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Real Hypersurfaces in Complex Two-Plane Grassmannians with Generalized Tanaka–Webster Invariant Shape Operator

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In this paper, we introduce a new notion of the generalized Tanaka–Webster invariant for a hypersurface M in $G_2(\mathbb{C}^{m+2})$, and give a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with generalized Tanaka–Webster invariant shape operator.

Key words: real hypersurfaces, complex two-plane Grassmannians, Hopf hypersurface, generalized Tanaka–Webster connection, Reeb parallel shape operator, $\mathfrak{D}^{\perp}$-parallel shape operator, Lie invariant shape operator.

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Introduction

The *generalized Tanaka–Webster* (in short, the *g-Tanaka–Webster*) *connection* for contact metric manifolds was introduced by Tanno [16] as a generalization of the well-known connection defined by Tanaka in [15] and, independently, by Webster in [17]. This connection coincides with the Tanaka–Webster connection if the associated CR-structure is integrable. The Tanaka–Webster connection is defined as the canonical affine connection on a non-degenerate pseudo-Hermitian CR-manifold. For a real hypersurface in a Kähler manifold with almost contact metric structure (ϕ, ξ, η, g), the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ for a non-zero

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real number k was given in [5] and [9]. In particular, if a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka–Webster connection.

Using the g-Tanaka–Webster connection, many geometers have studied some characterizations of real hypersurfaces in the complex space form $\tilde{M}_n(c)$ with constant holomorphic sectional curvature c. For instance, when $c > 0$, that is, $\tilde{M}_n(c)$ is a complex projective space $\mathbb{C}P^n$, Kon [9] proved that if the Ricci tensor $\hat{S}$ of the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ vanishes identically, then a real hypersurface in $\mathbb{C}P^n$ is locally congruent to a geodesic hypersphere with $k^2 \geq 4n(n-1)$.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It is the unique compact irreducible Riemannian manifold equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold. Then, naturally we could consider two geometric conditions for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ that a 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and a 3-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M (see Berndt and Suh [3]).

Here the almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ ($\nu = 1, 2, 3$), where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

By using these two geometric conditions and the results obtained by Alekseevskii [1], Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*, we say that the Reeb vector field ξ on M is Killing. This means that the metric tensor g is invariant under the Reeb flow of ξ on M. They gave a characterization of real hypersurfaces of type (A) in Theorem A in terms of the Reeb flow on M as follows (see Berndt and Suh [4]):

Theorem B. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

On the other hand, using Riemannian connection, in [12] Suh gave a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator. Moreover, Suh proved a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{F}$-parallel shape operator, where $\mathfrak{F} = [\xi] \cup \mathfrak{D}^{\perp}$(see [13]).

In particular, Jeong, Lee and Suh considered the g-Tanaka–Webster parallelism of A for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. In other words, the shape operator A is called *g-Tanaka–Webster parallel* if it satisfies $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any tangent vector fields X and Y on M. Using this notion, the authors gave a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ as follows (see [5]) :

Theorem C. *There does not exist any Hopf hypersurface in the complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel shape operator in the generalized Tanaka–Webster connection if $\alpha \neq 2k$.*

Moreover, Jeong, Kimura, Lee and Suh considered a more generalized notion weaker than a parallel shape operator in the g-Tanaka–Webster connection of M in $G_2(\mathbb{C}^{m+2})$. When the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ satisfies $(\hat{\nabla}^{(k)}_\xi A)Y = 0$ for any tangent vector field Y on M, we say that the shape operator is *g-Tanaka–Webster Reeb parallel*. Using this notion, the authors gave a characterization of the real hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$ as follows (see [6]) :

Theorem D. *Let M be a connected orientable Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the shape operator A is generalized Tanaka–Webster Reeb parallel, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Jeong, Lee and Suh introduced the notion of the *g-Tanaka–Webster $\mathfrak{D}^{\perp}$-parallel* shape operator for M in $G_2(\mathbb{C}^{m+2})$. It means that the shape operator A of M satisfies $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any X in $\mathfrak{D}^{\perp}$ and Y on M. Naturally, we can see that the notion of g-Tanaka–Webster $\mathfrak{D}^{\perp}$-parallel is weaker than the g-Tanaka–Webster parallelism. By using the notion of $\mathfrak{D}^{\perp}$-parallel for the g-Tanaka–Webster connection, we gave a characterization of the real hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$ as follows (see [7]) :

