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Interaction between "Accelerating-Packing" Flows for the Bryan–Pidduck Model

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The interaction between the "accelerating-packing" flows in a gas of rough spheres is studied. A bimodal distribution with the Maxwellian modes of special forms is used. Different sufficient conditions for the minimization of the uniform-integral error between the sides of the Bryan–Pidduk equation are obtained.

Key words: rough spheres, Bryan–Piddack equation, Maxwellian, "accelerating-packing" flows, error, bimodal distribution.

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1. Introduction

In the paper a model of rough spheres [1], first introduced by Bryan in 1894, is studied. The methods developed by Chapman and Enskog for general non-rotating spherical molecules were extended to Bryan's model by Pidduck in 1922. The advantage of this model over all other variably rotating models is that no variables are required to specify its orientation in the space.

These molecules are perfectly elastic and perfectly rough to be interpreted as follows. When two molecules collide, the velocities at the points of contact are not the same. It is supposed that the two spheres grip each other without slipping; first each sphere is strained by the other, and then the strain energy gets reconverted into kinetic energy of translation and rotation, no energy being lost. The effect is that the relative velocity of the spheres at the point of contact is reversed by the impact.

The Boltzmann equation for the model of rough spheres (or the Bryan–Pidduck equation) has the form [1–4]:

$$D(f) = Q(f, f); \tag{1}$$

$$D(f) \equiv \frac{\partial f}{\partial t} + V\frac{\partial f}{\partial x}; \tag{2}$$

$$Q(f,f) \equiv \frac{d^2}{2}\int\limits_{R^3} dV_1 \int\limits_{R^3} d\omega_1 \int\limits_{\Sigma} d\alpha B(V - V_1, \alpha) \\ \times \Big[f(t, V_1^*, x, \omega_1^*) f(t, V^*, x, \omega^*) - f(t, V, x, \omega) f(t, V_1, x, \omega_1) \Big]. \tag{3}$$

Here d is the diameter of the molecule associated with the moment of inertia I by the relation

$$I = \frac{bd^2}{4},$$

where b , $b \in \left(0, \frac{2}{3}\right]$, is the parameter characterizing the isotropic distribution of the matter inside the gas particle; t is the time; $x = (x^1, x^2, x^3) \in R^3$ is the spatial coordinate; $V = (V^1, V^2, V^3)$ and $w = (w^1, w^2, w^3) \in R^3$ are the linear and angular velocities of the molecule, respectively; $\frac{\partial f}{\partial x}$ is the gradient of the function f over the variable x; Σ is the unit sphere in the space R^3; α is the unit vector of R^3 directed along the line connecting the centers of the colliding molecules;

$$B\,(V - V_1, \alpha) = |(V - V_1, \alpha)| - (V - V_1, \alpha)$$

is the collision term.

The linear (V^*, V_1^*) and angular (w^*, w_1^*) molecular velocities after the collision can be expressed by the appropriate values before the collision:

$$V^* = V - \frac{1}{b+1}\left(b(V_1 - V) - \frac{bd}{2}\alpha \times (\omega + \omega_1) + \alpha(\alpha, V_1 - V) \right),$$

$$V_1^* = V_1 + \frac{1}{b+1}\left(b(V_1 - V) - \frac{bd}{2}\alpha \times (\omega + \omega_1) + \alpha(\alpha, V_1 - V) \right),$$

$$\omega^* = \omega + \frac{2}{d(b+1)}\left\{ \alpha \times (V - V_1) + \frac{d}{2}\left[\alpha(\omega + \omega_1, \alpha) - \omega - \omega_1\right] \right\},$$

$$\omega_1^* = \omega_1 + \frac{2}{d(b+1)}\left\{ \alpha \times (V - V_1) + \frac{d}{2}\left[\alpha(\omega + \omega_1, \alpha) - \omega - \omega_1\right] \right\}.$$

Exact Maxwell solutions of the Boltzmann equation for a more traditional and simpler model of hard spheres were found and classified in detail in [5–8]; their descriptions can also be found in [1, 9, 10]. The general form of Maxwell solutions (i.e., exact solutions of the system $D = Q = 0$) to the Bryan–Pidduck equation was firstly given in [11]. In particular, there was obtained the explicit form of the Maxwellian distribution describing "accelerating-packing" gas flow for this model.

The explicit approximate solutions of kinetic equations, which have a bimodal structure, were given by several authors. In particular, for the models of interaction between the molecules, we are interested in, they are to be found in [3, 4, 8, 12–15].

In [4], the interaction of two "screws" (stationary inhomogeneous Maxwellians) in a gas of rough spheres was studied, and the interaction of two "eddies" (non-stationary inhomogeneous Maxwellians) for the same Bryan–Pidduck model was described in [14]. Our goal is to study the interaction of two flows describing the motion of the "accelerating-packing" type. It should be noted that the hard-sphere model was solved and described in [15].

We use the following error firstly proposed in [4]:

$$\Delta = \sup_{(t,x)\in R^4} \int_{R^3}\int_{R^3} dV d\omega \Big| D(f) - Q(f,f) \Big|. \tag{4}$$

Next, we consider a bimodal distribution

$$f = \varphi_1 M_1 + \varphi_2 M_2, \tag{5}$$

where the functions $\varphi_i = \varphi_i(t,x)$, (here and below the index i takes only values 1 and 2), and the Maxwellians M_i correspond to the "accelerating-packing" movement and have the form

$$M_i = \rho_i I^{3/2} \left(\frac{\beta_i}{\pi}\right)^3 e^{-\beta_i\left(\left(V-\overline{V}_i\right)^2 + I\omega^2\right)}, \tag{6}$$

where ρ_i denotes the gas density

$$\rho_i = \rho_{0i} e^{\beta_i\left(\overline{V}_i^2 + 2\overline{u}_i x\right)}, \tag{7}$$

and

$$\overline{V}_i = \widehat{V}_i - \overline{u}_i t \tag{8}$$

is the mass velocity of molecules, $\beta_i = \frac{1}{2T}$ denotes the inverse temperature, and $\rho_{0i}, \overline{u}_i, \widehat{V}_i$ are arbitrary constants of the spaces R and R^3.

