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Integral Transforms with Non-separated Variables and Discontinuous Coefficients

O.E. Yaremko

Penza State University
37, Lermontov Str., Penza 440038, Russia
E-mail: yaremki@mail.ru

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Multidimensional integral transforms with non-separated variables for discontinuous coefficients problems are constructed in the case where the coefficient discontinuities are on the parallel hyperplanes. Explicit kernel formulas for ideal coupling conditions are obtained. The basic integral transform identity is proved.

Key words: integral transforms, non-separated variables, coupling conditions.

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1. Introduction

It is known that the structure of integral transforms, which can be used to solve the boundary value problems, is determined by the type of differential equation. A number of transforms have appeared in mathematical literature since the 70th of the last century in the works by Y.S. Uflyand [1, 2], M.P. Lenuk [3, 4], L.S. Nayda [4], V.S. Protsenko [5], etc. In particular, the author and I.I. Bavrin [7] have proposed integral transforms with non-separated variables for solving multidimensional problems.

Let $V \subset R^{n+1}$ be the half-space

$$V = \left\{(y_1, \ldots, y_n, x) \in R^{n+1} : x > 0\right\}.$$

Then the solution of the Dirichlet problem is expressed via the Poisson formula [8],

$$u(x,y) = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}} \int\limits_{y=0} \frac{x}{\left[(y-\eta)^2 + x^2\right]^{\frac{n+1}{2}}} f(\eta) d\eta.$$

Obviously the Poisson kernel has the form of the Laplace transform

$$\Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}\frac{x}{\left[x^2+(y-\eta)^2\right]^{\frac{n+1}{2}}}$$

$$=\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\int_0^\infty \lambda^{\frac{n}{2}}e^{-\lambda x}\frac{J_{\frac{n-1}{2}}\left(\lambda\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{n-2}{2}}}d\lambda,$$

where J_ν is the Bessel function of order ν [8]. Reproducing property of the Poisson kernel is obtained from the expansion of the function $f(y)$ with respect to the Laplace operator Δ eigenfunctions:

$$f(y)=\lim_{\tau\to 0}\int\limits_0^\infty \lambda^{\frac{n}{2}}e^{-\lambda\tau}\left(\frac{1}{\left(\sqrt{2\pi}\right)^n}\int\limits_{R^n}\frac{J_{\frac{n-2}{2}}\left(\lambda\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{n-2}{2}}}f\left(\eta\right)d\eta\right)d\lambda.$$

On the basis of this expansion we can conclude that the integral transforms with non-separated variables are defined as follows [7]: direct integral Fourier transform

$$F\left[f\right]\left(y,\lambda\right)=\frac{1}{\left(\sqrt{2\pi}\right)^n}\int\limits_{R^n}\frac{J_{\frac{n-2}{2}}\left(\lambda\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{n-2}{2}}}f\left(\eta\right)d\eta\equiv\hat{f}\left(y,\lambda\right), \tag{1}$$

inverse Fourier integral transform

$$F^{-1}[\hat{f}](y)=\lim_{\tau\to 0}\int\limits_0^\infty \lambda^{\frac{n}{2}}e^{-\lambda\tau}\hat{f}(y;\lambda)d\lambda\equiv f(y). \tag{2}$$

The goal of the paper is to construct multi-dimensional analogues of integral transforms (1), (2) appropriate for the differential equations with discontinuous coefficients.

2. One-dimensional Integral Transforms with Discontinuous Coefficients

In the paper, the integral transforms with discontinuous coefficients are constructed in accordance with author's work [10]. Let $\varphi\left(x,\lambda\right)$ and $\varphi^*\left(x,\lambda\right)$ be eigenfunctions of the direct and the dual Sturm–Liouville problems for the Fourier operator on piecewise-homogeneous axis I_n,

$$I_n=\left\{x:\ x\in\bigcup_{j=1}^{n+1}\left(l_{j-1},l_j\right),\ l_0=-\infty,\ l_{n+1}=\infty,\ l_j<l_{j+1},\ j=\overline{1,n}\right\}.$$