Theorem E. *Let M be a connected orientable Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the shape operator A is g-Tanaka–Webster $\mathfrak{D}^{\perp}$-parallel, then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$ where $m = 2n$.*

Specially, Suh asserted a characterization of the real hypersurfaces of type (A) in Theorem A by another geometric Lie invariant, that is, the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ is *invariant* under the Reeb flow on M as follows (see [14]) :

Theorem F. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M satisfies $\mathfrak{L}_\xi A = 0$ if and only if M is an open part of a tube around some totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Motivated by Theorem F, let us consider another Lie invariant of the shape operator in $G_2(\mathbb{C}^{m+2})$. First of all, we consider a new notion of the generalized Lie invariant shape operator related to the g-Tanaka–Webster connection of M in $G_2(\mathbb{C}^{m+2})$, namely, *the generalized Tanaka–Webster invariant* (in short, the g-Tanaka–Webster invariant) shape operator, that is, $(\hat{\mathfrak{L}}^{(k)}_X A)Y = 0$ for any vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$. Here $\hat{\mathfrak{L}}^{(k)}$ denotes the g-Tanaka–Webster Lie derivative induced from the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$. In general, the notion of the g-Tanaka–Webster invariant differs from the g-Tanaka–Webster parallel and gives us fruitful information rather than usual covariant parallelisms in the g-Tanaka–Webster connection.

By using this notion of Lie invariant for the g-Tanaka–Webster connection, we give a non-existence theorem for the real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ as follows:

Main Theorem. *There does not exist any Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with invariant shape operator in the generalized Tanaka–Webster connection if $\alpha \neq 2k$.*

1. Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2], [3] and [4]. By $G_2(\mathbb{C}^{m+2})$, we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K. Moreover, we equip it with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_o G_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its restriction to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way, $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\tilde{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ there exist three local one-forms q_1, q_2, q_3 such that

$$\tilde{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\Big\} \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\Big\},\end{aligned} \tag{1.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

Now we derive some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [3, 4, 10–13]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ will denote the Riemannian connection of (M, g). Let N be a local unit normal vector field of M, and A the shape operator of M with respect to N.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{1.3}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ in Sec. 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned}
&\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0,\\
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},\\
&\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned} \tag{1.4}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$ in Sec. 1 and (1.3), the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{aligned}
&\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu,\\
&\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi.
\end{aligned} \tag{1.5}$$

On the other hand, from the parallelism of the Kähler structure J, that is, $\tilde{\nabla}J = 0$ and the quaternionic Kähler structure J_ν, together with Gauss and Weingarten equations, it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{1.6}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{1.7}$$

$$\begin{aligned}
(\nabla_X\phi_\nu)Y = &- q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y\\
&+ \eta_\nu(Y)AX - g(AX, Y)\xi_\nu.
\end{aligned} \tag{1.8}$$

Using the above expression (1.2) for the curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi becomes

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\Big\} \\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\Big\} \\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\Big\}\xi_\nu\,.\end{aligned} \tag{1.9}$$

Now we introduce the notion of the g-Tanaka–Webster connection (see [9]).

As stated above, the Tanaka–Webster connection is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold (see [15], [17]). In [16], Tanno defined the g-Tanaka–Webster connection for the contact metric manifolds by the canonical connection. It coincides with the Tanaka–Webster connection if the associated CR-structure is integrable.

From now on, we introduce the g-Tanaka–Webster connection due to Tanno [16] for real hypersurfaces in Kähler manifolds by natural extending the canonical affine connection to a non-degenerate pseudo-Hermitian CR manifold.

Now let us recall the g-Tanaka–Webster connection $\hat{\nabla}$ defined by Tanno [16] for the contact metric manifolds as follows:

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y$$

for all vector fields X and Y (see [16]).

By taking (1.6) into account, the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ for the real hypersurfaces of Kähler manifolds is defined by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.10}$$

for a non-zero real number k (see [5] and [9]). (Note that $\hat{\nabla}^{(k)}$ is invariant under the choice of the orientation. Namely, we may take $-k$ instead of k in (1.10) for the opposite orientation $-N$.)

2. Key Lemmas

First, let us assume that the shape operator A is *invariant*, that is, $\mathfrak{L}_X A = 0$ for any tangent vector field X on M in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$.

