambda_0}{2} - t_1)]^2 dt_1 dt_2}{h_\Lambda(t_1; t_2) - z} = 0 \tag{22}$$

or

$$\Psi_\Lambda(z) = 0 \tag{23}$$

holds.

It is easy to see that equation (22) has the unique solution $\tilde{z} < m_\Lambda$ if $C > 0$; if $C < 0$, then this solution satisfies the condition $\tilde{z} > M_\Lambda$. If $C = 0$, equation (22) has no solution. Expressing the integrals in (23) via the integral

$$J^\star(z) = \int\limits_{T^2} \frac{dt_1 dt_2}{h_\Lambda(t_1; t_2) - z},$$

we obtain

$$\eta_\Lambda(z) J^\star(z) = \xi_\Lambda(z),$$

where

$$\eta_\Lambda(z) = C(z - 16sA)^2 + 16sA[2s(2s-1)B + C]$$
$$\times \cos^2 \frac{\Lambda_0}{2}(z - 16sA) + 512s^3(2s-1)A^2 B \cos^4 \frac{\Lambda_0}{2},$$

and

$$\xi_\Lambda(z) = -C(z - 16sA) + 16sA[2sA - C - 2s(2s-1)B] \cos^2 \frac{\Lambda_0}{2}.$$

In its turn, for $\eta_\Lambda(z) \neq 0$, the above last equation is equivalent to the equation

$$J^\star(z) = \frac{\xi_\Lambda(z)}{\eta_\Lambda(z)}. \tag{24}$$

Analyzing equation (24) outside the set G_Λ and taking into account that the function $J^\star(z)$ is monotonic for $z \notin [m_\Lambda; M_\Lambda]$, we obtain the statements similar to those of Theorems 5, 6.

For all other quasi-momenta, $\Lambda = (\Lambda_1; \Lambda_2), \Lambda_1 \neq \Lambda_2$, there exist the sets $G_j, j = \overline{0,5}$, of the parameters $J_n, n = \overline{1, 2s}$ and Λ such that in every set G_j the operator $\tilde{H}'_2$ has exactly j BS's (taking the multiplicity of energy levels into account) with the corresponding energy values $z_k, k = \overline{1,5}$, and $z_k \notin G_\Lambda$.

Indeed, in this case, for $\nu = 2$, the function $\Delta^\nu_\Lambda(z)$ has the form

$$\Delta^\nu_\Lambda(z) = \det D,$$

where

$$D = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \end{pmatrix}.$$

Here

$$d_{1,1} = 1-4s(2s-1)B\int\limits_{T^2} \frac{g_\Lambda(s)ds_1ds_2}{h_\Lambda(s)-z},\ d_{1,k+1} = -4C\int\limits_{T^2} \frac{f_{\Lambda_k}(s_k)}{h_\Lambda(s)-z}ds_1ds_2,\ k=1,2,$$

$$d_{k+1,1} = -4s(2s-1)B\int\limits_{T^2} \frac{\zeta_{\Lambda_k}(s_k)g_\Lambda(s)ds_1ds_2}{h_\Lambda(s)-z}, \quad k=1,2,$$

$$d_{k+1,k+1} = 1-4C\int\limits_{T^2} \frac{\zeta_{\Lambda_k}(s_k)f_{\Lambda_k}(s_k)ds_1ds_2}{h_\Lambda(s)-z}, \quad k=1,2,$$

$$d_{k+1,j+1} = -4C\int\limits_{T^2} \frac{\zeta_{\Lambda_k}(s_k)f_{\Lambda_j}(s_j)ds_1ds_2}{h_\Lambda(s)-z}, \quad k=1,2,\ j=1,2,\ k\neq j.$$