The eigenfunction $\varphi(x,\lambda)$,

$$\varphi(x,\lambda) = \sum_{k=2}^{n} \theta(x - l_{k-1})\, \theta(l_k - x)\, \varphi_k(x,\lambda)$$

$$+\theta(l_1 - x)\, \varphi_1(x,\lambda) + \theta(x - l_n)\, \varphi_{n+1}(x,\lambda),$$

is the solution of the system of separated differential equations

$$\left(a_m^2 \frac{d^2}{dx^2} + \lambda^2\right) \varphi_m(x,\lambda) = 0, \quad x \in (l_m, l_{m+1}); \quad m = 1, \dots, n+1,$$

the coupling conditions

$$\left[\alpha_{m1}^k \frac{d}{dx} + \beta_{m1}^k\right] \varphi_k = \left[\alpha_{m2}^k \frac{d}{dx} + \beta_{m2}^k\right] \varphi_{k+1},$$

$$x = l_k, \quad k = 1, \dots, n; \quad m = 1, 2,$$

and the boundary conditions

$$\varphi_1|_{x=-\infty} = 0, \quad \varphi_{n+1}|_{x=\infty} = 0.$$

Similarly, the eigenfunction $\varphi^*(x,\lambda)$,

$$\varphi^*(\xi,\lambda) = \sum_{k=2}^{n} \theta(\xi - l_{k-1})\, \theta(l_k - \xi)\, \varphi_k^*(\xi,\lambda)$$

$$+\theta(l_1 - \xi)\, \varphi_1^*(\xi,\lambda) + \theta(\xi - l_n)\, \varphi_{n+1}^*(\xi,\lambda),$$

is the solution of the system of separated differential equations

$$\left(a_m^2 \frac{d^2}{dx^2} + \lambda^2\right) \varphi_m^*(x,\lambda) = 0, \quad x \in (l_m, l_{m+1}); \quad m = 1, \dots, n+1,$$

the coupling conditions

$$\frac{1}{\Delta_{1,k}} \left[\alpha_{m1}^k \frac{d}{dx} + \beta_{m1}^k\right] \varphi_k^* = \frac{1}{\Delta_{2,k}} \left[\alpha_{m2}^k \frac{d}{dx} + \beta_{m2}^k\right] \varphi_{k+1}^*, \quad x = l_k,$$

where

$$\Delta_{i,k} = \det \begin{pmatrix} \alpha_{1i}^k & \beta_{1i}^k \\ \alpha_{2i}^k & \beta_{2i}^k \end{pmatrix} k = 1, \dots, n; \qquad i, m = 1, 2,$$

and the boundary conditions

$$\varphi_1|_{x=-\infty} = 0, \quad \varphi_{n+1}|_{x=\infty} = 0.$$

We normalize the eigenfunctions as follows:

$$\varphi_{n+1}(x,\lambda) = e^{ia_{n+1}^{-1}x\lambda}. \quad \varphi_{n+1}^{*}(x,\lambda) = e^{-ia_{n+1}^{-1}x\lambda}.$$

Let F_n and F_n^{-1} be the direct and the inverse Fourier transforms on the Cartesian axis with n division points defined as (see [10]) :

$$F_n[f](\lambda) = \sum_{m=0}^{n+1} \int_{l_m-1}^{l_m} u_m^{*}(\xi,\lambda)\, f_m(\xi)\, d\xi \equiv \hat{f}(\lambda), \tag{3}$$

$$f_k(x) = \frac{1}{\pi i} \int_{0}^{\infty} u_k(x,\lambda)\, \hat{f}(\lambda)\, \lambda d\lambda. \tag{4}$$

3. The Main Result

We will use the method of delta-like functions [4].

This means that we are looking for the solution of the problem, defined by the separated matrix systems $(n+1)$ of parabolic equations

$$\left(\frac{\partial}{\partial t} - A_j^2 \frac{\partial^2}{\partial x^2} - \Delta_y\right) U_j(t,x,y) = 0, \;\; (t,x,y) \in D_+ \times R^m, \;\; j = \overline{1,n+1} \tag{5}$$

bounded on the set $D \times R^m, D^+ = (0,\infty) \times I_n$, where

$$I_n = \left\{ x : \; x \in \mathop{U}_{j=1}^{n+1} (l_{j-1}, l_j), \;\; l_0 = -\infty, \;\; l_{n+1} = \infty, \;\; l_j < l_{j+1}, \;\; j = \overline{1,n} \right\}$$

$$\Delta_y = \frac{\partial^2}{\partial y_1^2} + \ldots + \frac{\partial^2}{\partial y_m^2},$$