From the definition of Lie derivative we have

$$\begin{aligned}(\mathfrak{L}_X A)Y &= \mathfrak{L}_X(AY) - A\mathfrak{L}_X Y \\ &= (\nabla_X A)Y - \nabla_{AY}X + A\nabla_Y X\end{aligned} \tag{2.1}$$

for any tangent vector fields X and Y on M.

By putting $X = \xi$ in (2.1), we obtain

$$(\mathfrak{L}_\xi A)Y = (\nabla_\xi A)Y - \nabla_{AY}\xi + A\nabla_Y \xi.$$

From Theorem F [14], if M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb invariant shape operator, that is, $\mathfrak{L}_\xi A = 0$, then M is locally congruent to a real hypersurface of type (A).

Now let us denote by M a real hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$. Then let us check whether the shape operator of type (A) is invariant in usual Levi–Civita connection. In order to solve this problem, we introduce a proposition due to Berndt and Suh [3] as follows:

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\},\\
T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \mathrm{Span}\{\xi_2,\ \xi_3\},\\
T_\lambda &= \{X \mid X \perp \mathbb{H}\xi,\ JX = J_1X\},\\
T_\mu &= \{X \mid X \perp \mathbb{H}\xi,\ JX = -J_1X\},
\end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$, respectively, denote real, complex and quaternionic spans of the structure vector field ξ, and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Applying $X = \xi_2$, $Y \in T_\lambda$ and $\xi = \xi_1 \in \mathfrak{D}^\perp$ in (2.1), we get

$$\begin{aligned}
0 &= (\nabla_{\xi_2} A)Y - \nabla_{AY}\xi_2 + A\nabla_Y\xi_2\\
&= (\nabla_{\xi_2} A)Y - \lambda\nabla_Y\xi_2 + A\nabla_Y\xi_2.
\end{aligned}$$

On the other hand, using (1.9) and $A\xi_2 = \beta\xi_2$, we have

$$\begin{aligned}(\nabla_{\xi_2} A)Y &= (\nabla_Y A)\xi_2 + \eta(\xi_2)\phi Y - \eta(Y)\phi\xi_2 - 2g(\phi\xi_2, Y)\xi \\ &\quad + \sum_{\nu=1}^{3} \Big\{\eta_\nu(\xi_2)\phi_\nu Y - \eta_\nu(Y)\phi_\nu\xi_2 - 2g(\phi_\nu\xi_2, Y)\xi_\nu\Big\} \\ &\quad + \sum_{\nu=1}^{3} \Big\{\eta_\nu(\phi\xi_2)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi\xi_2\Big\} \\ &\quad + \sum_{\nu=1}^{3} \Big\{\eta(\xi_2)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi\xi_2)\Big\}\xi_\nu \\ &= (\nabla_Y A)\xi_2 + \phi_2 Y - \phi_3\phi Y \\ &= -A\nabla_Y\xi_2 + \beta\nabla_Y\xi_2 + \phi_2 Y - \phi_3\phi Y. \end{aligned} \tag{2.2}$$

Thus we obtain

$$\begin{aligned}0 &= -A\nabla_Y\xi_2 + \beta\nabla_Y\xi_2 + \phi_2 Y - \phi_3\phi Y - \lambda\nabla_Y\xi_2 + A\nabla_Y\xi_2 \\ &= (\beta - \lambda)\nabla_Y\xi_2 \\ &= (\beta - \lambda)(q_1(Y)\xi_3 - q_3(Y)\xi_1 + \phi_2 AY).\end{aligned}$$

On the other hand, we know

$$\begin{aligned}\phi AY &= \nabla_Y\xi \\ &= \nabla_Y\xi_1 \\ &= q_3(Y)\xi_2 - q_2(Y)\xi_3 + \phi_1 AY.\end{aligned}$$

Taking the inner product with ξ_2, we have

$$g(\phi AY, \xi_2) = q_3(Y) + g(\phi_1 AY, \xi_2),$$

that is,

$$\begin{aligned}q_3(Y) &= g(\phi AY, \xi_2) - g(\phi_1 AY, \xi_2) \\ &= -g(AY, \phi\xi_2) + g(AY, \phi_1\xi_2) \\ &= 2g(AY, \xi_3) \\ &= 2\lambda g(Y, \xi_3) \\ &= 0.\end{aligned}$$

It yields

$$0 = (\beta - \lambda)q_1(Y)\xi_3 + \lambda(\beta - \lambda)\phi_2 Y.$$

Taking the inner product with $\phi_2 Y$ in the equation above, we have

$$\begin{aligned} 0 &= \lambda(\beta - \lambda) g(\phi_2 Y, \phi_2 Y) \\ &= \lambda(\beta - \lambda). \end{aligned}$$