In these formulas

$$g_\Lambda(s) = \sum_{k=1}^{2}[1+\cos\Lambda_k - 2\cos\frac{\Lambda_k}{2}\cos(\frac{\Lambda_k}{2}-s_k)],$$

$$f_{\Lambda_k}(s_k) = \cos(\frac{\Lambda_k}{2}-s_k) - \cos\frac{\Lambda_k}{2}, \quad k=1,2,$$

$$\zeta_{\Lambda_k}(s_k) = \cos(\frac{\Lambda_k}{2}-s_k), \quad k=1,2.$$

Expressing all integrals in the equation $\Delta_\Lambda^\nu(z)=0$ via $J^\star(z)$ and performing some algebraic transformations, we can reduce it to the form

$$\theta_\Lambda(z)J^\star(z) = \chi_\Lambda(z), \tag{25}$$

where $\theta_\Lambda(z)$ is the fifth-order polynomial in z, and $\chi_\Lambda(z)$ is the lower-order polynomial in z. Analyzing equation (25) outside the set G_Λ and taking into account that the function $J^\star(z)$ with $z \notin [m_\Lambda; M_\Lambda]$ is monotonic, we can easily verify that the equation has no more than five solutions outside the set G_Λ.

For an arbitrary $\nu \geq 3$ and $\Lambda = (\Lambda_1;\Lambda_2;\ldots;\Lambda_\nu) = (\Lambda_0;\Lambda_0;\Lambda_0;\ldots;\Lambda_0) \in T^\nu$, the change of the energy spectrum of the operator $\widetilde{H'}_2$ is similar to that observed in the case of $\nu=1$. In this case, if the parameters $J_1, J_2, \ldots, J_{2s}$ and Λ_0 satisfy the conditions of Theorems 5, 6, then there exist the statements of these theorems that are true. In this situation, the operator $\widetilde{H'}_2$ with $C \neq 0$ has only one additional BS with the energy z. Moreover, the energy level of this additional BS z degenerates $\nu - 1$ times, and $z < m_\Lambda$ $(z > M_\Lambda)$ if $C > 0$ $(C < 0)$. For all other values of the total quasi-momentum Λ, the operator $\tilde{H}_2'$ has at most $2\nu + 1$ BS's (taking the multiplicity of the energy levels into account) with the energy values lying outside the set G_Λ.

The proof of these statements is based on finding zeros of the function $\Delta_\Lambda^\nu(z)$. Expressing all integrals in $\Delta_\Lambda^\nu(z)$ via $J^\star(z)$, we can bring the equation $\Delta_\Lambda^\nu(z) = 0$ to the form

$$J^\star(z) = \frac{\mathscr{C}_\Lambda(z)}{\mathscr{D}_\Lambda(z)}, \tag{26}$$

where $\mathscr{D}_\Lambda(z)$ is the $(2\nu+1)$ th-order polynomial in z, and $\mathscr{C}_\Lambda(z)$ is also a polynomial in z whose order (with respect to $\mathscr{D}_\Lambda(z)$) is lower. The analyzing of equation (26) outside the set G_Λ leads to the proof of the above statements.

Theorem 7. *Let $A = 0$ and ν be arbitrary. Then the operator $\tilde{H}_2'$ has two BS's, φ_1 and φ_2, (not taking the multiplicity of energy levels into account) with the energy values $z_1 = -2C - 8s(2s-1)B\sum_{i=1}^{\nu}\cos^2\frac{\Lambda_i}{2}$ and $z_2 = -2C$. Moreover, z_1 is not degenerate, while z_2 is degenerative $\nu - 1$ times, and $z_i \notin G_\Lambda$, $i = 1, 2$, for all $\Lambda \in T^\nu$, i.e., the energy values of these BS's lie outside the continuous spectrum domain of the operator tilde$H_{2\Lambda}'$. When $B = 0$, this BS's vanishes because it is incorporated into the continuous spectrum.*

P r o o f. If $A = 0$, then $h_\Lambda(s) \equiv 0$, and

$$\Delta_\Lambda^\nu(z) = (1 + \frac{2C}{z})^{\nu-1}\{[1 + \frac{8s(2s-1)B\sum_{k=1}^{\nu}\cos^2\frac{\Lambda_k}{2}}{z}](1 + \frac{2C}{z}) - \frac{16s(2s-1)BC\sum_{k=1}^{\nu}\cos^2\frac{\Lambda_k}{2}}{z^2}\}.$$

Solving the equation $\Delta_\Lambda^\nu(z) = 0$, we prove the theorem.

