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Inverse Scattering Problem for One-Dimensional Schrödinger Equation with Discontinuity Conditions

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The direct and inverse scattering problems for the second order ordinary differential equation on the whole axis with discontinuity conditions at some point are considered.

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Introduction

Consider the differential equation

$$-y'' + q(x)\,y = \lambda^2 y, \qquad -\infty < x < +\infty, \tag{0.1}$$

with discontinuity conditions at a point $a \in (-\infty, +\infty)$

$$y(a-0) = \alpha y(a+0),$$

$$y'(a-0) = \alpha^{-1} y'(a+0), \tag{0.2}$$

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where $1 \neq \alpha > 0$, λ is a complex parameter, $q(x)$ is a real-valued function with

$$\int_{-\infty}^{+\infty} (1+|x|)\,|q(x)|\,dx < +\infty. \tag{0.3}$$

The aim of this paper is to study direct and inverse scattering problems for equation (0.1) with conditions (0.2). The inverse problem, where discontinuity conditions (0.2) are absent, i.e., $\alpha = 1$, was completely solved in [1–4]. The similar problem for the system of differential equations without discontinuity conditions was studied in [5–8]. Some aspects of direct and inverse problems for differential operators with discontinuity conditions were studied in [9–13].

Since the case $\alpha \neq 1$ is almost analogous to the case $\alpha = 1$, below we will consider the moments that differ these cases.

Notice that problem (0.1)–(0.2) can be rewritten in the form

$$-p(x)\left(\frac{1}{p^2(x)}\,(p(x)\,y)'\right)' + q(x)\,y = \lambda^2 y, \qquad -\infty < x < +\infty,$$

where $p(x) = \alpha$ for $x > a$ and $p(x) = 1$ for $x < a$.

1. Jost Type Solutions

The functions $e^{\pm}(x,\lambda)$ satisfying equation (0.1), conditions (0.2) and the condition

$$\lim_{x\to\pm\infty} e^{\pm}(x,\lambda)\,e^{\mp i\lambda x} = 1 \tag{1.1$_\pm$}$$

are called the Jost type solutions. It is not difficult to show that if $q(x) \equiv 0$, then the Jost solutions are

$$e_0^{\pm}(x,\lambda) = \begin{cases} e^{\pm i\lambda x}, & \pm x > \pm a, \\ Ae^{\pm i\lambda x} \pm Be^{\pm i\lambda(2a-x)}, & \pm x < \pm a, \end{cases}$$

where $A = \dfrac{1}{2}\left(\alpha + \dfrac{1}{\alpha}\right)$, $B = \dfrac{1}{2}\left(\alpha - \dfrac{1}{\alpha}\right)$.

Theorem 1.1. *Under condition* (0.3)*, equation* (0.1) *with discontinuity conditions* (0.2) *for all λ from the half-plane* $\mathrm{Im}\lambda \geq 0$ *has a solution $e^{\pm}(x,\lambda)$ which can be represented in the form*

$$e^{\pm}(x,\lambda) = e_0^{\pm}(x,\lambda) \pm \int_x^{\pm\infty} K^{\pm}(x,t)\,e^{\pm i\lambda t}\,dt, \tag{1.2$_\pm$}$$

where the kernels $K^{\pm}(x,t)$ satisfy the inequalities

$$\left|K^{\pm}(x,\,t)\right| \le \frac{C}{2}\sigma^{\pm}\left(\frac{x+t}{2}\right)\ e^{C\sigma_1^{\pm}(x)},\ 0<|x-a|<\pm(t-a),$$

$$\left|K^{\pm}(x,\,t)\right| \le \left\{\frac{C}{2}\sigma^{\pm}\left(\frac{x+t}{2}\right)+\frac{|B|}{2}\sigma^{\pm}\left(\frac{2a+x-t}{2}\right)\right\}e^{C\sigma_1^{\pm}(x)},$$
$$|t-a|<\pm(a-x), \tag{1.3$_\pm$}$$

where $C = A+|B|$, $\sigma^{\pm}(x) = \pm\int\limits_x^{\pm\infty}|q(s)|\,ds$, $\sigma_1^{\pm}(x)=\pm\int\limits_x^{\pm\infty}\sigma^{\pm}(s)ds$. *Moreover, the functions* $K^{\pm}(x,\,t)$ *are continuous at* $t\ne 2a-x$, $x\ne a$, *and the following relations are satisfied:*

$$K^{\pm}(x,\,x) = \pm\frac{A}{2}\int\limits_x^{\pm\infty}q(t)dt,\quad \pm x<\pm\,a,$$

$$K^{\pm}(x,\,x) = \pm\frac{1}{2}\int\limits_x^{\pm\infty}q(t)dt,\quad \pm\,x>\pm a,$$

$$K^{\pm}(x,\,2a-x+0)-K^{\pm}(x,\,2a-x-0)$$

$$=\pm\frac{B}{2}\left(\int\limits_a^{\pm\infty}q(t)dt-\int\limits_x^{a}q(t)dt\right),\quad \pm x<\pm\,a. \tag{1.4$_\pm$}$$

P r o o f. We give the proof of the theorem for the solution $e^+(x,\lambda)$. Problem (0.1), (0.2), $(1.1)_+$ is equivalent to the integral equation

$$e^+(x,\lambda) = e_0^+(x,\lambda)+\int\limits_x^{+\infty}S_0^+(x,t,\lambda)\,q(t)\,e^+(t,\lambda)\,dt, \tag{1.5}$$

where

$$S_0^+(x,t,\lambda)=\begin{cases}\dfrac{\sin\lambda(t-x)}{\lambda}, & a<x<t \text{ or } x<t<a,\\ A\dfrac{\sin\lambda(t-x)}{\lambda}+B\dfrac{\sin\lambda(t-2a+x)}{\lambda}, & x<a<t.\end{cases}$$

Substituting $(1.2)_+$ in (1.5) and using the uniqueness of the expansion in a Fourier integral, we obtain the equation for $K^+(x,t)$,

$$K^+(x,t) = K_0^+(x,t)+\frac{1}{2}\int\limits_x^{a}q(\xi)\int\limits_{t-\xi+x}^{t+\xi-x}K^+(\xi,s)\,ds\,d\xi$$

$$+\frac{A}{2}\int\limits_{a}^{+\infty} q(\xi) \int\limits_{t-\xi+x}^{t+\xi-x} K^{+}(\xi,s)\,ds\,d\xi$$

$$+\frac{B}{2}\int\limits_{a}^{+\infty} q(\xi) \int\limits_{t-\xi+2a-x}^{t+\xi-2a+x} K^{+}(\xi,s)\,ds\,d\xi, \quad x<a, \qquad (1.6)_{+}$$

$$K^{+}(x,t) = K_0^{+}(x,t) + \frac{1}{2}\int\limits_{x}^{+\infty} q(\xi) \int\limits_{t-\xi+x}^{t+\xi-x} K^{+}(\xi,s)\,ds\,d\xi, \quad x>a, \qquad (1.7)_{+}$$

where

$$K_0^{+}(x,t) = \frac{A}{2}\int\limits_{\frac{x+t}{2}}^{+\infty} q(\xi)\,d\xi + \frac{B}{2}\begin{cases} \int\limits_{\frac{2a+x-t}{2}}^{a} q(\xi)\,d\xi - \int\limits_{a}^{\frac{t+2a-x}{2}} q(\xi)\,d\xi, & x<t<2a-x, \\ \int\limits_{\frac{t+2a-x}{2}}^{+\infty} q(\xi)\,d\xi, & t>2a-x, \end{cases} \qquad (1.8)_{+}$$

for $x<a$, and

$$K_0^{+}(x,t) = \frac{1}{2}\int\limits_{\frac{x+t}{2}}^{+\infty} q(\xi)\,d\xi \qquad (1.9)_{+}$$

for $x>a$.

Thus, to finish the proof of the theorem for $e^{+}(x,\lambda)$, it is sufficient to show that for each fixed $x \neq a$ the system of equations $(1.6)_{+}, (1.7)_{+}$ has a solution $K^{+}(x,t)$ satisfying inequality $(1.3)_{+}$.

We put

$$K_n^{+}(x,t) = \frac{1}{2}\int\limits_{x}^{a} q(\xi) \int\limits_{t-\xi+x}^{t+\xi-x} K_{n-1}^{+}(\xi,s)\,ds\,d\xi + \frac{A}{2}\int\limits_{a}^{+\infty} q(\xi) \int\limits_{t-\xi+x}^{t+\xi-x} K_{n-1}^{+}(\xi,s)\,ds\,d\xi$$

$$+\frac{B}{2}\int\limits_{a}^{+\infty} q(\xi) \int\limits_{t-\xi+2a-x}^{t+\xi-2a+x} K_{n-1}^{+}(\xi,s)\,ds\,d\xi, \quad x<a,$$

$$K_n^{+}(x,t) = \frac{1}{2}\int\limits_{x}^{+\infty} q(\xi) \int\limits_{t-\xi+x}^{t+\xi-x} K_{n-1}^{+}(\xi,s)\,ds\,d\xi, \quad x>a, \quad n=1,2,\ldots,$$

where $K_0^{+}(x,t)$ is defined by $(1.8)_{+}, (1.9)_{+}$.