Consequently, we get $\lambda = 0$ or $\beta - \lambda = 0$, which contradicts the values of β and λ in Proposition A. From this, we conclude the following:

Proposition 2.1. *There does not exist a hypersurface in $G_2(\mathbb{C}^{m+2})$ with invariant shape operator.*

From this motivation, we consider a new notion of the g-Tanaka–Webster invariant shape operator. By using Lie invariant for the g-Tanaka–Webster connection, in Sec. 3 we will give a non-existence theorem for the real hypersurface in $G_2(\mathbb{C}^{m+2})$.

On the other hand, in [5] Jeong, Lee and Suh considered the notion of the g-Tanaka–Webster parallelism of the shape operator of a real hypersurface in the complex two-plane Grassmannians. Now in this section, let us give a new notion of the generalized Lie invariant of the shape operator for M in $G_2(\mathbb{C}^{m+2})$. As it is well known, the Lie derivative of Y with respect to X is defined by

$$\mathfrak{L}_X Y = \lim_{t \to 0} \frac{Y - (\varphi_t)_* Y}{t} = \nabla_X Y - \nabla_Y X,$$

where ∇ denotes the Levi–Civita connection of M in $G_2(\mathbb{C}^{m+2})$, and φ_t is a local 1-parameter group of the transformations generated by X. Similarly, we define the generalized Tanaka–Webster Lie derivative $\hat{\mathfrak{L}}^{(k)}_X$ for any direction X on M as follows:

$$\hat{\mathfrak{L}}^{(k)}_X Y = \hat{\nabla}^{(k)}_X Y - \hat{\nabla}^{(k)}_Y X,$$

where $\hat{\nabla}^{(k)}$ denotes the g-Tanaka–Webster connection of M in $G_2(\mathbb{C}^{m+2})$. Since $G_2(\mathbb{C}^{m+2})$ can be regarded as a Kähler manifold, the connection $\hat{\nabla}^{(k)}$ can be defined as in (1.10).

The shape operator A is said to be *generalized Tanaka–Webster invariant* if $(\hat{\mathfrak{L}}^{(k)}_X A)Y = 0$ for any tangent vector fields X and Y on M.

In this section, we will prove that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$ of M with g-Tanaka–Webster invariant shape operator.

From the definition of the g-Tanaka–Webster connection (1.10), we have

$$\begin{aligned} (\hat{\mathfrak{L}}^{(k)}_X A)Y = {} & (\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY \\ & - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y \\ & - \nabla_{AY} X - g(\phi A^2 Y, X)\xi + \eta(X)\phi A^2 Y + k\eta(AY)\phi X \\ & + A\nabla_Y X + g(\phi AY, X)A\xi - \eta(X)A\phi AY - k\eta(Y)A\phi X \end{aligned}$$

for any tangent vector fields X and Y on M.

Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with g-Tanaka–Webster invariant shape operator, that is, $(\hat{\mathfrak{L}}_X^{(k)}A)Y = 0$ and $A\xi = \alpha\xi$. This becomes

$$\begin{aligned} 0 &= (\hat{\mathfrak{L}}_X^{(k)}A)Y \\ &= (\nabla_X A)Y + g(\phi AX, AY)\xi - \alpha\eta(Y)\phi AX - k\eta(X)\phi AY \\ &\quad - \alpha g(\phi AX, Y)\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y \\ &\quad - \nabla_{AY}X - g(\phi A^2Y, X)\xi + \eta(X)\phi A^2Y + \alpha k\eta(Y)\phi X \\ &\quad + A\nabla_Y X + \alpha g(\phi AY, X)\xi - \eta(X)A\phi AY - k\eta(Y)A\phi X \end{aligned} \tag{2.3}$$

for any tangent vector fields X and Y on M.

