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On Analytic and Subharmonic Functions in Unit Disc Growing Near a Part of the Boundary

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In the paper there was found an analog of the Blaschke condition for analytic and subharmonic functions in the unit disc, which grow at most as a given function φ near some subset of the boundary.

Key words: analytic function, subharmonic function, Riesz measure, Blaschke condition.

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1. Introduction

It is well known that the zeroes $\{z_n\}$ of any bounded analytic function in the unit disk $\mathbb{D}$ satisfy the Blaschke condition

$$\sum(1-|z_n|)<\infty. \tag{1}$$

There have been a number of papers published where there can be found the analogous conditions for various classes of unbounded analytic and subharmonic functions (see, for example, [6–9]). In [5], there was studied the case of analytic functions in $\mathbb{D}$ of an exponential growth near a finite subset $E\subset\partial\mathbb{D}$. In [1], the case of an arbitrary compact subset $E\subset\partial\mathbb{D}$ was considered. Namely, there were considered subharmonic functions v in $\mathbb{D}$ such that

$$v(z)\le \operatorname{dist}(z,E)^{-q},\quad z\in\mathbb{D}, \tag{2}$$

for some $0<q<\infty$. The Riesz measures (generalized Laplacians) $\mu=\dfrac{1}{2\pi}\triangle v$ are proven to satisfy the condition

$$\int\limits_{\mathbb{D}}(1-|\lambda|)(\operatorname{dist}(\lambda,E))^{(q-\alpha)_+}d\mu(\lambda)<\infty \tag{3}$$

for α such that

$$\int_0^2 \frac{m\{s \in \partial\mathbb{D} : \operatorname{dist}(s,E) < t\}dt}{t^{\alpha+1}} < \infty. \tag{4}$$

Here $x_+ = \max\{0, x\}$, and m is the normalized Lebesgue measure on $\partial\mathbb{D}$, that is, $m(\partial\mathbb{D}) = 1$. If $m(E) > 0$, then (3) and (4) are valid for any $\alpha < 0$. If (4) is valid with some $\alpha > 0$ and $q \le \alpha$, then μ satisfies the Blaschke condition for bounded from above in $\mathbb{D}$ subharmonic functions

$$\int (1 - |\lambda|)d\mu(\lambda) < \infty. \tag{5}$$

It was also proved in [1] that (3) is not valid for the subharmonic function $v_0(z) = \operatorname{dist}(z,E)^{-q}$ in the case of divergent integral in (4).

If $f(z)$ is an analytic function, then $\log|f(z)|$ is a subharmonic function with the Riesz measure $\sum_n k_n\delta_{z_n}$, where δ_{z_n} are the unit masses in the zeros $\{z_n\}$ of $f(z)$, and k_n are the multiplicities of the zeros. Hence, if this is the case, (2) has the form

$$|f(z)| \le \exp\operatorname{dist}(z,E)^{-q}, \tag{6}$$

while (3) has the form

$$\sum_{z_n}(1 - |z_n|)\operatorname{dist}(z_n,E)^{(q-\alpha)_+} < \infty. \tag{7}$$

Note that condition (6) seems to be too restrictive to be applied to the operator theory [1]. It is natural to consider subharmonic (and analytic) functions such that

$$v(z) \leqslant \varphi(\operatorname{dist}(z,E)), \tag{8}$$

where $\varphi(t)$ is the monotonically decreasing on $\mathbb{R}^+$ function, $\varphi(t) \to +\infty$ as $t \to +0$. It is clear that for any subharmonic function v in $\mathbb{D}$, which grows as $\operatorname{dist}(z,E) \to 0$, condition (8) is valid for $\varphi(t) = \sup\{v(z) : \ \operatorname{dist}(z,E) \ge t\}$.

The Main Results

To formulate our results we need some notations. Let

$$\rho(z) = \operatorname{dist}(z,E), \quad F(t) = F_E(t) = m\{\zeta \in \partial\mathbb{D} : \rho(\zeta) < t\}.$$

Consider

$$I(\varphi,E) := \int_0^2 \varphi(s)dF(s).$$

Note that $F(t)$ is continuous on $[0, 2]$ and has a discontinuity at $t = 0$ if and only if $m(E) > 0$.