The next section contains the results of providing various sufficient conditions for the minimization of the residual (4) by a suitable choice of the coefficient functions φ_i and the parameters of the distribution.

2. Main Results

Theorem 1. *Let the functions φ_i in distribution* (5) *have the form*

$$\varphi_i(t,x) = \frac{D_i}{(1+t^2)^{\xi_i}} C_i \left(x + \overline{u}_i \frac{\left(\widehat{V}_i - \overline{u}_i t\right)^2}{2\overline{u}_i^2} \right), \tag{9}$$

where the constants D_i, ξ_i are as follows:

$$D_i > 0, \quad \xi_i \geqslant \frac{1}{2}, \tag{10}$$

and the functions C_i, which are nonnegative and belong to the space $C^1(R^3)$, have finite supports (i.e., finite functions) or are fast decreasing at infinity. Also let the following requirements be fulfilled:

$$\widehat{V}_i = \frac{\widehat{V}_{0i}}{\beta_i^{k_i}}, \quad \overline{u}_i = \frac{\overline{u}_{0i}}{\beta_i^{n_i}} \tag{11}$$

with the conditions

$$k_i \geqslant \frac{1}{2}, \quad n_i \geqslant 1, \quad k_i \geqslant \frac{1}{2}n_i, \tag{12}$$

and $\overline{u}_{0i}, \widehat{V}_{0i}$ be arbitrary fixed three-dimensional vectors.

Then the following assertion is true:

$$\begin{gathered} \forall\varepsilon > 0,\ \exists\delta > 0,\ \forall D_1,\ D_2 : 0 < D_1, D_2 < \delta, \\ \exists\beta_0,\ \forall\beta_i > \beta_0, \\ \Delta < \varepsilon. \end{gathered} \tag{13}$$

P r o o f. First we will show that there exists a value $\Delta^{'}$ such that

$$\Delta \leqslant \Delta^{'}, \tag{14}$$

and we have

$$\lim_{\beta_i \to +\infty} \Delta^{'} = K(\xi_1, \xi_2) \sum_{i=1}^{2} \rho_{0i} D_i \sup_{x \in R^3} \Big[\eta_i(x) C_i(x + a_i)\Big], \tag{15}$$

where the functions $\eta_i(x)$ are following:

$$\eta_i(x) = \left[\begin{array}{lll} 1, & n_i > 1; & k_i > \frac{1}{2}, \\ e^{2\overline{u}_{0i}x}, & n_i = 1; & k_i > \frac{1}{2}, \\ e^{\widehat{V}_{0i}^2 + 2\overline{u}_{0i}x}, & n_i = 1; & k_i = \frac{1}{2}. \end{array} \right. \tag{16}$$

$K(\xi_1, \xi_2)$ is a constant, and the vector constants a_i are equal to $\frac{\overline{u}_{0i}\widehat{V}_{0i}^2}{2\overline{u}_{0i}^2}$ if $k_i = \frac{1}{2}n_i$, and they are equal to zero if $k_i \neq \frac{1}{2}n_i$.

It is easy to show that for the function f of form (5) the following relations take place:

$$\begin{gathered} D(f) = M_1 D(\varphi_1) + M_2 D(\varphi_2) \\ = M_1 \left(\frac{\partial\varphi_1}{\partial t} + V \frac{\partial\varphi_1}{\partial x} \right) + M_2 \left(\frac{\partial\varphi_2}{\partial t} + V \frac{\partial\varphi_2}{\partial x} \right) \end{gathered}$$

and

$$Q(f,f) = \varphi_1\varphi_2\Big[Q\Big(M_1,M_2\Big)+Q\Big(M_2,M_1\Big)\Big].$$

It is well known that the right-hand side of the Bryan–Pidduck equation (3), considered as a bilinear operator on any two functions f, g, can be decomposed as follows:

$$Q(f,g) = G(f,g) - fL(g),$$

where

$$G(f,g) = \frac{d^2}{2}\int\limits_{R^3} dV_1 \int\limits_{R^3} d\omega_1 \int\limits_{\Sigma} d\alpha B(V-V_1,\alpha) f(t,x,V_1^*,\omega_1^*) g(t,x,V^*,\omega^*),$$

and

$$L(g) = \frac{d^2}{2}\int\limits_{R^3} dV_1 \int\limits_{R^3} d\omega_1 \int\limits_{\Sigma} d\alpha B(V-V_1,\alpha) g(t,x,V_1,\omega_1).$$

In [14], it was shown that

$$\int\limits_{R^3} dV \int\limits_{R^3} d\omega Q(M_i,M_j) = 0, \quad j=1,2,$$

and hence we have the relationship

$$\int\limits_{R^3} dV \int\limits_{R^3} d\omega G(M_i,M_j) = \int\limits_{R^3} dV \int\limits_{R^3} d\omega M_i L(M_j),$$

i.e., we can get the inequality

$$\Big|D(f) - Q(f,f)\Big| \leqslant M_1\Big(|D(\varphi_1)| + \varphi_1\varphi_2 L(M_2)\Big)$$
$$+M_2\Big(|D(\varphi_2)| + \varphi_1\varphi_2 L(M_1)\Big) + \varphi_1\varphi_2\Big(G(M_1,M_2) + G(M_2,M_1)\Big).$$

