

# Local Minimizers of the Magnetic Ginzburg–Landau Functional with $S^1$ -valued Order Parameter on the Boundary

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It was shown in [L. Berlyand and V. Rybalko, Solution with Vortices of a Semi-Stiff Boundary Value Problem for the Ginzburg–Landau Equation, *J. Eur. Math. Soc.* **12** (2010), 1497–1531] that in doubly connected domains there exist local minimizers of the simplified Ginzburg–Landau functional with modulus one and prescribed degrees on the boundary, unlike global minimizers that typically do not exist. We generalize the results and techniques of the aforementioned paper to the case of the magnetic Ginzburg–Landau functional.

*Key words:* superconductivity, Ginzburg–Landau functional, variational problems with lack of compactness.

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## 1. Introduction

The purpose of the paper is to generalize the results and methods of [6] (see also [13]) to the magnetic Ginzburg–Landau model. We consider the functional

$$F_\lambda[u, A] = \frac{1}{2} \int_G (|\nabla u - iAu|^2 + \frac{\lambda}{4} (|u|^2 - 1)^2) dx + \frac{1}{2} \int_\Omega (\operatorname{curl} A)^2 dx, \quad (1.1)$$

where  $G \subset \mathbb{R}^2$  is a bounded multiply connected domain and  $\Omega$  is the smallest simply connected domain containing  $G$ ,  $u \in H^1(G; \mathbb{C})$  is the order parameter,  $A \in H^1(\Omega; \mathbb{R}^2)$  is the vector potential of the induced magnetic field, and  $\lambda > 0$  is the coupling constant ( $\sqrt{\lambda/2}$  is the Ginzburg–Landau parameter). We assume

that the order parameter  $u$  takes values in  $S^1$  on the boundary  $\partial G$  and study critical points of the functional  $F_\lambda[u, A]$  in the space

$$(u, A) \in \mathcal{J} = \{u \in H^1(G; \mathbb{C}); |u| = 1 \text{ a.e. on } \partial G\} \times H^1(\Omega; \mathbb{R}^2). \quad (1.2)$$

For the sake of simplicity, we will consider doubly connected domains  $G$ ,

$$G = \Omega \setminus \bar{\omega}, \text{ where } \omega, \Omega \text{ are smooth simply connected domains } (\bar{\omega} \subset \Omega).$$

However, the results of the paper can be easily extended for general multiply connected domains.

Since the external magnetic field is zero in (1.1), global minimizers of the functional are trivial (up to a gauge transformation  $u \equiv \text{const} \in S^1$ ,  $A \equiv 0$ ), and we are interested in finding nontrivial local minimizers. A way to produce these nontrivial local minimizers is to minimize the functional in the class of pairs  $(u, A)$  with prescribed topological degree of  $u$  on the connected components of the boundary. Recall that, given a simple closed curve  $\gamma$ , the topological degree (winding number) of a map  $u \in H^{1/2}(\gamma; S^1)$  is an integer given by the classical formula (cf., e.g., [8])

$$\text{deg}(u, \gamma) = \frac{1}{2\pi} \int_{\gamma} u \wedge \frac{\partial u}{\partial \tau} \, ds,$$

where the integral is understood via  $H^{1/2} - H^{-1/2}$  duality, and  $\frac{\partial}{\partial \tau}$  is the tangential derivative with respect to the counterclockwise orientation of  $\gamma$ . The functional  $\text{deg}(\cdot, \gamma) : H^{1/2}(\gamma; S^1) \rightarrow \mathbb{R}$  is continuous with respect to the strong  $H^{1/2}$ -convergence. Therefore, for any prescribed  $p, q \in \mathbb{Z}$  minimizers of  $F_\lambda[u, A]$  over the set

$$\mathcal{J}_{pq} = \{(u, A) \in \mathcal{J}; \text{deg}(u, \partial\omega) = p, \text{deg}(u, \partial\Omega) = q\}$$

are local minimizers of  $F_\lambda[u, A]$  in  $\mathcal{J}$ . The problem however is that, in general, global minimizers of  $F_\lambda[u, A]$  in  $\mathcal{J}_{pq}$  do not exist ([4], see also [1],[3],[6]) because of the lack of continuity of  $\text{deg}(\cdot, \gamma) : H^{1/2}(\gamma; S^1) \rightarrow \mathbb{R}$  with respect to the weak  $H^{1/2}$ -convergence.

