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Analogs of Generalized Resolvents for Relations Generated by a Pair of Differential Operator Expressions One of which Depends on Spectral Parameter in Nonlinear Manner

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For the relations generated by a pair of differential operator expressions one of which depends on the spectral parameter in the Nevanlinna manner we construct the analogs of the generalized resolvents which are integro-differential operators.

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Dedicated to Academician Vladimir Aleksandrovich Marchenko on the occasion of his jubilee

Introduction

We consider either on a finite or an infinite interval the operator differential equation of arbitrary order

$$l_\lambda[y] = m[f],\ t \in \bar{\mathcal{I}},\ \mathcal{I} = (a, b) \subseteq \mathbb{R}^1 \tag{1}$$

in the space of the vector-functions with values in the separable Hilbert space $\mathcal{H}$, where

$$l_\lambda[y] = l[y] - \lambda m[y] - n_\lambda[y], \tag{2}$$

$l[y], m[y]$ are symmetric operator differential expressions. The order of $l_\lambda[y]$ is equal to $r > 0$. For the expression $m[y]$, the subintegral quadratic form $m\{y, y\}$ of its Dirichlet integral $m[y, y] = \int_{\mathcal{I}} m\{y, y\}dt$ is nonnegative for $t \in \bar{\mathcal{I}}$. The leading coefficient of the expression $m[y]$ may not have the inverse one from $B(\mathcal{H})$ for any $t \in \bar{\mathcal{I}}$ and it may even vanish on some intervals. For the operator differential expression $n_\lambda[y]$, the form $n_\lambda\{y, y\}$ depends on λ in the Nevanlinna manner for $t \in \bar{\mathcal{I}}$. Therefore the order $s \geq 0$ of $m[y]$ is even and $\leq r$.

In the Hilbert space $L^2_m(\mathcal{I})$ with metrics generated by the form $m[y, y]$, for equation (1)-(2) we construct the analogs $R(\lambda)$ of the generalized resolvents which in general are non-injective and which possess the following representation:

$$R(\lambda) = \int\limits_{\mathbb{R}^1} \frac{dE_\mu}{\mu - \lambda}, \tag{3}$$

where E_μ is a generalized spectral family for which E_∞ is less or equal to the identity operator. (Abstract operators possessing this representation were studied in [16].)

The construction is based on a special reduction of the equation

$$l[y] = m[f] \tag{4}$$

to the first order system with weight. Here l and m are the operator differential expressions which are not necessarily symmetric (in contrast to (2)). For the construction of $R(\lambda)$ we also introduce the characteristic operator of the equation

$$l_\lambda[y] = -\frac{(\Im l_\lambda)[f]}{\Im\lambda},\ t \in \bar{\mathcal{I}}, \tag{5}$$

where $(\Im l_\lambda)[f] = \frac{1}{2i}(l[f] - l^*[f])$.

In the case $r = 1$, $n_\lambda[y] = H_\lambda(t)y$ (here the mentioned reduction is not needed), the resolvents $R(\lambda)$ were constructed in [21].

Further we consider the boundary value problem obtained by adding to equation (1)–(2) the dissipative boundary conditions depending on a spectral parameter. We prove that for some boundary conditions, the solutions of these problems are generated by the operators $R(\lambda)$ if, in contrast to the case $s = 0$, $n_\lambda[y] = H_\lambda(t)y$, the boundary conditions contain the derivatives of the vector-function $f(t)$ that are taken at the ends of the interval.

In the case $n_\lambda[y] \equiv 0$, the results listed above are known [24], and $R(\lambda)$ is the generalized resolvent of the minimal relation generated by the pair of expressions $l[y]$ and $m[y]$. For this case, we show that in the regular case the set of all generalized resolvents coincide with the set of all operators $R(\lambda)$, and thereby by virtue of Theorem 3.2 their full description is given with the help of the boundary conditions. A review of other results for the case $n_\lambda[y] \equiv 0$ is contained in [23].

In [9] and [10], the conditions for holomorphy and continuous reversibility of the restrictions of maximal relations generated by $l_\lambda[y]$ (2) with $m[y] \equiv 0$, $n_\lambda[y] = H_\lambda(t)y$ in $L^2_{\Im H_{\lambda_0}(t)/\Im\lambda_0}$ ($\Im\lambda_0 \neq 0$) and also by the integral equation with the Nevanlinna matrix measure were studied (using some of the results from [22]). We remark that the relations inverse to those considered in [9, 10] do not possess representation (3). Also we note that the resolvent equation (1)–(2) is not reduced to the equations considered in [9, 10].

Many questions concerning differential operators and relations in the space of vector-functions are considered in the monographs [2, 4, 5, 18, 27, 28, 34, 35] containing an extensive literature review. The method of studying these operators and relations, based on the use of the abstract Weyl function and its generalization (Weyl family), was proposed in [12–14].

A preliminary version of the results obtained in this paper is contained in preprint [25]. The expansion formulae for homogeneous equation (1) will be obtained in our next paper.

We denote by (.) and $\|\cdot\|$ the scalar product and the norm in various spaces with special indices if it is necessary. For the differential operation l, we denote $\Re l = \frac{1}{2}(l + l^*)$, $\Im l = \frac{1}{2i}(l - l^*)$.

Let an interval $\Delta \subseteq \mathbb{R}^1$, $f(t)$ $(t \in \Delta)$ be a function with values in some Banach space B. The notation $f(t) \in C^k(\Delta, B)$, $k = 0, 1, \ldots$ (we omit the index k if $k = 0$) means that at any point of Δ, $f(t)$ has continuous in the norm $\|\cdot\|_B$ derivatives of order up to and including l that are taken in the norm $\|\cdot\|_B$; if Δ is either semi-open or closed interval, then on its ends belonging to Δ there exist one-sided continuous derivatives. The notation $f(t) \in C_0^k(\Delta, B)$ means that $f(t) \in C^k(\Delta, B)$ and $f(t) = 0$ in the neighbourhoods of the ends of Δ.

1. The Reduction of Equation (4) to the First Order System of Canonical Type with Weight. The Green Formula

In the separable Hilbert space $\mathcal{H}$ we consider Eq. (4), where $l[y]$ and $m[f]$ are differential expressions (that are not necessarily symmetric) with sufficiently smooth coefficients from $B(\mathcal{H})$ and of orders $r > 0$ and s, respectively. Here $r \geq s \geq 0$, s is even and these expressions are presented in the divergent form. Namely,

$$l[y] = \sum_{k=0}^{r} i^k l_k[y], \tag{6}$$

where $l_{2j} = D^j p_j(t) D^j$, $l_{2j-1} = \frac{1}{2} D^{j-1}\{D q_j(t) + s_j(t) D\} D^{j-1}$, $p_j(t)$, $q_j(t)$, $s_j(t) \in C^j(\bar{\mathcal{I}}, B(\mathcal{H}))$, $D = d/dt$; $m[f]$ is defined in a similar way with s instead of r, and $\tilde{p}_j(t)$, $\tilde{q}_j(t)$, $\tilde{s}_j(t) \in B(\mathcal{H})$ instead of $p_j(t)$, $q_j(t)$, $s_j(t)$.

In the case of even $r = 2n \geq s$, $p_n^{-1} \in B(\mathcal{H})$, we denote

$$Q(t,l) = \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix} = J/i, \;\; S(t,l) = Q(t,l), \tag{7}$$

$$H(t,\, l) = \|h_{\alpha\beta}\|_{\alpha,\,\beta=1}^2\,, \;\; h_{\alpha\beta} \in B(\mathcal{H}^n)\,, \tag{8}$$

where I_n is the identity operator in $\mathcal{H}^n$; h_{11} is a three-diagonal operator matrix whose elements under the main diagonal are equal to $\left(\frac{i}{2}q_1,\, \ldots,\, \frac{i}{2}q_{n-1}\right)$, the elements over the main diagonal are equal to $\left(-\frac{i}{2}s_1,\, \ldots,\, -\frac{i}{2}s_{n-1}\right)$, the elements on the main diagonal are equal to $\left(-p_0,\, \ldots,\, -p_{n-2},\, \frac{1}{4}s_n p_n^{-1} q_n - p_{n-1}\right)$; h_{12} is an operator matrix with the identity operators I_1 under the main diagonal, the elements on the main diagonal are equal to $\left(0,\, \ldots,\, 0,\, -\frac{i}{2}s_n p_n^{-1}\right)$, all the rest elements are equal to zero; h_{21} is an operator matrix with identity operators I_1 over the main diagonal, the elements on the main diagonal are equal to $\left(0,\, \ldots,\, 0,\, \frac{i}{2}p_n^{-1}q_n\right)$, all the rest elements are equal to zero; $h_{22} = \operatorname{diag}\left(0,\, \ldots,\, 0,\, p_n^{-1}\right)$.

In this case, we also denote*

$$W(t,\, l,\, m) = C^{*-1}(t,l)\left\{\|m_{\alpha\beta}\|_{\alpha,\,\beta=1}^2\right\} C^{-1}(t,l)\,, m_{\alpha\beta} \in B(\mathcal{H}^n)\,, \tag{9}$$

where m_{11} is a three-diagonal operator matrix whose elements under the main diagonal are equal to $\left(-\frac{i}{2}\tilde{q}_1, \ldots, -\frac{i}{2}\tilde{q}_{n-1}\right)$, the elements over the main diagonal are equal to $\left(\frac{i}{2}\tilde{s}_1, \ldots, \frac{i}{2}\tilde{s}_{n-1}\right)$, the elements on the main diagonal are equal to $(\tilde{p}_0, \ldots, \tilde{p}_{n-1})$; $m_{12} = \operatorname{diag}\left(0, \ldots, 0, \frac{i}{2}\tilde{s}_n\right)$, $m_{21} = \operatorname{diag}\left(0, \ldots, 0, -\frac{i}{2}\tilde{q}_n\right)$, $m_{22} = \operatorname{diag}(0, \ldots, 0, \tilde{p}_n)$.

The operator matrix $C(t,l)$ is defined by the condition

$$\begin{aligned} &C(t,l)\, col\left\{f(t)\,,\, f'(t)\,, \ldots,\, f^{(n-1)}(t)\,,\, f^{(2n-1)}(t)\,, \ldots,\, f^{(n)}(t)\right\} \\ &= col\left\{f^{[0]}(t|l)\,,\, f^{[1]}(t|l)\,, \ldots,\, f^{[n-1]}(t|l)\,,\, f^{[2n-1]}(t\,|l)\,, \ldots,\, f^{[n]}(t\,|l)\right\}, \end{aligned} \tag{10}$$

where $f^{[k]}(t\,|L)$ are quasi-derivatives of the vector-function $f(t)$ that correspond to the differential expression L.

The quasi-derivatives corresponding to l are equal (cf. [33]) to

$$y^{[j]}(t\,|l) = y^{(j)}(t)\,,\;\; j = 0,\, \ldots,\, \left[\frac{r}{2}\right] - 1, \tag{11}$$

* $W(t,l,m)$ is given for the case $s = 2n$. If $s < 2n$, one has to set the corresponding elements of the operator matrices $m_{\alpha\beta}$ to be equal to zero. In particular, if $s < 2n$, then $m_{12} = m_{21} = m_{22} = 0$, and therefore $W(t,l,m) = \operatorname{diag}(m_{11}, 0)$ in view of (14).

$$y^{[n]}\left(t\,|l\,\right)=\left\{\begin{array}{l} p_n y^{(n)}-\frac{i}{2}q_n y^{(n-1)},\ r=2n \\ -\frac{i}{2}q_{n+1}y^{(n)},\ r=2n+1 \end{array}\right. , \tag{12}$$

$$y^{[r-j]}\left(t\,|l\,\right)=-Dy^{[r-j-1]}\left(t\,|l\,\right)+p_j y^{(j)}+\frac{i}{2}\left[s_{j+1}y^{(j+1)}-q_j y^{(j-1)}\right], \tag{13}$$

$$j=0,\,\ldots,\,\left[\frac{r-1}{2}\right],\ q_0\equiv 0.$$

Then, $l\left[y\right]=y^{[r]}\left(t\,|l\,\right)$. The quasi-derivatives $y^{[k]}\left(t\,|m\,\right)$ corresponding to m are defined in the same way with even s instead of r, and $\tilde{p}_j,\tilde{q}_j,\tilde{s}_j$ instead of p_j,q_j,s_j.

It is easy to see that

$$C\left(t,l\right)=\left(\begin{array}{cc} I_n & 0 \\ C_{21} & C_{22} \end{array}\right),\ \ C_{\alpha\beta}\in B\left(\mathcal{H}^n\right), \tag{14}$$

where C_{21}, C_{22} are the upper triangular operator matrices with diagonal elements $\left(-\frac{i}{2}q_1,\,\ldots,\,-\frac{i}{2}q_n\right)$ and $\left((-1)^{n-1}p_n,\,(-1)^{n-2}p_n,\,\ldots,\,p_n\right)$, respectively.

In the case of odd $r=2n+1>s$, we denote

$$Q\left(t,l\right)=\begin{cases} J/i\oplus q_{n+1}, \\ q_1, \end{cases}\qquad S\left(t,l\right)=\begin{cases} J/i\oplus s_{n+1}, & n>0, \\ s_1, & n=0, \end{cases} \tag{15}$$

$$H\left(t,\,l\right)=\begin{cases} \left\|h_{\alpha\,\beta}\right\|^2_{\alpha,\,\beta=1}, & n>0, \\ p_0, & n=0, \end{cases} \tag{16}$$

where $B\left(\mathcal{H}^n\right)\ni h_{11}$ is a three-diagonal operator matrix whose elements under the main diagonal are equal to $\left(\frac{i}{2}q_1,\,\ldots,\,\frac{i}{2}q_{n-1}\right)$, the elements over the main diagonal are equal to $\left(-\frac{i}{2}s_1,\,\ldots,\,-\frac{i}{2}s_{n-1}\right)$, the elements on the main diagonal are equal to $(-p_0,\,\ldots,\,-p_{n-1})$, all the rest of the elements are equal to zero. $B\left(\mathcal{H}^{n+1},\,\mathcal{H}^n\right)\ni h_{12}$ is an operator matrix whose elements with numbers $j,\,j-1$ are equal to $I_1,\ j=2,\,\ldots,\,n$, the element with number $n,\,n+1$ is equal to $\frac{1}{2}s_n$, all the rest of the elements are equal to zero. $B\left(\mathcal{H}^n,\,\mathcal{H}^{n+1}\right)\ni h_{21}$ is an operator matrix whose elements with numbers $j-1,\,j$ are equal to $I_1,\ j=2,\,\ldots,\,n$, the element with number $n+1,\,n$ is equal to $\frac{1}{2}q_n$, all the rest elements are equal to zero. $B\left(\mathcal{H}^{n+1}\right)\ni h_{22}$ is an operator matrix whose last row is equal to $(0,\,\ldots,\,0,\,-iI_1,\,-p_n)$, last column is equal to $col\ (0,\,\ldots,\,0,\,iI_1,\,-p_n)$, all the rest elements are equal to zero.

In this case, we also denote*

$$W\left(t,\,l,\,m\right)=\left\|m_{\alpha\beta}\right\|^2_{\alpha,\,\beta=1}, \tag{17}$$

where m_{11} is defined in the same way as m_{11} (9). $B\left(\mathcal{H}^{n+1},\,\mathcal{H}^n\right)\ni m_{12}$ is an operator matrix whose elements with number $n,\,n+1$ are equal to $-\frac{1}{2}\tilde{s}_n$, all

* See the previous footnote

the rest elements are equal to zero. $B\left(\mathcal{H}^n,\,\mathcal{H}^{n+1}\right) \ni m_{21}$ is an operator matrix whose elements with number $n+1$, n are equal to $-\frac{1}{2}\tilde{q}_n$, all the rest elements are equal to zero. $B\left(\mathcal{H}^{n+1}\right) \ni m_{22} = \operatorname{diag}(0,\,\ldots,\,0,\,\tilde{p}_n)$.

Obviously, for $H\,(t,l)$ (8), (16) and $W\,(t,l,m)$ (9), (17) one has

$$H^*\,(t,l) = H\,(t,l^*)\,, W^*\,(t,l,\,m) = W\,(t,l,\,m^*)\,. \tag{18}$$

Lemma 1.1. *Let the order of $\Im l$ be even. Then*

$$\Im H\,(t,l) = W\,(t,l,\,-\Im l) = W\,(t,l^*,\,-\Im l)\,. \tag{19}$$

P r o o f. Let us prove the first equality in (19) for even $r = 2n$. Let us represent $H\,(t,l)$ (8) in the form

$$H\,(t,l) = A\,(t,l) + B\,(t,l)\,, \tag{20}$$

where $A(t,l) = H(t,l) - B(t,l)$ and

$$B\,(t,l) = \|B_{jk}\|^2_{j,\,k=1}\,,\;\; B_{jk} \in B\,(\mathcal{H}^n)\,, \tag{21}$$

$$B_{11} = \operatorname{diag}\left(0,\,\ldots,\,0,\,s_n p_n^{-1} q_n/4\right),\;\; B_{12} = \operatorname{diag}\left(0,\,\ldots,\,0,\,-is_n p_n^{-1}/2\right), \tag{22}$$

$$B_{21} = \operatorname{diag}\left(0,\,\ldots,\,0,\,ip_n^{-1} q_n/2\right),\;\; B_{22} = \operatorname{diag}\left(0,\,\ldots,\,0,\,p_n^{-1}\right). \tag{23}$$

In view of (14), (21)–(23), one has

$$B\,(t,l)\,C\,(t,l) = \|u_{jk}\|^{2n}_{j,\,k=1}\,,\;\; u_{jk} \in B\,(\mathcal{H})\,, \tag{24}$$

$u_{n2n} = -is_n/2$, $u_{2n\;2n} = I_1$, the rest $u_{jk} = 0$.

