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Lie Invariant Shape Operator for Real Hypersurfaces in Complex Two-Plane Grassmannians II

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A new notion of the generalized Tanaka–Webster $\mathfrak{D}^{\perp}$-invariant for a hypersurface M in $G_2(\mathbb{C}^{m+2})$ is introduced, and a classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with generalized Tanaka-Webster $\mathfrak{D}^{\perp}$-invariant shape operator is given.

Key words: real hypersurfaces, complex two-plane Grassmannians, Hopf hypersurface, generalized Tanaka–Webster connection, Reeb parallel shape operator, $\mathfrak{D}^{\perp}$-parallel shape operator, invariant shape operator, g-Tanaka–Webster invariant shape operator, g-Tanaka–Webster $\mathfrak{D}^{\perp}$-invariant shape operator.

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Introduction

The Tanaka–Webster connecton is a unique affine connection on a non-degenerate pseudo-Hermitian CR manifold which associates with the almost contact structure ([17, 18]). Tanno [17] introduced *the generalized Tanaka–Webster* (in short, the *g-Tanaka–Webster*) connection for contact Riemannian manifolds generalizing it for non-degenerate integrable CR manifolds. For a real hypersurface in Kähler manifolds with almost contact metric structure (ϕ, ξ, η, g), the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ for a non-zero real number k was given in [5] and [10].

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In particular, if a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka–Webster connection.

For a real hypersurface in complex space form $\tilde{M}_n(c)$ with constant holomorphic sectional curvature c, many geometers have studied some characterizations by using the g-Tanaka-Webster connection. For instance, when $c > 0$, that is, $\tilde{M}_n(c)$ is a complex projective space $\mathbb{C}P^n$, Kon [10] proved that if the Ricci tensor $\hat{S}$ of the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ vanishes identically, then a real hypersurface in $\mathbb{C}P^n$ is locally congruent to a geodesic hypersphere with $k^2 \geq 4n(n-1)$.

Now let us denote by the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ a set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space has a remarkable geometric structure. It is the unique compact irreducible Riemannian manifold equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact irreducible Kähler, quaternionic Kähler manifold which is not a hyper-Kähler manifold. The almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$. Then, naturally we could consider two geometric conditions for a hypersurface M in $G_2(\mathbb{C}^{m+2})$ that a 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$ and a 3-dimensional distribution $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M ([3]).

By using these two geometric conditions and the results of Alekseevskii [1], Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*, we say that the Reeb vector field ξ on M is Killing. This means that the metric tensor g is invariant under the Reeb flow of ξ on M. Berndt and Suh gave a characterization of real hypersurfaces of Type (A) in Theorem A in terms of the Reeb flow on M as follows (see [4]):

Theorem B. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Besides, Lee and Suh [11] gave a new characterization of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field ξ as follows:

Theorem C. *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

On the other hand, using the Riemannian connection, in [13] Suh gave a non-existence theorem of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator. Moreover, Suh proved a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with the $\mathfrak{F}$-parallel shape operator, where $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$(see [14]).

In particular, Jeong, Lee and Suh [5] considered a g-Tanaka–Webster parallel shape operator for a real hypersurface in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. In other words, the shape operator A is called *g-Tanaka-Webster parallel* if it satisfies $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any tangent vector fields X and Y on M. Using this notion, the authors gave a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Also, the authors considered a more generalized notion weaker than the parallel shape operator in the g-Tanaka–Webster connection of M. When the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ satisfies $(\hat{\nabla}^{(k)}_\xi A)Y = 0$ for any tangent vector field Y on M, we say that the shape operator is *g-Tanaka–Webster Reeb parallel.* Using such a notion, the authors gave a characterization of the real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ as follows (see [6]):

Theorem D. *Let M be a connected orientable Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the shape operator A is generalized Tanaka–Webster Reeb parallel, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Moreover, Jeong, Lee and Suh [7] introduced a notion of the *g-Tanaka–Webster $\mathfrak{D}^\perp$-parallel* shape operator for M in $G_2(\mathbb{C}^{m+2})$. It means that the shape operator A of M satisfies $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any X in $\mathfrak{D}^\perp$ and Y on M. Naturally, we can see that the g-Tanaka–Webster $\mathfrak{D}^\perp$-parallel is weaker than the g-Tanaka–Webster parallel. By using such a notion of $\mathfrak{D}^\perp$-parallel in the g-Tanaka–Webster connection, the authors gave a characterization of the real hypersurface of Type (B) in $G_2(\mathbb{C}^{m+2})$.

Specially, Suh [15] asserted a characterization of the real hypersurfaces of type (A) in Theorem A by another geometric Lie invariant, that is, the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ is *invariant* under the Reeb flow on M.

On the other hand, we considered another Lie invariant of the shape operator in $G_2(\mathbb{C}^{m+2})$, namely, a *g-Tanaka–Webster invariant shape operator*, that is,

$$(\hat{\mathfrak{L}}^{(k)}_X A)Y = 0$$

for any vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$, where $\hat{\mathfrak{L}}^{(k)}$ denotes the g-Tanaka–Webster Lie derivative induced from the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$. Usually, the notion of the g-Tanaka–Webster invariant is different from any Levi–Civita Lie invariants and gives us much more information than usual covariant parallelisms in the g-Tanaka–Webster connection. By using such a notion of Lie invariant in g-Tanaka–Webster connection, we gave a non-existence theorem for the real hypersurface in $G_2(\mathbb{C}^{m+2})$ as follows (see [9]):

Theorem E. *There does not exist any Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$ with g-Tanaka–Webster invariant shape operator.*

Meanwhile, we consider a new notion of *g-Tanaka–Webster Reeb invariant* shape operator for M in $G_2(\mathbb{C}^{m+2})$, that is, $(\hat{\mathfrak{L}}^{(k)}_\xi A)X = 0$ for any tangent vector field Y on M. Since $(\hat{\mathfrak{L}}^{(k)}_\xi A)X = (\hat{\nabla}^{(k)}_\xi A)X = 0$, from Theorem D we obtain the following Remark.