To construct local minimizers of  $F_\lambda[u, A]$  in  $\mathcal{J}$ , we consider the constrained minimization problem

$$m_\lambda(p, q, d) := \inf\{F_\lambda[u, A]; (u, A) \in \mathcal{J}_{pq}^{(d)}\}, \quad (1.3)$$

where  $p, q$  and  $d$  are given integers,

$$\mathcal{J}_{pq}^{(d)} = \{(u, A) \in \mathcal{J}_{pq}; d - 1/2 \leq \Phi(u, A, V_0) \leq d + 1/2\}, \quad (1.4)$$

and  $\Phi(u, A, V_0)$  is given by

$$\Phi(u, A, V_0) = \frac{1}{2\pi} \int_G \left( u \wedge \left( \left( \frac{\partial u}{\partial x_2} - iA_2 u \right) \frac{\partial V_0}{\partial x_1} - \left( \frac{\partial u}{\partial x_1} - iA_1 u \right) \frac{\partial V_0}{\partial x_2} \right) + A \cdot \nabla^\perp V_0 \right) dx, \tag{1.5}$$

with  $V_0$  being the unique solution of the boundary value problem

$$\begin{cases} -\Delta V_0 + V_0 = 1 & \text{in } G \\ V_0 = 1 & \text{on } \partial\Omega \\ V_0 = 0 & \text{on } \partial\omega. \end{cases} \tag{1.6}$$

The functional  $\Phi(u, A, V)$  (with harmonic  $V$  and  $A \equiv 0$ ) was introduced in [6] to find nontrivial local minimizers of the simplified Ginzburg–Landau functional with prescribed degrees. Here we will make use of  $\Phi(u, A, V)$  for various smooth functions  $V$ . Note that the functional  $\Phi(u, A, V_0)$  has the following properties (cf. [6], Section 3):

(a) for every  $u \in H^1(G; S^1)$  and  $A \in H^1(\Omega; \mathbb{R}^2)$  we have

$$\Phi(u, A, V_0) = \deg(u, \partial\omega) = \deg(u, \partial\Omega);$$

(b)  $\Phi(\cdot, V_0) : H^1(G; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R}$  is continuous with respect to the weak convergence.

Moreover, if we fix  $\Lambda > 0$  and consider pairs  $(u, A)$  from the sublevel set  $F_\lambda[u, A] \leq \Lambda$ , then  $\Phi(u, A, V_0)$  is never half-integer for sufficiently large  $\lambda > 0$ . More precisely, we prove (see Section )

**Proposition 1.** *Fix  $\Lambda > 0$ . There exists  $\lambda_0 = \lambda_0(\Lambda) > 0$  such that if  $\lambda \geq \lambda_0$ , then for every integer  $d$  and every  $(u, A) \in H^1(G; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$  satisfying  $F_\lambda[u, A] \leq \Lambda$  the closed constraint  $\Phi(u, A, V_0) \in [d - 1/2, d + 1/2]$  is equivalent to an open one, that is,*

$$d - 1/2 \leq \Phi(u, A, V_0) \leq d + 1/2 \iff d - 1/2 < \Phi(u, A, V_0) < d + 1/2. \tag{1.7}$$

Actually, it will be shown that  $\Phi(u, A, V_0)$  is close to integers uniformly in  $(u, A)$  satisfying  $F_\lambda[u, A] \leq \Lambda$  when  $\lambda$  is sufficiently large.

It follows from Proposition 1 that if

- the infimum  $m_\lambda(p, q, d)$  in (1.3) is attained and  $(u, A)$  is a minimizer,
- $m_\lambda(p, q, d) < \Lambda$  and  $\lambda \geq \lambda_0$ , where  $\lambda_0(= \lambda_0(\Lambda))$  is as in Proposition 1,

then  $(u, A)$  is a local minimizer of  $F_\lambda[u, A]$  in  $\mathcal{J}$ . However the attainability of  $m_\lambda(p, q, d)$  is still a nontrivial question due to the lack of continuity of  $\deg(u, \partial\Omega)$  and  $\deg(u, \partial\omega)$  with respect to the weak  $H^1$ -convergence.