Hence

$$C^*\,(t,l)\,B\,(t,l)\,C\,(t,l) = \|v_{jk}\|^{2n}_{j,k=1}\,,\;\; v_{jk} \in B\,(\mathcal{H})\,,$$

$v_{n\;2n} = -\frac{i}{2}\,(s_n - q_n^*)$, $v_{2n\;2n} = p_n^*$, rest $v_{jk} = 0$.

Thus, $C^*\,(t,l)\,\Im H\,(t,l)\,C\,(t,l) = C^*\,(t,l)\,W\,(t,l,-\Im l)\,C\,(t,l)$ in view of (8), (9), (10), (20) and the divergent form of the expression $-\Im l$ that follows from (6). The first equality in (19) for even r is proved. Its proof for odd r follows from (16), (17).

From the proof one can see that

$$W\,(t,l,\Im l) = -\Im H\,(t,l)\,. \tag{25}$$

The second equality in (19) is a corollary of (25) and (18). Lemma 1.1 is proved. ■

For the sufficiently smooth vector-function $f(t)$ we will denote (if $f(t)$ has a subscript, then the same subscript should be added to F)

$$\mathcal{H}^r \ni F(t,\, l, m)$$
$$= \begin{cases} \left(\sum\limits_{j=0}^{s/2} \oplus f^{(j)}(t)\right) \oplus 0 \oplus \ldots \oplus 0, \quad r = 2n, & r = 2n+1 > 1, s < 2n, \\ \left(\sum\limits_{j=0}^{n-1} \oplus f^{(j)}(t)\right) \oplus 0 \oplus \ldots \oplus 0 \oplus \left(-if^{(n)}(t)\right), & r = 2n+1 > 1, s = 2n, \\ f(t), & r = 1, \\ \left(\sum\limits_{j=0}^{n-1} \oplus f^{(j)}(t)\right) \oplus \left(\sum\limits_{j=1}^{n} \oplus f^{[r-j]}(t\,|l)\right), & r = s = 2n. \end{cases} \tag{26}$$

Similar notation (with corresponding capital letter) we will also use for vector-functions which are denoted by another letter and for another differential operations instead of l, m. For example for vector-function $y_1(t) \in \mathcal{H}$ and differential operations l_2, m_2 we denote by $Y_1(t, l_2, m_2)$ vector-function (26) with $y_1(t)$, l_2, m_2, r_2, s_2 instead of $f(t)$, l, m, r, s respectively, where r_2 and s_2 are the orders of l_2 and m_2 ($r_2 \geq \text{even}\, s_2$).

From now on in Eq. (4)

$$p_n^{-1}(t) \in B(\mathcal{H}) \quad (r = 2n)\,;\; (q_{n+1}(t) + s_{n+1}(t))^{-1} \in B(\mathcal{H}) \quad (r = 2n+1)\,.$$

Theorem 1.1. *Equation* (4) *is equivalent to the first-order system*

$$\frac{i}{2}\left((Q(t)\,\vec{y})' + S(t)\,\vec{y}\,'\right) + H(t)\,\vec{y} = W(t)\,F(t) \tag{27}$$

with the coefficients $Q(t) = Q(t,l)$, $S(t) = S(t,l)$ *(7), (15),* $H(t) = H(t,l)$ *(8), (16), weight* $W(t) = W(t,\, l^*,\, m)$, *and with* $F(t) = F(t,\, l^*, m)$ *that are obtained from (9), (17) and (26), respectively, with* l^* *instead of* l. *Namely, if* $y(t)$ *is a solution of equation (4), then*

$$\vec{y}(t) = \vec{y}(t,\, l,\, m,\, f)$$
$$= \begin{cases} \left(\sum\limits_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \left(\sum\limits_{j=1}^{n} \oplus \left(y^{[r-j]}(t\,|l) - f^{[s-j]}(t\,|m)\right)\right), & r = 2n, \\ \left(\sum\limits_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \left(\sum\limits_{j=1}^{n} \oplus \left(y^{[r-j]}(t\,|l) - f^{[s-j]}(t\,|m)\right)\right) & \\ \oplus \left(-iy^{(n)}(t)\right), & r = 2n+1 > 1, \\ \quad \text{(here } f^{[k]}(t\,|m) \equiv 0 \text{ as } k < \frac{s}{2}) & \\ y(t), & r = 1, \end{cases} \tag{28}$$

is a solution of (27) *with the coefficients, weight and* $F(t)$ *mentioned above. Any solution of Eq.* (27) *with these coefficients, weight and* $F(t)$ *is equal to* (28), *where* $y(t)$ *is some solution of Eq.* (4).

Let us notice that different vector-functions $f(t)$ can generate different right-hand sides of Eq. (27), but only the unique right-hand side of Eq. (4).

Below we will use notation similar to to (28) for another vector-functions instead of $y(t)$, $f(t)$ and another differential operations instead of l, m. For example for vector-functions $y_1(t)$, $g(t) \in \mathcal{H}$ and differential operations l_2, m_2 we denote by $\vec{y}_1(t, l_2, m_2, g)$ vector-function (28) with $y_1(t)$, $g(t)$, l_2, m_2, r_2, s_2 instead of $y(t)$, $f(t)$, l, m, r, s respectively, where r_2 and s_2 are the orders of l_2 and m_2 ($r_2 \geq \text{even}\, s_2$).

P r o o f. We need the following three lemmas.

Lemma 1.2. *Let* $L_\alpha[y]$ *be a differential expression of* $l[y]$ (6) *type and of order* α. *Let us add to* $L_\alpha[y]$ *the expressions of* $i^k l_k[y]$ *type, where* $k = \alpha + 1, \ldots, \beta$, *with the coefficients equal to zero. We obtain the expressions* $L_\beta[y]$ *which formally have order* β, *but in fact,* L_β *and* L_α *coincide. Then, for sufficiently smooth vector-function* $f(t)$,

$$f^{[\beta-j]}(t\,|L_\beta) = \begin{cases} f^{[\alpha-j]}(t\,|L_\alpha), & j = 0, \ldots, \left[\frac{a+1}{2}\right], \\ 0, & j = \left[\frac{a+1}{2}\right] + 1, \ldots, \left[\frac{\beta}{2}\right], \end{cases}$$

(*here* $f^{[0]}(t|L_1)$ *is defined by* (12) *with* $r = 1$).

P r o o f. The proof of Lemma 1.2 follows from formulae (12)–(13) for quasi-derivatives. ■

Lemma 1.3. *Let* $f(t) \in C^s([\alpha, \beta], \mathcal{H})$, $y(t)$ *be a solution of the corresponding Eq.* (4). *Then the sequence* $f_k(t) \in C^\infty([\alpha, \beta], \mathcal{H})$ *and the solutions* $y_k(t)$ *of equation* (4) *with* $f(t) = f_k(t)$ *exist such that*

$$f_k(t) \overset{C^s([\alpha,\beta],\mathcal{H})}{\longrightarrow} f(t), \quad y_k(t) \overset{C^r([\alpha,\beta],\mathcal{H})}{\longrightarrow} y(t).$$

This is a trivial consequence of the Weierstrass theorem for vector-functions [36] and formula (1.21) from [11].

Lemma 1.4. *Let the vector-function* $f(t) \in C^s\left(\bar{\mathcal{I}}, \mathcal{H}\right)$. *Then*

$$W(t, l^*, m)\, F(t, l^*, m)$$
$$= \begin{cases} \left(\sum\limits_{j=0}^{s/2-1} \oplus \left(f^{[s-j]}(t\,|m) + \left(f^{[s-j-1]}(t\,|m)\right)'\right)\right) & \\ \quad \oplus f^{[s/2]}(t\,|m) \oplus 0 \oplus \ldots \oplus 0, & r = 2n+1,\ r = 2n,\ 0 < s < 2n, \\ \left(\sum\limits_{j=0}^{s/2-1} \oplus \left(f^{[s-j]}(t\,|m) + \left(f^{[s-j-1]}(t\,|m)\right)'\right)\right) & \\ \quad \oplus 0 \oplus \ldots + \oplus 0 \oplus \left(-if^{[n]}(t\,|m)\right), & r = 2n+1,\ s = 2n > 0, \\ \tilde{p}_0(t)\, f(t) \oplus 0 \oplus \ldots \oplus 0, & s = 0, \\ \Bigg[\left(\sum\limits_{j=0}^{s/2-1} \oplus \left(f^{[s-j]}(t\,|m) + \left(f^{[s-j-1]}(t\,|m)\right)'\right)\right) & \\ \quad \oplus 0 \oplus \ldots \oplus 0\Bigg] + H(t,l)\left(0 \oplus \ldots \oplus 0 \oplus f^{[n]}(t\,|m)\right), & r = s = 2n. \end{cases} \tag{29}$$

Notice that $W(t, l^*, m)F(t, l^*, m)$ does not change if the null-components in $F(t, l^*, m)$ are changed by any $\mathcal{H}$-valued vector-functions.

P r o o f. Let us prove Lemma 1.4 for $r = s = 2n$. It is sufficient to verify that

$$\left(\|m_{\alpha\beta}(t)\|_{\alpha,\beta=1}^2\right) col\left\{f(t),\, f'(t),\, \ldots,\, f^{(n-1)}(t),\, f^{(2n-1)}(t),\, \ldots\, f^{(n)}(t)\right\}$$
$$= C^*(t, l^*)\left\{\left[\left(\sum_{j=0}^{n-1}\left(f^{[r-j]}(t\,|m) + \left(f^{[r-j-1]}(t\,|m)\right)'\right)\right) \oplus 0 \oplus \ldots \oplus 0\right]\right.$$
$$\left. + H(t\,|l)\left(0 \oplus 0 \oplus \ldots \oplus 0 \oplus f^{[n]}(t\,|m)\right)\right\}. \tag{30}$$

But in view of (9), (12), (13), the left-hand side of equality (30) is equal to

$$\left(\sum_{j=0}^{n-1} \oplus \left(f^{[r-j]}(t\,|m)\right) + \left(f^{[r-j-1]}(t\,|m)\right)'\right) \oplus \mathrm{O} \oplus \ldots \oplus \mathrm{O} \oplus f^{[n]}(t\,|m).$$

Hence equality (30) is true since $C(t, l^*)[\ldots] = [\ldots]$ and the last column of $C^*(t, l^*)\, H(t, l)$ is equal to $col\,(0, \ldots, 0, I_1)$ in view of (8), (14).

The proof for $r = 2n+1$, $s = 2n$ is carried out via direct calculation by using (17), (12), (13).

The proof for $s < 2n$ follows from the case $s = 2n$ considered above, Lemmas 1.2, 1.3 and the fact that the elements $u_{jk} \in B(\mathcal{H})$ of the matrix $W(t, l^*, m)$ are equal to zero if $s < 2n$ and $i > s/2$ or $j > s/2$. Lemma 1.4 is proved. ∎

Let us return to the proof of Theorem 1.1. Let $y\,(t)$ be a solution of Eq. (4). Then

$$\frac{i}{2}\left\{(Q\,(t,l)\,\vec{y}\,(t,l,m,0))^{\prime}+S\,(t,l)\,\vec{y}^{\,\prime}\,(t,l,m,0)\right\}-H\,(t,l)\,\vec{y}\,(t,l,m,0)$$
$$=\operatorname{diag}\left(y^{[r]}\,(t\,|l\,)\,,0,\ldots,0\right) \quad (31)$$

in view of formulae that are analogues to formulae (4.10), (4.11), (4.24), (4.25) from [26]. Using (31) and Lemma 1.4, via direct calculation we can show that $\vec{y}\,(t,l,m,f)$ (28) is a solution of (27) for $r=s=2n$, $r=2n+1$, $s=2n$. Therefore, in view of Lemmas 1.2, 1.3 , $\vec{y}\,(t,l,m,f)$ is a solution of (27) for $s<2n$.

Conversely, let $\vec{\tilde{y}}\,(t)=col\,(y_1,\ldots,y_r)$ be a solution of (27). Let $y\,(t)$ be a solution of the Cauchy problem obtained by adding the initial condition $\vec{y}\,(0,l,m,f)=\vec{\tilde{y}}\,(0)$ to Eq. (4). Then $\vec{\tilde{y}}\,(t)=\vec{y}\,(t,l,m,f)$ in view of the existence and uniqueness theorem. Theorem 1.1 is proved. ■

Notice that Theorem 1.1 remains valid if the null-components of $F(t,l^*,m)$ are changed by any $\mathcal{H}$-valued vector-functions.

For the differential expression $L[y]=\sum\limits_{k=0}^{R} i^k L_k[y]$, where $L_{2j}=D^jP_j\,(t)\,D^j$, $L_{2j-1}=\frac{1}{2}D^{j-1}\left\{DQ_j\,(t)+S_j\,(t)\;D\right\}D^{j-1}$, $P_j(t)$, $Q_j(t)$, $S_j(t)\in C^j(\overline{\mathcal{I}},B(\mathcal{H}))$ we denote by

$$L\,[f,\,g]=\int_{\mathcal{I}} L\,\{f,g\}\,dt \quad (32)$$

the bilinear form which corresponds to its Dirichlet integral. Here

$$L\,\{f,g\}=\sum_{j=0}^{[R/2]}\left(P_j\,(t)\,f^{(j)}\,(t)\,,\,g^{(j)}\,(t)\right)$$
$$+\frac{i}{2}\sum_{j=1}^{\left[\frac{R+1}{2}\right]}\left(S_j\,(t)\,f^{(j)}\,(t)\,,\,g^{(j-1)}\,(t)\right)-\left(Q_j\,(t)\,f^{(j-1)}\,(t)\,,g^{(j)}\,(t)\right). \quad (33)$$

Theorem 1.2. (On the relationships between bilinear forms) *Let $f\,(t)$, $y\,(t)$, $f_k\,(t)$, $y_k\,(t)$, $k=1,2$, be sufficiently smooth vector-functions. Then:*
1.
$$(W\,(t,\,l,\,m)\,F_1\,(t,l,m)\,,\,F_2\,(t,l,m))=m\,\{f_1,f_2\}\,. \quad (34)$$

2. *a) If the order of $\Im l$ is even, then*

$$(W\,(t,l,-\Im l)\,\vec{y}\,(t,l,m,f)\,,\,\vec{y}\,(t,\,l,\,m,\,f))$$
$$-\Im\,(W\,(t,\,l^*,m^*)\,\vec{y}\,(t,\,l,m,f)\,,\,F\,(t,l^*,m))$$
$$=-\,(\Im l)\,\{y,y\}-\Im\,(m^*\,\{y,f\})\,; \quad (35)$$

b)

$$m\{y_1,f_2\}-m\{f_1,y_2\}=\left(W(t,l,m)\vec{y}_1(t,l,m,f_1),F_2(t,l,m)\right)\\ -\left(W(t,l^*,m)F_1(t,l^*,m),\vec{y}_2(t,l^*,m^*,f_2)\right), \tag{36}$$

although for $r=s$ *the corresponding terms in the right- and left-hand sides of* (35) *and* (36) *do not coincide.*

P r o o f. Statement 1 follows from (9), (17), (26), (33).

2. Let $r=s=2n$. For more convenience, when using notations of (26) type, we omit the argument m. For example, by $F(t,l^*)$ we denote $F(t,l^*,m)$.

a) We denote

$$\mathcal{F}(t,m)=col\left\{0,\dots,0,f^{[2n-1]}(t\,|m),\dots,f^{[n]}(t\,|m)\right\}\in\mathcal{H}^r. \tag{37}$$

One has

$$\left(W(t,l,-\Im l)\vec{y}(t,l,m,f),\vec{y}(t,l,m,f)\right)=\left(W(t,l,-\Im l)Y(t,l),Y(t,l)\right)\\ -\left(W(t,l,-\Im l)Y(t,l),\mathcal{F}(t,m)\right)-\left(W(t,l,-\Im l)\mathcal{F}(t,m),Y(t,l)\right)\\ +\left(W(t,l,-\Im l)\mathcal{F}(t,m),\mathcal{F}(t,m)\right)-(\Im l)[y,y]\\ +2\Re\left(p_n^{*-1}y^{[n]}(t|\Im l),f^{[n]}(t|m)\right)+\Im\left(p_n^{-1}f^{[n]}(t|m),f^{[n]}(t|m)\right). \tag{38}$$

Here the last equality follows from (18), (34), (29), (8). On the other hand, we have

$$\Im\left(W(t,l^*,m^*)\vec{y}(t,l,m,f),F(t,l^*)\right)=\Im\left(W(t,l^*,m^*)Y(t,l^*),F(t,l^*)\right)\\ +\Im\left(W(t,l^*,m^*)\left((Y(t,l)-Y(t,l^*))-\mathcal{F}(t,m)\right),F(t,l^*)\right)=\Im\left(m^*\{y,f\}\right)\\ +2\Re\left(p_n^{*-1}y^{[n]}(t|\Im l),f^{[n]}(t|m)\right)+\Im\left(p_n^{-1}f^{[n]}(t|m),f^{[n]}(t|m)\right). \tag{39}$$

We prove the last equality similarly to (38) taking into account that $y^{[n]}(t|l)-y^{[n]}(t|l^*)=2iy^{[n]}(t|\Im l)$. Comparing (38) and (39), we obtain (35).

b) In view of (28), (34), (18) and Lemma 1.4, we have

$$\left(W(t,l,m)\vec{y}_1(t,l,m,f_1),F_2(t,l)\right)=m\{y_1(t,l,m,f_1),f_2\}\\ -\left(\mathcal{F}_1(t,m),H(t,l^*)\,col\left\{0,\,\dots,0,f_2^{[n]}(t\,|m^*)\right\}\right)\\ =m\{y_1,f_2\}-\left(p_n^{-1}f_1^{[n]}(t\,|m),f_2^{[n]}(t\,|m^*)\right), \tag{40}$$

where $\mathcal{F}_1(t,m)$ is defined by (37) with $f_1(t)$ instead of $f(t)$. Similarly,

$$\left(W(t,l^*,m)F_1(t,l^*),\vec{y}_2(t,l^*,m^*,f_2)\right)\\ =m\{f_1,y_2\}-\left(p_n^{-1}f_1^{[n]}(t\,|m),f_2^{[n]}(t\,|m^*)\right). \tag{41}$$

Comparing (40) and (41), we obtain (36).