It follows from the definition of $K_n^+(x,t)$ that

$$\left|K_n^+(x,t)\right| \le \frac{C}{2}\int\limits_x^{+\infty}|q(\xi)|\int\limits_{t-\xi+x}^{t+\xi-x}\left|K_{n-1}^+(\xi,s)\right| ds\, d\xi$$

or

$$\left|K_n^+(x,\,t)\right| \le \frac{C}{2}\int\limits_x^{\frac{x+t}{2}}|q(\xi)|\int\limits_{t-\xi+x}^{t+\xi-x}\left|K_{n-1}^+(\xi,\,s)\right| dsd\xi$$

$$+\frac{C}{2}\int\limits_{\frac{x+t}{2}}^{\infty}|q(\xi)|\int\limits_{\xi}^{t+\xi-x}\left|K_{n-1}^+(\xi,\,s)\right| ds\, d\xi \tag{1.10}$$

since $K(\xi,s)=0$ for $s<\xi$. Using $(1.8)_+,(1.9)_+$, we have

$$\left|K_0^+(x,\,t)\right| \le \frac{C}{2}\sigma^+\left(\frac{x+t}{2}\right), \qquad 0<|x-a|<t-a,$$

$$\left|K_0^+(x,\,t)\right| \le \frac{A}{2}\sigma^+\left(\frac{x+t}{2}\right)+\frac{|B|}{2}\sigma^+\left(\frac{2a+x-t}{2}\right), \qquad |t-a|<a-x.$$

Applying the principle of mathematical induction, from (1.10) we obtain

$$\left|K_n^+(x,\,t)\right| \le \frac{C}{2}\sigma^+\left(\frac{x+t}{2}\right)\frac{\left\{C\sigma_1^+(x)\right\}^n}{n!}, \qquad 0<|x-a|<t-a,$$

$$\left|K_n^+(x,\,t)\right| \le \left\{\frac{C}{2}\sigma^+\left(\frac{x+t}{2}\right)+\frac{|B|}{2}\sigma^+\left(\frac{2a+x-t}{2}\right)\right\}\frac{\left\{C\sigma_1^+(x)\right\}^n}{n!},$$
$$|t-a|<a-x.$$

Therefore the series $\sum\limits_{n=0}^{+\infty}K_n^+(x,t)$ uniformly converges on the set $t>x,\ t\neq 2a-x, x\neq a$ and its sum $K^+(x,t)$ is the solution of the system $(1.6)_+-(1.7)_+$ and satisfies inequality $(1.3)_+$. The validity of relations $(1.4)_+$ follows immediately from $(1.6)_+-(1.9)_+$.

The statement of the theorem for the solution $e^-(x,\,\lambda)$ can be established in a similar way. We give only integral equations for the kernel $K^-(x,\,t)$,

$$K^-(x,\,t)=K_0^-(x,\,t)+\frac{1}{2}\int\limits_a^x q(\xi)\int\limits_{t+\xi-x}^{t-\xi+x}K^-(\xi,s)ds\,d\xi$$

$$-\frac{A}{2}\int_{-\infty}^{a} q(\xi)\int_{t+\xi-x}^{t-\xi+x} K^{-}(\xi,s)ds\,d\xi-\frac{B}{2}\int_{-\infty}^{a} q(\xi)\int_{t+\xi-2a+x}^{t-\xi+2a-x} K^{-}(\xi,s)ds\,d\xi,$$

$$x>a, \qquad (1.6)_-$$

$$K^{-}(x,t)=K_0^{-}(x,t)+\frac{1}{2}\int_{-\infty}^{x} q(\xi)\int_{t+\xi-x}^{t-\xi+x} K^{-}(\xi,s)ds\,d\xi, \quad x<a, \qquad (1.7)_-$$

where

$$K_0^{-}(x,t)=\frac{A}{2}\int_{-\infty}^{\frac{x+t}{2}} q(\xi)d\xi$$

$$-\frac{B}{2}\begin{cases}\int_{a}^{\frac{2a+x-t}{2}} q(\xi)d\xi-\int_{\frac{t+2a-x}{2}}^{a} q(\xi)d\xi, & 2a-x<t<x,\\ \int_{-\infty}^{\frac{t+2a-x}{2}} q(\xi)d\xi, & t<2a-x,\end{cases} \qquad (1.8)_-$$

at $x>a$, and

$$K_0^{-}(x,t)=\frac{1}{2}\int_{-\infty}^{\frac{x+t}{2}} q(\xi)d\xi \qquad (1.9)_-$$

at $x<a$. ■

In virtue of formulas $(1.8)_\pm$ and $(1.9)_\pm$, there exist partial derivatives of the function $K_0^{\pm}(x,t)$ with respect to each argument at $t\neq 2a-x$ and $x\neq a$. Thus, it follows from equations $(1.6)_\pm$, $(1.7)_\pm$ that the functions $K^{\pm}(x,t)$ also have first partials with respect to both arguments at $t\neq 2a-x$ and $x\neq a$.

By differentiating equations $(1.5)_\pm$, $(1.6)_\pm$ and using estimations $(1.3)_\pm$, it is easy to prove the following lemma.

Lemma 1.1. *There exist partial derivatives of the function $K^{\pm}(x,t)$ with respect to both arguments at $t\neq 2a-x$ and $x\neq a$, moreover*

$$\left|\frac{\partial K^{\pm}(x_1,x_2)}{\partial x_i}\pm\frac{1}{4}q\left(\frac{x_1+x_2}{2}\right)\right|$$

$$\leq\frac{1}{2}\sigma^{\pm}(x_1)\,\sigma^{\pm}\left(\frac{x_1+x_2}{2}\right)e^{C\sigma_1^{\pm}(x_1)}, \quad \pm x_1>\pm a,$$

$$\left|\frac{\partial K^{\pm}(x_1,x_2)}{\partial x_i}\pm\frac{A}{4}q\left(\frac{x_1+x_2}{2}\right)\pm(-1)^i\frac{B}{4}q\left(\frac{x_2+2a-x_1}{2}\right)\right|$$

$$\leq \frac{C^2}{2}\sigma^{\pm}(x_1)\sigma^{\pm}\left(\frac{x_1+x_2}{2}\right)e^{C\sigma_1^{\pm}(x_1)}, \quad \pm x_2 \geq \pm(2a-x_1), \quad \pm x_1 < \pm a,$$

$$\Big|\frac{\partial K^{\pm}(x_1,\, x_2)}{\partial x_i} \pm \frac{A}{4}q\left(\frac{x_1+x_2}{2}\right) \pm (-1)^{i-1}\frac{B}{4}q\left(\frac{2a+x_1-x_2}{2}\right)$$

$$\mp(-1)^{i-1}\frac{B}{4}q\left(\frac{2a+x_2-x_1}{2}\right)\Big|$$

$$\leq \frac{C}{2}\left\{C\sigma^{\pm}\left(\frac{x_1+x_2}{2}\right) + |B|\,\sigma^{\pm}\left(\frac{2a+x_1-x_2}{2}\right)\right\}\sigma^{\pm}(x_1)e^{C\sigma_1^{\pm}(x_1)},$$

$$\pm x_1 \leq \pm x_2 \leq \pm(2a-x_1).$$

R e m a r k. The following relationships immediately follow from equations $(1.6)_{\pm}$, $(1.7)_{\pm}$:

$$K^{\pm}(a-0,\, t) = \alpha K^{\pm}(a+0,\, t), \qquad \pm t > \pm a,$$

$$K_x^{\pm'}(a-0,\, t) = \alpha^{-1}K_x^{\pm'}(a+0,\, t), \qquad \pm t > \pm a. \qquad (1.11)_{\pm}$$

Provided that $q(x)$ is differentiable, the functions $K^{\pm}(x,\, t)$ have the second partial derivatives, and we get the equation for them

$$\frac{\partial^2 K^{\pm}(x,\, t)}{\partial x^2} - \frac{\partial^2 K^{\pm}(x,\, t)}{\partial t^2} = q(x)K^{\pm}(x,t). \qquad (1.12)_{\pm}$$

On can show that conversely if the functions $K^{\pm}(x,\, t)$ satisfy equations $(1.12)_{\pm}$, relations $(1.4)_{\pm}$, $(1.11)_{\pm}$ and the conditions

$$\lim_{x+t\to\pm\infty}\frac{\partial K^{\pm}(x,\, t)}{\partial x} = \lim_{x+t\to\pm\infty}\frac{\partial K^{\pm}(x,\, t)}{\partial t} = 0$$

at infinity, then they are the solutions of equations $(1.6)_{\pm}$, $(1.7)_{\pm}$(see Remark in Appendix A). Therefore, the functions $e^{\pm}(x,\lambda)$, constructed by them by using formulas $(1.2)_{\pm}$, give the solutions of problem (0.1), (0.2) with the coefficients

$$q(x) = \begin{cases} \mp 2\frac{dK^{\pm}(x,x)}{dx}, & \pm x > \pm a, \\ \mp\frac{2}{A}\,\frac{dK^{\pm}(x,x)}{dx}, & \pm x < \pm a. \end{cases}$$

2. Direct Scattering Problem

Since the function $q(x)$ and the number α are real, the functions $\overline{e^+(x,\lambda)} \equiv e^+(x,-\lambda)$ and $\overline{e^-(x,\lambda)} \equiv e^-(x,-\lambda)$ are also the solutions of problem (0.1)–(0.2) together with $e^+(x,\lambda)$ and $e^-(x,\lambda)$ for real λ, and since the Wronskian of two solutions of problem (0.1)–(0.2) does not depend on x, we have

$$\begin{aligned} &W\left[e^+(x,\lambda),\ e^+(x,-\lambda)\right] \\ &= e^{+'}(x,\lambda)\, e^+(x,-\lambda) - e^+(x,\lambda)\, e^{+'}(x,-\lambda) = 2i\lambda, \\ &W\left[e^-(x,\lambda),\ e^-(x,-\lambda)\right] = -2i\lambda. \end{aligned} \tag{2.1}$$

Consequently, when $\lambda \neq 0$, the pairs $e^+(x,\lambda)$, $e^+(x,-\lambda)$ and $e^-(x,\lambda)$, $e^-(x,-\lambda)$ form two fundamental systems of the solutions for problem (0.1)–(0.2). Hence, for $\lambda \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$, the following representations are true:

$$e^+(x,\lambda) = b(\lambda)\, e^-(x,\lambda) + a(\lambda)\, e^-(x,-\lambda), \quad \lambda \in \mathbb{R}^*, \tag{2.2}$$

$$e^-(x,\lambda) = -b(-\lambda)\, e^+(x,\lambda) + a(\lambda)\, e^+(x,-\lambda), \quad \lambda \in \mathbb{R}^*. \tag{2.3}$$