The main result of the paper is

**Theorem 2.** *For any integers  $p, q$  and  $d > 0$  ( $d < 0$ ) with  $d \geq \max\{p, q\}$  ( $d \leq \min\{p, q\}$ ) there exists  $\lambda_1 = \lambda_1(p, q, d) > 0$  such that the infimum in (1.3) is always attained when  $\lambda \geq \lambda_1$  and any minimizer of (1.3) is a local minimizer of  $F_\lambda[u, A]$  in  $\mathcal{J}$ .*

Throughout the paper we will assume  $d > 0$  (the case  $d < 0$  follows by taking  $(\bar{u}, -A)$  instead of  $(u, A)$ ). As already mentioned, the principal part of Theorem 2 is to prove the attainability of the infimum in (1.3). To show this we follow essentially the same scheme as in [6]; namely, we first prove the attainability of the infimum in (1.3) for  $p = q = d$  and argue by induction, that is, pass to  $p = d, q = d - 1$  and  $p = d - 1, q = d$ , etc. The main technical result we use to pass from the prescribed degrees  $p, q$  to  $p, q - 1$  and  $p - 1, q$  is the following strict inequalities:

$$m_\lambda(p, q - 1, d) < m_\lambda(p, q, d) + \pi \quad \text{and} \quad m_\lambda(p - 1, q, d) < m_\lambda(p, q, d) + \pi \quad (1.8)$$

that hold provided that  $m_\lambda(p, q, d)$  is attained and  $\lambda$  is sufficiently large. Bounds (1.8) are proved by constructing testing pairs  $(u, A) \in \mathcal{J}_{p(q-1)}^{(d)}$  or  $(u, A) \in \mathcal{J}_{(p-1)q}^{(d)}$  with  $F_\lambda[u, A] < m_\lambda(p, q, d) + \pi$ , and this is the key point of the present paper. The construction of testing pairs completely differs from that of [6] and makes use of Bogomol’nyi’s representation of the functional [8] and the factorization idea of C. Taubes [23].

There is a vast body of literature on 2D Ginzburg–Landau type problems. Local minimizers of the simplified Ginzburg–Landau functional with the Dirichlet condition on the boundary were studied in [11, 16], more general results were obtained in [12]. Aforementioned works deal with large values of the Ginzburg–Landau parameter and essentially rely on the results of the pioneering work [7]; more specifically, they use the reduction to the renormalized energy functional introduced in [7].

Local minimizers with the Neumann boundary condition in multiply connected domains are related to the phenomenon of permanent currents, see, e.g., [18, 14]. However, it is not known whether these local minimizers can have zeros (vortices) in both simplified 2D model and magnetic Ginzburg–Landau 2D model with zero external field. One can find some results on nonexistence of local minimizers with vortices in [15] and [21]. In the case of nonzero external magnetic field, global minimizers and local ones can have vortices ([10, 19, 20, 22], see also references therein).

While the idea of local minimization is general, its implementation depends strongly on the concrete problem one is dealing with. The distinguishing feature of the problem studied in the present paper is its being a variational problem with a possible lack of compactness. Also, the local minimizers obtained in Theorem 2 have nonstandard behavior, their zeros are situated near the boundary and approach it as  $\lambda \rightarrow +\infty$  (this can be shown quite similarly to [6]).

In the paper, we use the following notation and conventions:

- Every closed curve is counterclockwise oriented. For such a curve,  $\tau$  and  $\nu$  stand for the unit tangent and unit normal vectors, respectively, that agree with the orientation ( $(\nu, \tau)$  is direct);
- The complex plane  $\mathbb{C}$  is identified with  $\mathbb{R}^2$  such that if  $x, y \in \mathbb{C}$ , then  $(x, y) = \frac{1}{2}(x\bar{y} + y\bar{x})$  and  $x \wedge y = \frac{i}{2}(x\bar{y} - y\bar{x})$  are the scalar and the wedge products, respectively;
- Given a fixed orthonormal frame  $(x_1, x_2)$  in  $\mathbb{R}^2$ ,  $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$  and  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$  denote the classical Cauchy operators. For a scalar (real-valued) function  $f$ ,  $\nabla^\perp f$  is the vector field given by  $\nabla^\perp f = (-\partial f / \partial x_2, \partial f / \partial x_1)$ .