For $r = 2n+1$, $s = 2n$ or $r = 2n+1 \vee 2n$, $s < 2n$, the corresponding terms in (35), (36) coincide in view of (9), (17), (26), (28), (34). For example, in these cases

$$\begin{aligned}(W(t,l,-\Im l)\,\vec{y}(t,l,m,f),\vec{y}(t,l,m,f)) &= ((W(t,l,-\Im l))\,Y(t,l),Y(t,l)) \\ &= -(\Im l)\{y,y\}.\end{aligned}$$

Theorem 1.2 is proved. ■

Notice that Theorem 1.2 remains valid if the null-components in $F_k(t,l,m)$, $F(t,l^*,m)$, $F_1(t,l^*,m)$ are changed by any $\mathcal{H}$-valued vector-functions.

Theorem 1.3. (The Green formula) *Let* $\mathrm{l}_k[y]$, $\mathrm{m}_k[y]$, $k = 1,2$, *be the differential expressions of* $l[y]$ (6), $m[y]$ *types, respectively. The orders of* l_k *are equal to* r, *the orders of* m_k *are different in general, even and are equal to* $s_k \le r$. *The coefficients of* l_k *at the highest derivative has the inverse from* $B(\mathcal{H})$ *for* $t \in [\alpha,\beta]$. *Let* $y_k(t) \in C^r([\alpha,\beta],\mathcal{H})$, $f_k(t) \in C^{s_k}([\alpha,\beta],\mathcal{H})$, *and* $\mathrm{l}_k[y_k] = \mathrm{m}_k[f_k]$, $k = 1, 2$. *Then*

$$\begin{aligned}&\int_\alpha^\beta \mathrm{m}_1\{f_1,y_2\}\,dt - \int_\alpha^\beta \mathrm{m}_2^*\{y_1,f_2\}\,dt - \int_\alpha^\beta (\mathrm{l}_1 - \mathrm{l}_2^*)\{y_1,y_2\}\,dt \\ &\qquad = \left(\frac{i}{2}\left(Q(t,\mathrm{l}_1) + Q^*(t,\mathrm{l}_2)\right)\vec{y}_1(t,\mathrm{l}_1,\mathrm{m}_1,f_1),\vec{y}_2(t,\mathrm{l}_2,\mathrm{m}_2,f_2)\right)\Bigg|_\alpha^\beta, \qquad (42)\end{aligned}$$

where $Q(t,\mathrm{l}_k)$, *are defined by* (7), (15) *with* l_k *instead of* l.

P r o o f. We need the following.

Lemma 1.5. *For the sufficiently smooth vector-functions* $g_1(t)$, $g_2(t)$, *one has*

$$\begin{aligned}&((H(t,\mathrm{l}_1) - H(t,\mathrm{l}_2^*))\,\vec{g}_1(t,\mathrm{l}_1,\mathrm{m}_1,0),\vec{g}_2(t,\mathrm{l}_2,\mathrm{m}_2,0)) \\ &= \begin{cases} -(\mathrm{l}_1 - \mathrm{l}_2^*)\{g_1,g_2\}, & r = 2n, \\ -(\mathrm{l}_1 - \mathrm{l}_2^*)\{g_1,g_2\} + \left(\mathrm{l}_{2n+1}^1 - \mathrm{l}_{2n+1}^{2^*}\right)\{g_1,g_2\}, & r = 2n+1,\end{cases} \qquad (43)\end{aligned}$$

where l_{2n+1}^k *are the analogs of* l_{2n+1} *for* l_k.

P r o o f. Let $r = 2n$. Then, in view of (20)–(24), (28), (10), (18), we have

$$((H(t,\mathrm{l}_1) - H(t,\mathrm{l}_2^*))\,\vec{g}_1(t,\mathrm{l}_1,\mathrm{m}_1,0), \vec{g}_2(t,\mathrm{l}_2,\mathrm{m}_2,0))$$
$$= ((A(t,\mathrm{l}_1) - A(t,\mathrm{l}_2^*))\,\vec{g}_1(t,\mathrm{l}_1,\mathrm{m}_1,0), \vec{g}_2(t,\mathrm{l}_2,\mathrm{m}_2,0))$$
$$+ \Big(C^*(t,\mathrm{l}_2)\,B(t,\mathrm{l}_1)\,C(t,\mathrm{l}_1)\,col\left\{g_1, g_1', \ldots, g_1^{(n-1)}, g_1^{(2n-1)}, \ldots, g_1^{(n)}\right\},$$
$$, col\left\{g_2, g_2', \ldots, g_2^{(n-1)}, g_2^{(2n-1)}, \ldots, g_2^{(n)}\right\}\Big) - \Big(col\left\{g_1, g_1', \ldots, g_1^{(2n-1)}, \ldots, g_1^{(n)}\right\},$$
$$, C^*(t,\mathrm{l}_1)\,B(t,\mathrm{l}_2)\,C(t,\mathrm{l}_2)\,col\left\{g_2, g_2', \ldots, g_2^{(n-1)}, g_2^{(2n-1)}, \ldots, g_2^{(n)}\right\}\Big)$$
$$= -(\mathrm{l}_1 - \mathrm{l}_2^*)\,\{f, g\}.$$

The proof of (43) for $r = 2n+1$ follows directly from (16), (28). Lemma 1.5 is proved. ■

Now the Green formula (42) can be obtained from the following Green formula for the Eq. (27) which correspond to the equations $\mathrm{l}_k[y] = \mathrm{m}_k[f]$:

$$\int_\alpha^\beta (W(t,\mathrm{l}_1^*,\mathrm{m}_1)\,F_1(t,\mathrm{l}_1^*,\mathrm{m}_1), \vec{y}_2(t,\mathrm{l}_2,\mathrm{m}_2,f_2))\,dt$$
$$- \int_\alpha^\beta (W(t,\mathrm{l}_2^*,\mathrm{m}_2^*)\,\vec{y}_1(t,\mathrm{l}_1,\mathrm{m}_1,f_1), F_2(t,\mathrm{l}_2^*,\mathrm{m}_2))\,dt$$
$$+ \int_\alpha^\beta ((H(t,\mathrm{l}_1) - H(t,\mathrm{l}_2^*))\,\vec{y}_1(t,\mathrm{l}_1,\mathrm{m}_1,f_1), \vec{y}_2(t,\mathrm{l}_2,\mathrm{m}_2 f_2))\,dt$$
$$- \int_\alpha^\beta \frac{i}{2}\Big\{\big((S(t,\mathrm{l}_1) - Q^*(t,\mathrm{l}_2))\vec{y}\,'_1(t,\mathrm{l}_1,\mathrm{m}_1,f_1), \vec{y}_2(t,\mathrm{l}_2,\mathrm{m}_2,f_2)\big)$$
$$- \big((Q(t,\mathrm{l}_1) - S^*(t,\mathrm{l}_2))\,\vec{y}_1(t,\mathrm{l}_1,\mathrm{m}_1,f_1), \vec{y}\,'_2(t,\mathrm{l}_2,\mathrm{m}_2,f_2)\big)\Big\}\,dt$$
$$= \left(\frac{i}{2}(Q(t,\mathrm{l}_1) + Q^*(t,\mathrm{l}_2))\,\vec{y}_1(t,\mathrm{l}_1,\mathrm{m}_1,f_1), \vec{y}_2(t,\mathrm{l}_2,\mathrm{m}_2,f_2)\right)\Bigg|_\alpha^\beta. \tag{44}$$

Let $r = s_k = 2n$. For more convenience, by $F_k(t,\mathrm{l}_k^*), Y_k(t,\mathrm{l}_k)$ we denote $F_k(t,\mathrm{l}_k^*,\mathrm{m}_k), Y_k(t,\mathrm{l}_k,\mathrm{m}_k)$, respectively. Then, in view of (8), (28), (29), (34), (43), one has:

$$(W(t,\mathrm{l}_1^*,\mathrm{m}_1)\,F_1(t,\mathrm{l}_1^*), \vec{y}_2(t,\mathrm{l}_2,\mathrm{m}_2,f_2)) = \mathrm{m}_1\{f_1, y_2\}$$
$$+ \Big(H(t,\mathrm{l}_1)\,col\left\{0, \ldots, 0, f_1^{[n]}(t\,|\mathrm{m}_1)\right\},$$
$$, col\left\{0, \ldots, 0, y_2^{[n]}(t\,|\mathrm{l}_2) - y_2^{[n]}(t\,|\mathrm{l}_1^*) - f_2^{[n]}(t\,|\mathrm{m}_2)\right\}\Big); \tag{45}$$

$$\begin{aligned}(W(t,\mathrm{l}_2^*,\mathrm{m}_2^*)\,\vec{y}_1(t,\mathrm{l}_1,\mathrm{m}_1,f_1)\,,F_2(t,\mathrm{l}_2^*)) = \mathrm{m}_2^*\{y_1,f_2\}\\ +\Big(H(t,\mathrm{l}_2^*)\,col\left\{0,\ldots,0,y_1^{[n]}(t\,|\mathrm{l}_1)-y_1^{[n]}(t\,|\mathrm{l}_2^*)-f_1^{[n]}(t\,|\mathrm{m}_1)\right\},\\ ,col\left\{0,\ldots,0,f_2^{[n]}(t\,|\mathrm{m}_2)\right\}\Big);\end{aligned} \tag{46}$$

$$\begin{aligned}((H(t,\mathrm{l}_1)-H(t,\mathrm{l}_2^*))\,\vec{y}_1(t,\mathrm{l}_1,\mathrm{m}_1,f_1)\,,\vec{y}_2(t,\mathrm{l}_2,\mathrm{m}_2,f_2)) = -(\mathrm{l}_1-\mathrm{l}_2^*)\{y_1,y_2\}\\ -((H(t,\mathrm{l}_1)-H(t,\mathrm{l}_2^*))\,Y_1(t,\mathrm{l}_1)\,,\mathcal{F}_2(t,\mathrm{m}_2))-((H(t,\mathrm{l}_1)-H(t,\mathrm{l}_2^*))\,\mathcal{F}_1(t,\mathrm{m}_1)\,,Y_2(t,\mathrm{l}_2))\\ +\Big((H(t,\mathrm{l}_1)-H(t,\mathrm{l}_2^*))\,col\left\{0,\ldots,0,f_1^{[n]}(t\,|\mathrm{m}_1)\right\},col\left\{0,\ldots,0,f_2^{[n]}(t\,|\mathrm{m}_2)\right\}\Big),\end{aligned} \tag{47}$$

where $\mathcal{F}_k(t,\mathrm{m}_k)$ are define by (37) with $f_k(t)$ instead of $f(t)$.

Let us denote by p_j^k,q_j^k,s_j^k the coefficients of l_k. In view of (8),

$$\begin{aligned}\Big(H(t,\mathrm{l}_1)\,col\left\{0,\ldots,0,f_1^{[n]}(t\,|\mathrm{m}_1)\right\},col\left\{0,\ldots,0,y_2^{[n]}(t\,|\mathrm{l}_2)-y_2^{[n]}(t\,|\mathrm{l}_1^*)\right\}\Big)\\ =\Big((p_n^1)^{-1}f_1^{[n]}(t\,|\mathrm{m}_1)\,,y_2^{[n]}(t\,|\mathrm{l}_2)-y_2^{[n]}(t\,|\mathrm{l}_1^*)\Big),\end{aligned} \tag{48}$$

and

$$\begin{aligned}\Big(col\left\{0,\ldots,0,y_1^{[n]}(t\,|\mathrm{l}_1)-y_1^{[n]}(t\,|\mathrm{l}_2^*)\right\},H(t,\mathrm{l}_2)\,col\left\{0,\ldots,0,f_2^{[n]}(t\,|\mathrm{m}_2)\right\}\Big)\\ =\Big(y_1^{[n]}(t\,|\mathrm{l}_1)-y_1^{[n]}(t\,|\mathrm{l}_2^*)\,,(p_n^2)^{-1}f_2^{[n]}(t\,|\mathrm{m}_2)\Big).\end{aligned} \tag{49}$$

On the other hand, in view of (8), (12), we have

$$\begin{aligned}-((H(t,\mathrm{l}_1)-H(t,\mathrm{l}_2^*))\,Y_1(t,\mathrm{l}_1)\,,\mathcal{F}_2(t,\mathrm{m}_2))\\ =-\Big(\Big(\big(i(p_n^1)^{-1}q_n^1/2-i(p_n^{2*})^{-1}s_n^{2*}/2\big)\,y_1^{(n-1)}+\big((p_n^1)^{-1}-(p_n^{2*})^{-1}\big)\,y_1^{[n]}(t\,|\mathrm{l}_1)\Big),\\ ,f_2^{[n]}(t\,|\mathrm{m}_2)\Big)=\Big(\big(p_n^{2*}\big)^{-1}\Big(y_1^{[n]}(t\,|\mathrm{l}_1)-y_1^{[n]}(t\,|\mathrm{l}_2^*)\Big),f_2^{[n]}(t\,|\mathrm{m}_2)\Big),\end{aligned} \tag{50}$$

where the last equality is a corollary of (12) and of its modification

$$(p_n^1)^{-1}y_1^{[n]}(t\,|\mathrm{l}_1)=y_1^{(n)}-\frac{i}{2}(p_n^1)^{-1}q_n^1y_1^{(n-1)}.$$

Analogously, it can be proved that

$$\begin{aligned}((H(t,\mathrm{l}_1)-H(t,\mathrm{l}_2^*))\,\mathcal{F}_1(t,\mathrm{m}_1)\,,Y_2(t,\mathrm{l}_2))\\ =\Big(f_1^{[n]}(t\,|\mathrm{m}_1)\,,(p_n^{1*})^{-1}\Big(y_2^{[n]}(t\,|\mathrm{l}_2)-y_2^{[n]}(t\,|\mathrm{l}_1^*)\Big)\Big).\end{aligned} \tag{51}$$

Comparing (44)–(51), we get (42) since the last $\int_\alpha^\beta$ in the left-hand side of (44) is equal to zero if $r = 2n$ in view of (7).

For $s_k < r = 2n$, the proof of (42) easy follows from (26), (28), (34), (43), (44) in view of the first footnote.

Now let $r = 2n+1$. Then the last $\int_\alpha^\beta$ in the left-hand side of (44) is equal to $\int_\alpha^\beta \left(\mathrm{l}_{2n+1}^1 - \mathrm{l}_{2n+1}^{2*}\right)\{y_1, y_2\}\,dt$. Hence the proof of (42) for $s_k \le 2n < r = 2n+1$ follows from (17), (26), (28), (34), (43), (44). Theorem 1.3 is proved. ■

R e m a r k 1.1. In view of Lemmas 1.2, 1.3 all results of this section are valid if the condition of eveness of s_k is changed by the condition $s_k \le 2\left[\frac{r}{2}\right]$.

2. Characteristic Operator

We consider an operator differential equation in the separable Hilbert space $\mathcal{H}_1$

$$\frac{i}{2}\Big(\left(Q(t)\,x(t)\right)' + Q^*(t)\,x'(t)\Big) - H_\lambda(t)\,x(t) = W_\lambda(t)\,F(t), \quad t \in \bar{\mathcal{I}}, \tag{52}$$

where $Q(t)$, $[\Re Q(t)]^{-1}$, $H_\lambda(t) \in B(\mathcal{H}_1)$, $Q(t) \in C^1\left(\bar{\mathcal{I}}, B(\mathcal{H}_1)\right)$; the operator function $H_\lambda(t)$ is continuous in t and is Nevanlinna's in λ. Namely, the following condition holds:

(**A**) The set $\mathcal{A} \supseteq \mathbb{C}\setminus\mathbb{R}^1$ exists, every its points has a neighbourhood independent of $t \in \bar{\mathcal{I}}$, in this neighbourhood, $H_\lambda(t)$ is analytic $\forall t \in \bar{\mathcal{I}}$; $\forall\lambda \in \mathcal{A}\; H_\lambda(t) = H^*_{\bar\lambda}(t) \in C\left(\bar{\mathcal{I}}, B(\mathcal{H}_1)\right)$; the weight $W_\lambda(t) = \Im H_\lambda(t)/\Im\lambda \ge 0\,(\Im\lambda \ne 0)$.