$A_j = \left(a_{kl}^{j}\right)$ is a positive-definite matrix $r \times r$ by the initial conditions

$$U_j(t,x,y)\,|_{t=0} = g_j(x,y), \;\; x \in I_n, y \in R^m, \tag{6}$$

by the edge conditions

$$U_1|_{x=-\infty} = 0, \;\; U_{n+1}|_{x=\infty} = 0, \tag{7}$$

and by the coupling conditions

$$\left[\alpha_{m1}^{k} \frac{\partial}{\partial x} + \beta_{m1}^{k}\right] U_k = \left[\alpha_{m2}^{k} \frac{\partial}{\partial x} + \beta_{m2}^{k}\right] U_{k+1}, \tag{8}$$

$$x = l_k, \quad k = \overline{1,n}; \quad m = \overline{1,\,2}.$$

Here $U_j(t,x,y)$ is an unknown vector-function, $g_j(x,y)$ is a given vector-function, $\alpha^k_{mi},\ \beta^k_{mi}, \gamma^k_{mi},\ \delta^k_{mi}$ are the matrices $r \times r$.

By using the Fourier integral with discontinuous coefficients of Section 2 and the Fourier integral with non-separated variables (1), (2), we obtain the representation for the solution of (3)–(6):

$$U_k\left(t,x,y\right) = -\frac{1}{\pi i}\frac{1}{\left(\sqrt{2\pi}\right)^m}\int\limits_{R^m}\sum_{j=1}^{n+1}\int\limits_{l_{j-1}}^{l_j}\lim_{\tau\to 0}\left(\int\limits_0^\infty \frac{J_{\frac{m-2}{2}}\left(\lambda\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}e^{-\lambda\tau}\lambda^{\frac{m}{2}}\,d\lambda\right.$$

$$\left.\times\int\limits_{-\infty}^{\infty} e^{-\beta^2 t}\varphi_k\left(x,\beta\right)\varphi_j^*\left(\xi,\beta\right)d\beta\right)f_j\left(\xi,\eta\right)d\xi d\eta, \quad k=\overline{1,n+1}, \tag{9}$$

where $\varphi_k\left(x,\beta\right), \varphi_j^*\left(\xi,\beta\right)$ are the eigenfunctions of the direct and the dual Sturm –Liouville problems, respectively.

We write the integral

$$\int\limits_0^\infty \frac{J_{\frac{m-2}{2}}\left(\lambda\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}e^{-\lambda\tau}\lambda^{\frac{m}{2}}\,d\lambda\int\limits_{-\infty}^{\infty} e^{-\beta^2 t}\varphi_k\left(x,\beta\right)\varphi_j^*\left(\xi,\beta\right)d\beta$$

in the polar coordinates

$$\lambda = \rho\sin\varphi, \quad \beta = \rho\cos\varphi, \quad 0 \le \rho < \infty, 0 \le \varphi \le \pi$$

to obtain

$$\int\limits_0^\infty \rho^{\frac{m}{2}}\rho d\rho\int\limits_0^\pi e^{-\rho^2 t\cos^2\alpha}\sin^{\frac{m}{2}}\alpha\frac{J_{\frac{m-2}{2}}\left(\rho\sin\alpha\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}e^{-\rho\tau\sin\alpha}$$

$$\varphi_k\left(x,\rho\cos\alpha\right)\varphi_j^*\left(\xi,\rho\cos\alpha\right)d\alpha.$$

We carry out the limit $\tau \to 0$ in (9) yielding

$$U_k\left(t,x,y\right) = -\frac{1}{\pi i}\left(\frac{1}{2\pi}\right)^{\frac{m}{2}}\int\limits_{R^m}\sum_{j=1}^{n+1}\int\limits_{l_{j-1}}^{l_j}\rho^{\frac{m}{2}}d\rho\left(\int\limits_0^\pi e^{-\rho^2 t\cos^2\alpha}\sin^{\frac{m}{2}}\alpha\right.$$

$$\left.\times\frac{J_{\frac{m-2}{2}}\left(\rho\sin\alpha\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}\phi_k\left(x,\rho\cos\alpha\right)\phi_j^*\left(\xi,\rho\cos\alpha\right)d\alpha\right)f_j\left(\xi,\eta\right)d\xi d\eta.$$