Remark. Let M be a connected orientable Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the shape operator A is generalized Tanaka–Webster Reeb invarint, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

In this paper, we consider a generalized condition named *g-Tanaka–Webster $\mathfrak{D}^\perp$-invariant shape operator*, that is, $\hat{\mathfrak{L}}^{(k)}_{\mathfrak{D}^\perp} A = 0$, where $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$. This condition is weaker than the Lie invariant in the g-Tanaka–Webster connection mentioned in Theorem E. By using such a notion of the g-Tanaka–Webster $\mathfrak{D}^\perp$-invariant, we give a classification theorem for the real hypersurface in $G_2(\mathbb{C}^{m+2})$ as follows:

Main Theorem. *Let M be a connected orientable Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the shape operator A is g-Tanaka–Webster $\mathfrak{D}^\perp$-invariant shape operator, then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$ with $\alpha = k$ and $q_i(X) = 0$ for any tangent vector field $X \in \mathfrak{D}$ and $i = 1, 2, 3$, where $m = 2n$.*

1. Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2, 3] and [4]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2)\times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K. Moreover, we equip it with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By the $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way, $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of the oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\tilde\nabla$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\tilde\nabla_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\Big\} \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\Big\},\end{aligned} \tag{1.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

Now we derive some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [3, 4], [11–14]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{1.3}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ from Sec. 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned} & \phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0, \\ & \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ & \phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ & \phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}\end{aligned} \tag{1.4}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$ from Sec. 1 and (1.3), the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{aligned} \phi\phi_\nu X &= \phi_\nu \phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\ \eta_\nu(\phi X) &= \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu \xi. \end{aligned} \tag{1.5}$$

On the other hand, from the Kähler structure J, that is, $\tilde{\nabla} J = 0$ and the quaternionic Kähler structure J_ν, together with the Gauss and Weingarten equations, it follows that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX, \tag{1.6}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{1.7}$$

$$\begin{aligned} (\nabla_X \phi_\nu)Y = &- q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y \\ &+ \eta_\nu(Y)AX - g(AX, Y)\xi_\nu. \end{aligned} \tag{1.8}$$

Using expression (1.2) for the curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi becomes:

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = &\ \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ &+ \sum_{\nu=1}^{3} \Big\{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu \Big\} \\ &+ \sum_{\nu=1}^{3} \Big\{ \eta_\nu(\phi X)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X \Big\} \\ &+ \sum_{\nu=1}^{3} \Big\{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X) \Big\}\xi_\nu\,. \end{aligned} \tag{1.9}$$

Now we introduce the notion of the g-Tanaka–Webster connection (see [10]).

As stated above, the Tanaka–Webster connection is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold (see [16, 18]). In [17], Tanno defined the g-Tanaka–Webster connection for contact metric manifolds by the canonical connection. It coincides with the Tanaka–Webster connection if the associated CR-structure is integrable.

From now on, we will introduce the g-Tanaka–Webster connection due to Tanno [17] for real hypersurfaces in Kähler manifolds by naturally extending the canonical affine connection to a non-degenerate pseudo-Hermitian CR manifold.

Now let us recall that the g-Tanaka-Webster connection $\hat{\nabla}$ was defined by Tanno [17] for contact metric manifolds as follows:

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y$$

for all vector fields X and Y.

By taking (1.6) into account, the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ for real hypersurfaces of Kähler manifolds is defined by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.10}$$

for a non-zero real number k (see [5] and [10]) (Note that $\hat{\nabla}^{(k)}$ is invariant under the choice of the orientation. Namely, we may take $-k$ instead of k in (1.10) for the opposite orientation $-N$).

2. Key Lemmas

In this section, we will prove that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$ for M in $G_2(\mathbb{C}^{m+2})$ with g-Tanaka–Webster $\mathfrak{D}^\perp$-invariant shape operator.

In [9], from the definition of the g-Tanaka–Webster connection (1.10), we have the following:

$$\begin{aligned}(\hat{\mathfrak{L}}^{(k)}_X A)Y = &(\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY\\ &- g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y\\ &- \nabla_{AY}X - g(\phi A^2Y, X)\xi + \eta(X)\phi A^2 Y + k\eta(AY)\phi X\\ &+ A\nabla_Y X + g(\phi AY, X)A\xi - \eta(X)A\phi AY - k\eta(Y)A\phi X\end{aligned}$$

for any tangent vector fields X and Y on M.

The shape operator A is said to be *generalized Tanaka–Webster* $\mathfrak{D}^\perp$*-invariant* if $(\hat{\mathfrak{L}}^{(k)}_X A)Y = 0$ for any tangent vector fields $X \in \mathfrak{D}^\perp$ and $Y \in TM$. Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with generalized Tanaka–Webster $\mathfrak{D}^\perp$-invariant shape operator. This becomes

$$\begin{aligned}0 &= (\hat{\mathfrak{L}}^{(k)}_X A)Y\\ &= (\nabla_X A)Y + g(\phi AX, AY)\xi - \alpha\eta(Y)\phi AX - k\eta(X)\phi AY\\ &\quad - \alpha g(\phi AX, Y)\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y\\ &\quad - \nabla_{AY}X - g(\phi A^2Y, X)\xi + \eta(X)\phi A^2Y + \alpha k\eta(Y)\phi X\\ &\quad + A\nabla_Y X + \alpha g(\phi AY, X)\xi - \eta(X)A\phi AY - k\eta(Y)A\phi X\end{aligned} \tag{2.1}$$

for any tangent vector fields X and Y on M.

Applying $X = \xi_\mu \in \mathfrak{D}^\perp$ and $Y = X$ in (2.1), we get

$$\begin{aligned}0 &= (\hat{\mathfrak{L}}^{(k)}_{\xi_\mu} A)X\\ &= (\nabla_{\xi_\mu} A)X + g(\phi A\xi_\mu, AX)\xi - \alpha\eta(X)\phi A\xi_\mu - k\eta(\xi_\mu)\phi AX\\ &\quad - \alpha g(\phi A\xi_\mu, X)\xi + \eta(X)A\phi A\xi_\mu + k\eta(\xi_\mu)A\phi X\\ &\quad - \nabla_{AX}\xi_\mu - g(\phi A^2X, \xi_\mu)\xi + \eta(\xi_\mu)\phi A^2X + \alpha k\eta(X)\phi\xi_\mu\\ &\quad + A\nabla_X\xi_\mu + \alpha g(\phi AX, \xi_\mu)\xi - \eta(\xi_\mu)A\phi AX - k\eta(X)A\phi\xi_\mu.\end{aligned} \tag{2.2}$$

Using (2.2), we can assert the following:

Lemma 2.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If M has the g-Tanaka–Webster $\mathfrak{D}^\perp$-invariant shape operator, then the principal curvature $\alpha = g(A\xi,\xi)$ is constant along the direction of ξ_μ, $\mu = 1,2,3$.*

P r o o f. Replacing X by ξ in (2.2), we have

$$\begin{aligned}
0 &= (\hat{\mathfrak{L}}^{(k)}_{\xi_\mu} A)\xi \\
&= (\nabla_{\xi_\mu} A)\xi + g(\phi A\xi_\mu, A\xi)\xi - \alpha\eta(\xi)\phi A\xi_\mu - k\eta(\xi_\mu)\phi A\xi \\
&\quad - \alpha g(\phi A\xi_\mu,\xi)\xi + \eta(\xi)A\phi A\xi_\mu + k\eta(\xi_\mu)A\phi\xi \\
&\quad - \nabla_{A\xi}\xi_\mu - g(\phi A^2\xi,\xi_\mu)\xi + \eta(\xi_\mu)\phi A^2\xi + \alpha k\eta(\xi)\phi\xi_\mu \\
&\quad + A\nabla_\xi\xi_\mu + \alpha g(\phi A\xi,\xi_\mu)\xi - \eta(\xi_\mu)A\phi A\xi - k\eta(\xi)A\phi\xi_\mu.
\end{aligned}$$

Then using $A\xi = \alpha\xi$, we obtain

$$\begin{aligned}
0 =&(\nabla_{\xi_\mu} A)\xi \\
&- \alpha\phi A\xi_\mu + A\phi A\xi_\mu - \alpha\nabla_\xi\xi_\mu + \alpha k\phi\xi_\mu + A\nabla_\xi\xi_\mu - kA\phi\xi_\mu \\
=& - A\phi A\xi_\mu + (\xi_\mu\alpha)\xi + \alpha\phi A\xi_\mu \\
&- \alpha\phi A\xi_\mu + A\phi A\xi_\mu - \alpha\nabla_\xi\xi_\mu + \alpha k\phi\xi_\mu + A\nabla_\xi\xi_\mu - kA\phi\xi_\mu \\
=&(\xi_\mu\alpha)\xi - \alpha\nabla_\xi\xi_\mu + \alpha k\phi\xi_\mu + A\nabla_\xi\xi_\mu - kA\phi\xi_\mu.
\end{aligned}$$

Taking inner product with ξ, we get

$$\xi_\mu\alpha = 0$$

for $\mu = 1,2,3$. Thus we have our assertion. ∎

Now we introduce the lemma as follows:

Lemma 2.2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If M has the g-Tanaka–Webster $\mathfrak{D}^\perp$-invariant shape operator, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

P r o o f. We assume that

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{*}$$

for some unit vector field $X_0 \in \mathfrak{D}$, and $\eta(\xi_1)\eta(X_0) \neq 0$.
By Berdnt and Suh (see [3], p. 6), under the assumption that M is Hopf, we know

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y) \tag{2.3}$$

for any tangent vector field Y on M. Applying $Y = \xi_\mu$, $\mu = 1, 2, 3$ in (2.3), we get

$$\xi_\mu \alpha = (\xi\alpha)\eta(\xi_\mu) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi\xi_\mu)$$

Using Lemma 2.1 and (*), this equation can be reduced to

$$(\xi\alpha)\eta(\xi_\mu) - 4\eta_1(\xi)\eta_1(\phi\xi_\mu) = 0. \tag{2.4}$$

On the other hand, we obtain

$$\begin{aligned}\eta_1(\phi\xi_\mu) &= -g(\xi_\mu, \phi_1(\eta(X_0)X_0 + \eta(\xi_1)\xi_1)) \\ &= \eta(X_0)g(\phi_1\xi_\mu, X_0) \\ &= 0\end{aligned}$$

because of $X_0 \in \mathfrak{D}$. Therefore, we rewrite (2.4) in the form

$$(\xi\alpha)\eta(\xi_\mu) = 0 \quad \text{for} \quad \mu = 1, 2, 3,$$

that is, $\xi\alpha = 0$ or $\eta(\xi_\mu) = 0$ for $\mu = 1, 2, 3$.

Case I: $\eta(\xi_\mu) = 0$ for $\mu = 1, 2, 3$.
Since the assumptions of (*), $\eta(\xi_2) = 0$ and $\eta(\xi_3) = 0$ are obvious.

Case II: $\xi\alpha = 0$.
Substituting X_0 for Y in (2.3) and using (*), we have

$$X_0\alpha = -4\eta_1(\xi)\eta_1(\phi X_0) = 0.$$

Thus we obtain $X_0\alpha = 0$.

Subcase II-1: $\alpha = 0$.
Applying $\alpha = 0$ and (*) in (2.3), we get

$$-4\eta_1(\xi)\eta_1(\phi Y) = 0.$$

Since $\eta_1(\xi) \neq 0$, we obtain

$$\begin{aligned}0 &= \eta_1(\phi Y) \\ &= -g(Y, \phi_1(\eta(X_0)X_0 + \eta(\xi_1)\xi_1)) \\ &= -\eta(X_0)g(Y, \phi_1 X_0)\end{aligned}$$

for any tangent vector field Y on M. Because of $\eta(X_0) \neq 0$, we have $\phi_1 X_0 = 0$. It gives us a contradiction.

Subcase II-2 : $\alpha \neq 0$.
Using (1.9) and (2.2), we get

$$\begin{aligned}
0 &= (\hat{\mathfrak{L}}^{(k)}_{\xi_\mu} A)X \\
&= (\nabla_X A)\xi_\mu + \eta(\xi_\mu)\phi X - \eta(X)\phi\xi_\mu - 2g(\phi\xi_\mu, X)\xi \\
&\quad + \sum_{\nu=1}^{3} \Big\{\eta_\nu(\xi_\mu)\phi_\nu X - \eta_\nu(X)\phi_\nu\xi_\mu - 2g(\phi_\nu\xi_\mu, X)\xi_\nu\Big\} \\
&\quad + \sum_{\nu=1}^{3} \Big\{\eta_\nu(\phi\xi_\mu)\phi_\nu\phi X - \eta_\nu(\phi X)\phi_\nu\phi\xi_\mu\Big\} \\
&\quad + \sum_{\nu=1}^{3} \Big\{\eta(\xi_\mu)\eta_\nu(\phi X) - \eta(X)\eta_\nu(\phi\xi_\mu)\Big\}\xi_\nu \\
&\quad + g(\phi A\xi_\mu, AX)\xi - \alpha\eta(X)\phi A\xi_\mu - k\eta(\xi_\mu)\phi AX \\
&\quad - \alpha g(\phi A\xi_\mu, X)\xi + \eta(X)A\phi A\xi_\mu + k\eta(\xi_\mu)A\phi X \\
&\quad - \nabla_{AX}\xi_\mu - g(\phi A^2X, \xi_\mu)\xi + \eta(\xi_\mu)\phi A^2X + \alpha k\eta(X)\phi\xi_\mu \\
&\quad + A\nabla_X\xi_\mu + \alpha g(\phi AX, \xi_\mu)\xi - \eta(\xi_\mu)A\phi AX - k\eta(X)A\phi\xi_\mu
\end{aligned} \tag{2.5}$$