In view of [22], $\forall\mu \in \mathcal{A}\bigcap\mathbb{R}^1 : W_\mu(t) = \partial H_\lambda(t)/\partial\lambda|_{\lambda=\mu}$ is a Bochner locally integrable function in the uniform operator topology.

For the convenience, we suppose that $0 \in \bar{\mathcal{I}}$ and denote $\Re Q(0) = G$.

Let $X_\lambda(t)$ be the operator solution of homogeneous equation (52) satisfying the initial condition $X_\lambda(0) = I$, where I is an identity operator in $\mathcal{H}_1$. Since $H_\lambda(t) = H^*_{\bar\lambda}(t)$, then

$$X^*_{\bar\lambda}(t)[\Re Q(t)]X_\lambda(t) = G, \;\; \lambda \in \mathcal{A}. \tag{53}$$

For any α, $\beta \in \bar{\mathcal{I}}$, $\alpha \le \beta$, we denote $\Delta_\lambda(\alpha,\beta) = \int_\alpha^\beta X^*_\lambda(t)\,W_\lambda(t)\,X_\lambda(t)\,dt$, $N = \{h \in \mathcal{H}_1\,|h \in Ker\Delta_\lambda(\alpha,\beta)\;\forall\alpha,\beta\}$, P is the ortho-projection onto $N^\perp$. N is independent of $\lambda \in \mathcal{A}$ [22].

For $x(t) \in \mathcal{H}_1$ or $x(t) \in B(\mathcal{H}_1)$, we denote $U[x(t)] = ([\Re Q(t)]\,x(t), x(t))$ or $U[x(t)] = x^*(t)\,[\Re Q(t)]\,x(t)$, respectively.

As in [21, 22], we introduce the following.

Definition 2.1. *An analytic operator-function* $M(\lambda) = M^*\left(\bar\lambda\right) \in B(\mathcal{H}_1)$ *of non-real* λ *is called a characteristic operator of Eq.* (52) *on* $\mathcal{I}$ *if for* $\Im\lambda \ne 0$ *and* *for any $\mathcal{H}_1$ — valued vector-function $F(t) \in L^2_{W_\lambda}(\mathcal{I})$ with compact support, the corresponding solution $x_\lambda(t)$ of Eq.* (52) *of the form*

$$x_\lambda(t,F) = \mathcal{R}_\lambda F = \int_{\mathcal{I}} X_\lambda(t)\left\{M(\lambda) - \frac{1}{2} sgn(s-t)(iG)^{-1}\right\} X^*_{\bar\lambda}(s) W_\lambda(s) F(s)\, ds \tag{54}$$

satisfies the condition

$$(\Im\lambda) \lim_{(\alpha,\beta)\uparrow\mathcal{I}} \left(U[x_\lambda(\beta,F)] - U[x_\lambda(\alpha,F)]\right) \le 0 \quad (\Im\lambda \neq 0). \tag{55}$$

Notice that in [22] the characteristic operator was defined when $Q(t) = Q^*(t)$. Our case is equivalent to this one since equation (52) coincides with equation of (52) type with $\Re Q(t)$ instead of $Q(t)$ and with $H_\lambda(t) - \frac{1}{2}\Im Q'(t)$ instead of $H_\lambda(t)$.

The properties of the characteristic operator and sufficient conditions for the existence of characteristic operators are obtained in [21, 22].

For the case $\dim\mathcal{H}_1 < \infty$, $Q(t) = \mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}$, $-\infty < a = 0$, the description of the characteristic operators was obtained in [31] (the results of [31] were specified and supplemented in [23]). For the case $\dim\mathcal{H}_1 = \infty$ and $\mathcal{I}$ is finite, the description of the characteristic operators was obtained in [22]. These descriptions were obtained under the condition that

$$\exists\lambda_0 \in \mathcal{A},\ [\alpha,\beta] \subseteq \overline{\mathcal{I}}:\ \Delta_{\lambda_0}(\alpha,\beta) \gg 0. \tag{56}$$

Definition 2.2. [21, 22] *Let $M(\lambda)$ be the characteristic operator of equation* (52) *on $\mathcal{I}$. We say that the corresponding condition* (55) *is separated for nonreal $\lambda = \mu_0$ if for any $\mathcal{H}_1$-valued vector function $f(t) \in L^2_{W_{\mu_0}(t)}(\mathcal{I})$ with compact support the following inequalities hold simultaneously for the solution $x_{\mu_0}(t)$* (54) *of equation* (52)*:*

$$\lim_{\alpha\downarrow a} \Im\mu_0 U[x_{\mu_0}(\alpha)] \ge 0, \quad \lim_{\beta\uparrow b} \Im\mu_0 U[x_{\mu_0}(\beta)] \le 0. \tag{57}$$

Theorem 2.1. [21, 22] *Let $P = I$, $M(\lambda)$ be the characteristic operator of equation* (52)*, $\mathcal{P}(\lambda) = iM(\lambda)G + \frac{1}{2}I$, such that we get the representation*

$$M(\lambda) = \left(\mathcal{P}(\lambda) - \frac{1}{2}I\right)(iG)^{-1}. \tag{58}$$

Then condition (55) *corresponding to $M(\lambda)$ is separated for $\lambda = \mu_0$ if and only if the operator $\mathcal{P}(\mu_0)$ is the projection, i.e.,*

$$\mathcal{P}(\mu_0) = \mathcal{P}^2(\mu_0). \tag{59}$$

Definition 2.3. [21, 22] *If the operator-function* $M(\lambda)$ *of the form* (58) *is the characteristic operator of equation* (52) *on* $\mathcal{I}$ *and, moreover,* $\mathcal{P}(\lambda) = \mathcal{P}^2(\lambda)$, *then* $\mathcal{P}(\lambda)$ *is called a characteristic projection of equation* (52) *on* $\mathcal{I}$.

The properties of the characteristic projections and sufficient conditions for their existence are obtained in [22]. Also, [22] contains the description of the characteristic projections and the abstract analogue of Theorem 2.1.

The following statement gives the necessary and sufficient conditions for the existence of the characteristic operator which corresponds to the separated boundary conditions such that the corresponding boundary condition at a regular point is self-adjoint. This statement follows from Theorem 2.1.

Let us denote by $\mathcal{H}_+$ ($\mathcal{H}_-$) the invariant subspace of the operator G which corresponds to the positive (negative) part of $\sigma(G)$.

Theorem 2.2. *Let* $-\infty < a$. *If* $P = I$, *then for the existence of the characteristic operator* $M(\lambda)$ *of equation* (52) *on* (a, b) *such that*

$$\exists \mu_0 \in \mathbb{C} \setminus \mathbb{R}^1 : \ U[x_{\mu_0}(a, F)] = U[x_{\bar{\mu}_0}(a, F)] = 0 \tag{60}$$

(*and therefore condition* (55) *is separated on* $\lambda = \mu_0$, $\lambda = \bar{\mu}_0$) *it is necessary that*

$$\dim \mathcal{H}_+ = \dim \mathcal{H}_-. \tag{61}$$

(*In* (60) $x_\lambda(t, F)$ *is a solution* (54) *of* (52) *which corresponds to the characteristic operator* $M(\lambda)$, $L^2_{w_{\mu_0}(t)}(a, b) \ni F = F(t)$ *is any* $\mathcal{H}_1$*-valued vector-function with compact support*).

If condition (56) *holds, then condition* (61) *is also sufficient for the existence of such characteristic operator.*

P r o o f. Necessity. Since $P = I$, we obtain

$$U[X_{\mu_0}(a)(I - \mathcal{P}(\mu_0))] = U[X_{\bar{\mu}_0}(a)(I - \mathcal{P}(\bar{\mu}_0))] = 0 \tag{62}$$

in view of the proof of n°2° of Theorem 1.1 from [22].

Let, for definiteness, $\Im \mu_0 > 0$. Then, in view of Theorem 2.4 and formula (1.69) from [22], (59), (62) and the fact that

$$\Im \lambda (X^*_\lambda(a)[\Re Q(a)] X_\lambda(a) - G) \le 0, \ \lambda \in \mathcal{A}, \tag{63}$$

we conclude that $X_{\mu_0}(a)(I - \mathcal{P}(\mu_0))\mathcal{H}_1$ and $X_{\bar{\mu}_0}(a)(I - \mathcal{P}(\bar{\mu}_0))\mathcal{H}_1$ are correspondingly maximal $\Re Q(a)$-nonnegative and maximal $\Re Q(a)$-nonpositive subspaces which are $\Re Q(a)$-neutral and $\Re Q(a)$-orthogonal in view of Remark 3.2 from [22], Theorem 2.1 and (53). Hence

$$(X_{\mu_0}(a)(I - \mathcal{P}(\mu_0))\mathcal{H}_1)^{[\perp]} = X_{\bar{\mu}_0}(a)(I - \mathcal{P}(\bar{\mu}_0))\mathcal{H}_1$$

in view of [3, p. 73] (here by $[\perp]$ we denote the $\Re Q(a)$-orthogonal complement). Therefore $X_{\mu_0}(a)(I-\mathcal{P}(\mu_0))\mathcal{H}_1$ is hypermaximal $\Re Q(a)$-neutral subspace in view of [3, p. 43]. Thus, in view of [3, p. 42], we obtain that $\dim\mathcal{H}_+(a)=\dim\mathcal{H}_-(a)$, where $\mathcal{H}_\pm(a)$ are analogs of $\mathcal{H}_\pm$ for $\Re Q(a)$. In view of (63), $X_{\mu_0}^{-1}(a)\mathcal{H}_+(a)$ and $X_{\bar{\mu}_0}^{-1}(a)\mathcal{H}_-(a)$ are correspondingly maximal uniformly G-positive and maximal uniformly G-negative subspaces. Therefore $\mathcal{H}_1$ is equal to the direct and G-orthogonal sum of these subspaces in view of (53) and [3, p. 75]. Hence we obtain (61) in view of the law of inertia [3, p. 54].

Sufficiency follows from Theorem 4.4. from [22]. The theorem is proved. ■

It is obvious that in Theorem 2.2 the point a can be replaced by the point b if $b<\infty$, but cannot be replaced by the point b if $b=\infty$ as the example of the operator id/dt on the semi-axis shows. Also this example shows that condition (60) is not necessary for the fulfilment of the condition $U[x_{\mu_0}(a,F)]=0$ only.

In the case of self-adjoint boundary conditions, the analogue of Theorem 2.2 for the regular differential operators in the space of vector-functions was proved in [32] (see also [34]). For the finite canonical systems depending on spectral parameter in a linear manner the analogue was proved in [29]. These analogs were obtained in a different way comparing with Theorem 2.2.

From this point and till the end of Remark 2.1 we suppose that $\mathcal{H}_1=\mathcal{H}^{2n}$, $Q(t)=J/i$ (7), $\mathcal{I}=(0,b)$, $b\le\infty$, and condition (56) hold. Let condition (55) be separated and $\mathcal{P}(\lambda)$ be a corresponding characteristic projection. In view of [22, p. 469], the Nevanlinna pair $\{-a(\lambda),\, b(\lambda)\}$, $a(\lambda),b(\lambda)\in B(\mathcal{H}^n)$ (see, for example, [13]) and the Weyl function $m(\lambda)\in B(\mathcal{H}^n)$ of equation (52) on $\mathcal{I}$ [22] exist such that

$$\mathcal{P}(\lambda)=\begin{pmatrix} I_n \\ m(\lambda)\end{pmatrix}\left(b^*\left(\bar\lambda\right)-a^*\left(\bar\lambda\right)m(\lambda)\right)^{-1}\left(a_2^*\left(\bar\lambda\right),\,-a_1^*\left(\bar\lambda\right)\right),\tag{64}$$

$$I-\mathcal{P}(\lambda)=\begin{pmatrix} a(\lambda) \\ b(\lambda)\end{pmatrix}\left(b(\lambda)-m(\lambda)\,a(\lambda)\right)^{-1}\left(-m(\lambda),I_n\right),\tag{65}$$

$$\left(b^*\left(\bar\lambda\right)-a^*\left(\bar\lambda\right)m(\lambda)\right)^{-1},\ \left(b(\lambda)-m(\lambda)\,a(\lambda)\right)^{-1}\in B(\mathcal{H}^n).$$

Conversely, $\mathcal{P}(\lambda)$ (64) is a characteristic projection for any Nevanlinna pair $(-a(\lambda),\, b(\lambda))$ and any Weyl function $m(\lambda)$ of equation (52) on $\mathcal{I}$.

Let the domain $D\subseteq\mathbb{C}_+$ be such that $\forall\lambda\in D:\ 0\in\rho\left(a(\lambda)-ib(\lambda)\right)$ (for example, $D=\mathbb{C}_+$ if $\exists\lambda_\pm\in\mathbb{C}_\pm$ such that $a^*(\lambda_\pm)\,b(\lambda_\pm)=b^*(\lambda_\pm)\,a(\lambda_\pm)$). Let the domain D_1 be symmetric to D with respect to the real axis. Then, in view of [22, p. 166] or Lemma 2.1 (see below), Corrolary 3.1 from [22] and (56), (64), (65), the operator $\mathcal{R}_\lambda F$ (54) for $\lambda\in D\bigcup D_1$ can be represented in the following form with using the operator solution $U_\lambda(t)\in B\left(\mathcal{H}^n,\,\mathcal{H}^{2n}\right)$ of Eq. (52), $(F=0)$ satisfying the accumulative (or dissipative) initial condition and the operator solution $V_\lambda(t)\in B\left(\mathcal{H}^n,\,\mathcal{H}^{2n}\right)$ of Weyl type of the same equation.

R e m a r k 2.1. *Let* $\lambda \in D \bigcup D_1$ *and* $\mathcal{H}_1$*-valid* $F(t) \in L^2_{W_\lambda}(\mathcal{I})^\star$. *Then solution* (54), (58), (64) *of Eq.* (52) *is equal to*

$$\mathcal{R}_\lambda F = \int_0^t V_\lambda (t) U^*_{\bar{\lambda}} (s) W_\lambda (s) F (s) ds + \int_t^b U_\lambda (t) V^*_{\bar{\lambda}} (s) W_\lambda (s) F (s) ds,$$

where the integrals converge strongly if the interval of integration is infinite. Here

$$U_\lambda (t) = X_\lambda (t) \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix}, \ V_\lambda (t) = X_\lambda (t) \begin{pmatrix} b(\lambda) \\ -a(\lambda) \end{pmatrix} K^{-1}(\lambda) + U_\lambda (t) m_{a,b}(\lambda), \tag{66}$$

where

$$K(\lambda) = a^*\left(\bar{\lambda}\right) a(\lambda) + b^*\left(\bar{\lambda}\right) b(\lambda), \ K^{-1}(\lambda) \in B(\mathcal{H}^n), \tag{67}$$

$$m_{a,b}(\lambda) = m^*_{a,b}\left(\bar{\lambda}\right) = K^{-1}(\lambda)\left(a^*\left(\bar{\lambda}\right) + b^*\left(\bar{\lambda}\right) m(\lambda)\right)\left(b^*\left(\bar{\lambda}\right) - a^*\left(\bar{\lambda}\right) m(\lambda)\right)^{-1}, \tag{68}$$

$$V_\lambda (t) h \in L^2_{W_\lambda(t)}(\mathcal{I}) \,\forall h \in \mathcal{H}^n. \tag{69}$$

Moreover, if $a(\lambda) = a\left(\bar{\lambda}\right), b(\lambda) = b\left(\bar{\lambda}\right)$ *as* $\Im\lambda \neq 0$, *then we can set* $D = \mathbb{C}_+$ *and* $\forall [0, \beta] \subseteq \bar{\mathcal{I}}$

$$\int_0^\beta V^*_\lambda (t) W_\lambda (t) V_\lambda (t) dt \le \frac{\Im m_{a,b}(\lambda)}{\Im\lambda} \ (\Im\lambda \neq 0).$$

For the construction of the solutions of Weyl type and the description of the Weyl function in various situations, see [1, 22] and references in [1].

Let us consider the operator differential expression $l_\lambda[y]$ of (6) type with the coefficients $p_j = p_j(t,\lambda)$, $q_j = q_j(t,\lambda)$, $s_j = s_j(t,\lambda)$ and of order r. Let $-l_\lambda$ depend on λ in Nevanlinna manner. Namely, from now on the condition below holds.

(**B**) The set $\mathcal{B} \supseteq \mathbb{C} \setminus \mathbb{R}^1$ exists, every its point has a neighbourhood independent of $t \in \bar{\mathcal{I}}$, in this neighbourhood, the coefficients $p_j = p_j(t,\lambda)$, $q_j = q_j(t,\lambda)$, $s = s_j(t,\lambda)$ of the expression l_λ are analytic $\forall t \in \bar{\mathcal{I}}$; $\forall \lambda \in \mathcal{B}$, $p_j(t,\lambda)$, $q_j(t,\lambda)$, $s_j(t,\lambda) \in C^j\left(\bar{\mathcal{I}}, B(\mathcal{H})\right)$ and

$$\begin{gathered} p_n^{-1}(t,\lambda) \in B(\mathcal{H}), \ r = 2n, \\ (q_{n+1}(t,\lambda) + s_{n+1}(t,\lambda))^{-1} \in B(\mathcal{H}), \ r = 2n+1, \ t \in \bar{\mathcal{I}}; \end{gathered} \tag{70}$$

these coefficients satisfy the following conditions:

$$p_j(t,\lambda) = p^*_j\left(t,\bar{\lambda}\right), q_j(t,\lambda) = s^*_j\left(t,\bar{\lambda}\right), \ \lambda \in \mathcal{B} \tag{71}$$

*The norms $\|\cdot\|_{L^2_{W_\lambda}(\mathcal{I})}$ are equivalent for $\lambda \in \mathcal{A}$ [22].