Using (2.3), we can assert the following:

Lemma 2.2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If M has the g-Tanaka–Webster invariant shape operator, then the principal curvature $\alpha = g(A\xi, \xi)$ is constant.*

P r o o f. Replacing Y by ξ in (2.3) and using $A\xi = \alpha\xi$, we have

$$\begin{aligned} 0 &= (\hat{\mathfrak{L}}_X^{(k)}A)\xi \\ &= (\nabla_X A)\xi - \alpha\phi AX + A\phi AX - \alpha\nabla_\xi X + \alpha k\phi X + A\nabla_\xi X - kA\phi X \\ &= -A\phi AX + (X\alpha)\xi + \alpha\phi AX - \alpha\phi AX + A\phi AX \\ &\quad - \alpha\nabla_\xi X + \alpha k\phi X + A\nabla_\xi X - kA\phi X. \end{aligned}$$

Then we have

$$0 = (X\alpha)\xi - \alpha\nabla_\xi X + \alpha k\phi X + A\nabla_\xi X - kA\phi X \tag{2.4}$$

for any tangent vector field X on M.

Taking the inner product of (2.4) with ξ, we get

$$\begin{aligned} 0 &= (X\alpha)g(\xi,\xi) - \alpha g(\nabla_\xi X, \xi) + \alpha k g(\phi X, \xi) + g(A\nabla_\xi X, \xi) - kg(A\phi X, \xi) \\ &= (X\alpha) - \alpha g(\nabla_\xi X, \xi) + \alpha g(\nabla_\xi X, \xi). \end{aligned}$$

Thus we have our assertion. ■

Now we introduce the lemma as follows:

Lemma 2.3. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If M has the g-Tanaka–Webster invariant shape operator, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

P r o o f. We assume that

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{*}$$

for some unit vector field $X_0 \in \mathfrak{D}$ and $\eta(\xi_1)\eta(X_0) \neq 0$.

Under the assumption that M is Hopf, Berdnt and Suh [3] gave

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$$

for any tangent vector field Y on M.

Using Lemma 2.2, we get

$$0 = \sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y).$$

From this, together with (*), we obtain

$$\begin{aligned} 0 &= \eta_1(\xi)\eta_1(\phi Y) \\ &= -\eta(\xi_1)g(\phi\xi_1, Y) \end{aligned}$$

for any tangent vector field Y on M. Because of $\eta(\xi_1) \neq 0$, we have

$$\begin{aligned} 0 &= \phi\xi_1 \\ &= \phi_1(\eta(X_0)X_0 + \eta(\xi_1)\xi_1) \\ &= \eta(X_0)\phi_1 X_0. \end{aligned}$$

Since $\eta(X_0) \neq 0$, we get $\phi_1 X_0 = 0$. This gives a contradiction.

Hence we complete the proof of this lemma. ■

3. The Proof of the Main Theorem

From now on, let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with g-Tanaka–Webster invariant shape operator. Then by Lemma 2.3, we consider the following two cases, that is, $\xi \in \mathfrak{D}^\perp$ and $\xi \in \mathfrak{D}$, respectively.

First, we consider the case $\xi \in \mathfrak{D}^\perp$. From this, without loss of generality, we may put $\xi = \xi_1$.

Lemma 3.1. *Let M be a Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with g-Tanaka–Webster invariant shape operator. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}^\perp$, then the shape operator A commutes with the structure tensor ϕ.*

P r o o f. Previously we obtained this equation

$$\begin{aligned} 0 &= (\hat{\mathfrak{L}}_X^{(k)} A)Y \\ &= (\hat{\nabla}_X^{(k)} A)Y - \hat{\nabla}_{AY}^{(k)}(X) + A\hat{\nabla}_Y^{(k)} X. \end{aligned}$$

By putting $X = \xi,\ Y = X$ and using (1.10) in the equation above, we have

$$\begin{aligned} 0 &= (\hat{\mathfrak{L}}_\xi^{(k)} A)X \\ &= (\hat{\nabla}_\xi^{(k)} A)X - \hat{\nabla}_{AX}^{(k)}(\xi) + A\hat{\nabla}_X^{(k)}\xi \\ &= (\hat{\nabla}_\xi^{(k)} A)X - \{\nabla_{AX}\xi + g(\phi A^2X, \xi)\xi - \eta(\xi)\phi A^2 X - k\eta(AX)\phi\xi\} \\ &\quad + A\{\nabla_X\xi + g(\phi AX, \xi)\xi - \eta(\xi)\phi AX - k\eta(X)\phi\xi\} \\ &= (\hat{\nabla}_\xi^{(k)} A)X - \phi A^2X + \phi A^2X + A\phi AX - A\phi AX \\ &= (\hat{\nabla}_\xi^{(k)} A)X \end{aligned} \tag{3.1}$$

for any tangent vector field X on M. So we can use the proof of the lemma ([6], Lemma 3.1). Since $\alpha \neq 2k$, we know that the shape operator A commutes with the structure tensor ϕ. ■

Due to Berdnt and Suh [4], the Reeb flow on M is *isometric* if and only if the structure tensor field ϕ commutes with the shape operator A of M. Thus, from Lemma 3.1 and Theorem B we have the following:

Lemma 3.2. Let M be a Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with g-Tanaka–Webster invariant shape operator. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}^\perp$, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now let us denote by M a real hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$. Then, using Lemma 3.2 and Proposition A due to Berndt and Suh [3], let us check whether the shape operator A of M is invariant for the g-Tanaka–Webster connection as follows:

Case A: $\xi \in \mathfrak{D}^\perp$.