for any tangent vector field X on M.

In [8], Jeong, Machado, Perez and Suh introduced the following

Lemma A. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. If the principal curvature α is constant along the direction of ξ, then the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the structure vector field ξ is invariant by the shape operator.*

Since $\xi\alpha = 0$, the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the structure vector field ξ is invariant by the shape operator. Thus we write

$$\begin{aligned}
\alpha(\eta(X_0)X_0 + \eta(\xi_1)\xi_1) &= \alpha\xi \\
&= A\xi \\
&= \eta(X_0)AX_0 + \eta(\xi_1)A\xi_1.
\end{aligned}$$

Therefore, we get

$$AX_0 = \alpha X_0 \quad \text{and} \quad A\xi_1 = \alpha\xi_1. \tag{2.6}$$

Applying $X = X_0$ and $\mu = 1$ in (2.5), we have

$$\begin{aligned}
0 &= (\hat{\mathfrak{L}}^{(k)}_{\xi_1} A)X_0 \\
&= (\nabla_{X_0} A)\xi_1 + \eta(\xi_1)\phi X_0 - \eta(X_0)\phi\xi_1 - 2g(\phi\xi_1, X_0)\xi \\
&\quad + \sum_{\nu=1}^{3} \Big\{\eta_\nu(\xi_1)\phi_\nu X_0 - \eta_\nu(X_0)\phi_\nu\xi_1 - 2g(\phi_\nu\xi_1, X_0)\xi_\nu\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\nu=1}^{3} \Big\{ \eta_\nu(\phi\xi_1)\phi_\nu\phi X_0 - \eta_\nu(\phi X_0)\phi_\nu\phi\xi_1 \Big\} \\
&+ \sum_{\nu=1}^{3} \Big\{ \eta(\xi_1)\eta_\nu(\phi X_0) - \eta(X_0)\eta_\nu(\phi\xi_1) \Big\}\xi_\nu \\
&+ g(\phi A\xi_1, AX_0)\xi - \alpha\eta(X_0)\phi A\xi_1 - k\eta(\xi_1)\phi AX_0 \\
&- \alpha g(\phi A\xi_1, X_0)\xi + \eta(X_0)A\phi A\xi_1 + k\eta(\xi_1)A\phi X_0 \\
&- \nabla_{AX_0}\xi_1 - g(\phi A^2X_0, \xi_1)\xi + \eta(\xi_1)\phi A^2X_0 + \alpha k\eta(X_0)\phi\xi_1 \\
&+ A\nabla_{X_0}\xi_1 + \alpha g(\phi AX_0, \xi_1)\xi - \eta(\xi_1)A\phi AX_0 - k\eta(X_0)A\phi\xi_1.
\end{aligned}
$$

Since $g(\phi\xi_1, X_0) = 0$, $\eta_\nu(\phi\xi_1) = \eta_\nu(\phi X_0) = 0$ for $\nu = 1, 2, 3$ and $\phi\xi_1 = \eta(X_0)\phi_1X_0$, by using (2.6), the above equation can be reduced to

$$
\begin{aligned}
0 = &(\nabla_{X_0}A)\xi_1 + \eta(\xi_1)\phi X_0 - \eta^2(X_0)\phi_1X_0 + \phi_1X_0 \\
&+ \alpha^2 g(\phi\xi_1, X_0)\xi - \alpha^2\eta^2(X_0)\phi_1X_0 - \alpha k\eta(\xi_1)\phi X_0 \\
&- \alpha^2 g(\phi\xi_1, X_0)\xi + \alpha\eta^2(X_0)A\phi_1X_0 + k\eta(\xi_1)A\phi X_0 \\
&- \alpha\nabla_{X_0}\xi_1 - \alpha^2 g(\phi X_0, \xi_1)\xi + \alpha^2\eta(\xi_1)\phi X_0 + \alpha k\eta^2(X_0)\phi_1X_0 \\
&+ A\nabla_{X_0}\xi_1 + \alpha^2 g(\phi X_0, \xi_1)\xi - \alpha\eta(\xi_1)A\phi X_0 - k\eta^2(X_0)A\phi_1X_0.
\end{aligned}
$$

Using the assumption $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ such that $\eta(X_0)\eta(\xi_1) \neq 0$, we get $\phi X_0 = -\eta(\xi_1)\phi_1X_0$. Then we rewrite

$$
\begin{aligned}
0 = &(\nabla_{X_0}A)\xi_1 - \eta^2(\xi_1)\phi_1X_0 - \eta^2(X_0)\phi_1X_0 + \phi_1X_0 \\
&- \alpha^2\eta^2(X_0)\phi_1X_0 + \alpha k\eta^2(\xi_1)\phi_1X_0 \\
&+ \alpha\eta^2(X_0)A\phi_1X_0 - k\eta^2(\xi_1)A\phi_1X_0 \\
&- \alpha\nabla_{X_0}\xi_1 - \alpha^2\eta^2(\xi_1)\phi_1X_0 + \alpha k\eta^2(X_0)\phi_1X_0 \\
&+ A\nabla_{X_0}\xi_1 + \alpha\eta^2(\xi_1)A\phi_1X_0 - k\eta^2(X_0)A\phi_1X_0.
\end{aligned}
$$

Because of $\eta^2(X_0) + \eta^2(\xi_1) = 1$, we get

$$
\begin{aligned}
0 &= (\nabla_{X_0}A)\xi_1 - \alpha^2\phi_1X_0 + \alpha k\phi_1X_0 + (\alpha - k)A\phi_1X_0 \\
&\quad - \alpha\nabla_{X_0}\xi_1 + A\nabla_{X_0}\xi_1 \\
&= -\alpha(\alpha - k)\phi_1X_0 + (\alpha - k)A\phi_1X_0 \\
&= (\alpha - k)\Big\{ -\alpha + \frac{\alpha^2 + 4\eta^2(X_0)}{\alpha} \Big\}\phi_1X_0,
\end{aligned}
$$

where $A\phi_1X_0 = \dfrac{\alpha^2 + 4\eta^2(X_0)}{\alpha}\phi_1X_0$, due to Berndt and Suh [4].