$\big((71) \iff l_\lambda = l^*_{\bar\lambda} \underset{\text{in view of (18)}}{\iff} H(t,l_\lambda) = H(t,l^*_{\bar\lambda}), \lambda \in \mathcal{B}\big)$. $\forall h_0, \ldots, h_{\left[\frac{r+1}{2}\right]} \in \mathcal{H}$:

$$\frac{\Im\left(\sum\limits_{j=0}^{[r/2]} (p_j(t,\lambda) h_j, h_j) + \frac{i}{2}\sum\limits_{j=1}^{\left[\frac{r+1}{2}\right]} \{(s_j(t,\lambda) h_j, h_{j-1}) - (q_j(t,\lambda) h_{j-1}, h_j)\}\right)}{\Im\lambda} \le 0,$$

$$t \in \bar{\mathcal{I}}, \;\; \Im\lambda \ne 0. \quad (72)$$

Thus the order of expression $\Im l_\lambda$ is even and therefore if $r = 2n+1$ is odd, then q_{m+1}, s_{m+1} are independent of λ and $s_{n+1} = q^*_{n+1}$.

Condition (72) is equivalent to the condition $(\Im l_\lambda)\{f,f\}/\Im\lambda \le 0$, $t \in \bar{\mathcal{I}}$, $\Im\lambda \ne 0$.

Hence $\frac{\Im H(t,l_\lambda)}{\Im\lambda} = W\left(t, l_\lambda, -\frac{\Im l_\lambda}{\Im\lambda}\right) \ge 0$, $t \in \bar{\mathcal{I}}$, $\Im\lambda \ne 0$ due to Lemma 1.1 and Theorem 1.2, and therefore $H(t,l_\lambda)$ satisfies condition **(A)** with $\mathcal{A} = \mathcal{B}$. Therefore $\forall\mu \in \mathcal{B} \cap \mathbb{R}^1$ $W(t, l_\mu, -\frac{\Im l_\mu}{\Im\mu}) = \left.\frac{\partial H(t,l_\lambda)}{\partial\lambda}\right|_{\lambda=\mu}$ is Bochner locally integrable in the uniform operator topology. Here, in view of (8), (16) $\forall\mu \in \mathcal{B} \bigcap \mathbb{R}^1$ $\exists \frac{\Im l_\mu}{\Im\mu} \overset{def}{=} \frac{\Im l_{\mu+i0}}{\Im(\mu+i0)} = \left.\frac{\partial l_\lambda}{\partial\lambda}\right|_{\lambda=\mu}$, where the coefficients $\frac{\partial p_j(t,\mu)}{\partial\lambda}$, $\frac{\partial q_j(t,\mu)}{\partial\lambda}$, $\frac{\partial s_j(t,\mu)}{\partial\lambda}$ of the expression $\partial l_\mu/\partial\mu$ are Bochner locally integrable in the uniform operator topology.

In $\mathcal{H}_1 = \mathcal{H}^r$, let us consider the equation

$$\frac{i}{2}\Big(\left(Q(t,l_\lambda)\vec{y}(t)\right)' + Q^*(t,l_\lambda)\vec{y}\,'(t)\Big) - H(t,l_\lambda)\vec{y}(t) = W\left(t, l_\lambda, -\frac{\Im l_\lambda}{\Im\lambda}\right) F(t). \tag{73}$$

This equation is an equation of (52) type. Equation (5) is equivalent to Eq. (73) with $F(t) = F\left(t, l_{\bar\lambda}, -\frac{\Im l_\lambda}{\Im\lambda}\right)$ due to Theorem 1.1 and (19).

Definition 2.4. *Every characteristic operator of Eq.* (73) *corresponding to Eq.* (5) *is said to be a characteristic operator of Eq.* (5) *on* $\mathcal{I}$.

Let $m[y]$ be the same as in the $n°1$ differential expression of even order $s \le r$ with the operator coefficients $\tilde p_j(t) = \tilde p^*_j(t)$, $\tilde q_j(t)$, $\tilde s_j(t) = \tilde q^*_j(t)$ that are independent of λ. Let $\forall h_0, \ldots, h_{\left[\frac{r+1}{2}\right]} \in \mathcal{H}$:

$$0 \le \sum_{j=0}^{s/2} (\tilde p_j(t) h_j, h_j) + \Im \sum_{j=1}^{s/2} (\tilde q_j(t) h_{j-1}, h_j)$$

$$\leq -\frac{\Im\left(\sum\limits_{j=0}^{[r/2]}\left(p_j\left(t,\lambda\right)h_j,h_j\right)+\frac{i}{2}\sum\limits_{j=1}^{\left[\frac{r+1}{2}\right]}\left(\left(s_j\left(t,\lambda\right)h_j,h_{j-1}\right)-\left(q_j\left(t,\lambda\right)h_{j-1},h_j\right)\right)\right)}{\Im\lambda},$$

$$t\in\bar{\mathcal{I}},\ \ \Im\lambda\neq 0. \quad (74)$$

Condition (74) is equivalent to the condition $0\leq m\left\{f,f\right\}\leq -(\Im l_\lambda)\left\{f,f\right\}/\Im\lambda$, $t\in\bar{\mathcal{I}}$, $\Im\lambda\neq 0$. Hence

$$0\leq W\left(t,l_\lambda,m\right)\leq W\left(t,l_\lambda,-\frac{\Im l_\lambda}{\Im\lambda}\right)=\frac{\Im H\left(t,l_\lambda\right)}{\Im\lambda}\quad t\in\bar{\mathcal{I}},\ \ \Im\lambda\neq 0, \qquad (75)$$

due to Theorem 1.2 and Lemma 1.1.

In view of Theorem 1.1, Eq. (1) is equivalent to the equation

$$\frac{i}{2}\left(\left(Q\left(t,l_\lambda\right)\vec{y}\left(t\right)\right)'+Q^*\left(t,l_\lambda\right)\vec{y}\,'\left(t\right)\right)-H\left(t,l_\lambda\right)\vec{y}\left(t\right)=W\left(t,l_{\bar\lambda},m\right)F\left(t,l_{\bar\lambda},m\right), \qquad (76)$$

where $Q\left(t,l_\lambda\right)$, $H\left(t,l_\lambda\right)$ are defined by (7), (8), (15), (16) with l_λ instead of l, and $W\left(t,l_{\bar\lambda},m\right)$, $F\left(t,l_{\bar\lambda},m\right)$ are defined by (9), (17) (26) with $l_{\bar\lambda}$ instead of l, and $\vec{y}\left(t\right)=\vec{y}\left(t,l_\lambda,m,f\right)$ is defined by (28) with l_λ instead of l.

In some cases we will suppose additionally that $\exists\lambda_0\in\mathcal{B};\ \alpha,\beta\in\bar{\mathcal{I}},\ 0\in[\alpha,\beta]$, the number $\delta>0$:

$$-\int\limits_\alpha^\beta\left(\frac{\Im l_{\lambda_0}}{\Im\lambda_0}\right)\left\{y\left(t,\lambda_0\right),y\left(t,\lambda_0\right)\right\}\,dt\geq\delta\left\|P\vec{y}\left(0,l_{\lambda_0},m,0\right)\right\|^2 \qquad (77)$$

for any solution $y\left(t,\lambda_0\right)$ of (5) as $\lambda=\lambda_0$, $f=0$, where $P\in B(\mathcal{H}^r)$ is the orthoprojection onto subspace $N^\perp$ which corresponds to Eq. (73). In view of Theorem 1.2, this condition is equivalent to the fact that for the equation (73)

$\exists\lambda_0\in\mathcal{A}=\mathcal{B};\ \alpha,\beta\in\bar{\mathcal{I}},\ 0\in[\alpha,\beta]$, the number $\delta>0$:

$$\left(\Delta_{\lambda_0}\left(\alpha,\beta\right)g,g\right)\geq\delta\left\|Pg\right\|^2,\quad g\in\mathcal{H}^r. \qquad (78)$$

Therefore, in view of [22], the fulfillment of (77) implies its fulfillment with $\delta\left(\lambda\right)>0$ instead of δ for all $\lambda\in\mathcal{B}$.

Lemma 2.1. *Let $M\left(\lambda\right)$ be a characteristic operator of Eq. (5) for which condition (77) holds with $P=I_r$ if $\mathcal{I}$ is infinite. Let $\Im\lambda\neq 0$, $\mathcal{H}^r$-valued $F\left(t\right)\in L^2_{W\left(t,l_{\bar\lambda},m\right)}\left(\mathcal{I}\right)$ (in particular, one can set $F\left(t\right)=F\left(t,l_{\bar\lambda},m\right)$, where*

$f(t) \in C^s(\mathcal{I}, \mathcal{H}), m[f,f] < \infty$). *Then the solution*

$$x_\lambda(t,F) = \mathcal{R}_\lambda F$$
$$= \int\limits_{\mathcal{I}} X_\lambda(t) \left\{ M(\lambda) - \frac{1}{2} sgn(s-t)(iG)^{-1} \right\} X_{\bar\lambda}^*(s) W(s, l_{\bar\lambda}, m) F(s)\, ds \tag{79}$$

of Eq. (76), *with* $F(t)$ *instead of* $F(t, l_{\bar\lambda}, m)$, *satisfies the following inequality:*

$$\|\mathcal{R}_\lambda F\|^2_{L^2_{W\left(t, l_\lambda, -\frac{\Im l_\lambda}{\Im\lambda}\right)}(\mathcal{I})} \le \Im(\mathcal{R}_\lambda F, F)_{L^2_{W\left(t, l_{\bar\lambda}, m\right)}(\mathcal{I})} / \Im\lambda, \ \Im\lambda \neq 0, \tag{80}$$

where $X_\lambda(t)$ *is the operator solution of homogeneous equation* (76) *such that* $X_\lambda(0) = I_r$, $G = \mathcal{R}Q(0, l_\lambda)$; *integral* (79) *converges strongly if* $\mathcal{I}$ *is infinite.*

P r o o f. Let us denote

$$K(t,s,\lambda) = X_\lambda(t) \left\{ M(\lambda) - \frac{1}{2} \operatorname{sgn}(s-t)(iG)^{-1} \right\} X_{\bar\lambda}^*(s).$$

If (77) holds with $P = I_r$ when $\mathcal{I}$ is infinite, then, in view of (75) and [22, p.166], there exists a locally bounded on s and on λ constant $k(s,\lambda)$ such that

$$\forall h \in \mathcal{H}^r: \quad \|K(t,s,\lambda) h\|_{L^2_{W\left(t, l_{\bar\lambda}, m\right)}(\mathcal{I})} \le k(s,\lambda) \|h\|. \tag{81}$$

Hence integral (79) converges strongly if $\mathcal{I}$ is infinite.

Let $F(t)$ have a compact support and $\operatorname{supp} F(t) \subseteq [\alpha, \beta]$.

Then, in view of (42),

$$\int\limits_\alpha^\beta \left(W\left(t, l_\lambda, -\frac{\Im l_\lambda}{\Im\lambda}\right) \mathcal{R}_\lambda F, \mathcal{R}_\lambda F \right) dt - \frac{\Im \int_\alpha^\beta (W(t, l_{\bar\lambda}, m) \mathcal{R}_\lambda F, F)\, dt}{\Im\lambda}$$
$$= \frac{1}{2} \frac{([\Re Q(t, l_\lambda)] \mathcal{R}_\lambda F, \mathcal{R}_\lambda F)}{\Im\lambda} \Bigg|_\alpha^\beta \le 0, \tag{82}$$

where the last inequality is a corollary of $n^\circ 2$ Theorem 1.1. from [22, p. 162] and the lemma below.

Lemma 2.2. *Let* $\mathcal{F}_\lambda$ *be the set of the* $\mathcal{H}^r$*-valued functions from* $L^2_{W(t, l_{\bar\lambda}, m)}(\alpha, \beta)$,

$$I_\lambda(\alpha, \beta) F = \int\limits_\alpha^\beta X_{\bar\lambda}^*(t) W(t, l_{\bar\lambda}, m) F(t)\, dt, \quad F(t) \in \mathcal{F}_\lambda. \tag{83}$$

Then

$$I_\lambda(\alpha,\beta)\,F\in\left\{Ker\int\limits_\alpha^\beta X^*_{\bar\lambda}(t)\,W(t,l_{\bar\lambda},m)\,X_{\bar\lambda}(t)\,dt\right\}^\perp\subseteq N^\perp. \qquad (84)$$

P r o o f. Let $h\in Ker\int_\alpha^\beta X^*_{\bar\lambda}(t)\,W(t,l_{\bar\lambda},m)\,X_{\bar\lambda}(t)\,dt\Rightarrow W(t,l_{\bar\lambda},m)\,X_{\bar\lambda}(t)\,h=0\Rightarrow I_\lambda(\alpha,\beta)\,F\perp h$. The second enclosure in (84) is a corollary of condition (75). Lemma 2.2 and inequality (82) are proved. ■

Thus Lemma 2.1 is proved if $\mathcal{I}$ is finite. Let us prove it for the infinite $\mathcal{I}$. Let the finite intervals $(\alpha_n,\beta_n)\uparrow\mathcal{I}$, $\quad F_n=\chi_nF$, where χ_n is a characteristic function of (α_n,β_n). If $(\alpha,\beta)\subseteq(\alpha_n,\beta_n)$, then

$$\|\mathcal{R}_\lambda F_n\|_{L^2_{W\left(t,l_\lambda,-\frac{\Im l_\lambda}{\Im\lambda}\right)}}(\alpha,\beta)\le\frac{\|F\|_{L^2_{W\left(t,l_{\bar\lambda},m\right)}}(\mathcal{I})}{|\Im\lambda|}$$

in view of (82), (75). But local uniformly on t: $(\mathcal{R}_\lambda F_n)(t)\to(\mathcal{R}_\lambda F)(t)$ in view of (81). Hence,

$$\|\mathcal{R}_\lambda F\|_{L^2_{W\left(t,l_\lambda,-\frac{\Im l_\lambda}{\Im\lambda}\right)}(\alpha,\beta)}\le\frac{\|F\|_{L^2_{W\left(t,l_{\bar\lambda},m\right)}}(\mathcal{I})}{|\Im\lambda|} \qquad (85)$$

for any finite (α,β). Thus (85) holds with $\mathcal{I}$ instead of (α,β). In view of the last fact, $\mathcal{R}_\lambda F_n\to\mathcal{R}_\lambda F$ in $L^2_{W\left(t,l_\lambda,-\frac{\Im l_\lambda}{\Im\lambda}\right)}(\mathcal{I})$. Hence (80) is proved since it is proved for F_n. Lemma 2.1 is proved. ■

Let us notice that in view of [22], $PM(\lambda)\,P$ is a characteristic operator of Eq. (5), if $M(\lambda)$ is its characteristic operator. Obviously the closures of operators $\mathcal{R}_\lambda$ corresponding to the characteristic operators $M(\lambda)$ and $PM(\lambda)\,P$ are equal in $B\left(L^2_{W(t,l_{\bar\lambda},m)}(\mathcal{I})\,,\,L_{W\left(t,l_\lambda,-\frac{\Im l_\lambda}{\Im\lambda}\right)}(\mathcal{I})\right)$.

Let us notice that in view of (74), l_λ can be a represented in the form of (2), where

$$l=\Re l_i,n_\lambda=l_\lambda-l-\lambda m;\Im n_\lambda\{f,f\}/\Im\lambda\ge0,t\in\bar{\mathcal{I}},\,\Im\lambda\ne0. \qquad (86)$$

From now on we will suppose that l_λ has representation (2), (86), and therefore the order of n_λ is even.

3. Main Results

We consider the pre-Hilbert spaces $\overset{\circ}{H}$ and H of the vector-functions $y(t) \in C_0^s(\bar{\mathcal{I}}, \mathcal{H})$ and $y(t) \in C^s(\bar{\mathcal{I}}, \mathcal{H})$, $m[y(t), y(t)] < \infty$, respectively, with a scalar product

$$(f(t), g(t))_m = m[f(t), g(t)],$$

where $m[f, g]$ is defined by (32) with the expression $m[y]$ from condition (74) instead of $L[y]$. Namely,

$$m[f, g] = \int_{\mathcal{I}} m\{f, g\}\, dt, \tag{87}$$

where

$$m\{f, g\} = \sum_{j=0}^{s/2} (\tilde{p}_j(t) f^{(j)}(t), g^{(j)}(t))$$

$$+\frac{i}{2}\sum_{j=1}^{s/2}\Big((\tilde{q}_j^*(t) f^{(j)}(t), g^{(j-1)}(t)) - (\tilde{q}_j(t) f^{(j-1)}(t), g^{(j)}(t))\Big).$$

The null elements of H are given by

Proposition 3.1. *Let $f(t) \in H$. Then*

$$m[f, f] = 0 \Leftrightarrow m[f] = f^{[s]}(t) = \dots = f^{[s/2]}(t) = 0, \ \ t \in \bar{\mathcal{I}}.$$

P r o o f. Let us denote by $m(t) \in B(\mathcal{H}^{n+1})$ the operator matrix corresponding to the quadratic form in the right-hand side of first inequality (74). Since $m(t) \geq 0$, one has in view of (12), (13)

$$m[f, f] = 0 \Leftrightarrow m(t)\, col\Big\{f(t), \dots, f^{(s/2)}, 0, \dots, 0\Big\} = 0 \Leftrightarrow f^{[s]}(t) = \dots = f^{[s/2]} = 0.$$

■

E x a m p l e 3.1. Let $\dim \mathcal{H} = 1$, $s = 2$, $\tilde{p}_1(t) > 0$, $|\tilde{q}_1(t)|^2 = 4\tilde{p}_1(t)\tilde{p}_0(t)$. Then for the expression $m[y]$ the first inequality (74) holds, and $m\{f_0, f_0\} \equiv 0$ for $f_0(t) = \exp\left(\frac{i}{2}\int_0^t \tilde{q}_1/\tilde{p}_1 dt\right) \neq 0$ in view of Proposition 3.1.