Moreover, according to (2.1),

$$a(\lambda) = \frac{1}{2i\lambda} W\left[e^+(x,\lambda),\ e^-(x,\lambda)\right], \quad \lambda \in \mathbb{R}^*, \tag{2.4}$$

$$b(\lambda) = -\frac{1}{2i\lambda} W\left[e^+(x,\lambda),\ e^-(x,-\lambda)\right], \quad \lambda \in \mathbb{R}^*. \tag{2.5}$$

We put

$$u^\pm(x,\lambda) = e^\mp(x,\lambda)\,\frac{1}{a(\lambda)}, \quad r^\pm(\lambda) = \mp\frac{b(\mp\lambda)}{a(\lambda)}, \quad t(\lambda) = \frac{1}{a(\lambda)}. \tag{2.6$_\pm$}$$

Then inequalities (2.2), (2.3) can be rewritten in the form

$$u^\pm(x,\lambda) = r^\pm(\lambda)\, e^\pm(x,\lambda) + e^\pm(x,-\lambda). \tag{2.7$_\pm$}$$

From $(2.6)_\pm$, $(2.7)_\pm$, using $(1.2)_\pm$, we obtain the asymptotic formulas

$$u^\pm(x,\lambda) = r^\pm(\lambda)\, e^{\pm i\lambda x} + e^{\mp i\lambda x} + o(1), \qquad x \to \pm\infty,$$

$$u^\pm(x,\lambda) = t(\lambda)\, e^{\mp i\lambda x} + o(1), \qquad x \to \mp\infty.$$

The solutions $u^\pm(x,\lambda)$ are called the eigenfunctions of the left $(u^-(x,\lambda))$ and the right $(u^+(x,\lambda))$ scattering problems, the coefficients $r^-(\lambda)$, $r^+(\lambda)$ are called the left and the right reflection coefficients, respectively, and $t(\lambda)$ is called the transmission coefficient.

Using formulas (2.1)–(2.5) and standard methods [3, §3.5], the following lemmas can be proved.

Lemma 2.1. *The functions* $a(\lambda)$, $b(\lambda)$ *defined by formulas* (2.4), (2.5), *admit the following representations* ($\lambda \in \mathbb{R}^*$)*:*

1) $a(\lambda) = A - \frac{d}{2i\lambda} + \frac{1}{2i\lambda} \int\limits_0^{+\infty} \varphi(t) e^{i\lambda t} dt,$

2) $b(\lambda) = Be^{2i\lambda a} + \frac{1}{2i\lambda} \int\limits_{-\infty}^{+\infty} \psi(t) e^{i\lambda t} dt,$

where $d = A \int\limits_{-\infty}^{+\infty} q(t)\, dt$, $\varphi(t) \in L_1(0,\infty)$, $\psi(t) \in L_1(-\infty,+\infty)$,

3) $|a(\lambda)|^2 - |b(\lambda)|^2 = 1.$

Lemma 2.2. *The function* $a(\lambda)$ *can have only a finite number of zeros in the half-plane* $\mathrm{Im}\lambda > 0$. *The zeros are simple and located on the imaginary half-axis, and the function* $a^{-1}(\lambda)$ *is bounded in some neighborhood of zero.*

In what follows, we will denote the zeros of the function $a(\lambda)$ by $i\chi_1, ..., i\chi_n$ $(a(i\chi_k) = 0,\ \ \chi_k > 0)$, and the inverses of the norms of eigenfunctions $u_k^{\pm} = e^{\pm}(x, i\chi_k)$ by $m_k^{\pm}$. Thus,

$$\left(m_k^{\pm}\right)^{-2} = \int\limits_{-\infty}^{\infty} \left|e^{\pm}(x, i\chi_k)\right|^2 dx.$$

The solutions $u_k^{+}(x)$ and $u_k^{-}(x)$ are linearly dependent

$$u_k^{\pm}(x) = c_k^{\pm} u_k^{\mp}(x).$$

Lemma 2.3. *The following relations hold:*

$$\left(m_k^{\pm}\right)^{-2} = ic_k^{\pm} \dot{a}(i\chi_k), \quad k = 1, 2, \ldots, n.$$

Lemma 2.4. *The function* $za(z)$ *is continuous on the closed upper half-plane, and* $\lim_{\lambda \to 0} \lambda a(\lambda)[r^{+}(\lambda) + 1] = 0$. *There exists* $C > 0$ *such that* $1 > (1 - |r^{+}(\lambda)|^2) > C\lambda^2(1+\lambda^2)^{-1}$.

The collections $\left\{r^{-}(\lambda),\ \ i\chi_k,\ m_k^{-}\right\}$ and $\left\{r^{+}(\lambda),\ \ i\chi_k,\ m_k^{+}\right\}$ are called the left and the right scattering data for problem (0.1), (0.2), respectively.

As in the case $\alpha = 1$, it is easy to show that one scattering data is uniquely defined by another one. Indeed, from formula $(2.6)_\pm$ and Lemma 2.3 there follow the equalities

$$r^-(\lambda) = -\frac{\overline{a(-\lambda)}}{a(\lambda)}, \quad \left(m_k^-\right)^{-2} = -\left(m_k^+\right)^2 \left[\dot{a}(i\chi_k)\right]^2 \tag{2.8}$$

by which the function $a(z)$ can be reconstructed

$$a(z) = A \exp\left\{ -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln\left[\left(1-\left|r^+(\lambda)\right|^2\right) A^2\right]}{\lambda - z} d\lambda \right\} \prod_{k=1}^{n} \frac{z - i\chi_k}{z + i\chi_k}. \tag{2.9}$$

The inverse scattering problem for problem (0.1)–(0.2) consists of the reconstruction of the potential $q(x)$ by the left or the right scattering data.

3. Main Equations of the Inverse Problem (Marchenko Equations)

In this section, we obtain the main equations of the inverse scattering problem. Note that according to $(2.6)_\pm$ and Lemma 2.1, we have

$$\left|r^\pm(\lambda)\right| < 1 \quad \text{for} \quad \lambda \in \mathbb{R}^*, \tag*{$(3.1)_\pm$}$$

$$r^\pm(\lambda) = r_0^\pm(\lambda) + O\left(\frac{1}{\lambda}\right) \quad \text{for} \quad |\lambda| \to +\infty,\ \lambda \in \mathbb{R}, \tag*{$(3.2)_\pm$}$$

where

$$r_0^\pm(\lambda) = \mp\frac{B}{A} e^{\mp 2i\lambda a}.$$

So, $r^\pm(\lambda) - r_0^\pm(\lambda) \in L_2(-\infty, +\infty)$ and, consequently, the function

$$R^\pm(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[r^\pm(\lambda) - r_0^\pm(\lambda)\right] e^{\pm i\lambda x} d\lambda \tag*{$(3.3)_\pm$}$$

also belongs to $L_2(-\infty, +\infty)$.

Theorem 3.1. *For each $x \neq a$, the kernels of representations $(1.2)_\pm$ satisfy the functional-integral equations (the main equations of the inverse problem)*

$$F_1^\pm(x, y) \pm \int_x^{\pm\infty} K^\pm(x,t) F^\pm(t+y)\, dt$$

$$+K^{\pm}(x,y) \mp \frac{B}{A}K^{\pm}(x,2a-y)=0, \quad \pm y > \pm x, \tag{3.4$_\pm$}$$

where

$$F_1^{\pm}(x,y)=\begin{cases} F^{\pm}(x+y), & \pm x > \pm a, \\ AF^{\pm}(x+y) \pm BF^{\pm}(2a-x+y), & \pm x < \pm a, \end{cases} \tag{3.5$_\pm$}$$

$$F^{\pm}(x)=R^{\pm}(x)+\sum_{k=1}^{n}\left(m_k^{\pm}\right)^2 e^{-\chi_k x}, \tag{3.6$_\pm$}$$

and the functions $R^{\pm}(x)$ *are defined by* $(3.3)_{\pm}$.

P r o o f. To obtain $(3.4)_+$, we use equality $(2.7)_+$ rewritten in the form

$$\left(\frac{1}{a(\lambda)}-\frac{1}{A}\right)e^{-}(x,\lambda)=\left(r^{+}(\lambda)-r_0^{-}(\lambda)\right)e^{+}(x,\lambda)+$$

$$+e^{+}(x,-\lambda)+r_0^{+}(\lambda)e^{+}(x,\lambda)-\frac{1}{A}e^{-}(x,\lambda).$$

Multiplying both sides of this equation by $\frac{1}{2\pi}e^{i\lambda y}$, where $y>x$, and integrating with respect to λ from $-\infty$ to $+\infty$, one can get

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{a(\lambda)}-\frac{1}{A}\right)e^{-}(x,\lambda)\,d\lambda=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(r^{+}(\lambda)-r_0^{-}(\lambda)\right)e^{+}(x,\lambda)\,e^{i\lambda y}d\lambda$$

$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(e^{+}(x,-\lambda)+r_0^{+}(\lambda)e^{+}(x,\lambda)-\frac{1}{A}e^{-}(x,\lambda)\right)e^{i\lambda y}d\lambda. \tag{3.7}$$

Then, using $(1.2)_+$ for the solution $e^{+}(x,\lambda)$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(r^{+}(\lambda)-r_0^{+}(\lambda)\right)e^{+}(x,\lambda)\,e^{i\lambda y}d\lambda=R_1^{+}(x,y)+\int_{x}^{\infty}K^{+}(x,t)\,R^{+}(t+y)\,dt,$$

where

$$R_1^{+}(x,y)=\begin{cases} R^{+}(x+y), & x>a, \\ AR^{+}(x+y)+BR^{+}(y+2a-x), & x<a. \end{cases}$$

From

$$e_0^{+}(x,-\lambda)+r_0^{+}(\lambda)e_0^{+}(x,\lambda)-\frac{1}{A}e_0^{-}(x,\lambda)\equiv 0,$$

for the second term in the right-hand side of (3.7) we obtain

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(e^{+}(x,-\lambda)+r_0^{+}(\lambda)\,e^{+}(x,\lambda)-\frac{1}{A}e^{-}(x,\lambda)\right)e^{i\lambda y}d\lambda$$

$$=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(\int\limits_{x}^{+\infty}K^{+}(x,t)\,e^{-i\lambda t}dt-\frac{B}{A}\int\limits_{x}^{+\infty}K^{+}(x,t)\,e^{i\lambda(t-2a)}dt\right.$$

$$\left.-\frac{1}{A}\int\limits_{-\infty}^{x}K^{-}(x,t)\,e^{-i\lambda t}dt\right)e^{i\lambda y}d\lambda$$

$$=K^{+}(x,y)-\frac{B}{A}K^{+}(x,2a-y)-\frac{1}{A}K^{-}(x,y)=K^{+}(x,y)-\frac{B}{A}K^{+}(x,2a-y)$$

since $K^{-}(x,y)=0$ for $y>x$.