Thus we have

$$
(\alpha - k)\frac{4\eta^2(X_0)}{\alpha}\phi_1X_0 = 0.
$$

Therefore we obtain

$$\alpha = k, \quad \text{where } k \text{ is a nonzero real number.} \tag{2.7}$$

Applying (2.7) in (2.3), we get

$$-4\eta_1(\xi)\eta_1(\phi Y) = 0$$

for any tangent vector field Y on M.

Then, by using the assumption $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ such that $\eta(\xi_1)\eta(X_0) \neq 0$, we write

$$\eta_1(\phi Y) = -g(\phi\xi_1, Y) = 0$$

for any tangent vector field Y on M. Thus we get

$$\phi\xi_1 = \eta(X_0)\phi_1 X_0 = 0,$$

that is, $\phi_1 X_0 = 0$. This gives a contradiction. Hence we complete the proof of this lemma. ∎

3. The Proof of the Main Theorem

From now on, let us assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with g-Tanaka–Webster $\mathfrak{D}^{\perp}$-invariant shape operator, that is $(\hat{\mathfrak{L}}^{(k)}_{\xi_\mu} A)X = 0$ for $\mu = 1, 2, 3$. Then, by Lemma 2.2, we consider the following two cases, that is, $\xi \in \mathfrak{D}^{\perp}$ or $\xi \in \mathfrak{D}$.

First, we consider the case $\xi \in \mathfrak{D}^{\perp}$. From this, without loss of generality, we may put $\xi = \xi_1$. By setting $\mu = 1$, we have

$$0 = (\hat{\mathfrak{L}}^{(k)}_{\xi_1} A)X = (\hat{\mathfrak{L}}^{(k)}_{\xi} A)X = (\hat{\nabla}^{(k)}_{\xi} A)X$$

for any tangent vector field X on M.

In [7], Jeong, Lee and Suh introduced the following:

Lemma B. *Let M be a Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with g-Tanaka-Webster $\mathfrak{D}^{\perp}$-parallel shape operator. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}^{\perp}$, then the shape operator A commutes with the structure tensor ϕ.*

Due to Berdnt and Suh [4], the Reeb flow on M is *isometric* if and only if the structure tensor field ϕ commutes with the shape operator A of M, that is, $A\phi = \phi A$. Thus, from Lemma B and Theorem B we have the following:

R e m a r k 3.1. Let M be a Hopf hypersurface, $\alpha \neq 2k$, in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with g-Tanaka–Webster $\mathfrak{D}^{\perp}$-invariant shape operator. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}^{\perp}$, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Then, by using Remark 3.1, we assume that M is a real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$. Then let us check whether the shape operator A of M is $\mathfrak{D}^{\perp}$-invariant in the g-Tanaka–Webster connection. In order to show this problem, we introduce a proposition due to Berndt and Suh [3] as follows:

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^{\perp}$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \operatorname{Span}\{\,\xi\,\} = \operatorname{Span}\{\,\xi_1\,\}, \\ T_\beta &= \mathbb{C}^{\perp}\xi = \mathbb{C}^{\perp}N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \operatorname{Span}\{\,\xi_2,\ \xi_3\,\}, \\ T_\lambda &= \{\,X \,|\, X \perp \mathbb{H}\xi,\ JX = J_1X\,\}, \\ T_\mu &= \{\,X \,|\, X \perp \mathbb{H}\xi,\ JX = -J_1X\,\}, \end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{H}\xi$ respectively denote real, complex and quaternionic spans of the structure vector field ξ, and $\mathbb{C}^{\perp}\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Case A : $\xi \in \mathfrak{D}^{\perp}$.
Applying $\mu = 2$ in (2.5), we get

$$\begin{aligned} 0 = {} & (\nabla_X A)\xi_2 + \eta(\xi_2)\phi X - \eta(X)\phi\xi_2 - 2g(\phi\xi_2, X)\xi \\ & + \sum_{\nu=1}^{3} \Big\{ \eta_\nu(\xi_2)\phi_\nu X - \eta_\nu(X)\phi_\nu\xi_2 - 2g(\phi_\nu\xi_2, X)\xi_\nu \Big\} \\ & + \sum_{\nu=1}^{3} \Big\{ \eta_\nu(\phi\xi_2)\phi_\nu\phi X - \eta_\nu(\phi X)\phi_\nu\phi\xi_2 \Big\} \\ & + \sum_{\nu=1}^{3} \Big\{ \eta(\xi_2)\eta_\nu(\phi X) - \eta(X)\eta_\nu(\phi\xi_2) \Big\}\xi_\nu \\ & + g(\phi A\xi_2, AX)\xi - \alpha\eta(X)\phi A\xi_2 - k\eta(\xi_2)\phi AX \\ & - \alpha g(\phi A\xi_2, X)\xi + \eta(X)A\phi A\xi_2 + k\eta(\xi_2)A\phi X \\ & - \nabla_{AX}\xi_2 - g(\phi A^2X, \xi_2)\xi + \eta(\xi_2)\phi A^2X + \alpha k\eta(X)\phi\xi_2 \\ & + A\nabla_X\xi_2 + \alpha g(\phi AX, \xi_2)\xi - \eta(\xi_2)A\phi AX - k\eta(X)A\phi\xi_2. \end{aligned}$$

By setting $X \in T_\lambda$ and $\xi = \xi_1 \in \mathfrak{D}^\perp$, we have

$$\begin{aligned} 0 = &(\nabla_X A)\xi_2 + \phi_2 X - \phi_3\phi X + \beta\lambda g(\phi\xi_2, X)\xi - \alpha\beta g(\phi\xi_2, X)\xi \\ &- \lambda\nabla_X\xi_2 - \lambda^2 g(\phi X, \xi_2)\xi + A\nabla_X\xi_2 + \alpha\lambda g(\phi X, \xi_2)\xi. \end{aligned}$$