By $\overset{\circ}{L}{}^2_m(\mathcal{I})$ and $L^2_m(\mathcal{I})$ we denote the completions of the spaces $\overset{\circ}{H}$ and H in the norm $\|\bullet\|_m = \sqrt{(\bullet, \bullet)_m}$, respectively. By $\overset{\circ}{P}$, we denote the orthoprojection in $L^2_m(\mathcal{I})$ onto $\overset{\circ}{L}{}^2_m(\mathcal{I})$.

Theorem 3.1. *Let $M(\lambda)$ be a characteristic operator of Eq.* (5) *for which condition* (77) *with $P = I_r$ holds if $\mathcal{I}$ is infinite. Let $\Im\lambda \neq 0$, $f(t) \in H$, and*

$$col\{y_j(t,\lambda,f)\} = \int\limits_{\mathcal{I}} X_\lambda(t)\left\{M(\lambda) - \frac{1}{2}\operatorname{sgn}(s-t)(iG)^{-1}\right\}$$
$$\times X^*_{\bar\lambda}(s)\,W(s,l_{\bar\lambda},m)\,F(s,l_{\bar\lambda},m)\,ds, \quad y_j \in \mathcal{H} \tag{88}$$

be a solution of Eq. (76) *which corresponds to Eq.* (1), *where $X_\lambda(t)$ is the operator solution of homogeneons Eq.* (76) *such that $X_\lambda(0) = I_r$; $G = \Re Q(0,l_\lambda)$ (if $\mathcal{I}$ is infinite, then integral* (88) *converges strongly). Then the first component of vector function* (88) *is a solution of Eq.* (1)*. It defines densely defined in $L^2_m(\mathcal{I})$ integro-differential operator*

$$R(\lambda)f = y_1(t,\lambda,f), \quad f \in H, \tag{89}$$

which has the following properties after closing:
1°

$$R^*(\lambda) = R(\bar\lambda), \ \ \Im\lambda \neq 0, \tag{90}$$

2°

$$R(\lambda) \ \textit{is holomorphic on}\ \mathbb{C}\setminus\mathbb{R}^1, \tag{91}$$

3°

$$\|R(\lambda)f\|^2_{L^2_m(\mathcal{I})} \le \frac{\Im(R(\lambda)f,f)_{L^2_m(\mathcal{I})}}{\Im\lambda}, \ \ \Im\lambda \neq 0, \ \ f \in L^2_m(\mathcal{I}). \tag{92}$$

Notice that the definition of the operator $R(\lambda)$ is correct. Indeed, if $f(t) \in H$, $m[f,f] = 0$, then $R(\lambda)f \equiv 0$ since $W(t,l_{\bar\lambda},m)F(t,l_{\bar\lambda},m) \equiv 0$ due to (34), (75).

P r o o f. In view of Lemma 2.1, integral (88) converges strongly if $\mathcal{I}$ is infinite. In view of Theorem 1.1, $y_1(t,\lambda,f)$ (89) is a solution of Eq. (1).

In view of (74), (35),

$$\|R(\lambda)f\|_{L^2_m(\alpha,\beta)} - \frac{\Im(R(\lambda)f,f)_{L^2_m(\alpha,\beta)}}{\Im\lambda} \le \|R(\lambda)f\|_{L^2_{-\frac{\Im l_\lambda}{\Im\lambda}}(\alpha,\beta)}$$
$$-\frac{\Im(R(\lambda)f,f)_{L^2_m(\alpha,\beta)}}{\Im\lambda} = \|\mathcal{R}_\lambda F(t,l_{\bar\lambda},m)\|_{L^2_{W\left(t,l_\lambda,-\frac{\Im l_\lambda}{\Im\lambda}\right)}(\alpha,\beta)}$$
$$-\frac{\Im(\mathcal{R}_\lambda F(t,l_{\bar\lambda},m), F(t,,l_{\bar\lambda},m))_{L^2_{W\left(t,l_{\bar\lambda},m\right)}(\alpha,\beta)}}{\Im\lambda}. \tag{93}$$

In view of Lemma 2.1, a nonnegative limit of the right-hand side of (93) exists, when $(\alpha,\beta)\uparrow\mathcal{I}$. Hence (92) is proved.

Let $\mathcal{H}^r$-valued $F(t)\in L^2_{W(t,l_{\bar\lambda},m)}(\mathcal{I})$. Then, in view of (75), Lemma 2.1, (19), one has

$$\|\mathcal{R}_\lambda F\|^2_{L^2_{W(t,l_\lambda,m)}(\mathcal{I})}\le\|\mathcal{R}_\lambda F\|^2_{L^2_{W\left(t,l_\lambda,-\frac{\Im l}{\Im\lambda}\right)}(\mathcal{I})}\le\frac{\Im\left(\mathcal{R}_\lambda F,F\right)_{L^2_{W\left(t,l_{\bar\lambda},m\right)}(\mathcal{I})}}{\Im\lambda},\tag{94}$$

$$\|\mathcal{R}_\lambda F\|^2_{L^2_{W\left(t,l_{\bar\lambda},m\right)}(\mathcal{I})}\le\|\mathcal{R}_\lambda F\|_{L^2_{W\left(t,l_{\bar\lambda},-\frac{\Im l_\lambda}{\Im\lambda}\right)}(\mathcal{I})}=\|\mathcal{R}_\lambda F\|_{L^2_{W\left(t,l_\lambda,-\frac{\Im l_\lambda}{\Im\lambda}\right)}(\mathcal{I})}.\tag{95}$$

In view of (94), (95), we have

$$\|\mathcal{R}_\lambda F\|_{L^2_{W(t,l_\lambda,m)}(\mathcal{I})}\le\|F\|_{L^2_{W\left(t,l_{\bar\lambda},m\right)}(\mathcal{I})}\Big/|\Im\lambda|,\tag{96}$$

$$\|\mathcal{R}_\lambda F\|_{L^2_{W\left(t,l_{\bar\lambda},m\right)}(\mathcal{I})}\le\|F\|_{L^2_{W\left(t,l_{\bar\lambda},m\right)}(\mathcal{I})}\Big/|\Im\lambda|.\tag{97}$$

Let $F\left(t\right)\in L^2_{W(t,l_{\bar\lambda},m)}\left(\mathcal{I}\right)$, $G\left(t\right)\in L^2_{W(t,l_\lambda,m)}\left(\mathcal{I}\right)$ be the $\mathcal{H}^r$-valued functions with a compact support. We have

$$\left(\mathcal{R}_\lambda F,G\right)_{L^2_{W(t,l_\lambda,m)}(\mathcal{I})}=\left(F,\mathcal{R}_{\bar\lambda},G\right)_{L^2_{W\left(t,l_{\bar\lambda},m\right)}(\mathcal{I})}\tag{98}$$

since $M\left(\lambda\right)=M^*\left(\bar\lambda\right)$. Due to inequalities (96), (97), equality (98) is valid for $F\left(t\right),G\left(t\right)$ with a non-compact support.

Now it follows from (36), (98) that $\forall f\left(t\right),g\left(t\right)\in H$,

$$\begin{aligned}m\left[R\left(\lambda\right)f,g\right]-m\left[f,R\left(\bar\lambda\right)g\right]&=\left(\mathcal{R}_\lambda F\left(t,l_{\bar\lambda},m\right),G\left(t,l_\lambda,m\right)\right)_{L^2_{W(t,l_\lambda,m)}(\mathcal{I})}\\&-\left(F\left(t,l_{\bar\lambda},m\right),\mathcal{R}_{\bar\lambda}G\left(t,l_\lambda,m\right)\right)_{L^2_{W\left(t,l_{\bar\lambda},m\right)}(\mathcal{I})}=0.\end{aligned}$$

Thus the closure of the operator $R\left(\lambda\right)f$ in $L^2_m\left(\mathcal{I}\right)$ possesses property (90).

Since in view of (92) for any $f\left(t\right),g\left(t\right)\in H$,

$$\left(R\left(\lambda\right)f,g\right)_{L^2_m(\alpha,\beta)}\to\left(R\left(\lambda\right)f,g\right)_{L^2_m(\mathcal{I})}\ \text{ as }\ (\alpha,\beta)\uparrow\mathcal{I}$$

uniformly in λ from any compact set from $\mathbb{C}\backslash\mathbb{R}^1$, we can see that, in view of the analyticity of the operator function $M\left(\lambda\right)$ and vector-function $W\left(t,l_{\bar\lambda},m\right)\,F\left(t,l_{\bar\lambda}\right)$ (see (29) with $l=l_\lambda$), the operator $R\left(\lambda\right)$ depends analytically on the non-real λ in view of [19, p. 195]. Theorem 3.1 is proved. ■

For $r = 1$, $n_\lambda[y] = H_\lambda(t)y$, Theorem 3.1 is known [21].

Notice that if $L^2_m(\mathcal{I}) = \overset{\circ}{L}{}^2_m(\mathcal{I})$, then Theorem 3.1 is valid with $f(t) \in \overset{\circ}{H}$ instead of $f(t) \in H$ and without condition (77) with $P = I_r$ for infinite $\mathcal{I}$.

The following theorem establishes a relationship between the resolvents $R(\lambda)$ given by Theorem 3.1 and the boundary value problems for Eq. (1), (2) with the boundary conditions depending on the spectral parameter. Similarly to the case $n_\lambda[y] \equiv 0$ [24], the pair $\{y, f\}$ satisfies the boundary conditions that contain both y and f derivatives of the corresponding orders at the ends of the interval.

Theorem 3.2. *Let the interval $\mathcal{I} = (a, b)$ be finite and condition (77) with $P = I_r$ hold.*

Let the operator-functions $\mathcal{M}_\lambda, \mathcal{N}_\lambda \in B(\mathcal{H}^r)$ depend analytically on the non-real λ,

$$\mathcal{M}^*_{\bar\lambda}\left[\Re Q(a, l_\lambda)\right]\mathcal{M}_\lambda = \mathcal{N}^*_{\bar\lambda}\left[\Re Q(b, l_\lambda)\right]\mathcal{N}_\lambda \quad (\Im\lambda \neq 0), \tag{99}$$

where $Q(t, l_\lambda)$ is the coefficient of Eq. (76) corresponding by Theorem 1.1 to Eq. (1),

$$\|\mathcal{M}_\lambda h\| + \|\mathcal{N}_\lambda h\| > 0 \quad (0 \neq h \in \mathcal{H}^r,\ \Im\lambda \neq 0), \tag{100}$$

the lineal $\{\mathcal{M}_\lambda h \oplus \mathcal{N}_\lambda h \,|h \in \mathcal{H}^r\} \subset \mathcal{H}^{2r}$ is a maximal $\mathcal{Q}$-nonnegative subspace if $\Im\lambda \neq 0$, where $\mathcal{Q} = (\Im\lambda)\,\mathrm{diag}\left(\Re Q(a, l_\lambda), -\Re Q(b, l_\lambda)\right)$ (and therefore

$$\Im\lambda\left(\mathcal{N}^*_\lambda\left[\Re Q(b, l_\lambda)\right]\mathcal{N}_\lambda - \mathcal{M}^*_\lambda\left[\Re Q(a, l_\lambda)\right]\mathcal{M}_\lambda\right) \leq 0 \quad (\Im\lambda \neq 0)). \tag{101}$$

Then:

1°. *For any $f(t) \in H$, the boundary problem obtained by adding the boundary conditions*

$$\exists h = h(\lambda, f) \in \mathcal{H}^r : \ \vec{y}(a, l_\lambda, m, f) = \mathcal{M}_\lambda h, \ \ \vec{y}(b, l_\lambda, m, f) = \mathcal{N}_\lambda h \tag{102}$$

to Eq. (1), where $\vec{y}(t, l_\lambda, m, f)$ is defined by (28) with l_λ instead of l, has the unique solution $R(\lambda) f$ in $C^r(\bar{\mathcal{I}}, \mathcal{H})$ as $\Im\lambda \neq 0$. It is generated by the resolvent $R(\lambda)$ constructed, as in Theorem 3.1, by using the characteristic operator

$$M(\lambda) = -\frac{1}{2}\left(X^{-1}_\lambda(a)\mathcal{M}_\lambda + X^{-1}_\lambda(b)\mathcal{N}_\lambda\right)\left(X^{-1}_\lambda(a)\mathcal{M}_\lambda - X^{-1}_\lambda(b)\mathcal{N}_\lambda\right)^{-1}(iG)^{-1}, \tag{103}$$

where

$$\left(X^{-1}_\lambda(a)\mathcal{M}_\lambda - X^{-1}_\lambda(b)\mathcal{N}_\lambda\right)^{-1} \in B(\mathcal{H}^r) \quad (\Im\lambda \neq 0),$$

$X_\lambda(t)$ is an operator solution of the homogeneous Eq. (76) such that $X_\lambda(0) = I_r$.

2°. *For any operator $R(\lambda)$ from Theorem 3.1, the vector-function $R(\lambda)f$ ($f \in H$) is a solution of some boundary problem as in 1°.*

Notice that if $f(t) \overset{H}{=} g(t)$, then in boundary conditions (102): $\vec{y}(t,l,m,f) = \vec{y}(t,l,m,g)$ in view of (28) and Proposition 3.1.

P r o o f. The proof of Theorem 3.2 follows from Theorems 1.1, 3.1 and from [22, Remark 1.1]. ∎

For the case $n_\lambda[y] \equiv 0$, Theorem 3.2 is known (see [22], [24]).

The example below shows that the following is possible: for some resolvent $R(\lambda)$ from Theorem 3.1 $\exists f_0(t) \overset{H}{\neq} 0$ such that $m[f_0] = 0$ and therefore the "resolvent" Eq. (1) for $R(\lambda) f_0$ is homogeneous but $R(\lambda) f_0 \overset{H}{\neq} 0$, $\Im\lambda \neq 0$.

E x a m p l e 3.2. Let $r = 2n$, $m[f]$ in (1) be an expression such that the equation $m[f] = 0$ has a solution $f_0(t) \overset{H}{\neq} 0$. Let in Theorem 3.2: $\mathcal{M}_\lambda = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$, $\mathcal{N}_\lambda = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}$, $R(\lambda)$ be the corresponding resolvent. Then $R(\lambda)f_0 \neq 0, \Im\lambda \neq 0$, while if $\mathcal{M}_\lambda = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}$, $\mathcal{N}_\lambda = \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}$, then for the corresponding resolvent $R(\lambda) f_0 \overset{H}{=} 0$, $\Im\lambda \neq 0$ (and therefore in view of [16, p. 87], $E_\infty f_0 = 0$ for the generalized spectral family E_μ which corresponds to $R(\lambda)$ by (3)).

It is known [16, p. 86] that the operator-function $R(\lambda)$ (90)–(92) can be represented in the form

$$R(\lambda) = (T(\lambda) - \lambda)^{-1}, \tag{104}$$

where $T(\lambda)$ is a linear relation such that

$$\Im T(\lambda) \leq 0 \ (\max), \ T(\bar{\lambda}) = T^*(\lambda), \ \lambda \in \mathbb{C}^+,$$

the Cayley transform $C_\mu(T(\lambda))$ defines a holomorphic function in $\lambda \in \mathbb{C}_+$ for some (and hence for all) $\mu \in \mathbb{C}_+$. The applications of abstract relations of $T(\lambda)$ type (Nevanlinna families) to the theories of boundary relations and of generalized resolvents are proposed in [12, 13].