Integrating the last estimation over the entire space of the linear and angular velocities, we obtain

$$\int\limits_{R^3} dV \int\limits_{R^3} d\omega \Big|D(f) - Q(f,f)\Big|$$
$$\leqslant \sum_{\substack{i,j=1\\ i\neq j}}^{2} \int\limits_{R^3} dV \int\limits_{R^3} d\omega \Big(|D(\varphi_i)| + \varphi_i\varphi_j L(M_j)\Big) M_i + 2\varphi_1\varphi_2 \int\limits_{R^3} dV \int\limits_{R^3} d\omega G(M_1,M_2)$$
$$\leqslant \sum_{i=1}^{2} \int\limits_{R^3} dV \int\limits_{R^3} d\omega |D(\varphi_i)| M_i + 4\varphi_1\varphi_2 \int\limits_{R^3} dV \int\limits_{R^3} d\omega G(M_1,M_2).$$

Then we use the formula, the proof of which is given in detail in [4],

$$\int\limits_{R^3} dV \int\limits_{R^3} d\omega G(M_1, M_2) = \frac{d^2\rho_1\rho_2}{\pi^2} \int\limits_{R^3} dq \int\limits_{R^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \overline{V}_1 - \overline{V}_2 \right|. \tag{17}$$

We extend the estimation taking into account (8) and (17),

$$\int\limits_{R^3} dV \int\limits_{R^3} d\omega \Big| D(f) - Q(f,f) \Big| \leqslant \sum_{i=1}^{2} \int\limits_{R^3} dV \int\limits_{R^3} d\omega |D(\varphi_i)| M_i + \mathbf{Y}$$

$$= \sum_{i=1}^{2} \int\limits_{R^3} dV \int\limits_{R^3} d\omega \left| \frac{\partial\varphi_i}{\partial t} + V \cdot \frac{\partial\varphi_i}{\partial x} \right| \rho_i I^{3/2} \left(\frac{\beta_i}{\pi} \right)^3 e^{-\beta_i \left(V - \overline{V}_i\right)^2 - \beta_i I \omega^2} + \mathbf{Y},$$

where the value $\mathbf{Y}$ is determined by the expression

$$\frac{4d^2\rho_1\rho_2\varphi_1\varphi_2}{\pi^2} \int\limits_{R^3} dq \int\limits_{R^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \widehat{V}_1 - \widehat{V}_2 + (\overline{u}_2 - \overline{u}_1)t \right|.$$

Thus, we can integrate over the space of angular velocities ω (three-dimensional Euler–Poisson integral) to get

$$\int\limits_{R^3} dV \int\limits_{R^3} d\omega \Big| D(f) - Q(f,f) \Big|$$

$$\leqslant \sum_{i=1}^{2} \int\limits_{R^3} dV \left| \frac{\partial\varphi_i}{\partial t} + V \cdot \frac{\partial\varphi_i}{\partial x} \right| \rho_i \left(\frac{\beta_i}{\pi} \right)^{3/2} e^{-\beta_i \left(V - \overline{V}_i\right)^2} + \mathbf{Y}.$$

By changing the variable

$$V = \frac{p}{\sqrt{\beta_i}} + \overline{V}_i,$$

whose Jacobian is $\beta_i^{-3/2}$, we can obtain the inequality, which will be often used in our further calculations,

$$\int\limits_{R^3} dV \int_{R^3} d\omega \Big| D(f) - Q(f,f) \Big| \leqslant \sum_{i=1}^{2} \frac{\rho_i}{\pi^{3/2}} \int\limits_{R^3} dp \left| \frac{\partial\varphi_i}{\partial t} + \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i - \overline{u}_i t \right) \cdot \frac{\partial\varphi_i}{\partial x} \right| e^{-p^2} + \mathbf{Y}. \tag{18}$$

For the existence of value (4) and the validity of inequality (14), as seen from estimation (18), it is sufficient to verify that the products of gas density (7) on the functions

$$\varphi_i;\ \frac{\partial \varphi_i}{\partial t};\ \left|\frac{\partial \varphi_i}{\partial x}\right|;\ \varphi_i t;\ \left(\overline{u}_i, \frac{\partial \varphi_i}{\partial x}\right) t \tag{19}$$

are bounded for any (t, x) from R^4.

In the representation of functions (9) let us introduce a redesignation

$$l = x + \overline{u}_i \frac{\left(\widehat{V}_i - \overline{u}_i t\right)^2}{2\overline{u}_i^2},$$

whence

$$\left(\widehat{V}_i - \overline{u}_i t\right)^2 = 2\overline{u}_i l - 2\overline{u}_i x,$$

and consequently we have

$$\varphi_i \rho_i = \rho_{0i} e^{2\overline{u}_i l \beta_i} \frac{D_i}{(1+t^2)^{\xi_i}} C_i(l). \tag{20}$$

The product (20) is a bounded function on $(t, x) \in R^4$ due to the properties of the function $C_i(l)$. It should be noted that the boundedness will remain true even after multiplying the value (20) by a variable t due to the expression $(1 + t^2)^{\xi_i}$, contained in the denominator, and condition (10). Similarly, we can prove the boundedness of the last three products by using the equations, obtained by direct differentiation of the function φ_i of the form (9) with respect to the time t and the position in space x,

$$\frac{\partial \varphi_i}{\partial t} = -\frac{D_i}{(1+t^2)^{\xi_i}} \left[\frac{2t\xi_i}{1+t^2} C_i(l) + \left(C_i^{'}(l), \overline{u}_i\right) \frac{\left(\widehat{V}_i, \overline{u}_i\right) - t\overline{u}_i^2}{\overline{u}_i^2}\right], \tag{21}$$

$$\frac{\partial \varphi_i}{\partial x} = \frac{D_i}{(1+t^2)^{\xi_i}} C_i^{'}(l). \tag{22}$$

