Some Applications of Meijer G-Functions as Solutions of Differential Equations in Physical Models

A. Pishkoo and M. Darus

School of Mathematical Sciences Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 Bangi, Selangor D. Ehsan, Malaysia

E-mail: apishkoo@yahoo.in
maslina@ukm.my

In this paper, we aim to show that the Meijer G-functions can serve to find explicit solutions of partial differential equations (PDEs) related to some mathematical models of physical phenomena, as for example, the Laplace equation, the diffusion equation and the Schrödinger equation. Usually, the first step in solving such equations is to use the separation of variables method to reduce them to ordinary differential equations (ODEs). Very often this equation happens to be a case of the linear ordinary differential equation satisfied by the G-function, and so, by proper selection of its orders $m; n; p; q$ and the parameters, we can find the solution of the ODE explicitly. We illustrate this approach by proposing solutions as: the potential function Φ, the temperature function T and the wave function Ψ, all of which are symmetric product forms of the Meijer G-functions. We show that one of the three basic univalent Meijer G-functions, namely $G_{0,2}^{1,0}$, appears in all the mentioned solutions.

Key words: Meijer G-functions; partial differential equations; Laplace equation; diffusion equation; Schrödinger equation; separation of variables.

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1.   Introduction

In the recent decades, the Meijer G-function has found various applications in different areas which are close to applied mathematics, such as mathematical physics (hydrodynamics, theory of elasticity, potential theory, etc.), theoretical physics, mathematical statistics, queuing theory, optimization theory, sinusoidal

signals, generalized birth and death processes and many others. Because of interesting and general properties of G-function, it is possible to represent the solutions of many problems in these fields in their terms. Stated in this way, the problems gain a much more general character due to the great freedom of choice of the orders $m; n; p; q$ and the parameters of G-functions in comparison to other special functions. Simultaneously, the calculations become simpler and more unified. Evidence showing the importance of the G-function is given by the fact that the basic elementary functions and most of the special functions of mathematical physics, including the generalized hypergeometric functions, follow as its particular cases. Therefore, each result concerning the G-function has become a key leading to numerous particular results for the Bessel functions, confluent hypergeometric functions, classical orthogonal polynomials and others (see [1]).

The Meijer G-function has been useful in mathematical physics because of its analytical properties, in particular, it can be expressed as a final sum of the generalized hypergeometric functions with the well-known series expansions. Some of the differential properties of Meijer G-functions were derived by E.E. Fitchard and V. Franco (see [2–5], [1, -Appendix]). The G-function is also relatively easy to compute numerically. Recently, K. Roach has discussed an algorithm for computing the formula representations of instances of the Meijer G-function [6]. The astrophysical thermonuclear functions $I_1(z, \nu)$ and $I_2(z, d, \nu)$ are expressed in terms of these functions [7]. The Meijer G-function is also used as the weight function to obtain the Gazeau–Klauder (photon-added) coherent states [8].

In the previous paper we have classified the univalent Meijer G-functions into three types. Three basic univalent Meijer G-functions are introduced, namely, $G_{0,2}^{1,0}; G_{1,2}^{1,1}; G_{1,1}^{1,1}$, and by the successive applications of fractional differintegral transformations, a number of univalent Meijer G-functions could be obtained and the Erdélyi–Kober operators ($m = 1, 2$) as the transformations preserving the univalence of the Meijer G-functions [9]. These classification and transformations are based on Kiryakova's studies in representing the generalized hypergeometric functions as fractional differintegral operators of three basic elementary functions [10, 11]. It is shown that the function $G_{0,2}^{1,0}$, as one of the three basic univalent Meijer G-functions, enters in all solutions of the three $PDEs$ considered here.

The starting point in this work is the question "Is it possible to represent some solutions of physical models explicitly in terms of the Meijer G-functions?" The question contains two important points that motivate us to answer it:

(i) Finding explicit solutions to mathematical models of various physical, statistical and even social events through Meijer G-functions was almost unknown as an idea till the 80s of the last century. But recently there appeared many books, surveys and papers emphasizing the role of the G-functions not only as kernels of some integrals.

(ii) Studying the ODEs satisfied by the Meijer G-functions hence treats and solves explicitly each particular ODE arising from a PDE by separation of variables.