The description of $T(\lambda)$ corresponding to $R(\lambda)$ from Theorem 3.1 in the regular case gives

Corollary 3.1. *Let $\mathcal{I}$ be finite and condition* (77) *with $P = I_r$ hold. Let us consider the relation $T(\lambda) = \overline{T'(\lambda)}$ as $\Im\lambda \neq 0$, where*

$$T'(\lambda) = \left\{ \left\{ \tilde{y}(t), \tilde{f}(t) \right\} \middle| \, \tilde{y}(t) \overset{L^2_m(\mathcal{I})}{=} y(t) \in C^r(\bar{\mathcal{I}}), \tilde{f}(t) \overset{L^2_m(\mathcal{I})}{=} f(t) \in H, \right.$$

$$(l - n_\lambda)[y] = m[f], \vec{y}(t, l - n_\lambda, m, f) \textit{ satisfy the boundary condition}$$
$$\exists h = h(\lambda, f) \in \mathcal{H}^r : \vec{y}(a, l - n_\lambda, m, f) = \mathcal{M}_\lambda h,\ \vec{y}(b, l - n_\lambda, m, f) = \mathcal{N}_\lambda h,$$
$$\textit{where the operators } \mathcal{M}_\lambda, \mathcal{N}_\lambda \textit{ satisfy the conditions of Theorem 3.2,}$$
$$\vec{y}(t, l - n_\lambda, m, f) \overset{def}{=} \vec{y}(t, l_\lambda, m, f)|_{m=0\, in\, l_\lambda} =$$
$$\left\{\begin{array}{ll} \left(\sum\limits_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \sum\limits_{j=1}^{n} \oplus \left(y^{[s'-j]}(t\,|l - n_\lambda) - f^{[s-j]}(t\,|m)\right), & r = 2n, \\ \left(\sum\limits_{j=0}^{n-1} \oplus y^{(j)}(t)\right) \oplus \left(\sum\limits_{j=1}^{n} \oplus \left(y^{[r-j]}(t\,|l - n_\lambda) - f^{[s-j]}(t\,|m)\right)\right) & \\ \oplus \left(-iy^{(n)}(t)\right), & r = 2n + 1 > 1, \\ \left(\textit{here } s' = \textit{order of expression } l - n_\lambda,\right. & \\ y^{[0]}(t|l - n_\lambda) = -\frac{i}{2}(q_1(t, \lambda) + \lambda \tilde{q}_1(t))y \textit{ as } s' = 1, & \\ \left. y^{[k']}(t\,|l - n_\lambda) \equiv 0 \textit{ as } k' < \left[\frac{s'+1}{2}\right],\ f^{[k]}(t\,|m) \equiv 0 \textit{ as } k < \frac{s}{2}\right) & \\ y(t), & r = 1, \end{array}\right\}. \tag{105}$$

Then:

1°. $(T(\lambda) - \lambda)^{-1}$ *is equal to the resolvent* $R(\lambda)$ (88), (89) *from Theorem 3.1 corresponding to the characteristic operator* $M(\lambda)$ (103).

2°. *Let* $R(\lambda)$ *be resolvent* (88), (89) *from Theorem 3.1. Then* $R(\lambda) = (T(\lambda) - \lambda)^{-1}$, *where* $T(\lambda)$ *is some relation as in item 1°.*

P r o o f. The proof follows from (28), Lemma 1.2, Theorem 3.2 and Remark 1.1 from [22]. ■

Let in (1), (2) $n_\lambda[y] \equiv 0$ i.e. $l_\lambda = l - \lambda m$, where $l = l^*$, $m = m^*$, and the coefficients of the expressions m satisfy first inequality (74).

In $L^2_m(\mathcal{I})$, we consider the linear relation

$$\mathcal{L}'_0 = \Big\{\{\tilde{y}(t), \tilde{g}(t)\}\, |\tilde{y}(t) \overset{L^2_m(\mathcal{I})}{=} y(t),\ \tilde{g}(t) \overset{L^2_m(\mathcal{I})}{=} g(t),\ y(t) \in C^r(\bar{\mathcal{I}}, \mathcal{H}),\ g(t) \in H,$$
$$l[y] = m[g], \vec{y}(t, l, m, g) \text{ is equal to zero in the end of } \mathcal{I} \text{ if this end is finite and}$$
$$\vec{y}(t, l, m, g) \text{ is equal to zero in the neighbourhood of the end of } \mathcal{I} \text{ if this end}$$
$$\text{is infinite}\Big\} \tag{106}$$

where $\vec{y}(t, l, m, g)$ is defined (in contrast to (28)) by (105) with $l_\lambda = l - \lambda m$, $f = g$.*

* Let us notice that the vector-function $g(t)$ in (106) may be non-equal to zero in the finite end or in the neighbourhood of the infinite end of $\mathcal{I}$.

Below we will assume that the relation $\mathcal{L}'_0$ consists of the pairs of $\{y,g\}$ type.

The relation $\mathcal{L}'_0$ is symmetric due to the following Green formula with $\lambda_k=0$:

Let $y_k(t)\in C^r([\alpha,\beta],\mathcal{H})$, $f_k(t)\in C^s([\alpha,\beta],\mathcal{H})$, $\lambda_k\in\mathbb{C}$, $l[y_k]-\lambda_k m[y_k]=m[f_k]$, $k=1,2$. Then

$$\int\limits_\alpha^\beta m\{f_1,y_2\}\,dt-\int\limits_\alpha^\beta m\{y_1,f_2\}\,dt+\left(\lambda_1-\bar\lambda_2\right)\int\limits_\alpha^\beta m\{y_1,y_2\}\,dt$$
$$=i\left([\Re Q(t,l_\lambda)]\vec{y}_1(t,l_{\lambda_1},m,f_1),\vec{y}_2(t,l_{\lambda_2},m,f_2)\right)\Big|_\alpha^\beta, \quad (107)$$

where $\vec{y}_k(t,l_{\lambda_k},m,f_k)$ for $\lambda_k\in\mathbb{R}^1$ is defined by (105) with $l_\lambda=l-\lambda m$ and $y_k(t)$, $f_k(t)$ instead of $y(t)$, $f(t)$.

This formula is a corollary of Theorem 1.3 if $\Im\lambda_k\neq0$. For its proof, for example, for the case $\lambda_1\in\mathbb{R}^1$, we need to modify (107) for the equation $l[y_1]-(\lambda_1+i\varepsilon)m[y]=m[f_1-i\varepsilon y_1]$ and then to pass to the limit in (107) as $\varepsilon\to+0$.

In general, the relation $\mathcal{L}'_0$ is not closed. We denote $\mathcal{L}_0=\bar{\mathcal{L}}'_0$.

Theorem 3.3. *Let $l_\lambda=l-\lambda m$ and the conditions of Theorem 3.1 hold. Then the operator $R(\lambda)$ from Theorem 3.1 is the generalized resolvent of the relation $\mathcal{L}_0$. Let $\mathcal{I}$ be finite and additionally the condition* (77) *hold. Then every generalized resolvent of relation $\mathcal{L}_0$ can be constructed as the operator $R(\lambda)$.*

Notice that Theorem 3.3 together with Theorem 3.2 give in the regular case the description of the set of all generalized resolvents of relation $\mathcal{L}_0$ with the help of the boundary conditions.

P r o o f. In view of [16] and taking into account properties (90)–(92) of the operator $R(\lambda)$, it is sufficiently to prove that $R(\lambda)(\mathcal{L}_0-\lambda)\subseteq\mathbf{I}$, where $\mathbf{I}$ is a graph of the identical operator in $L^2_m(\mathcal{I})$. But this proposition is proved similarly to [22, p. 453] taking into account (107) and the fact that in view of (105), (106) $\overrightarrow{(\tilde y-y)}(t,l-\lambda m,m,0)=\overrightarrow{\tilde y}(t,l-\lambda m,m,g-\lambda y)-\vec{y}(t,l,m,g)$ if $\{y,g\}\in\mathcal{L}'_0$, $\tilde y=R(\lambda)(g-\lambda y)$.

Conversely, let $\mathcal{I}$ be finite and R_λ be a generalized resolvent of the relation $\mathcal{L}_0$. We denote $N_\lambda=\{y(t)\in C^r(\mathcal{I},\mathcal{H})\,|\,l[y]-\lambda m[y]=0\}$ as $\lambda\in\mathcal{B}$. We need the following.

Lemma 3.3. *Let condition* (77) *hold. Then the lineal N_λ is closed in $L^2_m(\mathcal{I})$.*

P r o o f. The proof of Lemma 3.3 follows from (34). ■

Lemma 3.4. *Let $\lambda\in\mathcal{B}$. Then $\overline{R\left(\mathcal{L}'_0-\bar\lambda\right)}=N^\perp_\lambda$.*

P r o o f. Let $x(t) \in N_\lambda, f(t) \in H$, $y(t)$ be a solution of the following Cauchy problem:

$$l[y] - \bar{\lambda} m[y] = m[f], \quad \vec{y}(a, l_{\bar{\lambda}}, m, f) = 0. \tag{108}$$

Then

$$m[f,x] = i\left(\Re Q(b,l_\lambda)\,\vec{y}(b,l_{\bar{\lambda}},m,f),\, \vec{x}(b,l_\lambda,m,0)\right) \tag{109}$$

in view of Green formula (107). Therefore $\overline{R\left(\mathcal{L}_0' - \bar{\lambda}\right)} \subseteq N_{\bar{\lambda}}^{\perp}$.

Let $g(t) \in N_{\bar{\lambda}}^{\perp}$. Then $\exists\, H \ni g_n \overset{L^2_m(\mathcal{I})}{\to} g$, $g_n = x_n \oplus f_n$, $x_n \in N_\lambda$, $f_n \in N_{\bar{\lambda}}^{\perp} \Rightarrow f_n \in H$. Let y_n be a solution of problem (108) with f_n instead of f. In view of (109) with $f = f_n$, one has $\vec{y}_n(b, l_{\bar{\lambda}}, m, f_n) = 0 \Rightarrow f_n \in R\left(\mathcal{L}_0' - \bar{\lambda}\right)$. But $f_n \overset{L^2_m(\mathcal{I})}{\to} g$. Therefore $\overline{R\left(\mathcal{L}_0' - \bar{\lambda}\right)} \supseteq N_{\bar{\lambda}}^{\perp}$. Lemma 3.4 is proved. ∎

Lemma 3.5. *Let condition* (77) *hold,* $\lambda \in \mathcal{B}$. *Let* $\left\{\tilde{y}, \tilde{f}\right\} \in \mathcal{L}_0^* - \lambda$, $\tilde{f} \overset{L^2_m(\mathcal{I})}{=} f \in H$. *Then* $\tilde{y} \overset{L^2_m(\mathcal{I})}{=} y \in C^r\left(\bar{\mathcal{I}}, \mathcal{H}\right)$, *and* $y(t)$ *satisfies Eq.* (1).

P r o o f. Let $C^r\left(\bar{\mathcal{I}}, \mathcal{H}\right) \ni y_0$ be a solution of (1). Let $\{\varphi, \psi\} \in \mathcal{L}_0' - \bar{\lambda}$. Then $\vec{\varphi}(a, l_{\bar{\lambda}}, m, \psi) = \vec{\varphi}(b, l_{\bar{\lambda}}, m, \psi) = 0$ in view of (105), (106). Hence $m[\varphi, f] = m[\psi, y_0]$ due to Green formula (107). But $m[\varphi, f] = (\psi, \tilde{y})_{L^2_m(\mathcal{I})}$ in view of the definition of the adjoint relation. Hence $(\psi, \tilde{y} - y_0) \underset{L^2_m(\mathcal{I})}{=} 0$. Therefore $\tilde{y} - y_0 \overset{L^2_m(\mathcal{I})}{=} y - y_0 \in N_\lambda$ in view of Lemmas 3.3, 3.4. Hence $\tilde{y} \overset{L^2_m(\mathcal{I})}{=} y \in C^r\left(\bar{\mathcal{I}}, \mathcal{H}\right)$, and y is a solution of (1). Lemma 3.5 is proved. ∎

We return to the proof of Theorem 3.3.

Let $f \in H$. Then, in view of Lemma 3.5, $R_\lambda f \overset{L^2_m(\bar{\mathcal{I}})}{=} y \in C^r\left(\bar{\mathcal{I}}, \mathcal{H}\right)$, and y satisfies equation (1). Therefore, taking into account Theorem 1.1, [11, p. 148] and (53), we have

$$y(t) = [X_\lambda(t)]_1 \left\{ h - \frac{1}{2}(iG)^{-1} \left(\int_a^b sqn(s-t) X_{\bar{\lambda}}^*(s)\, W(s, l_{\bar{\lambda}}, m)\, F(s, l_{\bar{\lambda}}, m)\, ds \right) \right\}, \tag{110}$$

where $[X_\lambda(t)]_1 \in B(\mathcal{H}^r, \mathcal{H})$ is the first row of the operator solution $X_\lambda(t)$ from Theorem 3.1 written in the matrix form, $h = h_\lambda(f) \in N^{\perp}$ is defined in the unique way in view of (34) and condition (77).

Let us prove that h depends on $I_\lambda f \overset{def}{=} \int_a^b X_{\bar{\lambda}}^*(s)\, W(s, l_{\bar{\lambda}}, m)\, F(s, l_{\bar{\lambda}}, m)\, ds$ in the unique way. The operator $I_\lambda : H \to N^{\perp}$ in view of Lemma 2.2. Moreover, $I_\lambda N^{\perp} = N^{\perp}$, i.e., $\forall h_0 \in N^{\perp} \exists f_0 \in H : h_0 = I_\lambda f_0$. For example, we can set

$$f_0 = f_0(t,\lambda) = [X_{\bar{\lambda}}(t)]_1 \left\{ \Delta_{\bar{\lambda}}(\mathcal{I})\,|_{N^{\perp}} \right\}^{-1} h_0 \tag{111}$$

and utilize the equality

$$W\left(s,l_{\bar{\lambda}},m\right)F_0\left(s,l_{\bar{\lambda}},m\right)=W\left(s,l_{\bar{\lambda}},m\right)X_{\bar{\lambda}}\left(s\right)\{\ldots\}^{-1}h_0.$$

If $f\left(t\right),g\left(t\right)\in H$ are such functions that $I_\lambda f=I_\lambda g$, then, in view of (110),

$$\begin{aligned}&\Im\lambda\left(\left(\Re Q\left(t,l_\lambda\right)\right)\overrightarrow{\Delta y}\left(t,l_\lambda,m,f-g\right),\overrightarrow{\Delta y}\left(t,l_\lambda,m,f-g\right)\right)\Big|_\alpha^\beta\\&=\Im\lambda\left(\left(\Re Q\left(t,l_\lambda\right)\right)X_\lambda\left(t\right)\left(h_\lambda\left(f\right)-h_\lambda\left(g\right)\right),X_\lambda\left(t\right)\left(h_\lambda\left(f\right)-h_\lambda\left(g\right)\right)\right)\Big|_\alpha^\beta,\end{aligned}\tag{112}$$

where $\Delta y=R_\lambda\left[f-g\right]$. But in view of (107), the left-hand side of (112) is nonpositive since R_λ has a property of (92) type. The right-hand side of (112) is nonnegative in view of (42). Hence $h_\lambda\left(f\right)=h_\lambda\left(g\right)$ in view of (42), (77). Thus h depends on $I_\lambda f$ in the unique way and obviously in the linear way. Therefore

$$h=M\left(\lambda\right)I_\lambda f,\tag{113}$$

where $M\left(\lambda\right):N^\perp\to N^\perp$ is a linear operator, and thus $R_\lambda f$ $\left(f\in H\right)$ can be represented in the form (89).

Further, for definiteness, we will consider the most complicated case $r=s=2n$.

Let us prove that $M\left(\lambda\right)\in B\left(N^\perp\right)$, $\Im\lambda\neq0$. Let $h_0\in N^\perp$, $y=R_\lambda f_0$, where $f_0=f_0\left(t,\lambda\right)$ (111). Then, in view of (110) and Theorem 1.1, we have

$$X_\lambda\left(t\right)M\left(\lambda\right)h_0=Y\left(t,l_\lambda,m\right)-\mathcal{F}_0\left(t,m\right)-\frac{1}{2}X_\lambda\left(t\right)\left(iG\right)^{-1}\left(I_\lambda\left(a,t\right)-I_\lambda\left(t,b\right)\right)F_0,\tag{114}$$

where $Y\left(t,l_\lambda,m\right)$, $F_0=F_0\left(t,l_{\bar{\lambda}},m\right)$ are defined by (26), with y and f_0 instead of f, while $\mathcal{F}_0\left(t,m\right)$ is defined by (37) with f_0 instead of f and $I_\lambda\left(0,t\right)F_0$ is defined by (83). Therefore,

$$\begin{aligned}&\Delta_\lambda\left(a,b\right)M(\lambda)h_0\\&=I_{\bar{\lambda}}y-I_{\bar{\lambda}}\left(a,b\right)\left(\mathcal{F}_0\left(t,m\right)+\frac{1}{2}X_\lambda\left(t\right)\left(iG\right)^{-1}\left(I_\lambda\left(a,t\right)-I_\lambda\left(t,b\right)\right)F_0\right),\end{aligned}\tag{115}$$

where $I_{\bar{\lambda}}y$, $I_{\bar{\lambda}}\left(a,b\right)\left(\ldots\right)\in N^\perp$ in view of (84). But

$$\forall g\in\mathcal{H}^r:\ \left|\left(I_{\bar{\lambda}}y,g\right)\right|\leq\max_{t\in\bar{\mathcal{I}}}\left\|X_\lambda\left(t\right)\right\|\left\{\int_{\mathcal{I}}\left\|W\left(t,l_\lambda,m\right)\right\|dt\right\}^{1/2}\left\|R_\lambda f_0\right\|_{L_m^2(\mathcal{I})}\left\|g\right\|$$

in view of the Cauchy inequality and (34). Therefore,

$$\exists\text{ constant }c\left(\lambda\right):\ \left|\left(I_{\bar{\lambda}}y,g\right)\right|\leq c\left(\lambda\right)\left\|y\right\|\,\left\|g\right\|\tag{116}$$

since

$$\left\|R_\lambda f_0\right\|_{L_m^2(\mathcal{I})}\leq\left\|\Delta_{\bar{\lambda}}\left(a,b\right)\right\|^{1/2}\left\|\left(\Delta_{\bar{\lambda}}\left(a,b\right)|_{N^\perp}\right)^{-1}\right\|\left\|h_0\right\|/\left|\Im\lambda\right|$$

in view of (34), (116) and the inequality $\|R_\lambda f_0\|_{L^2_m(\mathcal{I})} \le \|f_0\|_{L^2_m(\mathcal{I})} \Big/ |\Im\lambda|$.