Taking into account assumptions (11), we have a low-temperature limit ($\beta_i \to +\infty$) of the density (7) depending on the numbers n_i and k_i

$$\lim_{\beta_i \to +\infty} \rho_i = \rho_{0i} \cdot \left[\begin{array}{lll} 1, & n_i > 1, & k_i > \frac{1}{2}; \\ e^{2\overline{u}_{0i} x} & n_i = 1, & k_i > \frac{1}{2}; \\ e^{\widehat{V}_{0i}^2 + 2\overline{u}_{0i} x} & n_i = 1, & k_i = \frac{1}{2}. \end{array}\right.$$

Taking into account equalities (11) and conditions (12), the following equation is evident:

$$\lim_{\beta_i\to+\infty}\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}+\widehat{V}_1-\widehat{V}_2+(\overline{u}_2-\overline{u}_1)t\right|=0. \tag{23}$$

As a result, substituting the calculated derivatives (21), (22) into (18), taking the supremum on both sides of (18) and performing a low-temperature limit by using the technique of [3, 4, 14, 15], we get the equality

$$\lim_{\beta_i\to+\infty}\Delta^{'}=\sum_{i=1}^{2}\rho_{0i}D_i\sup_{(t,x)\in R^4}\left\{\eta_i(x)\lim_{\beta_i\to+\infty}\frac{2|t|\xi_i C_i(l)}{(1+t^2)^{\xi_i+1}}\right\},$$

where the variable l is the sum of the variables x and r_i, the latter of which can be represented as follows:

$$r_i=\frac{\overline{u}_{0i}}{2\overline{u}_{0i}^2}\left(\frac{\widehat{V}_{0i}}{\beta_i^{k_i-\frac{1}{2}n_i}}-\frac{\overline{u}_{0i}t}{\beta_i^{\frac{1}{2}n_i}}\right)^2.$$

Then we have that

$$a_i=\lim_{\beta_i\to+\infty}(l-x)=\left[\begin{array}{ll}0, & k_i>\frac{1}{2}n_i;\\ \overline{u}_{0i}\frac{\widehat{V}_{0i}^2}{2\overline{u}_{0i}^2}, & k_i=\frac{1}{2}n_i.\end{array}\right.$$

As a result, we obtain

$$\begin{aligned}\lim_{\beta_i\to+\infty}\Delta^{'}&=\sum_{i=1}^{2}\rho_{0i}D_i\sup_{(t,x)\in R^4}\left\{\eta_i(x)\frac{2|t|\xi_i}{(1+t^2)^{\xi_i+1}}C_i(x+a_i)\right\}\\&\leqslant K(\xi_1,\xi_2)\sum_{i=1}^{2}\rho_{0i}D_i\sup_{x\in R^3}\Big[\eta_i(x)C_i(x+a_i)\Big],\end{aligned}$$

where the constant $K(\xi_1,\xi_2)$ is defined as follows:

$$K(\xi_1,\xi_2)=2\max_i\left\{\xi_i\sup_{t\in R}\frac{|t|}{(1+t^2)^{\xi_i+1}}\right\}.$$

Thus, we have shown that equality (15) is fulfilled, from which (with (14) being taken into account) the statement of our theorem, i.e. (13), follows. The theorem is proved. ■

Theorem 2. *Assume that the functions $\varphi_i(t,x)$ have the representation*

$$\varphi_i(t,x)=\psi_i(t,x)e^{-\beta_i\left(\left(\widehat{V}_i-\overline{u}_it\right)^2+2\overline{u}_ix\right)}, \tag{24}$$

where

$$\psi_i = D_i C_i(t),$$

here $D_i > 0$, and C_i *are finite functions.*

Let the condition

$$\overline{u}_i = \frac{\overline{u}_{0i}}{\beta_i^{n_i}}, \quad n_i \geqslant \frac{1}{2}, \tag{25}$$

hold.

Then:

a) *if the equality*

$$supp\, C_1 \cap supp\, C_2 = \varnothing$$

takes place, or

$$\widehat{V}_1 = \widehat{V}_2,$$

then assertion (13) *remains true.*

b) *In the case of arbitrary supports of the functions* (C_1 *and* C_2) *and the velocities* $\widehat{V}_1$, $\widehat{V}_2$, *Theorem 1 still holds if being complemented by the condition of the infinite smallness of the diameter of the gas particles* ($d < \delta$).

P r o o f. First we introduce and prove an auxiliary assertion that there exists a value of $\Delta^{'}$, and the inequality (14) is fulfilled, but if $n_i > \frac{1}{2}$, then

$$\begin{gathered}\lim_{\beta_i \to +\infty} \Delta^{'} = \sum_{i=1}^{2} \rho_{0i} \sup_{(t,x)\in R^4} \left| \frac{\partial \psi_i}{\partial t} + \widehat{V}_i \frac{\partial \psi_i}{\partial x} \right| \\ +4\pi d^2 \rho_{01}\rho_{02} \left| \widehat{V}_1 - \widehat{V}_2 \right| \sup_{(t,x)\in R^4} (\psi_1 \psi_2) = \mathbf{Z},\end{gathered} \tag{26}$$

while for $n_i = \frac{1}{2}$ we have

$$\lim_{\beta_i \to +\infty} \Delta^{'} = \mathbf{Z} + \frac{4}{\sqrt{\pi}} \sum_{i=1}^{2} \rho_{0i} |\overline{u}_{0i}| \sup_{(t,x)\in R^4} \psi_i. \tag{27}$$

If the assertion is true, then items (a) and (b) are also true.