Note. *In the theorem, the zero-order degeneracy corresponds to the case where there is no BS.*

Let $\widetilde{\pi} = (\pi; \pi; \ldots; \pi) \in T^\nu$.

Theorem 8. *Let $\Lambda = \widetilde{\pi}$, $\Lambda, \widetilde{\pi} \in T^\nu$ and $C \neq 0$. Then the operator $\tilde{H}_2'$ has only one BS φ with the energy value $z = 8sA\nu - 2C$, and this energy level is of multiplicity ν. In addition, if $C > 0$, then $z < m_\Lambda$, and if $C < 0$, then $z > M_\Lambda$. When $C = 0$, this BS vanishes because it is absorbed by the continuous spectrum.*

The proof is based on the equality $h_\Lambda(x) = 8sA\nu$ with $\Lambda = \widetilde{\pi}$ and also on the corresponding form of the function $\Delta_\Lambda^\nu(z) = (1 - \frac{2C}{8sA\nu - z})^\nu$ with $\Lambda = \widetilde{\pi}$.

Theorem 9. *Let $C = 0$, and ν be an arbitrary number. Then the operator $\tilde{H}_2'$ has at most one BS, the corresponding energy level is of multiplicity one, and $z \notin G_\Lambda$.*

P r o o f. If $C = 0$, the relations

$$h_{1\Lambda}(x;t) = -4s(2s-1)B\sum_{k=1}^{\nu}[1+\cos\Lambda_k - 2\cos\frac{\Lambda_k}{2}\cos(\frac{\Lambda_k}{2}-x_k)],$$

$$\Delta_{\Lambda}^{\nu}(z) = 1 - 4s(2s-1)B\int\limits_{T^\nu}\frac{g_\Lambda(s)ds}{h_\Lambda(s)-z},$$

where

$$g_\Lambda(s) = \sum_{k=1}^{\nu}[1+\cos\Lambda_k - 2\cos\frac{\Lambda_k}{2}\cos(\frac{\Lambda_k}{2}-s_k)],\ \Lambda\in T^\nu,\ s\in T^\nu,\ ds = ds_1 ds_2\dots ds_\nu,$$

hold. Using the form of the determinant $\Delta_{\Lambda}^{\nu}(z)$ and solving the corresponding equation, we get the proof of Theorem 9.

Besides, the qualitative pictures of the change of the energy spectrum of operator $\tilde{H}_2'$ in the cases for $s = 1/2$ and $s > 1/2$ are shown to be different. We also show that the energy spectrum of the system is the same either for integer and half-integer values of s or for odd and even values of s.

3. Structure of Essential Spectrum of Three-Particle System

We first determine the structure of the essential spectrum of a three-particle system consisting of two magnons and an impurity spin, and then estimate the number of thee-particle BS's in the system. Comparing formulas (2) and (7) and using the tensor products of the Hilbert spaces and the tensor products of the operators in Hilbert spaces [6], we can verify that the operator $\widetilde{H}_2$ can be represented in the form $\widetilde{H}_2 = \widetilde{H}_1 \bigotimes E + E \bigotimes \widetilde{H}_1 + K_1 + K_2$, where E is the unit operator in $\widetilde{\mathscr{H}_1}$, and K_1 and K_2 are the integral operators

$$(K_1 f)(x;y) = \int\limits_{T^\nu} h_1(x;y;t)f(t;x+y-t)dt,$$

$$(K_2 f)(x;y) = \int\limits_{T^\nu}\int\limits_{T^\nu} h_4(x;y;s;t)f(s;t)dsdt.$$