Theorem 1. *Let E be a closed subset of $\partial\mathbb{D}$, v be a subharmonic function in $\mathbb{D}$, $v \not\equiv -\infty$, and $v(z)$ satisfy* (8)*. If $I(\varphi, E) < \infty$, then the Riesz measure μ of the function $v(z)$ satisfies* (5)*.*

E x a m p l e 1. Let $E = \{\zeta_1, \dots, \zeta_k\}$, $\varphi(t) = t^{-1}\log^{-\alpha}(1/t)$, $\alpha > 1$. It is readily seen that $F(t) = 2kt/(2\pi) + o(1)$ as $t \to 1$. The assumption of Theorem 1 is fulfilled and the Riesz measure of any subharmonic function $v(z) \le \rho^{-1}(z)\log^{-\alpha}(1/\rho(z))$ satisfies the Blaschke condition (5).

Theorem 2. *Let E, v and μ satisfy the assumption of Theorem 1, $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an absolutely continuous nonnegative monotonically decreasing function, $\varphi(t) \to +\infty$ as $t \to +0$, and $\psi(t)$ be an absolutely continuous monotonically increasing function on $[0, 2]$ such that $\psi(0) = 0$. If*

$$\int\limits_0^{1/25} \psi(50y)\varphi'(y)F(y)dy > -\infty, \tag{9}$$

then

$$\int\limits_{\mathbb{D}} \psi(\rho(\lambda))(1 - |\lambda|)d\mu(\lambda) < \infty. \tag{10}$$

R e m a r k 1. In the case $m(E) > 0$, we always get $I(\varphi, E) = \infty$ and thus the assumption of Theorem 1 is not satisfied. Further, (9) takes the form

$$\int\limits_0^{1/25} \psi(50y)\varphi'(y)dy > -\infty.$$

In the case $E = \partial\mathbb{D}$, the result coincides with that obtained in [8].

E x a m p l e 2. Let $E = \{\zeta_1, \dots, \zeta_k\}$, $\varphi(t) = e^{1/t}$. The assumption of Theorem 2 is satisfied with $\psi(t) = e^{-c/t}$ for $c > 50$. Hence, if v is a subharmonic function $v(z)$ such that $v(z) \le \exp(1/\rho(z))$, then for $c > 50$ the integral

$$\int \exp(-c/\rho(z))(1 - |\lambda|)d\mu(\lambda)$$

converges.

The following theorem shows the accuracy of Theorem 2.

Theorem 3. *Let E, φ, ψ be the same as above and, in addition, $\varphi(1/t)$ be convex with respect to* $\log t$. *If*

$$\int_0^2 \psi(y)\varphi'(y)F(y)dy = -\infty, \tag{11}$$

then for the Riesz measure μ_0 of the function $v_0(z) = \varphi(\rho(z))$ we get

$$\int \psi(\rho(\lambda))(1-|\lambda|)d\mu_0(\lambda) = +\infty.$$

Now consider the case $v(z) = \log|f(z)|$ with the analytic function $f(z)$.

Theorem 1′. *Let E be a closed subset of $\partial\mathbb{D}$, $\varphi(t)$ be an absolutely continuous nonnegative monotonically decreasing function on $\mathbb{R}^+$, $\varphi(t) \to +\infty$ as $t \to +0$, $I(\varphi, E) < \infty$, f be an analytic function in $\mathbb{D}$ with zeros z_n. If*

$$|f(z)| \leqslant \exp(\varphi(\rho(z)),$$

then its zeros satisfy the Blaschke condition (1).

Theorem 2′. *Let E, f, φ be the same as in Theorem 1′, but the condition $I(\varphi, E) < \infty$ for some absolutely continuous monotonically increasing function ψ on* $[0, 2]$ *such that* $\psi(0) = 0$ *be replaced by* (9). *Then*

$$\sum_{z_n} \psi(\rho(z_n, E))(1-|z_n|) < \infty.$$

R e m a r k 2. For $\varphi(t) = t^{-q}$, $\psi(t) = t^s$, Theorems 1–3, 1′, 2′ were proved earlier in [1].

2. Demonstrations

We begin with the lemmas.