Note that inequality (18) remains true, therefore it is necessary to compute the derivatives of (24) by t and x

$$\frac{\partial \varphi_i}{\partial t} = e^{-\beta_i \left(\left(\widehat{V}_i - \overline{u}_i t \right)^2 + 2\overline{u}_i x \right)} \left\{ \frac{\partial \psi_i}{\partial t} + 2\beta_i \psi_i \left(\left(\widehat{V}_i, \overline{u}_i \right) - t\overline{u}_i^2 \right) \right\}, \tag{28}$$

$$\frac{\partial \varphi_i}{\partial x} = e^{-\beta_i \left(\left(\widehat{V}_i - \overline{u}_i t \right)^2 + 2\overline{u}_i x \right)} \left\{ \frac{\partial \psi_i}{\partial x} - 2\beta_i \overline{u}_i \psi_i \right\}. \tag{29}$$

Taking into account that the functions $\psi_i(t,x)$ are smooth and nonnegative, and basing on their type, it follows that the above expressions in (19) remain bounded. After replacing $\varphi_i(t,x)$ on $\psi_i(t,x)$, we can pass to the supremum in inequality (18), primarily substituting in it the expressions for derivatives (28) and (29). Thus we have

$$\Delta \leqslant \Delta'$$

$$= \sup_{(t,x)\in R^4} \sum_{i=1}^{2} \frac{\rho_i}{\pi^{3/2}} \int\limits_{R^3} dp \left| e^{-\beta_i\left(\left(\widehat{V}_i-\overline{u}_i t\right)^2+2\overline{u}_i x\right)} \left(\frac{\partial\psi_i}{\partial t} + 2\beta_i\psi_i\left(\left(\widehat{V}_i,\overline{u}_i\right) - t\overline{u}_i^2\right)\right)\right.$$

$$\left. + \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i - \overline{u}_i t\right) e^{-\beta_i\left(\left(\widehat{V}_i-\overline{u}_i t\right)^2+2\overline{u}_i x\right)} \left(\frac{\partial\psi_i}{\partial x} - 2\beta_i\overline{u}_i\psi_i\right)\right| e^{-p^2}$$

$$+4 \sup_{(t,x)\in R^4} \psi_1\psi_2 e^{-\beta_1\left(\left(\widehat{V}_1-\overline{u}_1 t\right)^2+2\overline{u}_1 x\right)-\beta_2\left(\left(\widehat{V}_2-\overline{u}_2 t\right)^2+2\overline{u}_2 x\right)} \frac{d^2\rho_1\rho_2}{\pi^2}$$

$$\times \int\limits_{R^3} dq \int\limits_{R^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \widehat{V}_1 - \widehat{V}_2 + (\overline{u}_2 - \overline{u}_1)t \right|.$$

Using the representation for density (7) and opening the parenthesis, after collecting similar terms, we can find that the value of Δ' is equal to the expression

$$\sup_{(t,x)\in R^4} \sum_{i=1}^{2} \frac{\rho_{0i}}{\pi^{3/2}} \int\limits_{R^3} dp \left| \frac{\partial\psi_i}{\partial t} + \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i - \overline{u}_i t\right)\frac{\partial\psi_i}{\partial x} - \frac{2\beta_i\overline{u}_i\psi_i}{\sqrt{\beta_i}} p \right| e^{-p^2}$$

$$+4 \sup_{(t,x)\in R^4} \frac{d^2\psi_1\psi_2\rho_{01}\rho_{02}}{\pi^2} \int\limits_{R^3} dq \int\limits_{R^3} dq_1 e^{-q^2-q_1^2} \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \widehat{V}_1 - \widehat{V}_2 + (\overline{u}_2 - \overline{u}_1)t \right|.$$

Next, by using condition (25), the value Δ' can be converted to the form

$$\sup_{(t,x)\in R^4} \sum_{i=1}^{2} \frac{\rho_{0i}}{\pi^{3/2}} \int\limits_{R^3} dp \left| \frac{\partial\psi_i}{\partial t} + \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i - \frac{\overline{u}_{0i}}{\beta_i^{n_i}} t\right) \cdot \frac{\partial\psi_i}{\partial x} - 2\beta_i^{\frac{1}{2}-n_i}\overline{u}_{0i}\psi_i p \right| e^{-p^2}$$

$$+4 \sup_{(t,x)\in R^4} \frac{d^2\psi_1\psi_2\rho_{01}\rho_{02}}{\pi^2} \int\limits_{R^3} dq \int\limits_{R^3} dq_1 e^{-q^2-q_1^2}$$

$$\times \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \widehat{V}_1 - \widehat{V}_2 + \left(\frac{\overline{u}_{02}}{\beta_2^{n_2}} - \frac{\overline{u}_{01}}{\beta_1^{n_1}}\right) t \right|,$$

which can be estimated by the following sum:

$$\sum_{i=1}^{2}\frac{\rho_{0i}}{\pi^{3/2}}\int\limits_{R^3}dp\sup_{(t,x)\in R^4}\left|\frac{\partial\psi_i}{\partial t}+\left(\frac{p}{\sqrt{\beta_i}}+\widehat{V}_i-\frac{\overline{u}_{0i}}{\beta_i^{n_i}}t\right)\cdot\frac{\partial\psi_i}{\partial x}\right|e^{-p^2}$$
$$+\frac{4\rho_{01}\rho_{02}d^2}{\pi^2}\sup_{(t,x)\in R^4}\psi_1\psi_2\int\limits_{R^3}dq\int\limits_{R^3}dq_1e^{-q^2-q_1^2}$$
$$\times\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}+\widehat{V}_1-\widehat{V}_2+\left(\frac{\overline{u}_{02}}{\beta_2^{n_2}}-\frac{\overline{u}_{01}}{\beta_1^{n_1}}\right)t\right|$$
$$+2\sum_{i=1}^{2}\frac{\rho_{0i}}{\pi^{3/2}}\int\limits_{R^3}dp\sup_{(t,x)\in R^4}\beta_i^{\frac{1}{2}-n_i}\left|\overline{u}_{0i}\psi_ip\right|e^{-p^2}.$$

Now, performing the limiting passage ($\beta_i\to+\infty$) under the sign of inequality and supremum, as in the proof of Theorem 1, we obtain equality (26) for $n_i>\frac{1}{2}$, and the validity of (27) can be proved by using the equality

$$\int\limits_{R^3}|p|e^{-p^2}dp=2\pi$$

obtained by direct integration in the spherical coordinates. Thus, we have shown the validity of the assertions of Theorem 2. ■

Theorem 3. *Let the functions φ_i in distribution* (5) *take the form*

$$\varphi_i(t,x)=\psi_i(t,x)\cdot e^{-\beta_i\left(\widehat{V}_i-\overline{u}_it\right)^2}, \tag{30}$$

and condition (25) *hold true, but now for $n_i\geqslant 1$.*

Then the assertion of Theorem 2 remains true if:

a) *the functions ψ_i have the form*

$$\psi_i(t,x)=\frac{D_i}{(1+t^2)^{\xi_i}}C_i\left(\left[x\times\widehat{V}_i\right]\right), \tag{31}$$

(10) *holds and, in addition to that, $\widehat{V}_i\perp\overline{u}_{0i}$;*

b) *there holds the representation*

$$\psi_i(t,x)=\frac{D_i}{(1+t^2)^{\xi_i}}C_i\left(x\right), \tag{32}$$

and on the functions C_i, used here, the same restrictions as in Theorem 1 are imposed.

P r o o f. Before proving Theorem 3, we will prove a proposition which states that there exists a value of Δ' such that (14) is fulfilled and there holds the equality

$$\lim_{\beta_i\to+\infty}\Delta' = \sum_{i=1}^{2}\rho_{0i}\sup_{(t,x)\in R^4}\mu_i(x)\left|\frac{\partial\psi_i}{\partial t}+\widehat{V}_i\frac{\partial\psi_i}{\partial x}\right|$$
$$+\,4\pi d^2\rho_{01}\rho_{02}\left|\widehat{V}_1-\widehat{V}_2\right|\sup_{(t,x)\in R^4}\left[\mu_1(x)\mu_2(x)\psi_1(t,x)\psi_2(t,x)\right] \tag{33}$$
$$+\,2\theta\sum_{i=1}^{2}\rho_{0i}\left|\left(\overline{u}_{0i},\widehat{V}_i\right)\right|\sup_{(t,x)\in R^4}\left[\mu_i(x)\psi_i(t,x)\right],$$

where at $n_i>1:\quad \theta=0$, and $\mu_i=1$; in the case $n_i=1:\quad \theta=1,\ \mu_i(x)=e^{2\overline{u}_{0i}x}$.

In the case of this theorem, estimation (18) also holds, so we again begin with calculation of the derivatives of (30) contained in the inequality. The derivative with respect to time t is expressed as follows:

$$\frac{\partial\varphi_i}{\partial t}=e^{-\beta_i\left(\widehat{V}_i-\overline{u}_i t\right)^2}\left\{\frac{\partial\psi_i}{\partial t}+2\beta_i\psi_i\left(\left(\overline{u}_i,\widehat{V}_i\right)-\overline{u}_i^2 t\right)\right\}, \tag{34}$$

and on the spatial coordinate x, it has the form

$$\frac{\partial\varphi_i}{\partial x}=\frac{\partial\psi_i}{\partial x}e^{-\beta_i\left(\widehat{V}_i-\overline{u}_i t\right)^2}. \tag{35}$$

Taking into account the conditions imposed on the functions $\psi_i(t,x)$ and derivatives (34), (35), we will pass to the supremum in inequality (18). Its existence follows from the conditions imposed on the functions C_i in the statement of the theorem. Then we transform the resulting expression using representation (7). The value Δ' gets the form

$$\sup_{(t,x)\in R^4}\sum_{i=1}^{2}\frac{\rho_{0i}e^{2\beta_i\overline{u}_i x}}{\pi^{3/2}}\int\limits_{R^3}dp\left|\frac{\partial\psi_i}{\partial t}+2\beta_i\psi_i\left(\left(\overline{u}_i,\widehat{V}_i\right)-\overline{u}_i^2 t\right)\right.$$
$$\left.+\left(\frac{p}{\sqrt{\beta_i}}+\widehat{V}_i-\overline{u}_i t\right)\frac{\partial\psi_i}{\partial x}\right|e^{-p^2}+\sup_{(t,x)\in R^4}\frac{4d^2\rho_{01}\rho_{02}\psi_1\psi_2 e^{2x(\beta_1\overline{u}_1+\beta_2\overline{u}_2)}}{\pi^2}$$
$$\times\int\limits_{R^3}dq\int\limits_{R^3}dq_1 e^{-q^2-q_1^2}\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}+\widehat{V}_1-\widehat{V}_2+(\overline{u}_2-\overline{u}_1)t\right|,$$