Therefore, (3.7) takes the form

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(\frac{1}{a(\lambda)}-\frac{1}{A}\right)e^{-}(x,\lambda)\,e^{i\lambda y}d\lambda=R_1^{+}(x,y)$$

$$+\int\limits_{x}^{\infty}K^{+}(x,t)\,R^{+}(t+y)\,dt+K^{+}(x,y)-\frac{B}{A}K^{+}(x,2a-y)\,. \tag{3.8}$$

Now we calculate the left-hand side of (3.8) with the help of contour integration. Since the function $\frac{1}{a(\lambda)}-\frac{1}{A}$ is regular in the upper half-plane $\mathrm{Im}\lambda>0$, except the finite number of points $i\chi_k$ (where it has simple poles), tends to zero for $|\lambda|\to\infty$, $\mathrm{Im}\lambda\geq 0$, and is bounded in some neighborhood of zero (see Lemmas 2.1 and 2.2) and the function $e^{-}(x,\lambda)\,e^{i\lambda y}$ for $y>x$ is uniformly bounded in the half-plane $\mathrm{Im}\lambda\geq 0$, by using Jordan's lemma, for $y>x$ we obtain

$$\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\left(\frac{1}{a(\lambda)}-\frac{1}{A}\right)e^{-}(x,\lambda)\,e^{i\lambda y}d\lambda$$

$$=i\sum_{k=1}^{n}\underset{\lambda=i\chi_k}{res}\left(\frac{1}{a(\lambda)}-\frac{1}{A}\right)e^{-}(x,\lambda)\,e^{i\lambda y}$$

$$=i\sum_{k=1}^{n}\frac{e^{-}(x,i\chi_k)\,e^{-\chi_k y}}{\dot{a}(i\chi_k)}=i\sum_{k=1}^{n}\frac{e^{+}(x,i\chi_k)\,e^{-\chi_k y}}{c_k^{+}\dot{a}(i\chi_k)}$$

$$= \sum_{k=1}^{n} m_k^2 \left\{ e_0^+(x, i\chi_k) e^{-\chi_k y} + \int_x^{\infty} K^+(x,t) e^{-\chi_k (t+y)} dt \right\}.$$

Substituting this into equality (3.8) and taking into account

$$e_0^+(x, i\chi_k) e^{-\chi_k y} = \begin{cases} e^{-\chi_k (x+y)}, & x > a, \\ Ae^{-\chi_k (x+y)} + Be^{-\chi_k (2a-x+y)}, & x < a, \end{cases}$$

we obtain equation $(3.4)_+$. It is also true for $y = x$ because of continuity. Equation $(3.4)_-$ can be obtained in a similar way by using equality $(2.7)_-$. ∎

4. Other Properties of the Scattering Data. Uniqueness Theorem for the Solution of the Inverse Problem

The main equations $(3.4)_\pm$ can be written in the form

$$F^{\pm}(x+y) + K^{\pm}(x,y) \pm \int_x^{\pm\infty} K^{\pm}(x,t) F^{\pm}(t+y) dt = 0,$$

$$\pm x > \pm a, \quad \pm y > \pm x, \tag{4.1$_\pm$}$$

$$AF^{\pm}(x+y) \pm BF^{\pm}(2a-x+y) + K^{\pm}(x,y) \mp \frac{B}{A} K^{\pm}(x, 2a-y)$$

$$\pm \int_x^{\pm\infty} K^{\pm}(x,t) F^{\pm}(t+y) dt = 0, \quad \pm x < \pm a, \quad \pm x < \pm y < \pm (2a-x), \tag{4.2$_\pm$}$$

$$AF^{\pm}(x+y) \pm BF^{\pm}(2a-x+y) + K^{\pm}(x,y)$$

$$\pm \int_x^{\pm\infty} K^{\pm}(x,t) F^{\pm}(t+y) dt = 0, \quad \pm x < \pm a, \quad \pm y > \pm (2a-x). \tag{4.3$_\pm$}$$

Equations $(4.1)_\pm$ coincide with the main equation in the case $\alpha = 1$ (see [3]). It implies that the functions $F^{\pm}(x)$ are absolutely continuous for $\pm x \geq 2a$, and

$$\int_{\pm 2a}^{+\infty} \left| F^{\pm}(\pm x) \right| dx < \infty, \qquad \int_{\pm 2a}^{+\infty} (1+|x|) \left| F^{\pm\prime}(\pm x) \right| dx < \infty.$$

It is clear that the functions $R^{\pm}(x)$ also have this property. From $(4.2)_\pm$, for $y \to x \pm +0$ we have $(\pm x < \pm a)$

$$AF^{\pm}(2x) \pm BF^{\pm}(2a \pm 0) + K^{\pm}(x,x) \mp \frac{B}{A} K^{\pm}(x, 2a - x \mp 0)$$

$$\pm \int_x^{\pm\infty} K^{\pm}(x,t) F^{\pm}(t+x) dt = 0.$$

From the above, taking into account the properties of the functions $K^{\pm}(x,t)$ (1), it is easy to show that the functions $F^{\pm}(x)$ are absolutely continuous when $x' \leq \pm x \leq \pm 2a$ and $\int\limits_{x'}^{\pm 2a} | F^{\pm}(\pm x) | \, dx < \infty$.

Now we pass to the limits in $(4.2)_{\pm}$ as $y \to 2a - x \mp 0$ and in $(4.3)_{\pm}$ as $y \to 2a - x \pm 0$ and subtract the obtained relations. Taking into account $(1.4)_{\pm}$, we have

$$F^{\pm}(2a+0) - F^{\pm}(2a-0) = \mp \frac{B}{A} \int\limits_{a}^{\pm\infty} q(t)\, dt.$$

Thus the scattering data of the considered problem satisfy the following conditions:

I. The reflection coefficients $r^{\pm}(\lambda)$ are continuous for real $\lambda \neq 0$, $r^{\pm}(-\lambda) = \overline{r^{\pm}(\lambda)}$, $|r^{\pm}(\lambda)| < 1$ and $r^{\pm}(\lambda) = r_0^{\pm}(\lambda) + O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \pm\infty$. Their Fourier transformations

$$R^{\pm}(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \left[r^{\pm}(\lambda) - r_0^{\pm}(\lambda) \right] e^{\pm i\lambda x} d\lambda$$

are real, absolutely continuous on any interval not containing the point $2a$, and at the point $x = 2a$ have finite limits $R^{\pm}(2a+0)$, $R^{\pm}(2a-0)$. Furthermore, the functions $R^{\pm}(x)$ belong to the space $L_2(-\infty,+\infty)$, and for any $x' > -\infty$,

$$\int\limits_{x'}^{+\infty} \left| R^{\pm}(\pm x) \right| dx < \infty, \qquad \int\limits_{x'}^{+\infty} (1+|x|) \left| R^{\pm'}(\pm x) \right| dx < \infty.$$

Theorem 4.1. *If conditions I are satisfied, equations* $(3.4)_{+}$ *and* $(3.4)_{-}$ *have the unique solutions* $K^{+}(x,\cdot) \in L_1(x,\infty)$ *and* $K^{-}(x,\cdot) \in L_1(-\infty,x)$ *for each fixed* $x > -\infty$ *and* $x < \infty$*, respectively.*

P r o o f. Notice that for each fixed $x > -\infty$, the operator

$$\left(M_x^{+} f \right)(y) = \begin{cases} f(y), & x > a \\ f(y) - \frac{B}{A} f(2a-y), & x < a, \end{cases}$$

acting in the space $L_1(x,+\infty)$ (and also in $L_2(x,+\infty)$), is invertible. Therefore the main equation $(3.5)_{+}$ is equivalent to

$$K^{+}(x,y) + \left(M_x^{+} \right)^{-1} F_1^{+}(x,y) + \left(M_x^{+} \right)^{-1} F^{+} K^{+}(x,\cdot)(y) = 0, \quad y > x,$$

i.e., to the equation with the compact operator $(M_x^+)^{-1} F^+$ (for the compactness of F^+, see [3, Lemma 3.3.1]). To prove the theorem, it is sufficient to show that the homogeneous equation

$$f_x(y) - \frac{B}{A} f_x(2a-y) + \int_x^{+\infty} f_x(t) F^+(t+y)\, dt = 0, \quad y > x, \tag{4.4$_+$}$$

has only the trivial solution $f_x(y) \in L_1(x, +\infty)$. By conditions I, the function $F^+(y)$ and the corresponding solution $f_x(y)$ are bounded in the half axis $x \le y < \infty$. Therefore, $f_x(\cdot) \in L_2(x, +\infty)$.