Applying $X = \xi_2$, $Y \in T_\lambda$ and $\xi = \xi_1 \in \mathfrak{D}^\perp$ in (2.3), we get

$$\begin{aligned} 0 = &(\nabla_{\xi_2} A)Y + g(\phi A\xi_2, AY)\xi - \alpha\eta(Y)\phi A\xi_2 - k\eta(\xi_2)\phi AY \\ &- \alpha g(\phi A\xi_2, Y)\xi + \eta(Y)A\phi A\xi_2 + k\eta(\xi_2)A\phi Y \\ &- \nabla_{AY}\xi_2 - g(\phi A^2Y, \xi_2)\xi + \eta(\xi_2)\phi A^2 Y + \alpha k\eta(Y)\phi\xi_2 \\ &+ A\nabla_Y\xi_2 + \alpha g(\phi AY, \xi_2)\xi - \eta(\xi_2)A\phi AY - k\eta(Y)A\phi\xi_2. \end{aligned}$$

Since $Y \in T_\lambda$, using $\phi T_\lambda \subset T_\lambda$, we have

$$\begin{aligned} g(\phi A\xi_2, AY) &= \lambda g(\phi A\xi_2, Y) \\ &= -\lambda^2 g(\phi Y, \xi_2) \\ &= \lambda^2 g(Y, \phi\xi_2) \\ &= 0. \end{aligned}$$

Similarly, we obtain $g(\phi A\xi_2, Y) = g(\phi A^2 Y, \xi_2) = g(\phi AY, \xi_2) = 0$. Then we have

$$\begin{aligned} 0 &= (\nabla_{\xi_2} A)Y - \nabla_{AY}\xi_2 + A\nabla_Y\xi_2 \\ &= (\nabla_{\xi_2} A)Y - \lambda\nabla_Y\xi_2 + A\nabla_Y\xi_2. \end{aligned}$$

Thus, using (2.2), we obtain

$$\begin{aligned} 0 &= -A\nabla_Y\xi_2 + \beta\nabla_Y\xi_2 + \phi_2 Y - \phi_3\phi Y - \lambda\nabla_Y\xi_2 + A\nabla_Y\xi_2 \\ &= (\beta - \lambda)\nabla_Y\xi_2 \\ &= (\beta - \lambda)(q_1(Y)\xi_3 - q_3(Y)\xi_1 + \phi_2 AY). \end{aligned}$$

Because of $q_3(Y) = 0$, taking the inner product with $\phi_2 Y$, we get

$$0 = \lambda(\beta - \lambda).$$

Consequently, we have $\lambda = 0$ or $\beta - \lambda = 0$. This gives a contradiction. So we give a proof of the Main Theorem for $\xi \in \mathfrak{D}^\perp$.

Now let us consider the following:

Case B: $\xi \in \mathfrak{D}$.

First of all, we introduce the proposition given by Berndt and Suh in [3] as follows:

Proposition B. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \mathrm{Span}\{\xi\}, \\ &T_\beta = \mathfrak{J}J\xi = \mathrm{Span}\{\xi_\nu \mid \nu = 1, 2, 3\}, \\ &T_\gamma = \mathfrak{J}\xi = \mathrm{Span}\{\phi_\nu\xi \mid \nu = 1, 2, 3\}, \\ &T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^\perp$ *is the orthogonal complement of* $\mathbb{H}\mathbb{C}\xi$*, where* $\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi$*.*

Applying $X = \xi$ in (2.3), we get

$$\begin{aligned} 0 &= (\hat{\mathfrak{L}}^{(k)}_\xi A)Y \\ &= (\hat{\nabla}^{(k)}_\xi A)Y \\ &= (\nabla_\xi A)Y - k\phi AY + kA\phi Y. \end{aligned}$$

Then we have

$$0 = \nabla_\xi(AY) - A\nabla_\xi Y - k\phi AY + kA\phi Y \tag{3.2}$$

for any tangent vector field Y on M.