which admits an upper bound by the following sum:

$$\sup_{(t,x)\in R^4}\sum_{i=1}^{2}\frac{\rho_{0i}e^{2\beta_i^{1-n_i}\overline{u}_{0i}x}}{\pi^{3/2}}\int\limits_{R^3}dp\left|\frac{\partial\psi_i}{\partial t}+\left(\frac{p}{\sqrt{\beta_i}}+\widehat{V}_i-\overline{u}_i t\right)\frac{\partial\psi_i}{\partial x}\right|e^{-p^2}$$
$$+\frac{4d^2\rho_{01}\rho_{02}}{\pi^2}\sup_{(t,x)\in R^4}\psi_1\psi_2 e^{2x\left(\beta_1^{1-n_1}\overline{u}_{01}+\beta_2^{1-n_2}\overline{u}_{02}\right)}$$
$$\times\int\limits_{R^3}dq\int\limits_{R^3}dq_1 e^{-q^2-q_1^2}\left|\frac{q}{\sqrt{\beta_1}}-\frac{q_1}{\sqrt{\beta_2}}+\widehat{V}_1-\widehat{V}_2+\left(\frac{\overline{u}_{02}}{\beta_2^{n_2}}-\frac{\overline{u}_{01}}{\beta_1^{n_1}}\right)t\right|$$
$$+2\sum_{i=1}^{2}\frac{\rho_{0i}e^{2\beta_i^{1-n_i}\overline{u}_{0i}x}}{\pi^{3/2}}\int\limits_{R^3}dp\sup_{(t,x)\in R^4}\psi_i\left|\left(\frac{\overline{u}_{0i}}{\beta_i^{n_i}},\widehat{V}_i\right)-\frac{\overline{u}_{0i}^2}{\beta_i^{2n_i}}t\right|e^{-p^2}.$$

Thus, if we now pass to the low-temperature limit, we will obtain assertion (33) with the corresponding values of θ and $\mu_i(x)$.

The verification of item (b), i.e., the functions of the form (32), is obvious enough, but (31) should be considered in more detail.

To begin with, we introduce a new orthogonal (due to the conditions of item (a)) basis consisting of the vectors $\overline{u}_i$, $\widehat{V}_i$ and $\left[\overline{u}_i\times\widehat{V}_i\right]$. In this basis, we expand an arbitrary vector x

$$x=x^1\overline{u}_i+x^2\widehat{V}_i+x^3\left[\overline{u}_i\times\widehat{V}_i\right].$$

Then, for the product $\psi_i e^{2\beta_i\overline{u}_i x}$, we get the representation

$$\frac{D_i}{(1+t^2)^{\xi_i}}C_i\left(x^1\left[\overline{u}_i\times\widehat{V}_i\right]-x^3\overline{u}_i\widehat{V}_i^2\right)e^{2\beta_i\overline{u}_i^2x^1},$$

which is constant on the second component of x^2. It is easy to see that on the remaining components of $\left(x^1,x^3\right)$, as well as on β_i, it is also bounded if we take into account the compact support property of C_i and requirement (25).

Next, let us compute the derivative with respect to x,

$$\frac{\partial\psi_i}{\partial x}=\frac{D_i}{(1+t^2)^{\xi_i}}\left[\widehat{V}_i\times C_i'\left(\left[x\times\widehat{V}_i\right]\right)\right],$$

and scalar multiply it by the vector $\widehat{V}_i$. It is obvious that the resulting product is equal to zero.

Taking this fact into account and using (33), it is easy to show the validity of the assertions for the functions of the form (31). The theorem is proved. ∎

Theorem 4. *Assume that the representation*

$$\varphi_i(t,x) = \psi_i(t,x) \cdot e^{-2\beta_i \overline{u}_i x} \tag{36}$$

takes place and requirement (11) *is retained, but with*

$$n_i \geqslant \frac{1}{2},\ k_i \geqslant \frac{1}{2}. \tag{37}$$

Let the function $\psi_i(t,x)$ *have the form*

$$\psi_i(t,x) = D_i C_i(t) E_i(x), \tag{38}$$

where $D_i > 0$, $C_i(t)$ *has the same properties as in the previous theorems, and the function* $E_i(x)$ *is nonnegative, finite or rapidly decreasing at infinity and bounded together with its gradient on* x. *Then* (13) *holds true.*

P r o o f. As in the previous theorems, we begin with the introduction of the auxiliary assertions. Again, we prove that there exists a value Δ' such that inequality (14) is fulfilled and its low-temperature limit is equal to:

a) at $n_i > \frac{1}{2}$, $k_i > \frac{1}{2}$:

$$\sum_{i=1}^{2} \rho_{0i} \sup_{(t,x)\in R^4} \left| \frac{\partial \psi_i}{\partial t} \right|; \tag{39}$$

b) in the case $n_i > \frac{1}{2}$, $k_i = \frac{1}{2}$:

$$\sum_{i=1}^{2} \rho_{0i} e^{\widehat{V}_{0i}^2} \sup_{(t,x)\in R^4} \left| \frac{\partial \psi_i}{\partial t} \right|; \tag{40}$$

c) if $n_i = \frac{1}{2}$, $k_i > \frac{1}{2}$:

$$\begin{aligned} &\sum_{i=1}^{2} \rho_{0i} \sup_{(t,x)\in R^4} \left\{ e^{t^2 \overline{u}_{0i}^2} \left| \frac{\partial \psi_i}{\partial t} + 2\psi_i t \overline{u}_{0i}^2 \right| \right\} \\ &+ \frac{4}{\sqrt{\pi}} \sum_{i=1}^{2} \rho_{0i} |\overline{u}_{0i}| \sup_{(t,x)\in R^4} \left(e^{t^2 \overline{u}_{0i}^2} \psi_i \right); \end{aligned} \tag{41}$$