It induces us to find a new method for doing this work. In fact, we will concentrate on G-function's ordinary linear differential equation (OLDE), and by proper selection of its orders $m; n; p; q$ and the parameters, we will equate G-function's OLDE and ODEs. By doing this, we can show that G-functions are the explicit solutions of the PDEs.

Helping idea in obtaining our results is that the Meijer G-functions include all elementary and special functions, and so the ODE for the G-function can include many cases of ordinary differential equations whose solutions are exactly these elementary and special functions. However, we have PDEs but not ODEs to solve, so the separation of variables is needed.

The laws of physics are almost exclusively written in the form of differential equations (PDEs). Depending on the geometry of the problem, these differential equations are separated into ODEs, each involving a single coordinate of a suitable coordinate system. With the Cartesian, cylindrical and spherical coordinates, the boundary conditions are important in determining the nature of the solutions of ODEs obtained from PDEs. These ODEs are usually of the Sturm–Liouville (S–L) type [12].

The contents of this paper is divided into three sections. In the first section, we recall the definition of the *Meijer G-function* and the G-function's ordinary linear differential equation. The second section introduces three well-known PDEs in physics and the method of separation of variables. The third section discusses the new representation of the solutions in terms of symmetric product forms of the univalent *Meijer G-functions*.

Definition 1.1. *A definition of the Meijer G-function is given by the following path integral in the complex plane, called of the Mellin–Barnes type* [1, 13–16]:

$$G_{p,q}^{m,n}\left(\begin{smallmatrix} a_1,...,a_p \\ b_1,...,b_q \end{smallmatrix} \Big| z\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \; \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \; \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s ds. \quad (1.1)$$

Here, an empty product means unity and the integers $m; n; p; q$ are called the orders of the G-function, or the components of the order $(m; n; p; q)$, while a_p and b_q are called the "parameters" and, in general, are complex numbers. The definition holds under the following assumptions: $0 \le m \le q$ and $0 \le n \le p$, where $m, n, p,$ and q are the integer numbers. $a_j - b_k \ne 1, 2, 3, \dots$ for $k = 1, \dots, n$ and $j = 1, 2, \dots, m$ imply that no pole of any $\Gamma(b_j - s), j = 1, \dots, m$ coincides with a pole of any $\Gamma(1 - a_k + s), k = 1, \dots, n$.

The Meijer G-function $y(z) = G_{p,q}^{m,n}(z|_{b_k}^{a_j})$ satisfies the linear ordinary differential equation of the generalized hypergeometric type

$$[(-1)^{p-m-n}z\prod_{j=1}^{p}(z\frac{d}{dz} - a_j + 1) - \prod_{k=1}^{q}(z\frac{d}{dz} - b_k)]y(z) = 0 \qquad (1.2)$$

whose order is equal to $\max(p, q)$, (see [3–5], [1, -Appendix]).

2. Three PDEs in Cartesian Coordinates System

A problem most suitable for the Cartesian coordinates has the boundaries with rectangular symmetry such as boxes or planes. Separation of variables leads to the ODEs in which certain constants (eigenvalues) appear. Different choices of signs for these constants can lead to different functional forms of general solution. The general form of the solution is indeterminate. However, once the boundary conditions are imposed, the unique solutions will emerge regardless of the initial functional form of the solution.

In electrostatics, where time-independent scalar fields such as potentials are studied, the law is described in vacuum by the **Laplace equation**

$$\nabla^2\Phi = 0.$$

Writing $\Phi(x, y, z)$ as a product of three functions, $\Phi(x, y, z) = X(x)Y(y)Z(z)$, yields three *ODEs* as follows:

$$\frac{d^2X}{dx^2} + \lambda X = 0, \frac{d^2Y}{dy^2} + \mu Y = 0, \frac{d^2Z}{dz^2} + \nu Z = 0, \qquad (2.1)$$

where $\lambda + \mu + \nu = 0$.

The Laplace equation describes not only electrostatics, but also heat transfer. When the transfer (diffusion) of heat takes place with the temperature being independent of time, the process is known as a steady-state heat transfer. The **diffusion equation** $\frac{\partial T}{\partial t} = a^2\nabla^2 T$ becomes the Laplace equation $\nabla^2 T = 0$, where T is the temperature and a is a constant characterizing the medium in which heat is flowing.