## 2. Critical Points of $F_\lambda[u, A]$ and Bogomol'nyi's Representation

One of the main properties of the functional  $F_\lambda[u, A]$  is its invariance under the gauge transformations  $u \mapsto e^{i\phi}u$ ,  $A \mapsto A + \nabla\phi$  (where  $\phi \in H^2(\Omega)$ ). It is easy to see that  $\Phi(u, A, V_0)$  also has the aforementioned property as well as  $\deg(u, \partial\Omega)$  and  $\deg(u, \partial\omega)$ . Thus, without loss of generality, we can assume that  $A$  is in the Coulomb gauge, i.e.,

$$\begin{cases} \operatorname{div} A = 0 \text{ in } \Omega \\ A \cdot \nu = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

The critical points of  $F_\lambda[u, A]$  in  $\mathcal{J}$ , in particular, the local minimizers, are the solutions of the system of Euler–Lagrange equations

$$-(\nabla - iA)^2 u + \frac{\lambda}{2}u(|u|^2 - 1) = 0 \text{ in } G, \tag{2.2}$$

$$-\nabla^\perp h = \begin{cases} j \text{ in } G \\ 0 \text{ in } \omega, \end{cases} \tag{2.3}$$

where  $h = \operatorname{curl} A$  is the magnetic field (scalar function), and

$$j = (iu, \nabla u - iAu)$$

is the current (note that  $h$  and  $j$  are gauge invariant). Additionally,  $h \in H^1(\Omega)$  and the boundary conditions

$$|u| = 1, j \cdot \nu = 0 \text{ on } \partial G, h = 0 \text{ on } \partial\Omega, \frac{\partial h}{\partial \tau} = 0 \text{ on } \partial\omega \quad (2.4)$$

are satisfied. We will assume that  $\partial G \in C^\infty$ , then every solution  $(u, A)$  of (2.2)–(2.4) satisfies  $u \in C^\infty(\bar{G}; \mathbb{C})$  and  $A \in C^\infty(\bar{G}; \mathbb{R}^2)$  (see [5]). We also have the pointwise inequality

$$|u| \leq 1 \text{ in } G$$

which is a consequence of the maximum principle applied to the equation for  $|u|^2$ ,

$$\Delta |u|^2 = \lambda |u|^2 (|u|^2 - 1) + 2 |\nabla u - iAu|^2 \text{ in } G. \quad (2.5)$$

Note that the equation  $-\nabla^\perp h = j$  can be written either in the form

$$\frac{\partial}{\partial z} \left( h - \frac{1}{2} (|u|^2 - 1) \right) = -\bar{u} \left( \frac{\partial u}{\partial z} - \frac{A_2 + iA_1}{2} u \right)$$

or

$$\frac{\partial}{\partial \bar{z}} \left( h + \frac{1}{2} (|u|^2 - 1) \right) = \bar{u} \left( \frac{\partial u}{\partial \bar{z}} + \frac{A_2 - iA_1}{2} u \right).$$

Then, taking  $\partial/\partial \bar{z}$  (or  $\partial/\partial z$ ) and using (2.5), we get

$$\Delta \left( h - \frac{1}{2} (|u|^2 - 1) \right) - |u|^2 h = -4 \left| \frac{\partial u}{\partial z} - \frac{A_2 + iA_1}{2} u \right|^2 - \frac{\lambda}{2} |u|^2 (|u|^2 - 1) \text{ in } G \quad (2.6)$$

and

$$\Delta \left( h + \frac{1}{2} (|u|^2 - 1) \right) - |u|^2 h = 4 \left| \frac{\partial u}{\partial \bar{z}} + \frac{A_2 - iA_1}{2} u \right|^2 + \frac{\lambda}{2} |u|^2 (|u|^2 - 1) \text{ in } G. \quad (2.7)$$

The representation valid for every  $(u, A) \in \mathcal{J}$ ,

$$F_\lambda[u, A] = \pm \pi (\deg(u, \partial\Omega) - \deg(u, \partial\omega)) + F^\pm[u, A] + \frac{1}{2} \int_\omega |\operatorname{curl} A|^2 dx + \frac{\lambda - 1}{8} \int_G (|u|^2 - 1)^2 dx, \quad (2.8)$$

where

$$F^+[u, A] = 2 \int_G \left| \frac{\partial u}{\partial \bar{z}} + \frac{A_2 - iA_1}{2} u \right|^2 dx + \frac{1}{2} \int_G \left| \operatorname{curl} A + \frac{|u|^2 - 1}{2} \right|^2 dx, \quad (2.9)$$

$$F^-[u, A] = 2 \int_G \left| \frac{\partial u}{\partial z} - \frac{A_2 + iA_1}{2} u \right|^2 dx + \frac{1}{2} \int_G \left| \operatorname{curl} A - \frac{|u|^2 - 1}{2} \right|^2 dx, \quad (2.10)$$

plays an important role in the analysis of the functional  $F_\lambda[u, A]$ . This representation is due to a remarkable observation of E.B. Bogomol'nyi [8], for a detailed derivation of (2.8) we refer to [9].