Lemma 1. *Let ν be a nonnegative finite measure on X, $g(x)$ be a Borel function on X, $\varphi(t)$ be a Borel function on $\mathbb{R}$. Then*

$$\int_X (\varphi \circ g)(x)d\nu(x) = \int_{-\infty}^{\infty} \varphi(y)dH(y),$$

where $H(y) = \nu\{x : g(x) < y\}$.

The proof of this lemma can be easily reduced to the case $\nu(X) = 1$ which is well known (see, for example, [2, formula (15.3.1)]).

Lemma 2. *For each $z \in \mathbb{D}$ and $\tau \in [0,1]$ we have $\rho(z) \leqslant 2\rho(\tau z)$.*

P r o o f. Consider $\zeta = z/|z|$ and $z' \in [0,\zeta]$ such that $\rho(z') = dist(E,[0,\zeta])$. The cases $z' = 0$ and $z \in [0,z']$ are trivial. Suppose $z \in (z',\zeta)$. We have

$$\rho(z) \leqslant \rho(z') + |z' - z| \leqslant \rho(z') + |z' - \zeta|.$$

Consider $\zeta' \in E$ such that $\rho(z') = |z' - \zeta'|$ and the triangle with vertexes on z', ζ, ζ'. The angle in z' is right, and the angle in ζ is greater than $\pi/4$. Thus, $|z' - \zeta| \leqslant |z' - \zeta'|$ and therefore $\rho(z) \leqslant 2\rho(z') \leq 2\rho(\tau z)$.

Note that the result of the lemma was used in [1, p.41] without proof.

P r o o f of Theorem 1. First, suppose $v(0) = 0$. Let $I(\varphi, E) < \infty$. Using Lemma 1 with the measure $m(\zeta)$ on $\partial\mathbb{D}$ and taking into account the equalities $F(y) \equiv 0$ for $y < 0$ and $F(y) \equiv 1$ for $y > 2$, we get

$$\int\limits_{\partial\mathbb{D}} (\varphi \circ \rho)(\zeta) dm(\zeta) = \int\limits_0^\infty \varphi(y) dF(y) = \int\limits_0^2 \varphi(y) dF(y) = I(\varphi, E) < \infty. \tag{12}$$

First prove that there exists a harmonic majorant for v in $\mathbb{D}$. Using the properties of the Poisson integral

$$U(z) = \int\limits_{\partial\mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \varphi(\rho(\zeta))\, dm(\zeta),$$

we obtain

$$\lim_{z \to \zeta} U(z) = \varphi(\rho(\zeta)), \quad \zeta \in \partial\mathbb{D} \backslash E.$$

Hence $\overline{\lim}_{z\to\zeta}(v(z) - U(z)) \leqslant 0$ for $\zeta \in \partial\mathbb{D}\backslash E$ and $\lim_{z\to\zeta\in E} U(z) = +\infty$.

Let $\Omega_t,\ t \in (0,1)$, be the connected component of the set $\{z \in \mathbb{D} : \rho(z) > t\}$ such that $0 \in \Omega_t$. Put

$$\Gamma_t = \{z \in \mathbb{D} : \rho(z) = t\}, \quad E_t = \{\zeta \in \partial\mathbb{D} :\ \rho(z) < t\}, \quad E_t^c = \partial\mathbb{D}\backslash E_t.$$

Since E_t is a finite union of disjoint open sets, it is seen that E_t^c is a finite union of disjoint closed sets. Moreover, $\partial\Omega_t \subset E_t^c \cup \Gamma_t$. Note that for the connected $\{z \in \mathbb{D} : \rho(z) > t\}$ we have $\partial\Omega_t = E_t^c \cup \Gamma_t$.

For $z \in \Gamma_t$, take $\zeta' \in E$ such that $|z - \zeta'| = t$. Put $\gamma_z = \{\zeta \in \partial\mathbb{D} : |\zeta - \zeta'| \leqslant t\}$. Clearly, $\gamma_z \subset \overline{E_t}$. Since the function $\varphi(t)$ is positive and $\varphi(\rho(\zeta)) \geq \varphi(t)$ on E_t , we get

$$U(z) \geq \int\limits_{\gamma_z} \frac{1 - |z|^2}{|\zeta - z|^2} \varphi(\rho(\zeta)) dm(\zeta) \geq \varphi(t)\omega(z, \gamma_z, \mathbb{D}),$$