The kernels of these operators have the forms

$$h_1(x;y;t) = -4s(2s-1)B\sum_{i=1}^{\nu}\{1+\cos(x_k+y_k) - 2\cos\frac{x_k+y_k}{2}\cos\frac{x_k-y_k}{2}\}$$

$$-4C\sum_{i=1}^{\nu}\{\cos\frac{x_k-y_k}{2} - \cos\frac{x_k+y_k}{2}\}\cos(\frac{x_k+y_k}{2}-t_k),\ \ x,y,t\in T^\nu,$$

and

$$h_4(x;y;s;t) = F\sum_{i=1}^{\nu}[1+\cos(x_i+y_i-s_i-t_i)+\cos(s_i+t_i)+\cos(x_i+y_i)$$

$$-\cos(x_i-s_i-t_i)-\cos(y_i-s_i-t_i)-\cos x_i-\cos y_i]+Q\sum_{i=1}^{\nu}[\cos(x_i-t_i)+\cos(y_i-s_i)]$$

$$+M\sum_{i=1}^{\nu}[\cos(x_i-s_i)+\cos(y_i-t_i)]+N\sum_{i=1}^{\nu}[\cos s_i+\cos t_i+\cos(x_i+y_i-s_i)$$

$$+\cos(x_i+y_i-t_i)],$$

here $B = J_2-(6s-1)J_3+(28s^2-10s+1)J_4-(120s^3-68s^2+14s-1)J_5+\dots$, $C = J_1+(4s^2-6s+1)J_2-(24s^3-32s^2+10s-1)J_3+(112s^4-160s^3+72s^2-14s+1)J_4-(480s^5-768s^4+448s^3-128s^2+18s-1)J_5+\dots$, $F = (2s-4s^2)(J_2^0-J_2)+(2s-16s^2+24s^3)(J_3^0-J_3)+\dots+\dots$, $Q = (-4s^2+2s)(J_2^0-J_2)+(-4s+20s^2-24s^3)(J_3^0-J_3)+\dots+\dots$, $M = 2[(J_1^0-J_1)-(1+5s+2s^2)(J_2^0-J_2)+(1-8s+22s^2-12s^3)(J_3^0-J_3)+\dots+\dots]$, $N = -(J_1^0-J_1)+(1-6s+4s^2)(J_2^0-J_2)-(1-10s+32s^2-24s^3)(J_3^0-J_3)+\dots+\dots$.

As we have already mentioned, for the fixed total quasi-momentum $x+y=\Lambda$ of the two-magnon subsystem, the operator H_2' and the space $\mathscr{H}_2$ can be decomposed into direct integrals $\widetilde{H'}_2 = \bigoplus\int_{T^\nu}\widetilde{H'}_{2\Lambda}d\Lambda$, $\widetilde{\mathscr{H}_2} = \bigoplus\int_{T^\nu}\widetilde{\mathscr{H}_2}_{\Lambda}d\Lambda$, such that the operators $K_{1\Lambda}$ become compact after the decomposition.

It can be seen from the expressions for the kernels of K_1 and K_2 that $K_{1\Lambda}$ and K_2 are finite-rank operators, i.e., finite-dimensional operators. Therefore, the essential spectra of $\widetilde{H}_2$ and $\widetilde{H}_1\bigotimes E+E\bigotimes\widetilde{H}_1$ coincide. A simple verification shows that the spectrum of $\widetilde{H}_1$ is independent of Λ, i.e., of λ and μ. The spectrum of $A\bigotimes E+E\bigotimes B$, where A and B are densely defined bounded linear operators, was studied in [6-8]. In these papers there were also given the explicit formulas expressing $\sigma_{ess}(A\bigotimes E+E\bigotimes B)$ and $\sigma_{disc}(A\bigotimes E+E\bigotimes B)$ in terms of $\sigma(A), \sigma_{disc}(A), \sigma(B)$, and $\sigma_{disc}(B)$:

$$\sigma_{disc}(A\bigotimes E+E\bigotimes B) = \{(\sigma(A)\backslash\sigma_{ess}(A))+(\sigma(B)\backslash\sigma_{ess}(B))\}\backslash\{(\sigma_{ess}(A)$$

$$+\sigma(B))\bigcup(\sigma(A)+\sigma_{ess}(B))\},$$

$$\sigma_{ess}(A\bigotimes E+E\bigotimes B) = (\sigma_{ess}(A)+\sigma(B))\bigcup(\sigma(A)+\sigma_{ess}(B)).$$

It is clear that $\sigma(A\bigotimes E+E\bigotimes B) = \{\lambda+\mu : \lambda\in\sigma(A), \mu\in\sigma(B)\}$.