### 3. Properties of Functional $\Phi(u, A, V)$

Let us rewrite  $\Phi(u, A, V)$  as the sum of two terms

$$\Phi(u, A, V) = \frac{1}{2\pi} \int_G u \wedge \left( \frac{\partial u}{\partial x_2} \frac{\partial V}{\partial x_1} - \frac{\partial u}{\partial x_1} \frac{\partial V}{\partial x_2} \right) dx + \frac{1}{2\pi} \int_G A \cdot \nabla^\perp V (1 - |u|^2) dx. \tag{3.1}$$

Then, using the results of [6] (Section 3) for the first term, we get that for every fixed  $V \in C^1(\overline{G})$  the functional  $\Phi(\cdot, V)$  is continuous with respect to the weak convergence in  $H^1(G; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ . If, in addition,  $V$  is such that  $V = 0$  on  $\partial\omega$  and  $V = 1$  on  $\partial\Omega$ , then  $\Phi(u, A, V) = \text{deg}(u, \partial\omega) = \text{deg}(u, \partial\Omega)$  for every  $u \in H^1(G; S^1)$ .

Fix now  $\Lambda > 0$ . Let us show that

$$\sup \{ \text{dist}(\Phi(u, A, V_0), \mathbb{Z}); F_\lambda[u, A] \leq \Lambda \} \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty. \tag{3.2}$$

Since  $\Phi(u, A, V_0)$  is invariant under gauge transformations, we can always assume (2.1). Then, given a sequence of pairs  $(u^\lambda, A^\lambda)$ ,  $\lambda \rightarrow \infty$ , satisfying the bound  $F_\lambda[u^\lambda, A^\lambda] \leq \Lambda$ , we can extract a subsequence converging weakly to a limit  $(u, A)$ . Moreover, since

$$\int_G (1 - |u^\lambda|^2)^2 dx \leq 8\Lambda/\lambda \rightarrow 0,$$

we have  $u \in H^1(G; S^1)$ . Thus  $\Phi(u, A, V_0) \in \mathbb{Z}$ , at the same time  $|\Phi(u, A, V_0) - \Phi(u^\lambda, A^\lambda, V_0)| \rightarrow 0$  as  $\lambda \rightarrow \infty$ , thanks to the continuity of  $\Phi(\cdot, V_0)$  with respect to the weak convergence. So far we have proven (3.2), which in turn implies Proposition 1.

### 4. Construction of Testing Pairs

As mentioned in Introduction, the main technical point in the proof of Theorem 2 is to show strict inequalities (1.8). In this Section we provide a detailed construction of testing pairs with energy control and prove (1.8) with their help.

Given a local minimizer  $(u, A) \in \mathcal{J}_{pq}^{(d)}$  of  $F_\lambda[u, A]$  in  $\mathcal{J}$ , we construct  $(w^{(\xi)}, B^{(\xi)}) \in \mathcal{J}_{p(q-1)}^{(d)}$  in the form

$$w^{(\xi)} = u\bar{a}e^{\phi/2}, \quad B^{(\xi)} = \begin{cases} A + \frac{1}{2}\nabla^\perp\phi, & \text{in } G, \\ A + \nabla^\perp\theta + \nabla\chi & \text{in } \omega, \end{cases} \tag{4.1}$$

where  $a$  is a conformal map from  $\Omega$  onto the unit disk with zero at  $\xi \in G$ , and  $\phi \in H^2(G)$ ,  $\theta, \chi \in H^2(\omega)$  are scalar functions. We assume that  $\phi, \theta, \chi$  depend on the parameter  $\xi$  omitted for brevity. In order to satisfy  $|w^{(\xi)}| = 1$  on  $\partial G$ , we have the following conditions:

$$\phi = 0 \text{ on } \partial\Omega, \quad \phi = -2 \log |a| \text{ on } \partial\omega. \tag{4.2}$$