From this, by putting $Y = \xi_2$, we obtain

$$\begin{aligned} 0 &= \nabla_\xi(A\xi_2) - A\nabla_\xi\xi_2 - k\phi A\xi_2 + kA\phi\xi_2 \\ &= \beta\nabla_\xi\xi_2 - A\nabla_\xi\xi_2 - k\beta\phi\xi_2 \\ &= \beta(q_1(\xi)\xi_3 - q_3(\xi)\xi_1 + \phi_2 A\xi) \\ &\quad - A(q_1(\xi)\xi_3 - q_3(\xi)\xi_1 + \phi_2 A\xi) - k\beta\phi\xi_2 \\ &= \alpha\beta\phi_2\xi - \alpha A\phi_2\xi - k\beta\phi_2\xi. \end{aligned}$$

Then we get

$$0 = \beta(\alpha - k)\phi_2\xi,$$

that is, $\beta = 0$ or $\alpha = k$.

Subcase 1: $\beta = 0$.

Since $\beta = \sqrt{2}\cot(\sqrt{2}r) > 0$ for $r \in (0, \pi/4)$, it gives us a contradiction.

Subcase 2: $\alpha = k$.

Using (2.3) and (1.9) , we have

$$\begin{aligned} 0 &= (\hat{\mathfrak{L}}^{(k)}_\xi A)Y \\ &= (\nabla_\xi A)Y - k\phi AY + kA\phi Y \\ &= -A\phi AY + (Y\alpha)\xi + (\alpha - k)\phi AY + kA\phi Y + \phi Y \\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)\phi_\nu Y - \eta_\nu(Y)\phi_\nu\xi + 3\eta_\nu(\phi Y)\xi_\nu\Big\} \end{aligned}$$

for any tangent vector field Y on M.

Applying $\xi \in \mathfrak{D}$ and $\alpha = k$ in this equation, we get

$$0 = -A\phi AY + \alpha A\phi Y + \phi Y + \sum_{\nu=1}^{3}\Big\{ -\eta_\nu(Y)\phi_\nu\xi + 3\eta_\nu(\phi Y)\xi_\nu\Big\}.$$

Combining $Y \in T_\lambda$ and $JT_\lambda = T_\mu$, we obtain

$$\begin{aligned}0 &= -\lambda A\phi Y + \alpha\mu\phi Y + \phi Y\\ &= -\lambda\mu\phi Y + \alpha\mu\phi Y + \phi Y\\ &= (-\lambda\mu + \alpha\mu + 1)\phi Y,\end{aligned}$$

that is,

$$\begin{aligned}0 &= -\lambda\mu + \alpha\mu + 1\\ &= -(\cot r)(-\tan r) + (-2\tan 2r)(-\tan r) + 1\\ &= 1 + 2\tan 2r\tan r + 1\\ &= 2(1 + \tan 2r\tan r).\end{aligned}$$

Thus we know

$$\begin{aligned}0 &= 1 + \tan 2r\tan r\\ &= 1 + \frac{2\tan r}{1-\tan^2 r}\tan r\\ &= \frac{1+\tan^2 r}{1-\tan^2 r} \quad \text{for} \quad r \in (0, \pi/4).\end{aligned}$$

Consequently, we have

$$1 + \tan^2 r = 0,$$

which contradicts $0 < \tan r < 1$.

Hence summing up all the cases, we have our Main Theorem from Introduction.

4. Generalized Tanaka–Webster Reeb Invariant for $\alpha = 2k$

In the proof of our Main Theorem, in Sec. 3 we assumed $\alpha \neq 2k$. But for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\alpha = 2k$ and $\xi \in \mathfrak{D}^\perp$, naturally the shape operator becomes Reeb parallel for the g-Tanaka–Webster connection. From this point of view, in this section we will show that the assumption of Reeb parallel for the g-Tanaka–Webster connection has no meaning for $\alpha = 2k$ and $\xi \in \mathfrak{D}^\perp$.