The **Schrödinger equation**, describing non-relativistic quantum phenomena, is as follows:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V(\mathbf{r})\Psi = -i\hbar\frac{\partial\Psi}{\partial t},$$

where m is the mass of a subatomic particle, $\hbar$ is the Plank constant (divided by 2π), V is the potential energy of the particle, and $|\Psi(\mathbf{r}, t)|^2$ is the probability density of finding the particle at $\mathbf{r}$ at time t.

In the next section, we obtain the solutions of these three equations in terms of the Meijer G-functions.

3. Results

The Meijer G-function $y(z) = G_{p,q}^{m,n}(z|_{b_k}^{a_j})$ satisfies the linear ordinary differential equation of the generalized hypergeometric type

$$[(-1)^{p-m-n}z\prod_{j=1}^{p}(z\frac{d}{dz} - a_j + 1) - \prod_{k=1}^{q}(z\frac{d}{dz} - b_k)]y(z) = 0 \qquad (3.1)$$

whose order is equal to $\max(p,q)$,(see [3–5], [1, Appendix]).

We here consider two cases when (3.1) reduces to first and second order ordinary differential equations, respectively:

Case 1. Setting $m = 1, n = 0, p = 0, q = 1$ in (3.1) yields

$$[-z - (z\frac{d}{dz} - b_1)]G_{0,1}^{1,0}(z|_{b_1}^{-}) = 0.$$

By changing z to $-z$ and dividing by z, we have

$$[\frac{d}{dz} - 1 + \frac{b_1}{z}]G_{0,1}^{1,0}(-z|_{b_1}^{-}) = 0.$$

On the other hand, changing from variable t to z gives

$$[\frac{d}{dz} - 1]T(z) = 0.$$

Equality condition for these two differential equations leads to $b_1 = 0$, and the solution in terms of the Meijer G-function is

$$T(z) = G_{0,1}^{1,0}(-z|_0^{-}) = e^z.$$

Case 2. Setting $m = 1, n = 0, p = 0, q = 2$ in (3.1) yields

$$[-z - (z\frac{d}{dz} - b_2)(z\frac{d}{dz} - b_1)]G_{0,2}^{1,0}(z|_{b_1,b_2}^{-}) = 0.$$

Changing variable from z to αz^2 gives

$$[-\alpha z^2 - (\frac{z}{2}\frac{d}{dz} - b_2)(\frac{z}{2}\frac{d}{dz} - b_1)]G_{0,2}^{1,0}(\alpha z^2|_{b_1,b_2}^{-}) = 0,$$

and dividing by z^2 gives

$$[\frac{d^2}{dz^2} + (1 - 2(b_1 + b_2))z\frac{d}{dz} + 4\alpha + 4\frac{b_1 b_2}{z^2}]G_{0,2}^{1,0}(\alpha z^2|_{b_1,b_2}^{-}) = 0. \qquad (3.2)$$

On the other hand,

$$[\frac{d^2}{dz^2} + 1]Z = 0.$$

Then an equality condition leads to

$$1 - 2(b_1 + b_2) = 0, 4\alpha = 1, b_1 b_2 = 0.$$

If $b_1 = \frac{1}{2}, b_2 = 0, \alpha = \frac{1}{4}$, then the first independent solution in terms of the Meijer G-function is

$$Z(z) = \sin z = G_{0,2}^{1,0}(\frac{1}{4}z^2|_{\frac{1}{2},0}^{-}).$$

If $b_1 = 0, b_2 = \frac{1}{2}, \alpha = \frac{1}{4}$, then the second independent solution in terms of the Meijer G-function is

$$Z(z) = \cos z = G_{0,2}^{1,0}(\frac{1}{4}z^2|_{0,\frac{1}{2}}^{-}).$$

Furthermore, an equality condition for (3.2) and the differential equation

$$[\frac{d^2}{dz^2} - 1]Z = 0$$

gives $1 - 2(b_1 + b_2) = 0, 4\alpha = -1, b_1 b_2 = 0$, that is,

$$b_1 = \frac{1}{2}, b_2 = 0, \alpha = -\frac{1}{4}.$$

If $b_1 = \frac{1}{2}, b_2 = 0, \alpha = -\frac{1}{4}$, then the first independent solution in terms of the Meijer G-function is

$$Z(z) = \sinh z = G_{0,2}^{1,0}(-\frac{1}{4}z^2|_{\frac{1}{2},0}^{-}).$$

If $b_1 = 0, b_2 = \frac{1}{2}, \alpha = -\frac{1}{4}$, then the second independent solution in terms of the Meijer G-function is

$$Z(z) = \cosh z = G_{0,2}^{1,0}(-\frac{1}{4}z^2|_{0,\frac{1}{2}}^{-}).$$

All the obtained solutions are shown in Table 1.