where

$$\omega(\lambda, \gamma_z, \mathbb{D}) = \int\limits_{\gamma_z} \frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} dm(\zeta)$$

is the harmonic measure at the point $\lambda = z$ with respect to the arc γ_z (see, for example, [3, §4.3]). On the other hand, the harmonic measure at any point $\lambda \in \mathbb{D}$ with respect to the arc γ_z equals $\beta/(2\pi)$, where $\beta = \beta(\lambda)$ is the length of the arc of the unit circle formed by the strait lines passing through λ and the ends of γ_z (see [10, chapter I, §5]). By direct calculations, $\beta(z) = \pi - 2\arcsin(t/2)$. Since $0 < t \le 1$, we obtain $\omega(z, \gamma_z, \mathbb{D}) \ge 1/3$ and

$$U(z) \geqslant \varphi(t)\, \omega(z, \gamma_z, \mathbb{D}) \geqslant \frac{1}{3}\, \varphi(t) \geqslant \frac{v(z)}{3}, \quad z \in \Gamma_t.$$

Therefore,

$$\overline{\lim}_{z\to\zeta}[v(z) - 3U(z)] \le 0,\ \zeta \in \Gamma_t, \qquad \overline{\lim}_{z\to\zeta}[v(z) - U(z)] \le 0,\ \zeta \in E_t^c.$$

Using the maximum principle, we get $v(z) \leqslant 3U(z)$ for all $z \in \Omega_t$.

Furthermore, by the Green formula [3, Theorem 4.5.4], we have

$$v(z) = u_t(z) - \int\limits_{\Omega_t} G_{\Omega_t}(z, \lambda) d\mu(\lambda), \tag{13}$$

where u_t is the least harmonic majorant of v in Ω_t. We have $u_t(z) \leqslant 3U(z)$ for $z \in \Omega_t$. The Green function $G_{\Omega_t}(z, \lambda)$ in Ω_t is equal to

$$G_{\Omega_t}(z, \lambda) = \log 1/(|z - \lambda|) - h_t(z, \lambda),$$

where $h_t(z, \lambda)$ is the harmonic function in Ω_t with the boundary values $\log 1/|z - \lambda|$. By [1, proof of Theorem 1], $G_{\Omega_t}(0, \lambda) \ge (1 - |\lambda|)/6$ for $\lambda \in \Omega_{kt}$ with $k = 25 > 6\pi + 3$. Consequently,

$$\begin{aligned} \int\limits_{\Omega_{kt}} (1 - |\lambda|) d\mu(\lambda) &\leqslant 6 \int\limits_{\Omega_t} G_{\Omega_t}(0, \lambda) d\mu(\lambda) = 6u_t(0) \\ &\leqslant 18U(0) = 18 \int_{\partial\mathbb{D}} \varphi(\rho(\zeta)) dm(\zeta) = 18I(\varphi, E). \end{aligned}$$

The later inequality is valid for all $t > 0$, hence we obtain the statement of the theorem for the case $v(0) = 0$.

If $v(0) > 0$, consider the function $v(z) - v(0)$ instead of $v(z)$.

If $-\infty < v(0) < 0$, consider the function $v_1(z) = \varphi(2)\dfrac{v(z) - v(0)}{\varphi(2) - v(0)}$. We have

$$v_1(z) \leqslant \varphi(\rho(z)) \frac{1 - v(0)/\varphi(\rho(z))}{1 - v(0)/\varphi(2)} \leqslant \varphi(\rho(z)).$$

Notice that the Riesz measure of the function $v_1(z)$ coincides with the Riesz measure of the function $v(z)$ up to a constant depending only on $v(0)$. Hence the Blaschke condition (5) for v_1 implies the same condition for v.