If, in addition, we assume that we can carry out the limit $t \to 0$ in the expansion of the eigenfunctions for multidimensional direct Sturm–Liouville problem $f_k(x,y)$ of Sec. 2, we obtain

$$f_k(x,y) = -\frac{1}{\pi i}\frac{1}{\left(\sqrt{2\pi}\right)^m}\int\limits_{R^m}\sum_{j=1}^{n+1}\int\limits_{l_{j-1}}^{l_j}\int\limits_0^{\infty}\rho^{\frac{m}{2}}\rho d\rho\left(\int\limits_0^{\pi} si\; n^{\frac{m}{2}}\alpha\frac{J_{\frac{m-2}{2}}\left(\rho\sin\alpha\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}\right.$$

$$\left.\times\varphi_k(x,\rho\cos\alpha)\varphi_j^*(\xi,\rho\cos\alpha)d\alpha\right)f_j(\xi,\eta)\,d\xi d\eta. \qquad (10)$$

Let us denote

$$\varphi_{k,j} \equiv \varphi_{k,j}\left(\rho,x,\xi,\left|y-\eta\right|\right) = \int\limits_0^{\pi}\sin^{\frac{m}{2}}\alpha\frac{J_{\frac{m-2}{2}}\left(\rho\sin\alpha\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}$$

$$\times\varphi_k\left(x,\rho\cos\alpha\right)\varphi_j^*\left(\xi,\rho\cos\alpha\right)d\alpha.$$

It is clear then that formula (10) can be written as

$$f_k(x,y) = \frac{1}{\pi}\int\limits_0^{\infty}\rho^{\frac{m+1}{2}}d\rho\frac{1}{\left(\sqrt{2\pi}\right)^m}\int\limits_{R^m}\sum_{j=1}^{n+1}\int\limits_{l_{j-1}}^{l_j}\varphi_{k,j}f_j(\xi,\eta)\,d\xi d\eta. \qquad (11)$$

Equation (11) allows us to write down the direct and the inverse multidimensional Fourier transforms with discontinuities on the planes $x = l_k$:

$$F_n[f](x,y,\lambda) = \frac{1}{\left(\sqrt{2\pi}\right)^m}\int\limits_{R^m}\sum_{j=1}^{n+1}\int\limits_{l_{j-1}}^{l_j}\varphi_{k,j}\left(\lambda,x,\xi,\left|y-\eta\right|\right)f_j(\xi,\eta)\,d\xi d\eta, \qquad (12)$$

$$f(x,y) = \int\limits_0^{\infty}\lambda^{\frac{m}{2}+1}F_n[f](x,y,\lambda)\,d\lambda. \qquad (13)$$

Now we can prove the basic integral identity for differential operator

$$B = \theta\left(l_1 - xt\right)\left(A_1^2\frac{d^2}{dx^2} + \Delta_y\right) + \sum_{k=1}^{n}\theta\left(x-l_{k-1}\right)\theta\left(l_k - x\right)\left(A_k^2\frac{d^2}{dx^2} + \Delta_y\right)$$

$$+\theta\left(x - l_n\right)\left(A_{n+1}^2\frac{d^2}{dx^2} + \Delta_y\right).$$

Theorem 1. *Let*

$$f\left(x,y\right)=\theta\left(l_1-x\right)\,f_1\left(x,y\right)+\sum_{k=2}^{n}\theta\left(x-l_{k-1}\right)\theta\left(l_k-x\right)\,f_k\left(x,y\right)$$
$$+\,\theta\left(x-l_n\right)\,f_{n+1}\left(x,y\right)$$

be a twice continuously differentiable on $D_+\times R^m$ *vector-function, in which*

$$f_{n+1}(x,y),\frac{\partial f_{n+1}(x,y)}{\partial x}$$

vanishes as $x\to+\infty$ *and* y *is fixed,*

$$f_1(x,y),\frac{\partial f_1(x,y)}{\partial x}$$

vanishes as $x\to-\infty$ *and* y *is fixed,*

$$f_i(x,y),\frac{\partial f_i(x,y)}{\partial y_j}$$

vanishes as $y_j\to\pm\infty$ *and* $x,y_1,y_2,\dots,y_{j-1},y_{j+1},\dots,y_m$ *are fixed.*

Assume also that the coupling conditions (8) are valid.