There are three kinds of boundary conditions that can be written in terms of the Meijer G-functions:

1. Separated boundary conditions (BCs):

$$\alpha_1 G_{p,q}^{m,n}|_a + \beta_1 G_{p,q}'^{m,n}|_a = 0, \qquad \alpha_2 G_{p,q}^{m,n}|_b + \beta_2 G_{p,q}'^{m,n}|_b = 0. \qquad (3.3)$$

2. Periodic BCs:

$$G_{p,q}^{m,n}|_a = G_{p,q}^{m,n}|_b, \qquad G_{p,q}'^{m,n}|_a = G_{p,q}'^{m,n}|_b. \qquad (3.4)$$

T a b l e 1. The general solutions in terms of the Meijer G-functions and elementary functions

Differential equation	Elementary functions	Meijer G-functions
$[\frac{d}{dz} - 1]T(z) = 0$	$T(z) = e^z$	$T(z) = G_{0,1}^{1,0}(-z\vert_0^{-})$
$[\frac{d^2}{dz^2} + 1]Z(z) = 0$	$Z(z) = A\sin z + B\cos z$	$Z(z) = AG_{0,2}^{1,0}(\frac{1}{4}z^2\vert_{\frac{1}{2},0}^{-})$
		$\quad + BG_{0,2}^{1,0}(\frac{1}{4}z^2\vert_{0,\frac{1}{2}}^{-})$
$[\frac{d^2}{dz^2} - 1]Z(z) = 0$	$Z(z) = C\sinh z + D\cosh z$	$Z(z) = CG_{0,2}^{1,0}(-\frac{1}{4}z^2\vert_{\frac{1}{2},0}^{-})$
		$\quad + DG_{0,2}^{1,0}(-\frac{1}{4}z^2\vert_{0,\frac{1}{2}}^{-})$

3. Generalization of separated and periodic BCs:

$$\alpha_{11}G_{p,q}^{m,n}(a\vert_{b_q}^{a_p}) + \alpha_{12}G'^{m,n}_{p,q}(a\vert_{b_q}^{a_p}) + \alpha_{13}G_{p,q}^{m,n}(b\vert_{b_q}^{a_p}) + \alpha_{14}G'^{m,n}_{p,q}(b\vert_{b_q}^{a_p}) = 0,$$

$$\alpha_{21}G_{p,q}^{m,n}(a\vert_{b_q}^{a_p}) + \alpha_{22}G'^{m,n}_{p,q}(a\vert_{b_q}^{a_p}) + \alpha_{23}G_{p,q}^{m,n}(b\vert_{b_q}^{a_p}) + \alpha_{24}G'^{m,n}_{p,q}(b\vert_{b_q}^{a_p}) = 0. \quad (3.5)$$

E x a m p l e 3.1. Steady-state heat conducting plate.
Let us consider a rectangular heat conducting plate with sides of length a and b. Three of the sides are held at $T = 0$, and the fourth side, at $y = b$, has a temperature variation $T = f(x)$. The flat faces are insulated so that they cannot lose heat to the surroundings. Assuming a steady-state heat transfer, the diffusion equation $\frac{\partial T}{\partial t} = a^2\nabla^2 T$ becomes the Laplace equation $\nabla^2 T = 0$. Let us calculate the variation of T over the plate.

The problem is two-dimensional. The separation of variables $T(x,y) = X(x)Y(y)$ leads to

$$\frac{d^2 X}{dx^2} + \lambda X = 0, \frac{d^2 Y}{dy^2} + \mu Y = 0, \quad (3.6)$$

where $\lambda + \mu = 0$.
The X equation (see Table 1) has a general solution as follows:

$$X(x) = AG_{0,2}^{1,0}(\frac{\lambda}{4}x^2\vert_{\frac{1}{2},0}^{-}) + BG_{0,2}^{1,0}(\frac{\lambda}{4}x^2\vert_{0,\frac{1}{2}}^{-}).$$

The boundary condition, $T(0,y) = 0$ for all y, implies that $X(0) = 0$. Therefore we get

$$X(x) = G_{0,2}^{1,0}(\frac{\lambda}{4}x^2\vert_{\frac{1}{2},0}^{-}) = \sin\sqrt{\lambda}x.$$

The X equation and two BCs, $T(0, y) = T(a, y) = 0$, form an S–L system for which we obtain the following eigenvalues and eigenfunctions:

- $\lambda_n = (\frac{n\pi}{a})^2$ and $X_n(x) = G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-}) = \sin(\frac{n\pi}{a})$ for $n = 1, 2, \ldots$.