It can be seen from the results of [1] that the spectrum of $\widetilde{H}_1$ consists of the continuous spectrum and at most three eigenvalues of multiplicity one, multiplicity $(\nu-1)$, and multiplicity ν.

First we prove the theorem on the finite-dimensional perturbations of bounded linear operators in Banach spaces.

Theorem 10. *Let A and B be the linear bounded self-adjoint operators with the difference of the self-adjoint operator with finite rank m. Then $\sigma_{ess}(A) = \sigma_{ess}(B)$, and at most m eigenvalues appear (taking into account their degeneration multiplicities).*

P r o o f. Let $C = A - B$. As C is a self-adjoint operator of rank m, the function $C(A-z)^{-1}$ is analytical and it has the value of the operator of rank at most m in $\mathbb{C}\backslash\sigma(A)$. It is meromorphic in $\mathbb{C}\backslash\sigma_{ess}(A)$ with finite-rank residues at points in $\sigma_{disc}(A)$. If $z \notin \sigma(A)$, then $(B-z)^{-1}$ exists if and only if there exists $(1 - C(A-z)^{-1})^{-1}$. We can conclude that in every component of $\mathbb{C}\backslash\sigma(A)$ the operator $(1 - C(A-z)^{-1})^{-1}$ is somewhere reversible. The components $\mathbb{C}\backslash\sigma(A)$ and $\mathbb{C}\backslash\sigma_{ess.}(A)$ coincide because of the discreteness of $\sigma_{disc}(A)$. By the Fredholm meromorphic theorem, the operator $(1 - C(A-z)^{-1})^{-1}$ exists on $\mathbb{C}\backslash\sigma_{ess}(A)$ everywhere, but the discrete set D' where it has finite rank residues. Here $D' = \sigma_{disc}(A)\bigcup D''$, where D'' consists of no more than m points, since the operator $C(A-z)^{-1}$ can have an eigenvalue equal to 1 with multiplicity no more than m. It follows that the operator B can have only a discrete spectrum in $\mathbb{C}\backslash\sigma_{ess}(A)$ such that $\sigma_{ess}(B) \subset \sigma_{ess}(A)$.

Every component of $\mathbb{C}\backslash\sigma_{ess}(B)$ has the points lying neither in $\sigma(A)$ nor in $\sigma(B)$. As C is a self-adjoint operator of rank m, the function $C(B-z)^{-1}$ is analytical and has the values of the operator of rank no more than m in $\mathbb{C}\backslash\sigma(B)$. It is meromorphic in $\mathbb{C}\backslash\sigma_{ess}(B)$ with the finite rank residues at the points of $\sigma_{disc}(B)$. If $z \notin \sigma(B)$, then $(A-z)^{-1}$ exists if and only if there exists $(1+C(B-z)^{-1})^{-1}$. One can conclude that in every component of $\mathbb{C}\backslash\sigma(B)$, the operator $(1+C(B-z)^{-1})^{-1}$ is somewhere reversible. The components $\mathbb{C}\backslash\sigma(B)$ and $\mathbb{C}\backslash\sigma_{ess}(B)$ coincide because of the discreteness $\sigma_{disc}(B)$. By the Fredholm meromorphic theorem, the operator $(1+C(B-z)^{-1})^{-1}$ exists in $\mathbb{C}\backslash\sigma_{ess}(B)$ everywhere except the discrete set D_1 where it has finite-rank residues. Here $D_1 = \sigma_{disc}(B)\bigcup D_2$, where D_2 consists of at most m points, since the operator $C(B-z)^{-1}$ can have an eigenvalue equal to -1 with the multiplicities at most m. Hence the operator A can have only a discrete spectrum in $\mathbb{C}\backslash\sigma_{ess}(B)$ such that $\sigma_{ess}(A) \subset \sigma_{ess}(B)$. Consequently, $\sigma_{ess}(A) = \sigma_{ess}(B)$. And we can conclude that when there are perturbations of self-adjoint operators with rank m, the essential spectrum of the operator exists, and at most m eigenvalues appear (taking into account their degeneration multiplicities).