If $v(0) = -\infty$, consider the harmonic function $h(z)$ in the disk $\{|z| < 1/2\}$ such that $h(z) = v(z)$ for $|z| = 1/2$ and put

$$v_1(z) = \begin{cases} \max(v(z), h(z)), |z| < \frac{1}{2} \\ v(z), |z| \geqslant \frac{1}{2}. \end{cases}$$

Clearly,

$$v_1(z) \leqslant \max_{|z|=1/2} v(z) \leqslant \varphi(1/2) \quad \text{for } |z| \leq 1/2 \quad \text{and} \quad v_1(0) \neq -\infty.$$

In addition, $v_1(z)$ is subharmonic in $\mathbb{D}$ (see [3, Theorem 2.4.5]) and the restriction of its Reisz measure μ_1 to the set $\{z \in \mathbb{D} : |z| > \frac{1}{2}\}$ is equal to μ. Applying the proved statement to the function $\varphi_1(z) = \max\{\varphi(z), \varphi(1/2)\}$, we obtain

$$\int_{\mathbb{D}} (1 - |\lambda|) d\mu_1(\lambda) < \infty.$$

Therefore the integral in (5) is also finite. Theorem 1 is proved.

P r o o f of Theorem 2. Arguing as above, we can consider only the case $v(0) = 0$.

Let $\Omega_t,\ \Gamma_t,\ E_t,\ E_t^c,\ \gamma_z,\ \omega(\lambda, \gamma_z, \mathbb{D})$ be the same as in the proof of Theorem 1. For $z \in \mathrm{D}$, put

$$V_t(z) = \int_{E_t^c} \frac{1 - |z|^2}{|\zeta - z|^2}\, \varphi(\rho(\zeta)) dm(\zeta) + \varphi(t) \int_{E_t} \frac{1 - |z|^2}{|\zeta - z|^2}\, dm(\zeta). \tag{14}$$

Since $\gamma_z \subset \overline{E_t}$, we get

$$V_t(z) \geqslant \varphi(t)\, \omega(z, \gamma_z, \mathbb{D}) \geqslant \frac{1}{3}\, \varphi(t).$$

Therefore,

$$\limsup_{z \to \zeta} v(z) \leqslant \lim_{z \to \zeta} 3V_t(z) = 3V_t(\zeta), \quad \zeta \in \partial\Omega_t.$$

Using the maximum principle, we get $v(z) \leqslant 3V_t(z)$ for all $z \in \Omega_t$, in particular, $v(0) \leqslant 3V_t(0)$. Clearly, we have

$$V_t(0) = \int_{\partial\mathbb{D}} V_t(\zeta) dm(\zeta) = \int_{E_t} \varphi(t) dm(\zeta) + \int_{E_t^c} \varphi(\rho(\zeta)) dm(\zeta)$$

$$= \varphi(t) F(t) + \int_{E_t^c} \varphi(\rho(\zeta)) dm(\zeta).$$

Using Lemma 1 with $g(\zeta) = \rho(\zeta)$ and taking into account the equality

$$H(y) = m\{\zeta : \rho(\zeta) < y\} - m\{\zeta : \rho(\zeta) \leqslant t\} = F(y) - F(t),$$

we get

$$\varphi(t)F(t) + \int\limits_{E_t^c} \varphi(\rho(\zeta))dm(\zeta) = \varphi(t)F(t) + \int\limits_t^2 \varphi(y)dF(y).$$

Integrating by parts, we get

$$\begin{aligned} V_t(0) &= \varphi(t)F(t) + \varphi(2)F(2) - \varphi(t)F(t) - \int\limits_t^2 \varphi'(y)F(y)dy \\ &= \varphi(2) - \int\limits_t^2 \varphi'(y)F(y)dy. \end{aligned} \tag{15}$$

Therefore, using the Green formula (13) and the estimate $u_t(z) \leq 3V_t(z)$ of the least harmonic majorant u_t in Ω_t, we get

$$\int\limits_{\Omega_t} G_{\Omega_t}(0,\lambda)d\mu(\lambda) = u_t(0) \leqslant 3V_t(0) = 3\left(\varphi(2) - \int\limits_t^2 \varphi'(y)F(y)dy\right). \tag{16}$$