Now let us multiply equation $(4.4)_+$ by $\overline{f_x(y)}$ and integrate with respect to y over the interval $(x, +\infty)$. Using $(3.3)_+$, $(3.5)_+$, $(3.6)_+$ and Parseval's identity

$$\int_x^\infty |f_x(y)|^2 dy = \frac{1}{2\pi} \int_{-\infty}^\infty \left|\tilde{f}(\lambda)\right|^2 d\lambda,$$

$$-\frac{B}{A} \int_x^\infty f_x(2a-y)\, \overline{f_x(y)} dy = \frac{1}{2\pi} \int_{-\infty}^\infty r_0^+(\lambda)\, \overline{\tilde{f}(\lambda)} \tilde{f}(-\lambda)\, d\lambda,$$

where $\tilde{f}_x(\lambda) = \int_x^\infty f_x(t)\, e^{-i\lambda t} dt$, we obtain

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left|\tilde{f}(\lambda)\right|^2 d\lambda + \sum_{k=1}^n \left(m_k^+\right)^2 \left|\tilde{f}(-i\chi_k)\right|^2 + \frac{1}{2\pi} \int_{-\infty}^\infty r^+(\lambda)\, \tilde{f}(-\lambda)\, \overline{\tilde{f}(\lambda)} d\lambda = 0.$$

Since $|r^+(\lambda)| = |r^+(-\lambda)|$, we obtain the estimate

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left|\tilde{f}(\lambda)\right|^2 d\lambda \le \frac{1}{2\pi} \int_{-\infty}^\infty |r^+(\lambda)| \left|\overline{\tilde{f}(-\lambda)}\right| \left|\tilde{f}(\lambda)\right| d\lambda$$

$$\le \frac{1}{2\pi} \int_{-\infty}^\infty |r^+(\lambda)| \frac{\left|\tilde{f}(-\lambda)\right|^2 + \left|\tilde{f}(\lambda)\right|^2}{2} d\lambda = \frac{1}{2\pi} \int_{-\infty}^\infty |r^+(\lambda)| \left|\tilde{f}(\lambda)\right|^2 d\lambda$$

or

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left\{1 - \left|r^+(\lambda)\right|\right\} \left|\tilde{f}(\lambda)\right|^2 d\lambda \le 0.$$

It follows from the above that $\tilde{f}(\lambda) \equiv 0$ since $1 - |r^+(\lambda)| > 0$ for all $\lambda \ne 0$. Thus, the main equation $(3.4)_+$ is uniquely solvable. The unique solvability of $(3.4)_-$ can be proved in a similar way. The theorem is proved. ■

Corollary. *The potential $q(x)$ from class (0.3) in problem* (0.1)–(0.2) *is uniquely defined by the right (left) scattering data, i.e., if the right (left) scattering data of two problems* (0.1)–(0.2) *with the potentials $q(x)$ and $\widetilde{q}(x)$ from class* (0.3) *coincide, then $q(x) = \widetilde{q}(x)$ a.e. on the whole axis.*

5. Solution of the Inverse Scattering Problem

In the next theorem we provide a solution to the inverse scattering problem from class (0.3).

Theorem 5.1. *For the collection $\{r^+(\lambda), i\chi_k, m_k^+\}$ to be the right scattering data of problem* (0.1)–(0.2) *with a real potential $q(x)$ satisfying* (0.3)*, the following necessary and sufficient conditions should be satisfied:*

1) *the function $r^+(\lambda)$ is continuous for all real $\lambda \neq 0$, $\overline{r^+(\lambda)} = r^+(-\lambda)$, $r^+(\lambda) = r_0^+(\lambda) + O\left(\frac{1}{\lambda}\right)$, $\lambda \to \pm\infty$, where $r_0^+(\lambda) = e^{-2i\lambda a}\frac{1-\alpha^2}{1+\alpha^2}$, and there exists $C > 0$ such that $1 - |r^+(\lambda)| \geq C\frac{\lambda^2}{1+\lambda^2}$;*

2) *the function $za(z)$, where*

$$a(z) = \frac{\alpha^2+1}{2\alpha} \exp\left\{ -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln\left[(1-|r^+(\lambda)|^2)\left(\frac{\alpha^2+1}{2\alpha}\right)^2\right]}{\lambda - z} d\lambda \right\} \prod_{k=1}^{n} \frac{z - i\chi_k}{z + i\chi_k},$$

is continuous on the closed upper half-plane, and

$$\lim_{\lambda \to 0} \lambda a(\lambda)[r^+(\lambda) + 1] = 0;$$

3) *the functions $R^+(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[r^+(\lambda) - r_0^+(\lambda)]e^{i\lambda x}d\lambda$ and $R^-(x) = -\frac{1}{2\pi} \times \int_{-\infty}^{+\infty}[\overline{r^+(\lambda)}\frac{a(-\lambda)}{a(\lambda)} - \frac{1-\alpha^2}{1+\alpha^2}e^{2i\lambda a}]e^{-i\lambda x}d\lambda$ are absolutely continuous on any segment not containing the point $2a$; there exist finite limits $R^\pm(2a+0)$, $R^\pm(2a-0)$, and the derivatives ${R^+}'(x)$, ${R^-}'(x)$ satisfy the inequalities*

$$\int_{\alpha'}^{+\infty} (1+|x|)\mid {R^+}'(x) \mid dx < \infty, \qquad \int_{-\infty}^{\beta'} (1+|x|)\mid {R^-}'(x) \mid dx < \infty$$

for all $\alpha' > -\infty$, $\beta' < +\infty$;

4) *the solutions $K^\pm(x,t)$ of the main equations $(3.4)_\pm$ satisfy the conditions*

$$K^\pm(x,x)\big|_{a\mp 0} = \frac{\alpha^2+1}{2\alpha}\, K^\pm(x,x)\big|_{a\pm 0}.$$

P r o o f. We give a short proof of sufficiency. The necessity was proved above. Basing on the given collection, we construct another collection $\{r^-(\lambda), i\chi_k, m_k^-\}$ using (2.8), (2.9) and show that these collections are, respectively, the right and left scattering data of problem (0.1)–(0.2) with a real potential $q(x)$ satisfying (0.3).

1. From the conditions of Theorem 5.1, we obtain that equations $(3.4)_+$ and $(3.4)_-$ reconstructed by the scattering data, have the unique solutions $K^+(x,y)$ and $K^-(x,y)$ according to Theorem 4.1. It is easy to show that these solutions satisfy the relations

$$A[F^{\pm}(2a+0) - F^{\pm}(2a-0)] + K^{\pm}(x, 2a-x+0)$$
$$-K^{\pm}(x, 2a-x-0) + \frac{B}{A}K^{\pm}(x,x) = 0, \quad \pm x < \pm a. \qquad (5.1)_{\pm}$$

2. Show that the functions $e^+(x,\lambda)$, $e^-(x,\lambda)$, constructed with the help of $K^+(x,t)$, $K^-(x,t)$ by formulas $(1.2)_+$ and $(1.2)_-$, satisfy the equations

$$-e^{\pm''}(x,\lambda) + q^{\pm}(x)e^{\pm}(x,\lambda) = \lambda^2 e^{\pm}(x,\lambda) \qquad (5,2)_{\pm}$$

and the discontinuity conditions

$$e^{\pm}(a-0,\lambda) = \alpha e^{\pm}(a+0,\lambda),$$
$$e^{\pm'}(a-0,\lambda) = \alpha^{-1} e^{\pm}(a+0,\lambda), \qquad (5,3)_{\pm}$$

moreover

$$\int_{x'}^{+\infty} (1+|x|)\left|q^+(x)\right| dx < \infty, \quad \int_{-\infty}^{x''} (1+|x|)\left|q^-(x)\right| dx < \infty. \qquad (5.4)$$

First, suppose that the functions $R^{\pm}(x)$ are twice continuously differentiable, and for all $\alpha' > -\infty$, $\beta' < +\infty$

$$\int_{\alpha'}^{+\infty} (1+|x|)\left|R^{+''}(x)\right| dx < \infty, \quad \int_{-\infty}^{\beta'} (1+|x|)\left|R^{-''}(x)\right| dx < \infty. \qquad (5.5)$$

Then the solutions $K^{\pm}(x,y)$ of the main equations $(3.4)_{\pm}$ are twice continuously differentiable for $t \neq 2a - x$ and $x \neq a$, moreover, for each x all first order and second order partial derivatives are summable with respect to y.

