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On the Perturbation of Self-Adjoint Operators with Absolutely Continuous Spectrum

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The perturbation of the linear self-adjoint operator with absolutely continuous spectrum is studied and the inverse problem on finding the perturbation by the given spectrum is solved.

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1. Preliminaries

The beginning of spectral theory ascends to the works of H. Weyl [9] (1909), F. Rellich [7] (1936) and K. Friedrichs [4] (1938). Particularly, the theorem on the invariance of the continuous part of the spectrum of self-adjoint operator under completely continuous perturbation belongs to H. Weyl [9]. In the works of T. Kato [6] and M. Rozenblum [8] it is shown that absolutely continuous part of the spectrum is invariant with respect to finite-dimensional perturbations.

In the paper, the perturbations of the linear self-adjoint operators in the Hilbert space are studied assuming that the spectrum of the original operator is absolutely continuous. It is shown how the spectral measure corresponding to absolutely continuous spectrum is changed. The inverse theorem on the recovering of the rank-one perturbation by the spectra of the original and perturbed operators is proved.

Let A be a linear self-adjoint operator in a Hilbert space H. We consider the perturbation of the operator A,

$$B - A = -\varphi^* \sigma \varphi, \tag{1}$$

where E is a Hilbert space, φ, σ are bounded operators, $\varphi : H \to E$, $\sigma : E \to E(\sigma = \sigma^*)$. Let us find the resolvent of the operator B. Obviously,

$$(B - \lambda I) = (A - \lambda I) - \varphi^* \sigma \varphi.$$

Assuming that λ does not belong to $\sigma(A) \cup \sigma(B)$ (where $\sigma(A)$ and $\sigma(B)$ are the spectra of the operators A and B, respectively), multiply the right-hand side of this equality by $R_\lambda(B) = (B - \lambda)^{-1}$ and its left-hand side by $R_\lambda(A) = (A - \lambda)^{-1}$ to obtain

$$R_\lambda(B) = R_\lambda(A) + R_\lambda(A)\varphi^* \sigma \varphi R_\lambda(B). \tag{2}$$

Applying the operator φ to both sides of the equality, we get

$$\varphi R_\lambda(A) = (I - \varphi R_\lambda(A)\varphi^* \sigma)\varphi R_\lambda(B).$$

Denote by $N(\lambda)$ the operator function in E

$$N(\lambda) = \varphi R_\lambda(A)\varphi^*. \tag{3}$$

We obtain

$$(I - N(\lambda)\sigma)^{-1}\varphi R_\lambda(A) = \varphi R_\lambda(B),$$

where λ is such that the operator $(I - N(\lambda)\sigma)^{-1}$ exists and it is bounded.

Substituting $\varphi R_\lambda(B)$ in (2), we get the representation of the resolvent of the operator B

$$R_\lambda(B) = R_\lambda(A) + R_\lambda(A)\varphi^* \sigma (I - N(\lambda)\sigma)^{-1}\varphi R_\lambda(A). \tag{4}$$

Thus, we proved the theorem.

Theorem 1. *Let A be a linear self-adjoint operator in a Hilbert space H, and the perturbed operator B have the form of* (1)*. Then the resolvent of the operator B has the form of* (4)*, where λ does not belong to $\sigma(A) \cup \sigma(B)$ and is such that the operator $(I - N(\lambda)\sigma)^{-1}$ exists and it is bounded.*

Denote by $W(\lambda)$ the operator function in E,

$$W(\lambda) = \sigma(I - N(\lambda)\sigma)^{-1}. \tag{5}$$

Theorem 2 *The function $W(\lambda)$ is the Nevanlinna function* (*the function of class N* [2, p. 120]) *admitting the representation $W(\lambda) = \sigma(I - N(\lambda)\sigma)^{-1} = \sigma + \int_{\mathbb{R}} \frac{dw_t}{t-\lambda}$, where $dw(t)$ is the finite measure.*

P r o o f. Obviously, $N(\lambda)$ is the Nevanlinna function and $sup\|yN(iy)\| < \infty$ when $y \geq 1$. Thus [3, p. 118],

$$N(\lambda) = \int_{\mathbb{R}} \frac{dF_x}{x-\lambda}, \tag{6}$$

where $F_x = \varphi E_x \varphi^*$, and E_x is the resolution of the identity of the operator A.