Notice that the problems on the finite rank perturbations for the compact operators were considered in [9–11].

The theorems below describe the structure of the essential spectrum of $\widetilde{H}_1 \bigotimes E + E \bigotimes \widetilde{H}_1$ and give lower and upper estimations for N, the number of points of discrete spectrum of the operator $\widetilde{H}_2$.

Theorem 11. *If* $\nu = 1$ *and* $\omega \in A_1 \bigcup A_7$*, then the essential spectrum of the operator* $\widetilde{H}_2$ *consists of a single interval* $\sigma_{ess.}(\widetilde{H}_2) = [0; 4p(s)]$ *or* $\sigma_{ess}(\widetilde{H}_2) = [4p(s); 0]$*, and the relation* $0 \le N \le 12$ *holds for the number* N *of three-particle BBs.*

Theorem 12. *If* $\nu = 1$ *and* $\omega \in A_6$ *or* $\omega \in A_5$*, then the essential spectrum of the operator* $\widetilde{H}_2$ *consists of the union of two intervals,* $\sigma_{ess}(\widetilde{H}_2) = [0; 4p(s)] \bigcup [z_1; z_1 + 2p(s)]$ *or* $\sigma_{ess}(\widetilde{H}_2) = [4p(s); 0] \bigcup [z_1; z_1 + 2p(s)]$*, and the relation* $1 \le N \le 13$ *holds for the number* N *of the three-particle operator.*

Theorem 13. *If* $\nu = 1$ *and* $\omega \in A_2 \bigcup A_3$ *or* $\omega \in A_4 \bigcup A_8$*, then the essential spectrum of the operator* $\widetilde{H}_2$ *consists of the union of three intervals,* $\sigma_{ess}(\widetilde{H}_2) = [0; 4p(s)] \bigcup [z_1; z_1 + 2p(s)] \bigcup [z_2; z_2 + 2p(s)]$*, or* $\sigma_{ess}(\widetilde{H}_2) = [4p(s); 0] \bigcup [z_1; z_1 + 2p(s)] \bigcup [z_2; z_2 + 2p(s)]$*, and the relation* $3 \le N \le 15$ *holds for the number* N *of the three-particle operator.*

Theorem 14. *If* $\nu = 2$ *and* $\omega \in B_1 \bigcup B_2$*, then the essential spectrum of the operator* $\widetilde{H}_2$ *consists of a single interval* $\sigma_{ess}(\widetilde{H}_2) = [0; 8p(s)]$*, or* $\sigma_{ess}(\widetilde{H}_2) = [8p(s); 0]$*, and the relation* $0 \le N \le 22$ *holds for the number* N *of the three-particle operator.*

Theorem 15. *If* $\nu = 2$ *and* $\omega \in B_3 \bigcup B_4$ *or* $\omega \in B_5 \bigcup B_6$*, then the essential spectrum of the operator* $\widetilde{H}_2$ *consists of the union of two intervals,* $\sigma_{ess}(\widetilde{H}_2) = [0; 8p(s)] \bigcup [z_1; z_1 + 4p(s)]$*, or* $\sigma_{ess}(\widetilde{H}_2) = [8p(s); 0] \bigcup [z_1; z_1 + 4p(s)]$*, and the relation* $1 \le N \le 23$ *holds for the number* N *of the three-particle operator.*

Theorem 16. *If* $\nu = 2$ *and* $\omega \in B_7 \bigcup B_8$ *or* $\omega \in B_9 \bigcup B_{10}$*, then the essential spectrum of the operator* $\widetilde{H}_2$ *consists of the union of three intervals,* $\sigma_{ess}(\widetilde{H}_2) = [0; 8p(s)] \bigcup [z_1; z_1 + 4p(s)] \bigcup [z_2; z_2 + 4p(s)]$*, or* $\sigma_{ess}(\widetilde{H}_2) = [8p(s); 0] \bigcup [z_1; z_1 + 4p(s)] \bigcup [z_2; z_2 + 4p(s)]$*, and the relation* $3 \le N \le 25$ *holds for the number* N *of the three-particle operator.*