Then the following holds true:

$$F_n\left[B\left(f\right)\right]=-\lambda^2F_n\left[f\right].$$

P r o o f. Integrate twice by parts with respect to each of the variables on the left, taking into account the conditions of the theorem. As a result, the operator B acts on the kernel:

$$F_n\left[B(f)\right](x,y,\lambda)=\frac{1}{\left(\sqrt{2\pi}\right)^m}\int\limits_{R^m}\sum_{j=1}^{n+1}\int\limits_{l_{j-1}}^{l_j}B_j\left[\varphi_{k,j}\left(\lambda,x,\xi,|y-\eta|\right)\right]f_j\left(\xi,\eta\right)d\xi d\eta.$$

Let us prove the equality $B_j[\varphi_{k,j}]=-\lambda^2\varphi_{k,j}$. We have

$$B_j[\varphi_{k,j}]=\int\limits_0^{\pi}\sin^{\frac{m}{2}}\alpha\Delta_\eta\left(\frac{J_{\frac{m-2}{2}}\left(\rho\sin\alpha\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}\right)$$
$$\times\varphi_k\left(x,\rho\cos\alpha\right)\left(\varphi_j^*\left(\xi,\rho\cos\alpha\right)\right)d\alpha$$
$$+\int\limits_0^{\pi}\sin^{\frac{m}{2}}\alpha\left(\frac{J_{\frac{m-2}{2}}\left(\rho\sin\alpha\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}\right)$$

$$\times\varphi_k\left(x,\rho\cos\alpha\right)a_j^2\frac{\partial^2}{\partial\xi^2}\left(\varphi_j^*\left(\xi,\rho\cos\alpha\right)\right)d\alpha$$

$$=-\rho^2\sin^2\alpha\int\limits_0^{\pi}\sin^{\frac{m}{2}}\alpha\left(\frac{J_{\frac{m-2}{2}}\left(\rho\sin\alpha\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}\right)$$

$$\times\varphi_k\left(x,\rho\cos\alpha\right)\left(\varphi_j^*\left(\xi,\rho\cos\alpha\right)\right)d\alpha$$

$$-\rho^2\cos^2\alpha\int\limits_0^{\pi}\sin^{\frac{m}{2}}\alpha\left(\frac{J_{\frac{m-2}{2}}\left(\rho\sin\alpha\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}\right)$$

$$\times\varphi_k\left(x,\rho\cos\alpha\right)\left(\varphi_j^*\left(\xi,\rho\cos\alpha\right)\right)d\alpha=-\rho^2\varphi_{k,j}.$$

We have used above that $\varphi_j^*\left(\xi,\rho\cos\alpha\right)$ are the eigenfunctions of the dual Sturm–Liouville problems and the relation

$$\Delta_\eta\left(\frac{J_{\frac{m-2}{2}}\left(\rho\sin\alpha\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}\right)=-\rho^2\sin^2\alpha\left(\frac{J_{\frac{m-2}{2}}\left(\rho\sin\alpha\left|y-\eta\right|\right)}{\left|y-\eta\right|^{\frac{m-2}{2}}}\right).$$

By using the basic identity [9], we conclude

$$\frac{\rho^{\frac{m}{2}}J_{\frac{m-2}{2}}\left(\rho\left|y\right|\right)}{\left|y\right|^{\frac{m-2}{2}}}=\frac{1}{(2\pi)^{\frac{m}{2}}}\int\limits_{S_\rho}e^{i<y,\xi>}dS_\rho.$$

This completes the proof. ■

The above formulas for the direct and the inverse Fourier transforms with non-separated variables are significantly simpler in the case of ideal coupling conditions on one surface. This case is widely known in engineering practice. Consider, for the sake of simplicity, the scalar case, assuming that the ideal coupling conditions are in the plane $x=0$,

$$\varphi_1\left(x,y\right)=\varphi_2\left(x,y\right),x=0,y\in R^m;$$

$$\varphi'_{1x}\left(x,y\right)=\nu\varphi'_{2x}\left(x,y\right),x=0,y\in R^m;\nu=\frac{\lambda_2}{\lambda_1}.$$