Since the (exactly one) zero  $\xi$  of  $a$  lies in  $G$ , we have

$$\deg(w^{(\xi)}, \partial\omega) = p, \quad \deg(w^{(\xi)}, \partial\Omega) = q - 1. \tag{4.3}$$

We calculate  $F_\lambda[w^{(\xi)}, B^{(\xi)}]$  using (2.8), (2.10),

$$\begin{aligned} F_\lambda[w^{(\xi)}, B^{(\xi)}] = & \pi + F_\lambda[u, A] + 2 \int_G \left| \frac{\partial u}{\partial z} - \frac{A_2 + iA_1}{2} u \right|^2 (|a|^2 e^\phi - 1) \, dx \\ & + \frac{1}{2} \int_G \left( v(\Delta\phi - |u|^2(|a|^2 e^\phi - 1)) + \frac{1}{4} (\Delta\phi - |u|^2(|a|^2 e^\phi - 1))^2 \right) \, dx \\ & + \frac{\lambda - 1}{8} \int_G \left( (|u|^2 |a|^2 e^\phi - 1)^2 - (|u|^2 - 1)^2 \right) \, dx \\ & + \frac{1}{2} \int_\omega \left( (\Delta\theta + \operatorname{curl} A)^2 - (\operatorname{curl} A)^2 \right) \, dx, \end{aligned} \tag{4.4}$$

where  $v = \operatorname{curl} A - (|u|^2 - 1)/2$ . Then we expand the integrands in the two last terms of (4.4) and use the pointwise equality

$$\begin{aligned} & 4 \left| \frac{\partial u}{\partial z} - \frac{A_2 + iA_1}{2} u \right|^2 (|a|^2 e^\phi - 1) \\ = & (-\Delta v + |u|^2 v)(|a|^2 e^\phi - 1) - \frac{\lambda - 1}{2} |u|^2 (|u|^2 - 1)(|a|^2 e^\phi - 1) \end{aligned}$$

(cf. (2.6)) to get

$$\begin{aligned} F_\lambda[w^{(\xi)}, B^{(\xi)}] = & \pi + F_\lambda[u, A] + \frac{1}{2} \int_G (-\Delta v + |u|^2 v)(|a|^2 e^\phi - 1) \, dx \\ & + \frac{1}{2} \int_G \left( v(\Delta\phi - |u|^2(|a|^2 e^\phi - 1)) + \frac{1}{4} (\Delta\phi - |u|^2(|a|^2 e^\phi - 1))^2 \right) \, dx \\ & + \frac{\lambda - 1}{8} \int_G |u|^4 (|a|^2 e^\phi - 1)^2 \, dx + \frac{1}{2} \int_\omega \left( (\Delta\theta)^2 + 2\Delta\theta \operatorname{curl} A \right) \, dx. \end{aligned} \tag{4.5}$$

Set  $\phi \in H^2(G)$  to be the unique solution of the equation

$$-\Delta\phi + |u|^2(|a|^2e^\phi - 1) = 0 \quad \text{in } G \tag{4.6}$$

subject to the boundary conditions (4.2) (this problem has the unique solution  $\phi \in H^2(G)$ , see, e.g., [9], Theorem 4.3), then (4.5) simplifies to

$$\begin{aligned} F_\lambda[w^{(\xi)}, B^{(\xi)}] = & \pi + F_\lambda[u, A] + \frac{1}{2} \int_G (-\Delta v + |u|^2 v)(|a|^2 e^\phi - 1) \, dx \\ & + \frac{\lambda - 1}{8} \int_G |u|^4 (|a|^2 e^\phi - 1)^2 \, dx + \frac{1}{2} \int_\omega ((\Delta\theta)^2 + 2\Delta\theta \operatorname{curl} A) \, dx. \end{aligned} \tag{4.7}$$