Let us estimate that the function $W(\lambda)$ is the Nevanlinna function

$$\frac{W(\lambda)-W^*(\lambda)}{\lambda-\bar{\lambda}} = \frac{\sigma(I-N(\lambda)\sigma)^{-1}-(I-\sigma N^*(\lambda))^{-1}\sigma}{\lambda-\bar{\lambda}}$$

$$= (I-\sigma N^*(\lambda))^{-1}\left\{\frac{(I-\sigma N^*(\lambda))\sigma-\sigma(I-N(\lambda)\sigma)}{\lambda-\bar{\lambda}}\right\}(I-N(\lambda)\sigma)^{-1}$$

$$= (I-\sigma N^*(\lambda))^{-1}\sigma\left\{\frac{N(\lambda)-N^*(\lambda)}{\lambda-\bar{\lambda}}\right\}\sigma(I-N(\lambda)\sigma)^{-1}$$

$$= W^*(\lambda)\left\{\frac{N(\lambda)-N^*(\lambda)}{\lambda-\bar{\lambda}}\right\}W(\lambda).$$

Since $N(\lambda)$ is the Nevanlinna function, then $W(\lambda)$ is also the Nevanlinna function. Besides

$$W(\lambda)-\sigma = \sigma-\sigma(I-N(\lambda)\sigma)^{-1} = (\sigma(I-N(\lambda)\sigma)-\sigma)(I-N(\lambda)\sigma)^{-1}$$

$$= -\sigma N(\lambda)\sigma(I-N(\lambda)\sigma)^{-1}.$$

For $y \geq 1$,

$$sup\,\|y(W(iy)-\sigma)\| \leq sup\|\sigma y N(iy)\sigma\|\|(1-N(iy)\sigma)^{-1}\| < \infty.$$

Therefore

$$W(\lambda) = \sigma(I-N(\lambda)\sigma)^{-1} = \sigma + \int_{\mathbb{R}} \frac{dw_t}{t-\lambda}. \tag{7}$$

■

In (2), multiplying by φ^* the right-hand side of the equality, and by $\sigma\varphi$ its left-hand side, we obtain

$$\sigma\varphi R_\lambda(B)\varphi^* = \sigma\varphi R_\lambda(A)\varphi^* + \sigma\varphi R_\lambda(A)\varphi^*\sigma\varphi R_\lambda(B)\varphi^*.$$

Transforming the expression, we get

$$-I + (I+\sigma\varphi R_\lambda(B)\varphi^*) = \sigma\varphi R_\lambda(A)\varphi^*(I+\sigma\varphi R_\lambda(B)\varphi^*),$$

$$(I + \sigma\varphi R_\lambda(B)\varphi^*)(I - \sigma\varphi R_\lambda(A)\varphi^*) = I,$$

$$(I + \sigma\varphi R_\lambda(B)\varphi^*) = (I - \sigma N(\lambda))^{-1}.$$

Multiplying by σ the right-hand side of the equality, we obtain

$$(I + \sigma\varphi R_\lambda(B)\varphi^*)\sigma = W(\lambda).$$

Thus, the "restriction" of the resolvent of the operator B on E by the operator φ can be expressed as the operator function $W(\lambda)$ (5).

Let us recall the well-known fact [1, p. 95].

Theorem (M. Riesz). *Suppose the function $f_t(-\infty < t < \infty)$ is measurable and belongs to $L^p(-\infty, \infty)$. Then the principal value of the integral*

$$\widetilde{f}_x = v.p. \int\limits_{\mathbb{R}} \frac{f_t}{x - t} dt$$

exists and $\widetilde{f}_x \in L^p(-\infty, \infty)$.

2. Direct problem

Theorem 3. *Suppose F_t from (6) is a completely continuous operator function and σ is an operator of trace class and it is reversible. Then the function w_t from (7) is completely continuous, and if it holds $\int\limits_{\mathbb{R}} \|F_t'\|^2 dt < \infty$, $\int\limits_{\mathbb{R}} \|w_t'\|^2 dt < \infty$, then w_t satisfies $W_t^+ F_t' W_t^+ - w_t' = 2\pi i w_t' F_t' W_t^+$, where W_x^+ is the limit value of W_λ on the real axis in the semiplane $\mathbb{C}_+$.*

P r o o f. Obviously, the operator function w_t from (7) is completely continuous because F_t is a completely continuous operator function and the perturbation is an operator of trace class [2, p. 345].