Consider the region $\pm x < \pm a$, $\pm x < \pm y < \pm(2a - x)$. Then the main equations $(3.4)_\pm$ become like $(4.2)_\pm$. After twice differentiating this equation with respect to y and integrating by parts, we get

$$AF^{\pm''}(x+y) \pm BF^{\pm''}(2a-x+y) + K^{\pm}_{yy}{}''(x,y) \mp \frac{B}{A}K^{\pm}_{yy}{}''(x,2a-y)$$

$$\mp K^{\pm''}(x,y)F^{\pm'}(x+y) - \left[K^{\pm}(x,\,t)\Big|_{t=2a-x-0}^{2a+x+0}\right] F^{\pm'}(2a-x+y)$$

$$\pm K^{+'}_t(x,t)\Big|_{t=x} F^{+}(x+y) + \left[K'_t(x,t)\Big|_{t=2a-x-0}^{2a-x+0}\right] F^{+}(2a-x+y)$$

$$\pm \int_x^{\pm\infty} K^{\pm}_{tt}{}''(x,t)F^{+}(t+y)dt = 0$$

By differentiating equations $(4.2)_\pm$ two times with respect to x, we have

$$AF^{\pm''}(x+y) \pm BF^{\pm''}(2a-x+y) + K^{\pm}_{xx}{}''(x,y) \mp \frac{B}{A}K^{\pm}_{xx}{}''(x,2a-y)$$

$$\mp K^{\pm''}(x,x)F^{\pm}(x+y) \mp K^{\pm}(x,x)F^{\pm'}(x+y)$$

$$+\left[K^{\pm}(x,t)\Big|_{t=2a-x-0}^{2a-x+0}\right]' F^{\pm}(2a-x+y)$$

$$\pm\left[K^{+}(x,t)\Big|_{2a-x-0}^{2a-x+0}\right] F^{\pm'}(2a-x+y) \mp K^{\pm'}_x(x,y)\Big|_{y=x} F^{\pm}(x+y)$$

$$\mp\left[K^{\pm'}_x(x,t)'\Big|_{t=2a-x-0}^{2a-x+0}\right] F^{\pm}(2a-x+y) \pm \int_x^{\pm\infty} K^{\pm}_{xx}{}''(x,t)F^{\pm}(t+y)dt = 0.$$

Subtracting from the latter equation the previous one, we obtain

$$K^{\pm}_{xx}{}''(x,y) \mp \frac{B}{A}K^{\pm}_{xx}{}''(x,2a-y) - K^{\pm}_{yy}{}''(x,y) \pm \frac{B}{A}K^{\pm}_{yy}{}''(x,2a-y)$$

$$\mp 2K^{\pm'}(x,x)F^{\pm}(x+y) + 2\left[K^{\pm}(x,t)\Big|_{t=2a-x-0}^{2a-x+0}\right] F^{\pm}(2a-x+y)$$

$$\pm \int_x^{\pm\infty} \left(K^{\pm}_{xx}{}''(x,t) - K^{\pm}_{tt}{}''(x,t)\right) F^{\pm}(t+y)dt = 0. \qquad (5.5')_\pm$$

By virtue of $(5.1)_\pm$ and the main equation $(4.2)_\pm$, we get

$$\pm 2K^{\pm'}(x,x)F^{\pm}(x+y) + 2\left[K^{\pm}(x,y)\Big|_{y=2a-x-0}^{2a-x+0}\right] F^{\pm}(2a-x+y)$$

$$= -q^{\pm}(x)\left[AF^{\pm}(x+y) \pm BF^{\pm}(2a-x+y)\right]$$

$$= q^{\pm}(x)\left[K^{\pm}(x,y) \mp \frac{B}{A}K^{\pm}(x,2a-y)\right] \pm \int\limits_{x}^{\infty} K^{\pm}(x,\,t)F^{\pm}(t+y)dt. \quad (5.5'')_{\pm}$$

From $(5.5')_{\pm}$ and $(5.5'')_{\pm}$ it follows that

$$K_{xx}^{\pm\,''}(x,y) - K_{yy}^{\pm\,''}(x,y) - q^{\pm}(x)K^{\pm}(x,y)$$

$$\mp\frac{B}{A}\left\{K_{xx}^{\pm\,''}(x,2a-y) - K_{yy}^{\pm\,''}(x,2a-y) - q^{\pm}(x)K^{\pm}(x,2a-y)\right\}$$

$$\pm\int\limits_{x}^{\pm\infty}\left\{K_{xx}^{\pm\,''}(x,t) - q^{\pm}(x)K^{\pm}(x,\,t) - K_{tt}^{\,''}(x,t)\right\}F^{\pm}(t+y)dt = 0,$$

i.e., the functions

$$h_x^{\pm}(y) = K_{xx}^{\pm\,''}(x,y) - q^{\pm}(x)K^{\pm}(x,y) - K_{yy}^{\pm\,''}(x,y)$$

are summable solutions of homogeneous equations which correspond to $(4.2)_{\pm}$. In the similar way as for equations $(4.1)_{\pm}$ and $(4.3)_{\pm}$, we obtain that the solutions $K^{\pm}(x,y)$ of the main equations $(3.4)_{\pm}$ satisfy the equation

$$K_{xx}^{\pm\,''} - q^{\pm}(x)K^{\pm} - K_{yy}^{\pm\,''} = 0 \qquad (*)$$

according to Theorem 4.1.

In virtue of condition 4), $(5.1)_{\pm}$ yields that the functions $K^{\pm}(x,y)$ satisfy relations $(1.4)_{\pm}$.

By our assumptions (see (5.5)), it can be easily shown that the relations

$$\lim_{x+y\to\pm\infty} K_x^{\pm\,'}(x,\,y) = \lim_{x+y\to\pm\infty} K_y^{\pm\,'}(x,\,y) = 0 \qquad (**)$$

also hold.

Now we will show that the functions $K^{\pm}(x,y)$ satisfy the conditions

$$K^{\pm}(a-0,\,y) - \alpha K^{\pm}(a+0,\,y) = 0, \qquad (5.6)_{\pm}$$

$$K_x^{\pm\,'}(a-0,\,y) - \alpha^{-1}K_x^{\pm\,'}(a+0,\,y) = 0. \qquad (5.7)_{\pm}$$

Take $x = a\pm 0$ and $x = a\mp 0$ in the main equations $(4.1)_{\pm}$ and $(4.3)_{\pm}$, respectively. Subtracting from the first obtained relation the second one multiplied by α, we get that the differences $K^{\pm}(a-0,y) - \alpha K^{\pm}(a+0,y)$ are the solutions of homogeneous equations corresponding to the main equations $(4.1)_{\pm}$ at $x = a$. Thus, according to Theorem 4.1, we obtain $(5.6)_{\pm}$.

Prove that conditions $(5.7)_\pm$ also hold. Notice that for the solutions of the main equations the relationships below are true:

$$K^\pm(x,2a-x\pm 0)\big|_{a\mp 0}=\alpha^{\pm 1}K^\pm(x,x)\big|_{a\pm 0}\,,$$
$$K^\pm(x,2a-x\mp 0)\big|_{a\mp 0}=K^\pm(x,x)\big|_{a\mp 0}\,. \qquad (5.8)_\pm$$

Indeed, in equations $(4.1)_\pm$ set first $y=x$, then $x=a\pm 0$, and in equations $(4.3)_\pm$ first $y=2a-x\pm 0$, then $x=a\mp 0$. Multiply the first obtained relations by $\alpha^{\pm 1}$ and subtract the second ones. As a result, according to $(5.6)_\pm$, we get the first equalities from $(5.8)_\pm$. Supposing first $y=2a-x-0,\ \ x=a-0$ and then $y=x,\ \ x=a-0$ in equations $(4.2)_\pm$, it is easy to obtain the second relations from $(5.8)_\pm$.

Now differentiate equations $(4.1)_\pm$ and $(4.3)_\pm$ with respect to the variable x and assume $x=a\pm 0$ and $x=a\mp 0$, respectively. As a result, we have

$$F^{\pm'}(a+y)+K_x^{\pm'}(a\pm 0,y)\mp K^\pm(x,x)\big|_{a\pm 0}F^\pm(a+y)$$
$$\pm\int_a^{\pm\infty}K_x^{\pm'}(a\pm 0,t)F^\pm(t+y)dt=0, \qquad (5.9)_\pm$$

$$(A\mp B)F^{\pm'}(a+y)+K_x^{\pm'}(a\mp 0,y)\pm[K^\pm(x,2a-x+0)$$
$$-K^\pm(x,2a-x-0)]\big|_{x=a\mp 0}\cdot F^\pm(a+y)\mp K^\pm(x,x)\big|_{a\mp 0}F^\pm(a+y)$$
$$\pm\int_a^{\pm\infty}K_x^{\pm'}(a\mp 0,t)F^\pm(t+y)dt=0. \qquad (5.10)_\pm$$

Multiply $(5.9)_\pm$ by $\alpha^{\mp 1}$ and subtract $(5.10)_\pm$. By virtue of $(5.8)_\pm$, using condition 4) of the theorem, we get that the differences $\alpha^{\mp 1}K_x^{\pm'}(a\pm 0,y)-K_x^{\pm'}(a\mp 0,y)$ also satisfy homogeneous equations corresponding to $(4.1)_\pm$ at $x=a$. Thus,

$$\alpha^{\mp 1}K_x^{\pm'}(a\pm 0,y)-K_x^{\pm'}(a\mp 0,y)=0$$

and, consequently, conditions $(5.7)_\pm$ are also satisfied.

Therefore, if conditions (5.5) hold, then the solutions $K^\pm(x,y)$ of the main equations $(3.4)_\pm$ satisfy equation $(*)$, relations $(1.4)_\pm$ (where the functions $q(x)$ must correspond to the functions $q^\pm(x)$), $(5.6)_\pm$, $(5.7)_\pm$ and conditions $(**)$ at infinity. Hence, according to Remark from Section 1, the functions $e^\pm(x,\lambda)$ satisfy equations $(5.2)_\pm$ and conditions $(5.3)_\pm$.

The case, when only conditions 3) of the theorem are satisfied, can be considered by passing to limit (see [3], p. 212).

Finally show that conditions (5.4) hold. Since at $\pm x > \pm a$ the main equations $(3.4)_\pm$ become like $(4.1)_\pm$, namely they have the form analogous to the case $\alpha = 1$, and conditions 3) of Theorem 5.1 are the same as in the case $\alpha = 1$, then it is not difficult to show that if $x' \geq a$, $x'' \leq a$, then (5.4) are true (see [3], p. 209). It should be shown that $q^+(x)$ $(q^-(x))$ are summable in the interval (x', a) $((a, x''))$ for every $x' > -\infty$ $(x'' < +\infty)$. But it is easy to establish these facts by means of the formula (which is equivalent to equation $(4.2)_\pm$)

$$K^\pm(x,y) = \frac{A^2}{A^2 - B^2}\left[\varphi^\pm(x,y) \pm \varphi^\pm(x, 2a - y)\right],$$

where

$$\varphi^\pm(x,y) = -AF^\pm(x+y) \mp BF^\pm(2a - x + y) \mp \int\limits_x^{\pm\infty} K^\pm(x,t)F^\pm(t+y)dt,$$

using conditions 3) of the theorem and summability of partial derivatives $K_x^{\pm'}$, $K_t^{\pm'}$.