Since $X \in T_\lambda$, $g(\phi X, \xi_2) = -g(X, \phi\xi_2) = 0$.
Using $(\nabla_X A)\xi_2 + A\nabla_X\xi_2 = \beta\nabla_X\xi_2$, we obatin

$$\begin{aligned} 0 &= (\beta - \lambda)\nabla_X\xi_2 \\ &= (\beta - \lambda)(q_1(X)\xi_3 - q_3(X)\xi_1 + \phi_2 AX). \end{aligned} \tag{3.1}$$

On the other hand, we know that

$$\begin{aligned} \phi AX &= \nabla_X\xi \\ &= \nabla_X\xi_1 \\ &= q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX. \end{aligned}$$

Taking inner product with ξ_2, we have

$$g(\phi AX, \xi_2) = q_3(X) + g(\phi_1 AX, \xi_2),$$

that is,

$$q_3(X) = 2\lambda g(X, \xi_3) = 0.$$

Because of $q_3(Y) = 0$, equation (3.1) reduces to

$$(\beta - \lambda)(q_1(X)\xi_3 + \lambda\phi_2 X) = 0. \tag{3.2}$$

Taking inner product with ξ_3 in (3.2), we rewrite

$$(\beta - \lambda)q_1(X) = 0.$$

Since $\beta - \lambda > 0$ by Proposition A, $q_1(X) = 0$. Consequently, from (3.2) we get

$$(\beta - \lambda)\lambda\phi_2 X = 0,$$

that is, $\phi_2 X = 0$. This gives a contradiction. So we give a proof of our main theorem for $\xi \in \mathfrak{D}^\perp$.

On the other hand, from Theorem C we have the following:

R e m a r k 3.2. Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with g-Tanaka–Webster $\mathfrak{D}^\perp$- invariant shape operator. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}$, then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

Now let us consider that M is a Hopf hypersurface of Type (B) in $G_2(\mathbb{C}^{m+2})$. Then, using Remark 3.2 and Proposition B due to Berndt and Suh [3], we can check whether the shape operator A of M satisfies $\mathfrak{D}^{\perp}$-invariant in the g-Tanaka–Webster connection. First of all, we introduce the proposition given by Berndt and Suh in [3] as follows:

Proposition B. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \mathrm{Span}\{\,\xi\,\},\\ &T_\beta = \mathfrak{J}J\xi = \mathrm{Span}\{\,\xi_\nu \mid \nu = 1,2,3\,\},\\ &T_\gamma = \mathfrak{J}\xi = \mathrm{Span}\{\,\phi_\nu\xi \mid \nu = 1,2,3\,\},\\ &T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^{\perp}, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^{\perp}$ is the orthogonal complement of $\mathbb{H}\mathbb{C}\xi$, where $\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi$.

Case B : $\xi \in \mathfrak{D}$.
Applying $\xi \in \mathfrak{D}$ in (2.5), we get

$$\begin{aligned} 0 &= (\hat{\mathfrak{L}}^{(k)}_{\xi_\mu} A)X \\ &= (\nabla_X A)\xi_\mu - \eta(X)\phi\xi_\mu - 2g(\phi\xi_\mu, X)\xi + \phi_\mu X \\ &\quad + \sum_{\nu=1}^{3} \Big\{ -\eta_\nu(X)\phi_\nu\xi_\mu - 2g(\phi_\nu\xi_\mu, X)\xi_\nu - \eta_\nu(\phi X)\phi_\nu\phi\xi_\mu \Big\} \\ &\quad + g(\phi A\xi_\mu, AX)\xi - \alpha\eta(X)\phi A\xi_\mu - \alpha g(\phi A\xi_\mu, X)\xi + \eta(X)A\phi A\xi_\mu \\ &\quad - \nabla_{AX}\xi_\mu - g(\phi A^2X, \xi_\mu)\xi + \alpha k\eta(X)\phi\xi_\mu \\ &\quad + A\nabla_X\xi_\mu + \alpha g(\phi AX, \xi_\mu)\xi - k\eta(X)A\phi\xi_\mu \end{aligned} \tag{3.3}$$

for any tangent vector field X on M.

Case B-I: $X = \xi \in T_\alpha$.
By putting $X = \xi$ in (3.3), we have

$$\begin{aligned}0 &= (\nabla_\xi A)\xi_\mu - \phi\xi_\mu + \phi_\mu\xi - \alpha\phi A\xi_\mu + A\phi A\xi_\mu \\ &\quad - \nabla_{A\xi}\xi_\mu + \alpha k\phi\xi_\mu + A\nabla_\xi\xi_\mu - kA\phi\xi_\mu.\end{aligned}$$

Using $A\xi = \alpha\xi$, $A\xi_\mu = \beta\xi_\mu$ and $A\phi\xi_\mu = \gamma\phi\xi_\mu = 0$, it can be reduced to

$$(\nabla_\xi A)\xi_\mu - \alpha\beta\phi\xi_\mu - \alpha\nabla_\xi\xi_\mu + \alpha k\phi\xi_\mu + A\nabla_\xi\xi_\mu = 0.$$

Since $(\nabla_\xi A)\xi_\mu + A\nabla_\xi\xi_\mu = \beta\nabla_\xi\xi_\mu$ and $\nabla_\xi\xi_\mu = q_{\mu+2}(\xi)\xi_{\mu+1} - q_{\mu+1}(\xi)\xi_{\mu+2} + \phi_\mu A\xi$, we rewrite

$$(\beta-\alpha)\Big\{q_{\mu+2}(\xi)\xi_{\mu+1} - q_{\mu+1}(\xi)\xi_{\mu+2}\Big\} + \alpha(k-\alpha)\phi_\mu\xi = 0.$$

Consequently, we get

$$(\beta-\alpha)q_{\mu+1}(\xi) = 0,\ (\beta-\alpha)q_{\mu+2}(\xi) = 0 \ \text{ and } \ \alpha(k-\alpha) = 0.$$

From constant principal curvatures of Proposition B, that is, $\beta - \alpha > 0$ and $\alpha < 0$, we obtain

$$q_{\mu+1}(\xi) = 0, \quad q_{\mu+2}(\xi) = 0 \text{ and } \alpha = k,$$

that is, $\alpha = k$ and $q_i(\xi) = 0,\ i = 1,2,3$.