The one-dimensional components of eigenfunctions are given in [4]:

$$\varphi_1\left(x,\lambda\right)=\left(\cos\lambda\frac{x}{a_1}+i\frac{1}{\sqrt{\delta_0}}\sin\lambda\frac{x}{a_1}\right)\left(1+\delta_0\right);$$

$$\varphi_2\left(x,\lambda\right)=\left(\cos\lambda\frac{x}{a_2}+i\sqrt{\delta_0}\sin\lambda\frac{x}{a_2}\right)\left(1+\delta_0\right);$$

$$\varphi_k^*(x,\lambda) = r_k\overline{\varphi_k(x,\lambda)},\ k=1,2,\ r_1 = \frac{a_2}{\nu_0 a_1^2},\ r_2 = \frac{1}{a_2},\ \delta_0 = \frac{a_2}{\nu_0 a_1}.$$

The multidimensional components of eigenfunctions with non- separated variables φ_{kj} have the form:

$$\varphi_{11} = \frac{1+\delta_0}{a_1}\frac{J_{\frac{m-1}{2}}\left(\rho\sqrt{\frac{(x-\xi)^2}{a_1^2}+|y-\eta|^2}\right)}{\left(\frac{(x-\xi)^2}{a_1^2}+|y-\eta|^2\right)^{\frac{m-1}{2}}} - \frac{1-\delta_0}{a_1}\frac{J_{\frac{m-1}{2}}\left(\rho\sqrt{\frac{(x+\xi)^2}{a_1^2}+|y-\eta|^2}\right)}{\left(\frac{(x+\xi)^2}{a_1^2}+|y-\eta|^2\right)^{\frac{m-1}{2}}},$$

$$\varphi_{12} = \frac{1+\delta_0}{a_2\sqrt{\delta_0}}\frac{J_{\frac{m-1}{2}}\left(\rho\sqrt{\left(\frac{x}{a_2}-\frac{\xi}{a_1}\right)^2+|y-\eta|^2}\right)}{\left(\left(\frac{x}{a_2}-\frac{\xi}{a_1}\right)^2+|y-\eta|^2\right)^{\frac{m-1}{2}}}$$

$$+\frac{1-\delta_0}{a_2\sqrt{\delta_0}}\frac{J_{\frac{m-1}{2}}\left(\rho\sqrt{\left(\frac{x}{a_2}+\frac{\xi}{a_1}\right)^2+|y-\eta|^2}\right)}{\left(\left(\frac{x}{a_2}+\frac{\xi}{a_1}\right)^2+|y-\eta|^2\right)^{\frac{m-1}{2}}},$$

$$\varphi_{21} = \sqrt{\delta_0}\frac{1+\delta_0}{a_1}\frac{J_{\frac{m-1}{2}}\left(\rho\sqrt{\left(\frac{x}{a_1}-\frac{\xi}{a_2}\right)^2+|y-\eta|^2}\right)}{\left(\left(\frac{x}{a_1}-\frac{\xi}{a_2}\right)^2+|y-\eta|^2\right)^{\frac{m-1}{2}}}$$

$$+\sqrt{\delta_0}\frac{1-\delta_0}{a_1}\frac{J_{\frac{m-1}{2}}\left(\rho\sqrt{\left(\frac{x}{a_1}+\frac{\xi}{a_2}\right)^2+|y-\eta|^2}\right)}{\left(\left(\frac{x}{a_1}+\frac{\xi}{a_2}\right)^2+|y-\eta|^2\right)^{\frac{m-1}{2}}},$$

$$\varphi_{22} = \frac{1+\delta_0}{a_2\delta_0}\frac{J_{\frac{m-1}{2}}\left(\rho\sqrt{\frac{(x-\xi)^2}{a_2^2}+|y-\eta|^2}\right)}{\left(\frac{(x-\xi)^2}{a_2^2}+|y-\eta|^2\right)^{\frac{m-1}{2}}} - \frac{1-\delta_0}{a_2\delta_0}\frac{J_{\frac{m-1}{2}}\left(\rho\sqrt{\frac{(x+\xi)^2}{a_2^2}+|y-\eta|^2}\right)}{\left(\frac{(x+\xi)^2}{a_2^2}+|y-\eta|^2\right)^{\frac{m-1}{2}}}.$$

Now the integral transforms given by formulas (12), (13) are constructed.

4. Conclusion

Let us remark that integral transforms (12), (13) can be used to solve problems of mathematical physics by using the standard algorithm: find the solution in the images then return to the originals. An advantage of our formulas is that they involve just one spectral parameter contained in the final formulas while the integral transforms with separated variables contain m parameters.

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