Therefore, a general solution $X(x)$ can be written as

$$X(x) = \sum_{n=1}^{\infty} A_n G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-}).$$

On the other hand, the Y equation does not form an $S - L$ system. Nevertheless, we can solve the equation $Y'' - (\frac{n\pi}{a})^2 Y = 0$ to obtain a general solution

$$Y(y) = C \sinh y + D \cosh y = C G_{0,2}^{1,0}(-\frac{1}{4}y^2|_{\frac{1}{2},0}^{-}) + D G_{0,2}^{1,0}(-\frac{1}{4}y^2|_{0,\frac{1}{2}}^{-}).$$

The boundary condition $T(x, 0) = 0$ for all x implies that $Y(0) = 0$. So,

$$Y = \sum_{n'=1}^{\infty} C_{n'} \sinh \frac{n'\pi y}{a} \delta_{nn'} = \sum_{n'=1}^{\infty} C_{n'} G_{0,2}^{1,0}(-\frac{n'^2\pi^2}{4a^2}y^2|_{\frac{1}{2},0}^{-}) \delta_{nn'},$$

where $\sin \sqrt{\mu} y = \sin \sqrt{-\lambda} y$ (see (3.6)) which means $\sin \sqrt{\mu} y = \sinh \sqrt{\lambda} y$.

Thus, the most general solution consistent with the three BCs is

$$T(x, y) = X(x)Y(y) = \sum_{n=1}^{\infty} E_n G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-}) G_{0,2}^{1,0}(-\frac{n^2\pi^2}{4a^2}y^2|_{\frac{1}{2},0}^{-}).$$

The fourth BC gives

$$f(x) = T(x, b) = \sum_{n=1}^{\infty} [E_n G_{0,2}^{1,0}(-\frac{n^2\pi^2}{4a^2}b^2|_{\frac{1}{2},0}^{-})] G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-})$$

$$= \sum_{n=1}^{\infty} F_n G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-})$$

whose coefficients can be determined from

$$F_n = \frac{2}{a} \int_0^a G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-}) f(x) dx.$$

E x a m p l e 3.2. Rectangular conducting box.
Consider a rectangular conducting box with sides a, b and c. All faces are held at zero potential except the top face whose potential is given by a function $f(x, y)$ at $z = c$. Let us find the potential at all points inside the box.

The relevant PDE for this situation is the Laplace equation $\nabla^2\Phi = 0$. The problem is three-dimensional. Writing $\Phi(x, y, z) = X(x)Y(y)Z(z)$ yields three ODEs

$$\frac{d^2X}{dx^2} + \lambda X = 0, \frac{d^2Y}{dy^2} + \mu Y = 0, \frac{d^2Z}{dz^2} + \nu Z = 0, \qquad (3.7)$$

where $\lambda + \mu + \nu = 0$.

The vanishing of Φ at $x = 0$ and $x = a$ gives

- $\Phi(0, y, z) = X(0)Y(y)Z(z) = 0$ for all y, z, that is, $X(0) = 0$,

- $\Phi(a, y, z) = X(a)Y(y)Z(z) = 0$ for all y, z, so $X(a) = 0$.

The X equation and two BCs, $X(0) = X(a) = 0$, form the S–L system whose eigenvalues and eigenfunctions are

- $\lambda_n = (\frac{n\pi}{a})^2$ and $X_n(x) = G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-})$ for $n = 1, 2, ...$

Similarly, the second equation in (3.7) means that

- $\mu_m = (\frac{m\pi}{b})^2$ and $Y_m(y) = G_{0,2}^{1,0}(\frac{m^2\pi^2}{4b^2}y^2|_{\frac{1}{2},0}^{-})$ for $m = 1, 2, ...$

Furthermore, the third equation in (3.7) does not lead to an S–L system, whose eigenvalues and eigenfunctions consistent with the boundary condition $Z(0) = 0$ are

- $(\gamma_{mn})^2 = (\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2$ and $Z(z) = A_{mn}G_{0,2}^{1,0}(-\frac{\gamma_{mn}}{4}z^2|_{\frac{1}{2},0}^{-})$.