By [1, proof of Theorem 1], $G_{\Omega_t}(0,\lambda) \geq (1-|\lambda|)/6$ for $\lambda \in \Omega_{kt}$ with $k = 25 > 6\pi + 3$. Therefore,

$$\int\limits_{\Omega_{kt}} (1-|\lambda|)d\mu(\lambda) \leqslant 18\left(\varphi(2) - \int\limits_t^2 \varphi'(y)F(y)dy\right) \tag{17}$$

if $kt \in (0,1)$. In particular, the measure $(1-|\lambda|)d\mu(\lambda)$ of the set $\{\lambda \in \mathbb{D} : \rho(\lambda) \geqslant \varepsilon\}$ is finite for each $\varepsilon > 0$. Applying Lemma 1 with the function $g = \rho$ and taking into account that $\rho(\lambda) \leq 2$, we get

$$\int\limits_{\{\lambda\in\mathbb{D}:\rho(\lambda)\geqslant\varepsilon\}} \psi(\rho(\lambda))(1-|\lambda|)d\mu(\lambda) = \int\limits_\varepsilon^2 \psi(t)dG(t)$$

with $G(t) = \int\limits_{\{\lambda\in\mathbb{D}:\varepsilon\leqslant\rho(\lambda)<t\}} (1-|\lambda|)d\mu(\lambda)$. We have

$$\int\limits_{\varepsilon}^{2} \psi(t)dG(t) = -\int\limits_{\varepsilon}^{2} \psi(t)d\left(\int\limits_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)d\mu(\lambda)\right)$$

$$= \psi(\varepsilon)\int\limits_{\{\lambda:\rho(\lambda)\geq\varepsilon\}} (1-|\lambda|)d\mu(\lambda) + \int\limits_{\varepsilon}^{2} \psi'(t)\left(\int\limits_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)d\mu(\lambda)\right)dt. \quad (18)$$

We claim that under the condition

$$\int\limits_{0}^{2} \psi'(t)\left(\int\limits_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)d\mu(\lambda)\right)dt < \infty$$

we have

$$\psi(\varepsilon)\int\limits_{\{\lambda:\rho(\lambda)\geq\varepsilon\}} (1-|\lambda|)d\mu(\lambda) \to 0 \quad \text{as} \quad \varepsilon\to 0. \quad (19)$$

Indeed, for any $\eta > 0$ and sufficiently small positive $\delta < \varepsilon < 2$,

$$\int\limits_{\delta}^{\varepsilon} \psi'(t)\left(\int\limits_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)d\mu(\lambda)\right)dt \leqslant \eta.$$

Therefore,

$$(\psi(\varepsilon)-\psi(\delta))\int\limits_{\{\lambda:\rho(\lambda)\geq\varepsilon\}} (1-|\lambda|)d\mu(\lambda) = \int\limits_{\delta}^{\varepsilon} \psi'(t)dt \int\limits_{\{\lambda:\rho(\lambda)\geq\varepsilon\}} (1-|\lambda|)d\mu(\lambda)$$

$$\leqslant \int\limits_{\delta}^{\varepsilon} \psi'(t)\left(\int\limits_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)d\mu(\lambda)\right)dt \leqslant \eta.$$

Passing to the limit $\delta\to 0$, we obtain

$$\psi(\varepsilon)\int\limits_{\{\lambda:\rho(\lambda)\geq\varepsilon\}} (1-|\lambda|)d\mu(\lambda) \leqslant \eta,$$

which proves (19).

By Lemma 2, if $\rho(z) > t$, then $\rho(\tau z) > t/2$ for all $z \in \mathbb{D}$ and $0 < \tau < 1$. Hence the interval $[0, z]$ belongs to the set $\{\zeta : \rho(\zeta) > t/2\}$, and $\{z : \rho(z) > t\} \subset \Omega_{t/2}$. Therefore,

$$\int\limits_{\varepsilon}^{2} \psi'(t) \left(\int\limits_{\{\rho(\lambda)>t\}} (1-|\lambda|)d\mu(\lambda) \right) dt \leqslant \int\limits_{\varepsilon}^{2} \psi'(t) \left(\int\limits_{\Omega_{t/2}} (1-|\lambda|)d\mu(\lambda) \right) dt.$$

By (18), to prove the convergence of the integral

$$\int\limits_{\mathbb{D}} \psi(\rho(\lambda))(1-|\lambda|)d\mu(\lambda),$$

it is sufficient to show the convergence of the integral

$$\int\limits_{0}^{2} \psi'(t) \left(\int\limits_{\Omega_{t/2}} (1-|\lambda|)d\mu(\lambda) \right) dt = 2k \int\limits_{0}^{1/k} \psi'(2kt) \left(\int\limits_{\Omega_{kt}} (1-|\lambda|)d\mu(\lambda) \right) dt.$$