Case B-II: $X \in T_\beta$, where $T_\beta = \text{Span}\{\,\xi_i \mid i = 1,2,3\,\}$.
By setting $X = \xi_i,\ i = 1,2,3$ in (3.3), we have

$$\begin{aligned}0 &= (\nabla_{\xi_i} A)\xi_\mu - \eta(\xi_i)\phi\xi_\mu - 2g(\phi\xi_\mu,\xi_i)\xi + \phi_\mu\xi_i \\ &\quad + \sum_{\nu=1}^{3}\Big\{-\eta_\nu(\xi_i)\phi_\nu\xi_\mu - 2g(\phi_\nu\xi_\mu,\xi_i)\xi_\nu - \eta_\nu(\phi\xi_i)\phi_\nu\phi\xi_\mu\Big\} \\ &\quad + g(\phi A\xi_\mu, A\xi_i)\xi - \alpha\eta(\xi_i)\phi A\xi_\mu - \alpha g(\phi A\xi_\mu,\xi_i)\xi + \eta(\xi_i)A\phi A\xi_\mu \\ &\quad - \beta\nabla_{\xi_i}\xi_\mu - g(\phi A^2\xi_i,\xi_\mu)\xi + \alpha k\eta(\xi_i)\phi\xi_\mu \\ &\quad + A\nabla_{\xi_i}\xi_\mu + \alpha g(\phi A\xi_i,\xi_\mu)\xi - k\eta(\xi_i)A\phi\xi_\mu \\ &= (\nabla_{\xi_i} A)\xi_\mu + \phi_\mu\xi_i + \sum_{\nu=1}^{3}\Big\{-\eta_\nu(\xi_i)\phi_\nu\xi_\mu - 2g(\phi_\nu\xi_\mu,\xi_i)\xi_\nu\Big\} \\ &\quad - \beta\nabla_{\xi_i}\xi_\mu + A\nabla_{\xi_i}\xi_\mu.\end{aligned}$$

Since $(\nabla_{\xi_i} A)\xi_\mu + A\nabla_{\xi_i}\xi_\mu = \beta\nabla_{\xi_i}\xi_\mu$, it can be reduced to

$$\phi_\mu\xi_i + \sum_{\nu=1}^{3}\Big\{-\eta_\nu(\xi_i)\phi_\nu\xi_\mu - 2g(\phi_\nu\xi_\mu,\xi_i)\xi_\nu\Big\} = 0. \tag{3.4}$$

Subcase II-1 : $i = \mu$ in (3.4).

$$\phi_\mu \xi_\mu + \sum_{\nu=1}^{3} \Big\{ -\eta_\nu(\xi_\mu)\phi_\nu\xi_\mu - 2g(\phi_\nu\xi_\mu, \xi_\mu)\xi_\nu \Big\} = 0.$$

Subcase II-2 : $i = \mu + 1$ in (3.4).

$$\begin{aligned} \phi_\mu \xi_{\mu+1} + \sum_{\nu=1}^{3} \Big\{ -\eta_\nu(\xi_{\mu+1})\phi_\nu\xi_\mu - 2g(\phi_\nu\xi_\mu, \xi_{\mu+1})\xi_\nu \Big\} \\ = \xi_{\mu+2} - \phi_{\mu+1}\xi_\mu - 2\xi_{\mu+2} \\ = 0. \end{aligned}$$

Subcase II-3 : $i = \mu + 2$ in (3.4).

$$\begin{aligned} \phi_\mu \xi_{\mu+2} + \sum_{\nu=1}^{3} \Big\{ -\eta_\nu(\xi_{\mu+2})\phi_\nu\xi_\mu - 2g(\phi_\nu\xi_\mu, \xi_{\mu+2})\xi_\nu \Big\} \\ = -\xi_{\mu+1} - \phi_{\mu+2}\xi_\mu + 2\xi_{\mu+1} \\ = 0. \end{aligned}$$

Summing up the above three subcases, we note that the shape operator A of M is $\mathfrak{D}^\perp$-invariant on T_β in the g-Tanaka–Webster connection.

Case B-III : $X \in T_\gamma$, where $T_\gamma = \mathrm{Span}\{ \phi_i\xi \mid i = 1, 2, 3 \}$.
By putting $X = \phi_i\xi$ in (3.3), we have

$$\begin{aligned} 0 = & (\nabla_{\phi_i\xi}A)\xi_\mu - \eta(\phi_i\xi)\phi\xi_\mu - 2g(\phi\xi_\mu, \phi_i\xi)\xi + \phi_\mu\phi_i\xi \\ & + \sum_{\nu=1}^{3} \Big\{ -\eta_\nu(\phi_i\xi)\phi_\nu\xi_\mu - 2g(\phi_\nu\xi_\mu, \phi_i\xi)\xi_\nu - \eta_\nu(\phi\phi_i\xi)\phi_\nu\phi\xi_\mu \Big\} \\ & + g(\phi A\xi_\mu, A\phi_i\xi)\xi - \alpha\eta(\phi_i\xi)\phi A\xi_\mu - \alpha g(\phi A\xi_\mu, \phi_i\xi)\xi + \eta(\phi_i\xi)A\phi A\xi_\mu \\ & - \nabla_{A\phi_i\xi}\xi_\mu - g(\phi A^2\phi_i\xi, \xi_\mu)\xi + \alpha k\eta(\phi_i\xi)\phi\xi_\mu \\ & + A\nabla_{\phi_i\xi}\xi_\mu + \alpha g(\phi A\phi_i\xi, \xi_\mu)\xi - k\eta(\phi_i\xi)A\phi\xi_\mu. \end{aligned}$$

Since $\gamma = 0$, $(\nabla_{\phi_i\xi}A)\xi_\mu + A\nabla_{\phi_i\xi}\xi_\mu = \beta\nabla_{\phi_i\xi}\xi_\mu$ and $\nabla_{\phi_i\xi}\xi_\mu = q_{\mu+2}(\phi_i\xi)\xi_{\mu+1} - q_{\mu+1}(\phi_i\xi)\xi_{\mu+2} + \phi_\mu A\phi_i\xi$, this equation reduces to

$$\begin{aligned} \beta\Big\{q_{\mu+2}(\phi_i\xi)\xi_{\mu+1} - q_{\mu+1}(\phi_i\xi)\xi_{\mu+2}\Big\} - 2g(\phi\xi_\mu, \phi_i\xi)\xi \\ + \phi_\mu\phi_i\xi - \sum_{\nu=1}^{3} \eta_\nu(\phi\phi_i\xi)\phi_\nu\phi\xi_\mu - \alpha\beta g(\phi\xi_\mu, \phi_i\xi)\xi = 0. \end{aligned} \tag{3.5}$$