Consequently, putting everything together, we obtain

$$\Phi(x, y, z) = X(x)Y(y)Z(z)$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-})G_{0,2}^{1,0}(\frac{m^2\pi^2}{4b^2}y^2|_{\frac{1}{2},0}^{-})G_{0,2}^{1,0}(-\frac{\gamma_{mn}}{4}z^2|_{\frac{1}{2},0}^{-}).$$

To specify Φ completely, we must determine the arbitrary constants A_{mn}. Imposing the last BC, $\Phi(x, y, c) = f(x, y)$, yields

$$
\begin{aligned}
f(x, y) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-})G_{0,2}^{1,0}(\frac{m^2\pi^2}{4b^2}y^2|_{\frac{1}{2},0}^{-})G_{0,2}^{1,0}(-\frac{\gamma_{mn}}{4}c^2|_{\frac{1}{2},0}^{-}) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-})G_{0,2}^{1,0}(\frac{m^2\pi^2}{4b^2}y^2|_{\frac{1}{2},0}^{-}),
\end{aligned}
$$

where $B_{mn} = A_{mn}G_{0,2}^{1,0}(-\frac{\gamma_{mn}}{4}c^2|_{\frac{1}{2},0}^{-})$.

E x a m p l e 3.3. Conducting heat in a rectangular plate.

Consider a rectangular heat-conducting plate with sides of length a and b all held at $T = 0$. Assume that at time $t = 0$ the temperature has a distribution function $f(x, y)$. Let us find the variation of the temperature for all points (x, y) at all times $t > 0$.

The diffusion equation for this problem is

$$\frac{\partial T}{\partial t} = k^2 \nabla^2 T = k^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right).$$

A separation of variables $T(x, y, t) = X(x)Y(y)g(t)$ leads to three ODEs:

$$\frac{d^2 X}{dx^2} + \lambda X = 0, \frac{d^2 Y}{dy^2} + \mu Y = 0, \frac{dg}{dt} + k^2(\lambda + \mu)g = 0. \tag{3.8}$$

The BCs, $T(0, y, t) = T(a, y, t) = T(x, 0, t) = T(x, b, t) = 0$, together with the three $ODEs$, give rise to two $S - L$ systems. The solutions to both are easy to find:

- $\lambda_n = \left(\frac{n\pi}{a} \right)^2$ and $X_n(x) = G_{0,2}^{1,0} \left(\frac{n^2 \pi^2}{4a^2} x^2 \big|_{\frac{1}{2}, 0}^{-} \right)$ for $n = 1, 2, \ldots,$

- $\mu_m = \left(\frac{m\pi}{b} \right)^2$ and $Y_m(y) = G_{0,2}^{1,0} \left(\frac{m^2 \pi^2}{4b^2} y^2 \big|_{\frac{1}{2}, 0}^{-} \right)$ for $m = 1, 2, \ldots.$

These give rise to the general solution

$$X(x) = \sum_{n=1}^{\infty} A_n G_{0,2}^{1,0} \left(\frac{n^2 \pi^2}{4a^2} x^2 \big|_{\frac{1}{2}, 0}^{-} \right), \quad Y(y) = \sum_{m=1}^{\infty} B_m G_{0,2}^{1,0} \left(\frac{m^2 \pi^2}{4b^2} y^2 \big|_{\frac{1}{2}, 0}^{-} \right)$$

with $\gamma_{mn} = k^2(\lambda_n + \mu_m)$.

The solution to the g equation can be expressed as

$$g(t) = C_{mn} e^{-\gamma_{mn} t} = C_{mn} G_{0,1}^{1,0} (\gamma_{mn} t \big|_0^{-}).$$

Consequently, the most general solution can be expressed as follows:

$$T(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} G_{0,1}^{1,0}(\gamma_{mn} t \big|_0^{-}) G_{0,2}^{1,0} \left(\frac{n^2 \pi^2}{4a^2} x^2 \big|_{\frac{1}{2}, 0}^{-} \right) G_{0,2}^{1,0} \left(\frac{m^2 \pi^2}{4b^2} y^2 \big|_{\frac{1}{2}, 0}^{-} \right),$$

where $A_{mn} = A_n B_m C_{mn}$ is an arbitrary constant.