We have

$$\int\limits_{0}^{1/k} \psi'(2kt) \left(\int\limits_{t}^{2} \varphi'(y)F(y)dy \right) dt = \frac{1}{2k}\psi(2) \int\limits_{1/k}^{2} \varphi'(y)F(y)dy$$

$$- \lim_{t\to 0} \frac{\psi(2kt)}{2k} \int\limits_{t}^{2} \varphi'(y)F(y)dy + \frac{1}{2k} \int\limits_{0}^{1/k} \psi(2ky)\varphi'(y)F(y)dy$$

$$\geqslant \text{const} + \frac{1}{2k} \int\limits_{0}^{1/k} \psi(2ky)\varphi'(y)F(y)dy.$$

By the condition of the theorem, the last integral is finite. The proof is complete.

P r o o f of Theorem 3. Note that the function $-\log\rho(z)$ is subharmonic, hence the function $\varphi(\rho(z)) = \varphi\left(\frac{1}{e^{-\log\rho(z)}}\right)$ is subharmonic as well.

Using the Green formula (13) for the function $v_0(z)$ in the domain Ω_t, we get

$$\varphi(1) = v_0(0) = u_t^0(0) - \int\limits_{\Omega_t} (\log 1/|\lambda| - h_t(0,\lambda))\, d\mu_0(\lambda), \tag{20}$$

where $u_t^0(z)$ is the least harmonic majorant for $v_0(z)$ in Ω_t, and $h_t(0, \lambda)$ is the solution of Dirichlet problem in Ω_t with the boundary values $\log 1/|\lambda|$. Clearly,

$h_t(0,\lambda) \geqslant 0$. On the other hand, if $V_t(z)$ is the defined in (14) harmonic function in $\mathbb{D}$, then $\lim_{z\to\zeta} V_t(z) = v_0(\zeta)$ for $\zeta \in E_t^c$ and $V_t(z) \leqslant \varphi(t)$ in $\mathbb{D}$. Hence, $V_t(z) \leq v_0(z)$ on Γ_t and $V_t(z) \leqslant u_t^0(z)$ in Ω_t. Combining equality (15) for $V_t(0)$ with (20), we get

$$\varphi(2) - \int_t^2 \varphi'(y)F(y)dy \leqslant \varphi(1) + \int_{\Omega_t} \log\frac{1}{|\lambda|} d\mu_0(\lambda).$$

Note that $\Omega_t \subset \{\lambda : \rho(\lambda) > t\}$ and $\log\dfrac{1}{|\lambda|} \leqslant 2(1-|\lambda|)$ for $|\lambda| \geq 1/2$. Hence,

$$-\int_t^2 \varphi'(y)F(y)dy \leqslant 2 \int_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)d\mu_0(\lambda) + \text{const}. \tag{21}$$

On the other hand, arguing as in the proof of Theorem 2, we get

$$\int_{\{\lambda:\rho(\lambda)\geq\varepsilon\}} \psi(\rho(\lambda))(1-|\lambda|)d\mu_0(\lambda)$$

$$= \psi(\varepsilon) \int_{\{\lambda:\rho(\lambda)\geq\varepsilon\}} (1-|\lambda|)d\mu_0(\lambda) + \int_\varepsilon^2 \psi'(t) \left(\int_{\{\lambda:\rho(\lambda)>t\}} (1-|\lambda|)d\mu_0(\lambda) \right) dt.$$

Using (21), we get

$$-\psi(\varepsilon)\int_\varepsilon^2 \varphi'(y)F(y)dy - \int_\varepsilon^2 \psi'(t)\left(\int_t^2 \varphi'(y)F(y)dy\right)dt$$
$$\leqslant 2 \int_{\{\lambda:\rho(\lambda)\geq\varepsilon\}} \psi(\rho(\lambda))(1-|\lambda|)d\mu_0(\lambda) + \text{const}.$$

Integrating by parts in t, we obtain

$$-\int_\varepsilon^2 \psi(t)\varphi'(t)F(t)dt \leqslant 2 \int_{\{\lambda:\rho(\lambda)\geq\varepsilon\}} \psi(\rho(\lambda))(1-|\lambda|)d\mu_0(\lambda) + \text{const}.$$

Proceeding here to the limit as $\varepsilon \to 0$ and using the assumption of the theorem, we complete the proof.

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