d) and with $k_i = n_i = \frac{1}{2}$:

$$\begin{aligned} &\sum_{i=1}^{2} \rho_{0i} \sup_{(t,x)\in R^4} \left\{ e^{\left(\widehat{V}_{0i} - \overline{u}_{0i} t\right)^2} \left| \frac{\partial \psi_i}{\partial t} + 2\psi_i t \overline{u}_{0i}^2 \right| \right\} \\ &+ 2 \sum_{i=1}^{2} \rho_{0i} \left(\frac{2|\overline{u}_{0i}|}{\sqrt{\pi}} + \left| \left(\overline{u}_{0i}, \widehat{V}_{0i} \right) \right| \right) \sup_{(t,x)\in R^4} \left\{ e^{\left(\widehat{V}_{0i} - \overline{u}_{0i} t\right)^2} \psi_i \right\}. \end{aligned} \tag{42}$$

Using inequality (18), remaining true, given that only the form of the function φ_i is changed, let us find the derivatives of function (36) with respect to t and x

$$\frac{\partial \varphi_i}{\partial t} = e^{-2\beta_i \overline{u}_i x} \frac{\partial \psi_i}{\partial t}, \tag{43}$$

$$\frac{\partial \varphi_i}{\partial x} = e^{-2\beta_i \overline{u}_i x} \left(\frac{\partial \psi_i}{\partial x} - 2\beta_i \overline{u}_i \psi_i \right). \tag{44}$$

Now, calculating the supremum of both sides of (18) and using the boundedness of all the terms, we substitute the expressions for the derivatives (43), (44). As a result, for the value Δ', we will have the expression

$$\sup_{(t,x)\in R^4} \sum_{i=1}^{2} \frac{\rho_{0i} e^{\beta_i \left(\widehat{V}_i - \overline{u}_i t\right)^2}}{\pi^{3/2}} \int\limits_{R^3} dp \left| \frac{\partial \psi_i}{\partial t} + \left(\frac{p}{\sqrt{\beta_i}} + \widehat{V}_i - \overline{u}_i t \right) \left(\frac{\partial \psi_i}{\partial x} - 2\beta_i \overline{u}_i \psi_i \right) \right| e^{-p^2}$$

$$+ \frac{4d^2 \rho_{01} \rho_{02}}{\pi^2} \sup_{(t,x)\in R^4} \psi_1 \psi_2 e^{\beta_1 \left(\widehat{V}_1 - \overline{u}_1 t\right)^2 + \beta_2 \left(\widehat{V}_2 - \overline{u}_2 t\right)^2}$$

$$\times \int\limits_{R^3} dq \int\limits_{R^3} dq_1 e^{-q^2 - q_1^2} \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \widehat{V}_1 - \widehat{V}_2 + (\overline{u}_2 - \overline{u}_1) t \right|.$$

This sum under conditions (11) and (37) can be rewritten as follows:

$$\sup_{(t,x)\in R^4} \sum_{i=1}^{2} \frac{\rho_{0i} e^{\beta_i \left(\frac{\widehat{V}_{0i}}{\beta_i^{k_i}} - \frac{\overline{u}_{0i}}{\beta_i^{n_i}} t \right)^2}}{\pi^{3/2}} \int\limits_{R^3} dp \left| \frac{\partial \psi_i}{\partial t} + \left(\frac{p}{\sqrt{\beta_i}} + \frac{\widehat{V}_{0i}}{\beta_i^{k_i}} - \frac{\overline{u}_{0i}}{\beta_i^{n_i}} t \right) \right.$$

$$\left. \times \left(\frac{\partial \psi_i}{\partial x} - 2\beta_i^{1-n_i} \overline{u}_{0i} \psi_i \right) \right| e^{-p^2}$$

$$+ \frac{4d^2 \rho_{01} \rho_{02}}{\pi^2} \sup_{(t,x)\in R^4} \psi_1 \psi_2 e^{\beta_1 \left(\frac{\widehat{V}_{01}}{\beta_1^{k_1}} - \frac{\overline{u}_{01}}{\beta_1^{n_1}} t \right)^2 + \beta_2 \left(\frac{\widehat{V}_{02}}{\beta_2^{k_2}} - \frac{\overline{u}_{02}}{\beta_2^{n_2}} t \right)^2}$$

$$\times \int\limits_{R^3} dq \int\limits_{R^3} dq_1 e^{-q^2 - q_1^2} \left| \frac{q}{\sqrt{\beta_1}} - \frac{q_1}{\sqrt{\beta_2}} + \frac{\widehat{V}_{01}}{\beta_1^{k_1}} - \frac{\widehat{V}_{02}}{\beta_2^{k_2}} + \left(\frac{\overline{u}_{02}}{\beta_2^{n_2}} - \frac{\overline{u}_{01}}{\beta_1^{n_1}} \right) t \right|.$$

Now, performing, as before, the limiting passage and some simple transformations, we can obtain the values of the low-temperature limits of the value Δ' specified in (39)–(42).

Then, calculating the derivative of (38) with respect to t and substituting it into (39)–(42), we can see that assertion (13) is true. Hence the theorem is proved. ■

In the present paper, we obtained the results similar to those described in [15] for the hard-sphere model. Thus, under the assumptions mentioned in Theorems 1–4, we can extend the results, previously known for the hard-sphere model, on the model of rough spheres.

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