Theorem 17. *If* $\nu = 2$ *and* $\omega \in B_{11} \bigcup B_{12}$ *or* $\omega \in B_{13} \bigcup B_{14}$*, then the essential spectrum of the operator* $\widetilde{H}_2$ *consists of the union of four intervals,* $\sigma_{ess}(\widetilde{H}_2) = [0; 8p(s)] \bigcup [z_1; z_1 + 4p(s)] \bigcup [z_2; z_2 + 4p(s)] \bigcup [z_3; z_3 + 4p(s)]$*, or* $\sigma_{ess}(\widetilde{H}_2) = [8p(s); 0] \bigcup [z_1; z_1 + 4p(s)] \bigcup [z_2; z_2 + 4p(s)] \bigcup [z_3; z_3 + 4p(s)]$*, and the relation* $6 \le N \le 28$ *holds for the number* N *of the three-particle operator.*

Theorem 18. *If* $\nu = 3$ *and* $\omega \in Q_1 \bigcup Q_2 \bigcup Q_3 \bigcup Q_4$*, then the essential spectrum of the operator* $\widetilde{H}_2$ *consists of a single interval* $\sigma_{ess}(\widetilde{H}_2) = [0; 12p(s)]$ *or* $\sigma_{ess}(\widetilde{H}_2) = [12p(s); 0]$*, and the relation* $0 \le N \le 32$ *holds for the number* N *of three-particle BBs.*

Theorem 19. *If* $\nu = 3$ *and* $\omega \in Q_5 \bigcup Q_6$ *or* $\omega \in Q_7 \bigcup Q_8$*, then the essential spectrum of the operator* $\widetilde{H}_2$ *consists of the union of two intervals,* $\sigma_{ess}(\widetilde{H}_2) =$

$[0;12p(s)]\bigcup[z_1;z_1+6p(s)]$, or $\sigma_{ess}(\widetilde{H}_2)=[12p(s);0]\bigcup[z_1;z_1+6p(s)]$, and the relation $1\le N\le 33$ holds for the number N of the three-particle operator.

Theorem 20. *If $\nu=3$ and $\omega\in Q_9\bigcup Q_{10}$ or $\omega\in Q_{11}\bigcup Q_{12}$, then the essential spectrum of the operator $\widetilde{H}_2$ consists of the union of three intervals, $\sigma_{ess}(\widetilde{H}_2)=[0;12p(s)]\bigcup[z_1;z_1+6p(s)]\bigcup[z_2;z_2+6p(s)]$, or $\sigma_{ess}(\widetilde{H}_2)=[12p(s);0]\bigcup[z_1;z_1+6p(s)]\bigcup[z_2;z_2+6p(s)]$, and the relation $3\le N\le 35$ holds for the number N of the three-particle operator.*

Theorem 21. *If $\nu=3$ and $\omega\in Q_{13}\bigcup Q_{14}$ or $\omega\in Q_{15}\bigcup Q_{16}$, then the essential spectrum of the operator $\widetilde{H}_2$ consists of the union of four intervals, $\sigma_{ess}(\widetilde{H}_2)=[0;12p(s)]\bigcup[z_1;z_1+6p(s)]\bigcup[z_2;z_2+6p(s)]\bigcup[z_3;z_3+6p(s)]$, or $\sigma_{ess}(\widetilde{H}_2)=[12p(s);0]\bigcup[z_1;z_1+6p(s)]\bigcup[z_2;z_2+6p(s)]\bigcup[z_3;z_3+6p(s)]$, and the relation $6\le N\le 38$ holds for the number N of the three-particle operator.*

P r o o f. The proofs of Theorems 11-21 are similar. Therefore we prove one of the theorems. As an example, we prove Theorem 21. From Theorem 3 (in statement (iv)) from [1], it is seen that for $\omega\in Q_{13}\bigcup Q_{14}$ $(\omega\in Q_{15}\bigcup Q_{16})$ the operator $\widetilde{H}_1$ has exactly three LIS's, φ_1,φ_2 and φ_3, with the energies z_1,z_2 and z_3 (z_4,z_5 and z_6) satisfying the inequalities $z_i<m_3$, $i=1,2,3$ ($z_j>M_3$, $j=4,5,6$). Moreover, the level z_1 (z_4) is of multiplicity one, the level z_2 (z_5) is of multiplicity two and the level z_3 (z_6) is of multiplicity three.