Obviously, $|(I_{\bar\lambda}\,(a,b)\,(\ldots)\,,g)|$ satisfies the estimate of type (116). Therefore $M\,(\lambda) \in B(N^\perp)$.

Now we have to prove that $M(\lambda)$ is a characteristic operator of Eq. (73).

To prove that $M\,(\lambda)$ is strongly continuous for nonreal λ, it is enough to prove that $(a,b)\,M\,(\lambda)$ is strongly continuous for Δ_λ. The strong continuity of the last one obviously follows from the strong continuity of the vector-function $I_{\bar\lambda} R_\lambda f_0(t,\lambda)$.

In view of (34), we have $\forall g \in \mathcal{H}^r$

$$\begin{gathered}(I_{\bar\lambda} R_\lambda f_0\,(t,\lambda) - I_\mu R_{\bar\mu} f_0\,(t,\mu)\,,g)\\ = m\,[R_\lambda f_0\,(t,\lambda)\,,[X_\lambda\,(t)]_1\,g] - m\left[R_\mu f_0\,(t,\mu)\,,[X_\mu\,(t)]_1\,g\right].\end{gathered}$$

Then the required statement can be derived from the equality

$$\begin{gathered}m\left\{[X_\lambda\,(t) - X_\mu\,(t)]_1\,g,\,[X_\lambda\,(t) - X_\mu\,(t)]_1\,g\right\}\\ = \Big(W\,(t,l_\lambda,m)\,((X_\lambda\,(t) - X_\mu\,(t))\,g + (\lambda-\mu)\,\mathcal{F}(t,m))\,,\\ (X_\lambda\,(t) - X_\mu\,(t))\,g + (\lambda-\mu)\,\mathcal{F}(t,m)\Big),\end{gathered}$$

where $\mathcal{F}(t,m)$ is defined by (37) with $f(t) = [X_\mu\,(t)]_1\,g$, $\|X_\lambda\,(t) - X_\mu\,(t)\| \underset{\mu\to\lambda}{\to} 0$ uniformly in $t \in [a,b]$, and from the analogous equality for $m\{f_0(t,\lambda) - f_0(t,\mu),\ f_0(t,\lambda) - f_0(t,\mu)\}$.

Let us prove that $M\,(\lambda)$ is analytic for non real λ. To prove this fact, it is enough, in view of the strong continuity of $M(\lambda)$, to prove the analyticity of $(I_{\lambda\mu} M\,(\lambda)\,I_\lambda f, g)$ in λ, where $f\,(t) \in C^r\left(\bar{\mathcal{I}},\mathcal{H}\right)$, $g \in \mathcal{H}^r$, $(\Im\lambda)(\Im\mu) > 0$,

$$I_{\lambda\mu} = \int\limits_a^b X_\mu^*\,(t)\,W\,(t,l_\mu,m)\,X_\lambda\,(t)\,dt \;\in\; B\left(N^\perp\right),$$

$I_{\lambda\mu}^{-1} \in B\left(N^\perp\right)$ if $|\lambda-\mu|$ is sufficiently small. In view of (115), (89), Theorem 1.1, (34), (29), (8), we have

$$\begin{aligned}(I_{\lambda\mu} M\,(\lambda)\,I_\lambda f, g) &= m\left[R_\lambda f, [X_\mu\,(t)]_1\,g\right] + (\lambda-\mu)\int\limits_a^b \Big(\left(R_\lambda f\right)^{[n]}(t\,|m)\,,\,g^{(n)}\,(t)\Big)\,dt\\ &\quad + \text{terms independent of } R_\lambda f \text{ and analytic in } \lambda, \quad (117)\end{aligned}$$

where $g^{(n)}\,(t) \overset{def}{=} (p_n - \bar\mu\tilde p_n)^{-1}\left([X_\mu]_1\,g\right)^{[n]}(t\,|m)$.

For the scalar or vector function $F(\lambda)$, let us denote

$$\Delta_{km}F(\lambda) = \frac{F(\lambda + \Delta_k\lambda) - F(\lambda)}{\Delta_k\lambda} - \frac{F(\lambda + \Delta_m\lambda) - F(\lambda)}{\Delta_m\lambda}.$$

Let us also denote

$$\mathrm{R}_n(\lambda) = \int_a^b \left(\tilde{p}_n (R_\lambda f)^{[n]} (t|m), g^{(n)}\right) dt.$$

In view of (12), (74), we have

$$|\Delta_{km}\mathrm{R}_n(\lambda)| \le (m[\Delta_{km}R_\lambda f,\, \Delta_{km}R_\lambda f])^{\frac{1}{2}} \left(\int_a^b (\tilde{p}_n g^{(n)}, g^{(n)})dt\right)^{1/2}. \qquad (118)$$

Therefore $\mathrm{R}_n(\lambda)$ depends analytically on nonreal λ in view of the analyticity of R_λ, and thus the analyticity of $M(\lambda)$ is proved in view of (117).

Let us consider the solution $x_\lambda(t, F) = \mathcal{R}_\lambda F$ (79) of Eq. (73). Let us prove that $x_\lambda(t, F)$ satisfies condition (55). Let us denote $y(t) = R_\lambda f$. Then, in view of Green formula (42),

$$m[y, y] - \frac{\Im m[y, f]}{\Im\lambda} = \frac{1}{2}\left([\Re Q(t, l_\lambda)]\vec{y}(t, l_\lambda, m, f), \vec{y}(t, l_\lambda, m, f)\right)\Big|_a^b /\Im\lambda. \quad (119)$$

But the left-hand side of (119) is ≤ 0 since $R_\lambda f$ is a generalized resolvent. Consequently,

$$\forall f \in H: \ \left([\Re(Q(t, l_\lambda)]\vec{y}(t, l_\lambda, m, f), \vec{y}(t, l_\lambda, m, f)\right)\big|_a^b /\Im\lambda \le 0. \qquad (120)$$

But in view of (77), (84) for every $\mathcal{H}^r$-valued $F(t) \in L^2_{W(t,l_{\bar\lambda},m)}\left(\bar{\mathcal{I}}\right)$, there exists a vector-function $f(t) \in H$ such that $x_\lambda(a, F) = \vec{y}(a, l_\lambda, m, f)$, $x_\lambda(b, F) = \vec{y}(b, l_\lambda m, f)$. Finally, (55) is proved in view of (120).

To prove that $M(\lambda)$ is a characteristic operator of Eq. (73) it remains to show that $M(\bar\lambda) = M^*(\lambda)$.

Let us consider the following operator $\tilde{M}(\lambda) \in B\left(N^\perp\right)$:

$$\tilde{M}(\lambda) = M(\lambda), \ \tilde{M}\left(\bar\lambda\right) = M^*(\lambda), \ \Im\lambda > 0,$$

which is a characteristic operator of equation (73) in view of [22]. By Theorem 3.1, this characteristic operator generates the operator $R(\lambda)$ (89).

But $R(\lambda) = R_\lambda$, $\Im\lambda > 0 \Rightarrow R(\bar\lambda) = R^*(\lambda) = R^*_\lambda = R_{\bar\lambda}$, $\Im\lambda > 0 \Rightarrow \forall f \in H$:

$$\left\| [X_{\bar\lambda}(t)]_1 \left(M^*(\lambda) - M(\bar\lambda)\right) \int_a^b X^*_\lambda(s)\, W(s, l_\lambda, m)\, F(s, l_\lambda, m)\, ds \right\|_m = 0$$

$$\Rightarrow \forall h \in N^\perp : \ \Delta_{\bar\lambda}(a,b) \left(M(\bar\lambda) - M^*(\lambda)\right) h = 0 \Rightarrow M(\bar\lambda) = M^*(\lambda).$$

Theorem 3.3 is proved. ∎

Let $\mathcal{I}_k, k = 1, 2$ be finite intervals, $\mathcal{I}_1 \subset \mathcal{I}_2$. Then, in spite of the fact that $f(t) \in C^s(\bar{\mathcal{I}}_2, \mathcal{H})$ but $\chi_{\mathcal{I}_1} f(t) \notin C^s(\bar{\mathcal{I}}_2, \mathcal{H})$, where $\chi_{\mathcal{I}_1}$ is the characteristic function of $\mathcal{I}_1$, one has.

Corollary 3.2. *Let* $0 \in \mathcal{I}_1$ *and condition* (77) *with* $\mathcal{I} = \mathcal{I}_2$ *hold. Let* R_λ *be the generalized resolvent of the relation* $\mathcal{L}_0$ *in* $L^2_m(\mathcal{I})$ *with* $\mathcal{I} = \mathcal{I}_2$*. Then, by Theorems 3.1, 3.3, there exists the characteristic operator* $M(\lambda)$ *of Eq.* (5) *such that* $R_\lambda f = y_1(t, \lambda, f)$ (88), $t \in \mathcal{I} = \mathcal{I}_2$, $f \in H\,(= H(\mathcal{I}_2))$*. Let us define the operator* $y_1^1(t, \lambda, f) = R^1_\lambda f, t \in \mathcal{I} = \mathcal{I}_1$, $f \in H\,(= H(\mathcal{I}_1))$ *by the same formula* (88) *as operator* $R_\lambda f$*, but with* $\mathcal{I} = \mathcal{I}_1$ *instead of* $\mathcal{I} = \mathcal{I}_2$*. Then this operator is (after closing) the generalized resolvent of the relation* $\mathcal{L}_0$ *in* $L^2_m(\mathcal{I})$ *with* $\mathcal{I} = \mathcal{I}_1$*.*

For the generalized resolvents of differential operators, a representation of (89) type was obtained in [37] for the scalar case, and in [6] for the case of the operator coefficients. For the generalized resolvents for (1), (2) with $s = 0$, $n_\lambda[y] \equiv 0$, the representation of this type was obtained in [7, 8, 20].

Therefore the characteristic operator of equation (5) is an analogue of the characteristic matrix from [37].

The resolvents of the self-adjoint scalar differential operator in [17, p. 528], [30, p. 280] are represented in another form. Let us transform (89) to the form analogous to [17, p. 528], [30, p. 280]. (The integrals in (121), (122) converge strongly if the interval of integration is infinite.)

R e m a r k 3.1. *Let us represent the characteristic operator* $M(\lambda)$ *from Theorem 3.1 in the form* (58)*. Then* $R(\lambda)f$ (89) *can be represented in the form*

$$R(\lambda) f = \int_a^t \sum_{j=1}^{r} y_j(t,\lambda) \sum_{k=0}^{s/2} \left(x_j^{(k)}(s,\bar\lambda)\right)^* \mathrm{m}_k[f(s)]\, ds + \int_t^b \sum_{j=1}^{r} x_j(t,\lambda) \sum_{k=0}^{s/2} \left(y_j^{(k)}(s,\bar\lambda)\right)^* \mathrm{m}_k[f(s)]\, ds, \quad (121)$$

where $x_j(t,\lambda), y_j(t,\lambda) \in B(\mathcal{H})$ *are the operator solutions of equation* (1) *as* f=0 *such that* $(x_1(t,\lambda),\ldots,x_r(t,\lambda))$ *is the first row* $[X_\lambda(t)]_1 \in B(\mathcal{H}^r,\mathcal{H})$ *of the operator matrix* $X_\lambda(t)$, $(y_1(t,\lambda),\ldots,y_r(t,\lambda)) = [X_\lambda(t)]_1 \mathcal{P}(\lambda)(iG)^{-1}$, $\mathrm{m}_k[f(s)] = \tilde{p}_k(s) f^{(k)}(s) + \frac{i}{2}\left(\tilde{q}^*_{k+1}(s) f^{(k+1)}(s) - \tilde{q}_k(s) f^{(k-1)}(s)\right) \left(\tilde{q}_0 \equiv 0,\ \tilde{q}_{\frac{s}{2}+1} \equiv 0\right)$.

P r o o f. In view of Theorem 1.2, one has

$$\begin{aligned}\forall h \in \mathcal{H}^r: \left(X^*_{\bar\lambda}(t) W_{\bar\lambda}(t) F_{\bar\lambda}(t), h\right) &= m\left\{f(t), [X_{\bar\lambda}(t)]_1 h\right\} \\ &= \left(\left([X_\lambda(t)]_1{}^*, [X_\lambda(t)]_1'{}^*, \ldots, [X_\lambda(t)]_1^{(s/2)*}\right)\right. \\ &\quad \times \left. col\left\{\mathrm{m}_0[f(t)], \mathrm{m}_1[f(t)], \ldots, \mathrm{m}_{s/2}[f(t)]\right\}, h\right).\end{aligned}$$

Now Remark 3.1 follows from (88)–(89) since $(\mathcal{P}(\lambda) - I_r)(iG)^{-1} = \left(\mathcal{P}(\bar\lambda)(iG)^{-1}\right)^*$ in view of [22, p. 451]. ■

Remark 3.1 shows that $(\mathcal{P}(\lambda) - I_r)(iG)^{-1}$ is an analogue of the matrix transposed to the matrix $\left\|\theta^-_{ij}(\lambda)\right\|$ from [17, p. 528] and is an analogue of the characteristic matrix from [30, p. 280], $(\mathcal{P}(\lambda)(iG)^{-1}$ is an analogue of the matrix transposed to the matrix $\left\|\theta^+_{ij}(\lambda)\right\|$ from [17, p. 528]).

If r is even, $\mathcal{I} = (0,b)$, $b \le \infty$, and condition (55) is separated, then formula (89) can be transformed to the form analogues to that from [30, p. 275–279].

R e m a r k 3.2. *Let* $r = 2n$, $\mathcal{I} = (0,b)$, $b \le \infty$ *and condition* (78) *hold with* $P = I_r$. *(Therefore, for Eq.* (73) *condition* (56) *holds.) Let for the characteristic operator* $M(\lambda)$ *of Eq.* (5) *condition* (55) *be separated. (Therefore* $M(\lambda)$ *has representation* (58), *where characteristic projection* $\mathcal{P}(\lambda)$ *can be represented in the form* (64), (65) *with the help of some Nevanlinna pair* $\{-a(\lambda), b(\lambda)\}$ *and some Weyl function* $m(\lambda)$ *of equation* (76)*; this equation with* $F(t) = 0$ *has an operator solution* $U_\lambda(t), V_\lambda(t)$ (66)–(68)*). Let the domains* D, D_1 *be the same as in Remark 2.1. Then* $R(\lambda)f$ (89) *for* $\lambda \in D \bigcup D_1$ *can be represented in the form*

$$\begin{aligned}R(\lambda) f = &\int_0^t \sum_{j=1}^n v_j(t,\lambda) \sum_{k=0}^{s/2} \left(u_j^{(k)}(s,\bar\lambda)\right)^* \mathrm{m}_k[f(s)]\, ds \\ &+ \int_t^b \sum_{j=1}^n u_j(t,\lambda) \sum_{k=0}^{s/2} \left(v_j^{(k)}(s,\bar\lambda)\right)^* \mathrm{m}_k[f(s)]\, ds, \qquad (122)\end{aligned}$$

where $u_j(t,\lambda), v_j(t,\lambda) \in B(\mathcal{H})$ *are the operator solutions of equation* (1) *as* $f=0$ *such that* $(u_1(t,\lambda),\dots u_n(t,\lambda)) = [X_\lambda(t)]_1 \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix}$,

$$(v_1(t,\lambda),\dots,v_n(t,\lambda))$$
$$= [X_\lambda(t)]_1 \begin{pmatrix} b(\lambda) \\ -a(\lambda) \end{pmatrix} K^{-1}(\lambda) + (u_1(t,\lambda),\dots,u_n(t,\lambda))\, m_{a,b}(\lambda), \tag{123}$$

$K(\lambda)$, $m_{a,b}(\lambda)$ *see* (67), (68)*;* $(v_1(t,\lambda),\dots,v_n(t,\lambda))\,h \in L^2_m(\mathcal{I})\ \forall h \in \mathcal{H}^n$.

Moreover, if $a(\lambda) = a(\bar{\lambda})$, $b(\lambda) = b(\bar{\lambda})$ *as* $\Im\lambda \neq 0$, *then we can set* $D = \mathbb{C}_+$ *and*

$$\|(v_1(t,\lambda),\dots,v_n(t,\lambda))\,h\|^2_m \leq \frac{\Im(m_{a,b}(\lambda)h,h)}{\Im\lambda} \quad (\Im\lambda \neq 0).$$

P r o o f. The proof of Remark 3.2 follows from Remark 2.1 and Theorem 1.2. ■

Remark 3.2 shows that the operator-function $m_{a,b}(\lambda)$ from (122), (123) is an analogue of the characteristic matrix from [30, p. 278] since for any self-adjoint operator initial condition (in particular, the condition of the type given in [30, p. 277]), the resolvent (122) exists such that the solution-row $(u_1(t,\lambda),\dots,u_n(t,\lambda))$ satisfies this condition. For example, if $a(\lambda) = I_n$, $b(\lambda) = b = b^*$, then $m_{I_n,b}(\lambda)$ is equal to the characteristic matrix of [30, p. 276] type minus $b(I_n + b^2)^{-1}$.

Let us note that the connection between the generalized resolvents of the minimal operator, corresponding to the self-adjoint extension in the Krein space, and the boundary value problem with boundary conditions depending on the spectral parameter locally holomorphic in some set $\subset \mathbb{C} \setminus \mathbb{R}^1$ was studied in [15] for the scalar symmetric Sturm–Liouville operator on the semi-axis in the limit point case.

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