For $\forall\lambda \in \mathbb{C} \setminus \mathbb{R}$ from (7) we obtain

$$\sigma = \left(\sigma + \int\limits_{\mathbb{R}} \frac{dw_t}{t - \lambda}\right)\left(1 - \int\limits_{\mathbb{R}} \frac{dF_t}{t - \lambda}\sigma\right). \tag{8}$$

Then

$$0 = -\int\limits_{\mathbb{R}} \frac{d\sigma F_t \sigma}{t - \lambda} + \int\limits_{\mathbb{R}} \frac{dw_t}{t - \lambda} - \int\limits_{\mathbb{R}} \frac{dw_t}{t - \lambda} \int\limits_{\mathbb{R}} \frac{dF_s}{s - \lambda}\sigma.$$

Transform the double integral. According to Fubini's theorem,

$$\int_{\mathbb{R}} \frac{dw_t}{t-\lambda} \int_{\mathbb{R}} \frac{dF_s}{s-\lambda}\sigma = \iint_{\mathbb{R}^2} \frac{dw_t dF_s \sigma}{(t-\lambda)(s-\lambda)}.$$

By the conditions of the theorem $\int_{\mathbb{R}} \|F_t'\|^2 dt < \infty$, $\int_{\mathbb{R}} \|w_t'\|^2 dt < \infty$, so it holds by Riesz's theorem

$$\iint_{\mathbb{R}^2} \frac{dw_t dF_s \sigma}{(t-\lambda)(s-\lambda)} = \iint_{\mathbb{R}^2} \frac{dw_t dF_s \sigma}{(s-t)(t-\lambda)} - \iint_{\mathbb{R}^2} \frac{dw_t dF_s \sigma}{(s-t)(s-\lambda)}$$

$$= \int_{\mathbb{R}} \frac{dw_t}{t-\lambda}\Phi_t + \int_{\mathbb{R}} \Psi_s \frac{dF_s \sigma}{s-\lambda},$$

where

$$\Phi_t = \int_{\mathbb{R}} \frac{dF_s \sigma}{s-t}, \ \Psi_s = \int_{\mathbb{R}} \frac{dw_t}{t-s}.$$

The integrals Φ_t and Ψ_t are treated as the principal value of the improper integral and exist by the Riesz theorem.

Hence from (8) we obtain

$$\int_{\mathbb{R}} \frac{d(\sigma F_t \sigma - w_t)}{t-\lambda} = -\int_{\mathbb{R}} \frac{dw_t}{t-\lambda}\Phi_t - \int_{\mathbb{R}} \Psi_s \frac{dF_s \sigma}{s-\lambda},$$

$$\int_{\mathbb{R}} \frac{d_t \left[\sigma F_t \sigma + \int_0^t w_s' \Phi_s ds + \int_0^t \Psi_s F_s' \sigma ds\right]}{t-\lambda} = \int_{\mathbb{R}} \frac{dw_t}{t-\lambda}.$$

Thereby, taking into account the properties of the Hilbert transform [3], we have

$$\sigma F_t' \sigma + w_t' \Phi_t + \Psi_t F_t' \sigma = w_t'$$

or

$$(\sigma + \Psi_t) F_t' \sigma = w_t'(I - \Phi_t). \tag{9}$$

From $W_\lambda = \sigma(I - N_\lambda \sigma)^{-1}$ it follows that $W_\lambda(I - N_\lambda \sigma) = \sigma$. Direct λ to the real axis in the semiplane $\mathbb{C}_+$. By the conditions of the theorem, there exist W_x^+, N_x^+ which are the limits of W_λ and N_λ, respectively. Thus, $W_x^+(I - N_x^+ \sigma) = \sigma$ [5, p. 57].

By the Sokhotski formulas, we get

$$W_x^+ = \pi i w_x + v.p.\int_{\mathbb{R}} \frac{w_t'}{t-x}dt,$$

$$N_x^+ = \pi i F_x + v.p.\int_{\mathbb{R}} \frac{F_t'}{t-x}dt.$$

By the definition of Φ_t and Ψ_t, we have

$$\sigma + \Psi_t = W_t^+ - \pi i w_t',$$

$$I - \Phi_t = I - N_t^+\sigma + \pi i F_t'\sigma.$$

Substituting the obtained expression into (9), we get

$$(W_t^+ - \pi i w_t')F_t'\sigma = w_t'(I - N_t^+\sigma + \pi i F_t'\sigma).$$

Since σ is reversible, then ${W_x^+}^{-1}\sigma = I - N_x^+\sigma$ and we obtain

$$W_t^+F_t'\sigma - \pi i w_t'F_t'\sigma = w_t'{W_x^+}^{-1}\sigma + \pi i w_t'F_t'\sigma.$$

Finally we get the expression which connects two measures

$$W_t^+F_t' - w_t'{W_t^+}^{-1} = 2\pi i w_t'F_t'. \tag{10}$$

■

3. Inverse problem

Suppose $dim\,E = 1$ and the operator $\varphi : H \to E$ acts in the following way: $\varphi h =< h, g > f$, where $f \in E$, $\|f\| = 1$, $g \in H$, $h \in H$.