The continuous spectrum of the operator $\widetilde{H}_1$ consists of the interval $[0;6p(s)]$ or $[6p(s);0]$. Therefore, the essential spectrum of the operator $\widetilde{H}_2$ consists of a set $[0;6p(s)]+\{[0;6p(s)],z_1,z_2,z_3\}$, i.e., $\sigma_{ess}(\widetilde{H}_2)=[0;12p(s)]\bigcup[z_1;z_1+6p(s)]\bigcup[z_2;z_2+6p(s)]\bigcup[z_3;z_3+6p(s)]$. The numbers $2z_1,2z_2,2z_3,z_1+z_2,z_1+z_3,z_2+z_3$ are the eigenvalues of the operator $\widetilde{H}_1\bigotimes E+E\bigotimes\widetilde{H}_1$ and are outside the domain of the essential spectrum of $\widetilde{H}_1\bigotimes E+E\bigotimes\widetilde{H}_1$. It is clear that the multiplicity of their eigenvalues is at most $3\times 3=9$. Consequently, these six eigenvalues of the operator $\widetilde{H}_1\bigotimes E+E\bigotimes\widetilde{H}_1$ belong to the discrete spectrum of the considering three-particle operator.

Then, the operator $K_{1\Lambda}$ in the three-dimensional case is the seven-rank operator, while the rank of the operator K_2 is equal to 25. Consequently, as follows from Theorem 10, the number N of the points of discrete spectrum of the three-particle operator is not less than 6 and not more than $6+7+25=38$.

Theorem 22. *Let ν be an arbitrary number, $p(s)\equiv 0$, and $J_n\ne 0, n=1,2,\ldots,2s$. Then the essential spectrum of the operator $\widetilde{H}_2$ consists of three points, $\sigma_{ess}(\widetilde{H}_2)=\{0;\frac{q(s)}{2};\frac{2\nu+1}{2}q(s)\}$, and the relation $3\le N\le 10\nu+5$ holds for the number N of the points of discrete spectrum of the three-particle operator.*

P r o o f. When ν is an arbitrary number, $p(s)\equiv 0$, and $J_n\ne 0$, $n=1,2,\ldots,2s$, by Theorem 4 from [1], the operator $\widetilde{H}_1$ has two eigenvalues equal to $z_1=\frac{q(s)}{2}$ and $z_2=\frac{2\nu+1}{2}q(s)$, where z_1 is of multiplicity $(2\nu-1)$, while z_2 is of multiplicity one. The essential (continuous) spectrum of the operator $\widetilde{H}_1$ consists of a single point 0. Therefore, $\sigma_{ess}(\widetilde{H}_2) = \{0; \frac{q(s)}{2}; \frac{2\nu+1}{2}q(s)\}$, and the points $q(s); (2\nu+1)q(s); (\nu+1)q(s)$ are the eigenvalues of the operator $\widetilde{H}_1 \bigotimes E + E \bigotimes \widetilde{H}_1$. Now, taking into account that the operators $K_{1\Lambda}$ and K_2 are of ranks $2\nu+1$ and $8\nu+1$, respectively, we immediately obtain the proof of Theorem 22.

It should be noticed that if $h(x;y)$ is an arbitrary 2π-periodic continuous function, $h_2(x;s) = h_3(x;s)$ is an arbitrary degenerated 2π-periodic continuous kernel, and $h_1(x;y;t)$ and $h_4(x;y;s;t)$ are also arbitrary degenerated 2π-periodic continuous kernels, i.e., the operators $K_{1\Lambda}$ and K_2 are arbitrary finite-dimensional operators, then the analogous results are true.

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