Subcase III-1 : $i = \mu$ in (3.5).

$$\begin{aligned}&\beta q_{\mu+2}(\phi_\mu\xi)\xi_{\mu+1} - \beta q_{\mu+1}(\phi_\mu\xi)\xi_{\mu+2} - 2\xi + \phi_\mu^2\xi + \phi_\mu^2\xi - \alpha\beta\xi\\ &= \beta q_{\mu+2}(\phi_\mu\xi)\xi_{\mu+1} - \beta q_{\mu+1}(\phi_\mu\xi)\xi_{\mu+2} - (\alpha\beta+4)\xi = 0.\end{aligned}$$

Since $\beta > 0$ and $\alpha\beta + 4 = 0$, we have

$$q_{\mu+1}(\phi_\mu\xi) = 0 \quad \text{and} \quad q_{\mu+2}(\phi_\mu\xi) = 0,\ \mu = 1,2,3.$$

Subcase III-2 : $i = \mu + 1$ in (3.5).

$$\begin{aligned}&\beta q_{\mu+2}(\phi_{\mu+1}\xi)\xi_{\mu+1} - \beta q_{\mu+1}(\phi_{\mu+1}\xi)\xi_{\mu+2} + \phi_\mu\phi_{\mu+1}\xi + \phi_{\mu+1}\phi_\mu\xi\\ &= \beta q_{\mu+2}(\phi_{\mu+1}\xi)\xi_{\mu+1} - \beta q_{\mu+1}(\phi_{\mu+1}\xi)\xi_{\mu+2} = 0,\end{aligned}$$

because of $\phi_\mu\phi_{\mu+1}\xi = \phi_{\mu+2}\xi + \eta_{\mu+1}(\xi)\xi_\mu$ and $\phi_{\mu+1}\phi_\mu\xi = -\phi_{\mu+2}\xi + \eta_\mu(\xi)\xi_{\mu+1}$. Since $\beta > 0$, we obtain

$$q_{\mu+1}(\phi_{\mu+1}\xi) = 0 \quad \text{and} \quad q_{\mu+2}(\phi_{\mu+1}\xi) = 0,\ \mu = 1,2,3.$$

Subcase III-3 : $i = \mu + 2$ in (3.5).

$$\begin{aligned}&\beta q_{\mu+2}(\phi_{\mu+2}\xi)\xi_{\mu+1} - \beta q_{\mu+1}(\phi_{\mu+2}\xi)\xi_{\mu+2} + \phi_\mu\phi_{\mu+2}\xi + \phi_{\mu+2}\phi_\mu\xi\\ &= \beta q_{\mu+2}(\phi_{\mu+2}\xi)\xi_{\mu+1} - \beta q_{\mu+1}(\phi_{\mu+2}\xi)\xi_{\mu+2} = 0.\end{aligned}$$

Since $\beta > 0$, we rewrite

$$q_{\mu+1}(\phi_{\mu+2}\xi) = 0 \quad \text{and} \quad q_{\mu+2}(\phi_{\mu+2}\xi) = 0,\ \mu = 1,2,3.$$

From the above three subcases, we get $q_i(X) = 0,\ i = 1,2,3$ for any tangent vector field $X \in T_\gamma$.

Case B-IV : $X \in T_\lambda$.
By putting $X \in T_\lambda$ in (3.3), we have

$$\begin{aligned}0 &= (\nabla_X A)\xi_\mu + \phi_\mu X - \lambda\nabla_X\xi_\mu + A\nabla_X\xi_\mu\\ &= \beta\nabla_X\xi_\mu + \phi_\mu X - \lambda\nabla_X\xi_\mu\\ &= (\beta-\lambda)\Big\{q_{\mu+2}(X)\xi_{\mu+1} - q_{\mu+1}(X)\xi_{\mu+2} + \phi_\mu AX\Big\} + \phi_\mu X\\ &= (\beta-\lambda)q_{\mu+2}(X)\xi_{\mu+1} - (\beta-\lambda)q_{\mu+1}(X)\xi_{\mu+2} - (\lambda^2 - \beta\lambda - 1)\phi_\mu X.\end{aligned}$$

Since $\beta - \lambda = 2\cot(2r) - \cot(r) = -\tan(r) = \mu < 0$ with some $r \in (0, \frac{\pi}{4})$ and $\lambda^2 - \beta\lambda - 1 = 0$, we obtain

$$q_{\mu+1}(X) = 0 \quad \text{and} \quad q_{\mu+2}(X) = 0,\ \mu = 1,2,3,$$

that is, $q_i(X) = 0,\ i = 1,2,3$ for any tangent vector field $X \in T_\lambda$.

Case B-V : $X \in T_\mu$.
By setting $X \in T_\mu$ in (3.3), we get

$$\begin{aligned} 0 &= (\nabla_X A)\xi_\mu + \phi_\mu X - \mu \nabla_X \xi_\mu + A \nabla_X \xi_\mu \\ &= \beta \nabla_X \xi_\mu + \phi_\mu X - \mu \nabla_X \xi_\mu \\ &= (\beta - \mu) q_{\mu+2}(X) \xi_{\mu+1} - (\beta - \mu) q_{\mu+1}(X) \xi_{\mu+2} - (\mu^2 - \beta\mu - 1)\phi_\mu X. \end{aligned}$$

Since $\beta - \mu = \lambda = \cot(r) > 0$ with some $r \in (0, \frac{\pi}{4})$ and $\mu^2 - \beta\mu - 1 = 0$, we have

$$q_{\mu+1}(X) = 0 \quad \text{and} \quad q_{\mu+2}(X) = 0, \ \mu = 1, 2, 3,$$

that is, $q_i(X) = 0, \ i = 1, 2, 3$ for any tangent vector field $X \in T_\mu$.

Hence, summing up all the cases mentioned above, we give a complete proof of our Main Theorem in Introduction.

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