Summing up the above situations, we assert the following:

Proposition 4.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, such that $\alpha = 2k$ and $\xi \in \mathfrak{D}^\perp$. Then the shape operator A is g-Tanaka–Webster Reeb parallel.*

P r o o f. From the definition of the g-Tanaka–Webster connection (1.10), we get

$$\begin{aligned}(\hat{\nabla}^{(k)}_X A)Y &= \hat{\nabla}^{(k)}_X (AY) - A\hat{\nabla}^{(k)}_X Y \\ &= (\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY \\ &\quad - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y\end{aligned}$$

for any tangent vector fields X and Y on M.

Putting $X = \xi, Y = X$ in this equation, we have

$$\begin{aligned}(\hat{\nabla}^{(k)}_\xi A)X = (\nabla_\xi A)X + g(\phi A\xi, AX)\xi - \eta(AX)\phi A\xi - k\eta(\xi)\phi AX \\ - g(\phi A\xi, X)A\xi + \eta(X)A\phi A\xi + k\eta(\xi)A\phi X.\end{aligned}$$

Since M is a Hopf hypersurface of $G_2(\mathbb{C}^{m+2})$, we obtain

$$(\hat{\nabla}^{(k)}_\xi A)X = (\nabla_\xi A)X - k\phi AX + kA\phi X$$

for any tangent vector field X on M.

Using (1.9), we have

$$\begin{aligned}(\hat{\nabla}^{(k)}_\xi A)X = (\nabla_X A)\xi + \phi X + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)\phi_\nu X - \eta_\nu(X)\phi_\nu\xi - 3g(\phi_\nu\xi, X)\xi_\nu\Big\} \\ - k\phi AX + kA\phi X. \qquad (4.1)\end{aligned}$$

Applying $\alpha = 2k$ and $\xi = \xi_1 \in \mathfrak{D}^\perp$ in (4.1), we get

$$\begin{aligned}(\hat{\nabla}^{(k)}_\xi A)X &= (\nabla_X A)\xi + \phi X + \phi_1 X - \eta_2(X)\phi_2\xi - \eta_3(X)\phi_3\xi \\ &\quad - 3g(\phi_2\xi, X)\xi_2 - 3g(\phi_3\xi, X)\xi_3 - \frac{\alpha}{2}\phi AX + \frac{\alpha}{2}A\phi X \\ &= -A\phi AX + \alpha\phi AX + \phi X + \phi_1 X + \eta_2(X)\xi_3 - \eta_3(X)\xi_2 \\ &\quad + 3\eta_3(X)\xi_2 - 3\eta_2(X)\xi_3 - \frac{\alpha}{2}\phi AX + \frac{\alpha}{2}A\phi X.\end{aligned}$$

Thus we have

$$\begin{aligned}(\hat{\nabla}^{(k)}_\xi A)X &= -A\phi AX + \frac{\alpha}{2}\phi AX + \phi X + \phi_1 X \\ &\quad - 2\eta_2(X)\xi_3 + 2\eta_3(X)\xi_2 + \frac{\alpha}{2}A\phi X. \qquad (4.2)\end{aligned}$$

On the other hand, we know from Berdnt and Suh [4],

$$\begin{aligned}2A\phi AX &= \alpha A\phi X + \alpha\phi AX + 2\phi X + 2\phi_1 X \\ &\quad - 4\eta_2(X)\xi_3 + 4\eta_3(X)\xi_2 \qquad (4.3)\end{aligned}$$

for any tangent vector field X on M. Then (4.2) can be rearranged as follows:

$$2(\hat{\nabla}^{(k)}_{\xi}A)X = -2A\phi AX + \alpha\phi AX + 2\phi X + 2\phi_1 X \\ - 4\eta_2(X)\xi_3 + 4\eta_3(X)\xi_2 + \alpha A\phi X.$$

Therefore, from (4.3), we obtain

$$(\hat{\nabla}^{(k)}_{\xi}A)X = 0$$

for any tangent vector field X on M. ∎

R e m a r k 4.2. In the paper [6] due to Jeong, Kimura, Lee and Suh, Proposition 4.1 is also remarked.

R e m a r k 4.3. From Proposition 4.1 together with (3.1), for the case $\alpha = 2k$, it can be easily verified that

$$(\hat{\mathfrak{L}}^{(k)}_{\xi}A)Y = 0$$

for any tangent vector field Y on M. Thus the assumption of Reeb invariant for $\alpha = 2k$ has no meaning.

Accordingly, if we consider that $(\hat{\mathfrak{L}}^{(k)}_{\xi}A)Y = 0$, that is, the g-Tanaka–Webster Reeb invariant shape operator, it should be natural to consider the condition that $\alpha \neq 2k$.

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