To determine it, we impose the initial condition $T(x, y, 0) = f(x, y)$. This yields the following:

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} G_{0,2}^{1,0} \left(\frac{n^2 \pi^2}{4a^2} x^2 \big|_{\frac{1}{2}, 0}^{-} \right) G_{0,2}^{1,0} \left(\frac{m^2 \pi^2}{4b^2} y^2 \big|_{\frac{1}{2}, 0}^{-} \right),$$

which determines the coefficients A_{mn}

$$A_{mn} = \frac{4}{ab} \int\limits_0^a dx \int\limits_0^b dy f(x,y) G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-}) G_{0,2}^{1,0}(\frac{m^2\pi^2}{4b^2}y^2|_{\frac{1}{2},0}^{-}).$$

E x a m p l e 3.4. Quantum particle in a box.
The behaviour of an atomic particle of mass μ confined in a rectangular box with sides a, b and c (an infinite three- dimensional potential well is governed by the Schrödinger equation for a free particle

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}),$$

and the BC that $\psi(x,y,z)$ vanishes at all sides of the box for all time.

A separation of variable $\psi(x,y,z,t) = X(x)Y(y)Z(z)T(t)$ yields the ODEs

$$\frac{d^2X}{dx^2} + \lambda X = 0, \qquad \frac{d^2Y}{dy^2} + \sigma Y = 0, \qquad \frac{d^2Z}{dz^2} + \nu Z = 0, \qquad (3.9)$$

$$\frac{dT}{dt} + i\omega T = 0, \qquad (3.10)$$

where $\omega = \frac{\hbar}{2\mu}(\lambda + \sigma + \nu)$.

Vanishing of ψ, BCs at $x = 0$ and $x = a$, for all y, z; at $y = 0$ and $y = b$ for all x, z; at $z = 0$ and $z = c$ for all x, y, gives

- $\psi(0,y,z,t) = \psi(a,y,z,t) = 0$, which means $X(0) = X(a) = 0$,

- $\psi(x,0,z,t) = \psi(x,b,z,t) = 0$, which means $Y(0) = Y(b) = 0$,

- $\psi(x,y,0,t) = \psi(x,y,c,t) = 0$, which means $Z(0) = Z(C) = 0$,

leads to three S–L systems whose solutions (see Table 1) are easily found:

- $X_n(x) = G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^{-})$, $\lambda_n = (\frac{n\pi}{a})^2$ for $n = 1,2,\ldots$,

- $Y_m(y) = G_{0,2}^{1,0}(\frac{m^2\pi^2}{4b^2}y^2|_{\frac{1}{2},0}^{-})$, $\sigma_m = (\frac{m\pi}{b})^2$ for $m = 1,2,\ldots$,

- $Z_l(z) = G_{0,2}^{1,0}(\frac{l^2\pi^2}{4c^2}z^2|_{\frac{1}{2},0}^{-})$, $\nu_m = (\frac{l\pi}{c})^2$ for $l = 1,2,\ldots$.

On the other hand, the time equation has the solution (Table 1) of the following form:

$$T(t) = C_{lmn} G_{0,1}^{1,0}(i\omega_{lmn}t|_0^-), \qquad \text{where} \qquad \omega_{lmn} = \frac{\hbar}{2\mu}[(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 + (\frac{l\pi}{c})^2].$$

Therefore, the solution of the *Schrödinger* equation consistent with the BCs is

$$\psi(x,y,z,t)$$
$$= \sum_{l,m,n=1}^{\infty} A_{lmn} G_{0,1}^{1,0}(i\omega_{lmn}t|_0^-) G_{0,2}^{1,0}(\frac{n^2\pi^2}{4a^2}x^2|_{\frac{1}{2},0}^-) G_{0,2}^{1,0}(\frac{m^2\pi^2}{4b^2}y^2|_{\frac{1}{2},0}^-) G_{0,2}^{1,0}(\frac{l^2\pi^2}{4c^2}z^2|_{\frac{1}{2},0}^-).$$

The constants $A_{l,m,n}$ are determined by the initial shape $\psi(x,y,z,0)$ of the wave function.

4. Conclusions

In this article, we illustrated that the Meijer G-functions have many applications in explicit solutions of the three well-known $PDEs$, namely the Laplace equation, the diffusion equation and the Schrödinger equation. These solutions are very nice because of their symmetric product forms. We believe that the G-function will be used as a *global* function in physics and engineering for unification.

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