3. To prove the theorem, it is sufficient to show that for the real values $\lambda \neq 0$, the functions $e^+(x, \lambda)$ and $e^-(x, \lambda)$ are related by the equalities

$$r^\pm(\lambda)e^\pm(x,\lambda) + \overline{e^\pm(x,\lambda)} = \frac{1}{a(\lambda)}e^\mp(x,\lambda). \qquad (5.11)_\pm$$

In fact, by virtue of $(5.2)_\pm$, it follows from $(5.11)_\pm$ that

$$q_+(x) = q_-(x) \overset{def}{=} q(x), \quad -\infty < x < +\infty,$$

and according to (5.4),

$$\int\limits_{-\infty}^{\infty} (1 + |x|)\,|q(x)|\,dx < +\infty.$$

Show that then $\{r^+(\lambda), i\chi_k, m_k^+\}$ and $\{r^-(\lambda), i\chi_k, m_k^-\}$ are the right and left scattering data of the constructed problem (0.1), (0.2).

Denote by $\{\tilde r^+(\lambda), i\tilde\chi_k, \tilde m_k^+\}$ and $\{\tilde r^-(\lambda), i\tilde\chi_k, \tilde m_k^-\}$ the right and left scattering data of the constructed problem (0.1), (0.2). The functions $e^+(x, \lambda)$ and $e^-(x, \lambda)$ will be Jost type solutions of the constructed problem (0.1), (0.2). Thus, by virtue of the results of direct problem of scattering theory (see Section 2), we can write

$$\tilde r^\pm(\lambda)e^\pm(x,\lambda) + \overline{e^\pm(x,\lambda)} = \frac{1}{\tilde a(\lambda)}e^\mp(x,\lambda). \qquad (5.12)_\pm$$

From $(5.11)_{\pm}$ and $(5.12)_{\pm}$ we have

$$a(\lambda)r^{+}(\lambda)e^{+}(x,\lambda) + a(\lambda)\overline{e^{+}(x,\lambda)} = e^{-}(x,\lambda),$$

$$\tilde{a}(\lambda)\tilde{r}^{+}(\lambda)e^{+}(x,\lambda) + \tilde{a}(\lambda)\overline{e^{+}(x,\lambda)} = e^{-}(x,\lambda),$$

respectively. Subtracting this relations, we get

$$\left\{a(\lambda)r^{+}(\lambda) - \tilde{a}(\lambda)\tilde{r}^{+}(\lambda)\right\} e^{+}(x,\lambda) + \{a(\lambda) - \tilde{a}(\lambda)\}\, \overline{e^{+}(x,\lambda)} = 0.$$

Since for $\lambda \neq 0$, $e^{+}(x,\lambda)$ and $\overline{e^{+}(x,\lambda)}$ are linearly independent, then from the last identity it follows that

$$a(\lambda)r^{+}(\lambda) - \tilde{a}(\lambda)\tilde{r}^{+}(\lambda) = 0, \quad a(\lambda) = \tilde{a}(\lambda),$$

i.e., $a(\lambda) = \tilde{a}(\lambda)$, $r^{+}(\lambda) = \tilde{r}^{+}(\lambda)$.

Analogously, relations $(5.11)_{-}$ and $(5.12)_{-}$ yield $r^{-}(\lambda) = \tilde{r}^{-}(\lambda)$. Consequently, the zeros of the functions $a(\lambda)$ and $\tilde{a}(\bar{\lambda})$ coincide: $i\chi_k = i\tilde{\chi}_k$. Thus,

$$(m_k^{\pm})^{-2} = \int_{-\infty}^{\infty} \left|e^{\pm}(x, i\chi_k)\right|^2 dx = \int_{-\infty}^{\infty} \left|e^{\pm}(x, i\tilde{\chi}_k)\right|^2 dx = (\tilde{m}_k^{\pm})^{-2}.$$

Therefore, the collection of the quantities $\{\tilde{r}^{+}(\lambda),\, i\tilde{\chi}_k,\, \tilde{m}_k^{+}\}$ and $\{\tilde{r}^{-}(\lambda),\, i\tilde{\chi}_k,\, \tilde{m}_k^{-}\}$ are the right and the left scattering data of the constructed problem (0.1), (0.2).

4. Now, turn to the proof of relations $(5.11)_{\pm}$. Suppose

$$\Phi^{\pm}(x,y) = R_1^{\pm}(x,y) \pm \int_{x}^{\pm\infty} K^{\pm}(x,t)R^{\pm}(t+y)dt,$$

where

$$R_1^{\pm}(x,y) = \begin{cases} R^{\pm}(x+y), & \pm x > \pm a, \\ AR^{\pm}(x+y) \pm BR^{\pm}(2a - x + y), & \pm x < \pm a, \end{cases}$$

$$A = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right), \qquad B = \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right).$$

Since $R^{\pm}(y) \in L_2(-\infty, +\infty)$, then for each fixed x $\Phi^{\pm}(x,y) \in L_2(-\infty, +\infty)$. We have

$$\lim_{N\to\infty} \int_{-N}^{N} \Phi^{\pm}(x,y)e^{\mp i\lambda y}dy = [r^{\pm}(\lambda) - r_0^{\pm}(\lambda)]\, \left[e_0^{\pm}(x,\lambda)\right.$$

$$\left.\pm\int_{x}^{\pm\infty}K^{\pm}(x,t)e^{\pm i\lambda t}dt\right]=\left[r^{\pm}(\lambda)-r_0^{\pm}(\lambda)\right]e^{\pm}(x,\lambda). \qquad (5.13)_{\pm}$$

On the other hand, by virtue of equations $(3.4)_{\pm}$,

$$\Phi^{\pm}(x,y)=-K^{\pm}(x,y)\pm\frac{B}{A}K^{\pm}(x,2a-y)-\sum_{k=1}^{n}\left(m_k^{\pm}\right)^2e^{\pm}(x,i\chi_k),\quad \pm y>\pm x.$$

Hence,

$$\lim_{N\to\infty}\int_{-N}^{N}\Phi^{\pm}(x,y)e^{\mp i\lambda y}dy=\lim_{N\to\infty}\left\{\pm\int_{\mp N}^{x}\Phi^{\pm}(x,y)e^{\mp i\lambda y}dy\right\}$$

$$\mp\int_{x}^{\pm\infty}K^{\pm}(x,y)e^{\mp i\lambda y}dy+\frac{B}{A}\int_{x}^{\pm\infty}K^{\pm}(x,2a-y)e^{\mp i\lambda y}dy$$

$$-\sum_{k=1}^{n}\left(m_k^{\pm}\right)^2e^{\pm}(x,i\chi_k)\frac{e^{\mp(\chi_k+i\lambda)x}}{\chi_k+i\lambda}=\lim_{N\to\infty}\left\{\pm\int_{\mp N}^{x}\Phi^{\pm}(x,y)e^{\mp i\lambda y}dy\right\}$$

$$+e_0^{\pm}(x,-\lambda)-e^{\pm}(x,-\lambda)-r_0^{\pm}(\lambda)\left[-e_0^{\pm}(x,-\lambda)+e^{\pm}(x,\lambda)\right]$$

$$-\sum_{k=1}^{n}\left(m_k^{\pm}\right)^2e^{\pm}(x,i\chi_k)\frac{e^{\mp(\chi_k+i\lambda)x}}{\chi_k+i\lambda}. \qquad (5.14)_{\pm}$$

Taking into consideration $(5.13)_{\pm}$, $(5.14)_{\pm}$ and the formulas

$$r_0^{\pm}(\lambda)e_0^{\pm}(x,\lambda)+e_0^{\pm}(x,-\lambda)=\frac{1}{A}e_0^{\mp}(x,\lambda),$$

we get the equality

$$r^{\pm}(\lambda)e^{\pm}(x,\lambda)+e^{\pm}(x,-\lambda)=\frac{1}{a(\lambda)}\,h^{\mp}(x,\lambda), \qquad (5.15)_{\pm}$$

where

$$h^{\pm}(x,\lambda)=a(\lambda)\left\{\frac{1}{A}e_0^{\pm}(x,\lambda)+\lim_{N\to\infty}\left(\pm\int_{\mp N}^{x}\Phi^{\mp}(x,y)e^{\pm i\lambda y}dy\right)\right.$$

$$\left.-\sum_{k=1}^{n}\left(m_k^{\mp}\right)^2e^{\mp}(x,i\chi_k)\,\frac{e^{\pm(\chi_k+i\lambda)x}}{\chi_k+i\lambda}\right\}. \qquad (5.16)_{\pm}$$

Now it is sufficient to show that $h^{\pm}(x,\lambda) = e^{\pm}(x,\lambda)$.

If we use expressions $(5.15)_{\pm}$ and $(5.16)_{\pm}$ for the functions $h^{\pm}(x,\lambda)$ and conditions 2) of the theorem, the proof of this equality will completely coincide with the proof of analogous assertion at $\alpha = 1$ (see [3], p. 277–278). For this reason we do not derive it here.

R e m a r k. Condition 4) of Theorem 5.1 is necessary. The function $r^{+}(\lambda) = -\frac{B+\frac{\beta}{2i\lambda}}{A+\frac{\beta}{2i\lambda}} e^{-2i\lambda a}$ for $\alpha\beta < 0$ satisfies all conditions of the theorem, except condition 4), and is not a right reflection coefficient of the problem of the form (0.1)–(0.2). Indeed, in this case the main equations $(3.4)_{\pm}$ have the solutions

$$K^{\pm}(x,t) = \begin{cases} 0, & \text{if } \pm x > \pm a,\ \pm t > \pm x \text{ or } \pm x < \pm a,\ \pm t > \pm(2a - x), \\ -\frac{\beta}{2}, & \text{if } \pm x < \pm a,\ \pm x < \pm t < \pm(2a - x). \end{cases}$$

Therefore the Jost solutions satisfy equations (0.1) with $q(x) = 0$, and conditions (0.2) are not satisfied. But if $\beta = 0$, then condition 4) is also satisfied, and in this case the inverse problem has a solution: $r^{+}(\lambda) = r_0^{+}(\lambda)$ is the right reflection coefficient of problem (0.1)–(0.2) with the potential $q(x) \equiv 0$.