The next theorem allows to recover the perturbation by the two measures of the operator functions N_λ and W_λ for the rank-one perturbation.

Theorem 4. *Let A be a linear self-adjoint operator with absolutely continuous spectrum and let there be given two completely continuous nondecreasing scalar functions F_t,w_t such that $\int_{\mathbb{R}}|F_t'|^2dt < \infty$, $\int_{\mathbb{R}}|w_t'|^2dt < \infty$. If there exists a solution W_t^+ of (10) in the class of the Nevanlinna functions such that $W_t^+ - \pi i w_t' + \int_{\mathbb{R}}\frac{dw_s}{s-t} \equiv$ const and $F_t'w_t' \leq 1/\pi^2$, then there exists a perturbation of the form of (1), where $dim\,E = 1$, $\varphi h =< h, g > f$, $f \in E$, $\|f\| = 1$, $g, h \in H$, moreover*

$$N(\lambda) = \int_{\mathbb{R}} \frac{dF_t}{t-\lambda} = \varphi R_\lambda(A)\varphi^*,$$

$$W(\lambda) = \sigma + \int\limits_{\mathbb{R}} \frac{dw_t}{t-\lambda} = \sigma(I - N(\lambda)\sigma)^{-1}.$$

P r o o f. Let us take $\forall f : \|f\| = 1$. Define the function $N(\lambda) = \int\limits_{\mathbb{R}} \frac{dF_t}{t-\lambda}$.

Let g be the desired vector in the representation for φ. Then the following relation holds:

$$< E_t g, g >= F_t,$$

where E_t is the resolution of the identity of the operator A.

Since the spectrum of the operator A is simple, then there exists a generating vector $\xi \in H$ such that

$$\int\limits_{-\infty}^{x} |g_t|^2 d\rho_t = F_x,$$

where $\rho_t =< E_t\xi, \xi >$ [2, p. 282].

Taking the derivative of the equality, we obtain

$$|g_x|^2 \rho'_x = F'_x.$$

Therefore for $\forall x \in \mathbb{R} : \rho'_x \neq 0$ we have $|g_x|^2 = \frac{F'_x}{\rho'_x}$, for other $x's$ one can assume $g_x = 0$.

Taking an arbitrary function g_t, which satisfies the obtained condition, we get

$$g = \int\limits_{\mathbb{R}} g_t dE_t \xi.$$

Thus the function $N(\lambda) =< R_\lambda(A)g, g >= \varphi R_\lambda(A)\varphi^*$.

By the conditions of the theorem, there exists a solution W_t^+ of (10),

$${W_t^+}^2 F'_t - w'_t = 2\pi i w'_t F'_t W_t^+.$$

Solving the equation, we obtain

$$W_t^+ = \pi i w'_t \pm \sqrt{\frac{w'_t}{F'_t} - \pi^2 {w'_t}^2}. \tag{11}$$

Since $F'_t w'_t \leq 1/\pi^2$, then

$$\mathrm{Im} W_t^+ = \pi w'_t \geq 0.$$

Define σ by the following expression:

$$\sigma = W_t^+ - \pi i w'_t + \int\limits_{\mathbb{R}} \frac{dw_s}{s-t}.$$

Note that the defined σ is a constant by the conditions of the theorem and $\sigma \in \mathbb{R}$.

Define the function $W(\lambda) = \sigma(I - N(\lambda)\sigma)^{-1}$. By Theorem 2, it holds $W(\lambda) = \sigma + \int\limits_{\mathbb{R}} \frac{d\nu_t}{t-\lambda}$, where $d\nu_t$ is some finite measure.

From the conditions on the limit value of the function $W(\lambda)$ on real axis (11), by using the Stieltjes–Perron inversion formula [3], we can get that $d\nu_t = dw_t$ and, consequently,

$$W(\lambda) = \sigma + \int\limits_{\mathbb{R}} \frac{dw_t}{t - \lambda} = \sigma(I - N(\lambda)\sigma)^{-1}.$$

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