A. On one Problem for Hyperbolic Equation with Discontinuity Conditions

Introduce the regions $D_1 = \{(x,t) : x > a,\ t > x)\}$, $D_2 = \{(x,t) : x < a,\ t > 2a - x)\}$, $D_3 = \{(x,t) :\ x < a,\ x < t < 2a - x)\}$ and consider the following problem:
Find the functions $U(x,t)$ satisfying the equation

$$U_x'' - U_{tt}'' = f(x,t), \quad (x,t) \in D_1 \bigcup D_1 \bigcup D_3, \tag{A.1}$$

and the conditions

$$U(x,x) = \varphi_1(x), \quad a < x < \infty, \tag{A.2}$$

$$U(x,x) = \varphi_2(x), \quad -\infty < x < a, \tag{A.3}$$

$$U(x, 2a - x + 0) - U(x, 2a - x - 0) = \psi(x), \quad -\infty < x < a, \tag{A.4}$$

$$U(a - 0, t) = \alpha U(a + 0, t), \quad a < t < +\infty, \tag{A.5}$$

$$U_x'(a - 0, t) = \alpha^{-1} U_x'(a + 0, t), \quad a < t < +\infty, \tag{A.6}$$

$$\lim_{x+t\to+\infty} \Big(U_x'(x,t) - U_t'(x,t)\Big) = 0. \tag{A.7}$$

Theorem A. *Let the function $f(x,t)$ be differentiable, $f(x,t)=0$ for $t<x$ and for each fixed $x\in(-\infty,\,\infty)$*

$$\int_x^{\infty} d\tau \int_{\tau}^{+\infty} |f(\tau,\xi)|\, d\xi < +\infty,$$

and the functions $\varphi_1(x)$, $\varphi_2(x)$, $\psi(x)$ be twice differentiable, and

$$A\varphi_1(a)=\varphi_2(a), \quad \alpha\varphi_1(a)-\varphi_2(a)=\psi(a). \qquad (A_*)$$

Then the solution of problem $(A.1)$–$(A.7)$ can be represented as

$$U(x,t)=\varphi_1\left(\frac{x+t}{2}\right)+\frac{1}{2}\int_x^{+\infty} d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau,\xi)d\xi, \quad (x,t)\in D_1, \qquad (A.8)$$

$$U(x,t)=A\varphi_1\left(\frac{x+t}{2}\right)+B\varphi_1\left(\frac{t-x}{2}+a\right)+\frac{1}{2}\int_x^{a} d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau,\xi)d\xi$$

$$+\frac{A}{2}\int_a^{+\infty} d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau,\xi)d\xi+\frac{B}{2}\int_a^{+\infty} d\tau \int_{t-\tau-x+2a}^{t+\tau+x-2a} f(\tau,\xi)d\xi, \ (x,t)\in D_2, \qquad (A.9)$$

$$U(x,t)=\varphi_2\left(\frac{x+t}{2}\right)+B\varphi_1\left(a+\frac{t-x}{2}\right)-\psi\left(a-\frac{t-x}{2}\right)$$

$$+\frac{1}{2}\int_x^{a} d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau,\xi)d\xi+\frac{A}{2}\int_a^{+\infty} d\tau \int_{t-\tau+x}^{t+\tau-x} f(\tau,\xi)d\xi+\frac{B}{2}\int_a^{+\infty} d\tau \int_{t-\tau-x+2a}^{t+\tau+x-2a} f(\tau,\xi)d\xi,$$

$$(x,t)\in D_3, \qquad (A.10)$$

where $A=\frac{1}{2}\left(\alpha+\frac{1}{\alpha}\right)$, $B=\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right)$.

P r o o f. Let $(x,t)\in D_1$. In new variables $\xi=\frac{t+x}{2}$, $\eta=\frac{t-x}{2}$, equation $(A.1)$ takes the form $-\tilde{U}''_{\xi\eta}=\tilde{f}(\xi,\eta)$. By integrating this equation with respect to ξ and taking into consideration condition $(A.7)$ we get

$$\tilde{U}'_{\eta}=\int_{\xi}^{+\infty}\tilde{f}(\xi',\eta)d\xi'.$$

From which, integrating with respect to η by virtue of condition $(A.2)$, we get

$$\tilde{U}(\xi,\eta)=\varphi_1(\xi)+\int_{\xi}^{+\infty} d\xi' \int_0^{\eta}\tilde{f}(\xi',\eta')d\eta'.$$

Turning again to the variables x and y, we get $(A.8)$.

Now, let $(x,t) \in D_2$. The general solution of equation $(A.1)$ one can write as

$$U(x,t) = \theta_1\left(\frac{t+x}{2}\right) + \theta_2\left(\frac{t-x}{2}\right) + \frac{1}{2}\int\limits_{x}^{a} d\tau \int\limits_{t-\tau+x}^{t+\tau-x} f(\tau,\xi)d\xi. \qquad (A.11)$$

Then, according to conditions $(A.5)$ and $(A.6)$, for determining twice differentiable arbitrary functions θ_1 and θ_2, we get the relations

$$\theta_1\left(\frac{t+a}{2}\right) + \theta_2\left(\frac{t-a}{2}\right) = \alpha\left[\varphi_1\left(\frac{t+a}{2}\right) + \frac{1}{2}\int\limits_{a}^{+\infty} d\tau \int\limits_{t-\tau+a}^{t+\tau-a} f(\tau,\xi)d\xi\right],$$

$$\frac{1}{2}\theta_1'\left(\frac{t+a}{2}\right) - \frac{1}{2}\theta_2\left(\frac{t-a}{2}\right) = \alpha^{-1}\left[\frac{1}{2}\varphi_1'\left(\frac{t+a}{2}\right) - \right.$$

$$\left. - \frac{1}{2}\int\limits_{a}^{+\infty}[f(\tau,t+\tau-a) - f(\tau,t-\tau+a)]d\tau\right]. \qquad (A.12)$$

From which we find

$$\theta_1(s) = A\varphi_1(s) + \frac{B}{2}\int\limits_{a}^{+\infty} d\tau \int\limits_{c_1}^{2s-2a+\tau} f(\tau,\xi)d\xi$$

$$+\frac{A}{2}\int\limits_{a}^{+\infty} d\tau \int\limits_{2c-\tau}^{c_2} f(\tau,\xi)d\xi + c_3,$$

$$\theta_2(s) = B\varphi_1(s+a) + \frac{A}{2}\int\limits_{a}^{+\infty} d\tau \int\limits_{c_2}^{2s+\tau} f(\tau,\xi)d\xi$$

$$+\frac{B}{2}\int\limits_{a}^{+\infty} d\tau \int\limits_{2s-\tau+2a}^{c_1} f(\tau,\xi)d\xi + c_4, \qquad (A.13)$$

where $c_1,\ c_2,\ c_3,\ c_4$ are arbitrary constants. Moreover, in virtue of $(A.12)$,

$$c_3 + c_4 = 0. \qquad (A.14)$$

Substituting expressions $(A.13)$ for the functions θ_1 and θ_2 into formula $(A.11)$ and taking into consideration condition $(A.14)$, we get representation $(A.9)$.

Finally, assume that $(x,t) \in D_3$. We will look for the solution $U(x,t)$ in this region in the form

$$U(x,t) = \theta_3\left(\frac{t+x}{2}\right) + \theta_4\left(\frac{t-x}{2}\right) + \frac{1}{2}\int\limits_x^a d\tau \int\limits_{t-\tau+x}^{t-\tau-x} f(\tau,\xi)d\xi. \qquad (A.15)$$

For determining the arbitrary functions θ_3, θ_4, make use of conditions $(A.3)$, $(A.4)$ and equality $f(\tau,\xi) = 0$ for $\xi < \tau$. Hence,

$$\theta_3(x) + \theta_4(0) = \varphi_2(x),$$

$$A\varphi_1(a) + B\varphi_1(2a-x) + \frac{1}{2}\int\limits_x^a d\tau \int\limits_{2a-\tau}^{2a-2x+\tau} f(\tau,\xi)d\xi + \frac{A}{2}\int\limits_a^{+\infty} d\tau \int\limits_{2a-\tau}^{2a-2x+\tau} f(\tau,\xi)d\xi$$

$$-\left[\theta_3(a) + \theta_4(a-x) + \frac{1}{2}\int\limits_x^a d\tau \int\limits_{2a-\tau}^{2a-2x+\tau} f(\tau,\xi)d\xi\right] = \psi(x).$$

From which we define the functions $\theta_3(x), \theta_4(x)$ and substitute the obtained expressions into $(A.15)$. As a result, according to conditions (A_*), we come to representation $(A.10)$. ■

R e m a r k. In the case when $q(x)$ is differentiable, the kernel $K^+(x,t)$ is the solution of problem $(A.1)$–$(A.7)$, where $f(x,t) = q(x)K^+(x,t)$, $\varphi_1(x) = \frac{1}{2}\int\limits_x^\infty q(\xi)d\xi$, $\varphi_2(x) = \frac{A}{2}\int\limits_x^\infty q(\xi)d\xi$, $\psi(x) = \frac{B}{2}\left(\int\limits_a^\infty q(\xi)d\xi - \int\limits_x^a q(\xi)d\xi\right)$ (see Sec. 1). Thus, applying representations $(A.8)$–$(A.10)$, we get integral equations $(1.6)_+$, $(1.7)_+$.

In a similar way, one can also get integral equations $(1.6)_-$, $(1.7)_-$.

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