We set the requirement that

$$\frac{\partial\theta}{\partial\nu} = \frac{1}{2} \frac{\partial\phi}{\partial\nu} \quad \text{on } \partial\omega, \tag{4.8}$$

which leads, after integrating by parts in (4.7), to

$$\begin{aligned} F_\lambda[w^{(\xi)}, B^{(\xi)}] = & \pi + F_\lambda[u, A] + \frac{1}{2} \int_G v(-\Delta(|a|^2 e^\phi) + |u|^2(|a|^2 e^\phi - 1)) \, dx \\ & + \frac{\lambda - 1}{8} \int_G |u|^4 (|a|^2 e^\phi - 1)^2 \, dx + \frac{1}{2} \int_\omega (\Delta\theta)^2 \, dx, \end{aligned} \tag{4.9}$$

where we have also used the facts that  $v = \operatorname{curl} A$  on  $\partial\omega$ ,  $\operatorname{curl} A = \operatorname{const}$  in  $\bar{\omega}$  and

$$\int_{\partial\omega} \frac{\partial|a|^2}{\partial\nu} \frac{ds}{|a|^2} = \int_\omega \Delta \log |a|^2 \, dx = 0$$

( $a$  is a holomorphic function without zeros in  $\omega$ ).

Now set  $\theta$  to be a solution of the equation

$$\Delta\theta = \frac{1}{2|\omega|} \int_{\partial\omega} \frac{\partial\phi}{\partial\nu} \, ds \quad \text{in } \omega$$

subject to the boundary condition (4.8). In order to have  $B^{(\xi)} \in H^1(\Omega; \mathbb{R}^2)$ , we define  $\chi \in H^2(G)$  as a function satisfying the boundary conditions  $\chi = 0$  on  $\partial\omega$ ,  $\frac{\partial\chi}{\partial\nu} = -\frac{1}{2} \frac{\partial\phi}{\partial\tau} + \frac{\partial\theta}{\partial\tau}$  on  $\partial\omega$  (for definiteness, we can assume  $\Delta^2\chi = 0$  in  $\omega$ ). Thus, for every  $\xi \in G$  we have  $(w^{(\xi)}, B^{(\xi)}) \in \mathcal{J}_{p(q-1)}$ , and (4.9) yields

$$\begin{aligned} F_\lambda[w^{(\xi)}, B^{(\xi)}] = & \pi + F_\lambda[u, A] + \frac{1}{2} \int_G v(-\Delta(|a|^2 e^\phi) + |u|^2(|a|^2 e^\phi - 1)) \, dx \\ & + \frac{\lambda - 1}{8} \int_G |u|^4 (|a|^2 e^\phi - 1)^2 \, dx + \frac{1}{8|\omega|} \left( \int_{\partial\omega} \frac{\partial\phi}{\partial\nu} \, ds \right)^2. \end{aligned} \tag{4.10}$$

Next we show that  $F_\lambda[w^{(\xi)}, B^{(\xi)}] < \pi + F_\lambda[u, A]$  when  $\xi$  is sufficiently close to  $\partial\Omega$ . To this end, we first study the asymptotic behavior of the conformal map  $a$  and the solution of  $\phi$  of problem (4.2), (4.6) as  $\xi \rightarrow \partial\Omega$ .

**Lemma 3.** *Let  $\tilde{\xi}$  be the nearest point projection of  $\xi \in G$  on  $\partial\Omega$  and let  $\delta = |\xi - \tilde{\xi}|$  be sufficiently small. Then*

- (i)  $|\frac{\partial a}{\partial z}(x)| \leq C\delta/(\delta + |x - \tilde{\xi}|)^2$  and  $\|1 - |a|^2\|_{L^2(G)} \leq C\delta|\log \delta|^{1/2}$ ;
- (ii)  $\|\phi\|_{W^{2,r}(G)} \leq C(r)\delta$  for every  $1 < r < 2$  and  $\|\phi\|_{W^{2,2}(G)} \leq C\delta|\log \delta|^{1/2}$ ,

where  $a$  is the unique (up to a constant factor with modulus one) conformal map from  $\Omega$  onto the unit disk with prescribed zero at the point  $\xi$ , and  $\phi$  is the unique solution of problem (4.2), (4.6).

**P r o o f.** To see (i), we should note that  $a$  can be written as  $a = \sigma(\mathcal{F}(x) - \mathcal{F}(\xi))/(1 - \overline{\mathcal{F}(\xi)}\mathcal{F}(x))$ , where  $\sigma \in S^1$  is a constant and  $\mathcal{F}$  is a fixed conformal map from  $\Omega$  onto the unit disk. Then the proof of the first bound in (i) is straightforward. The second bound in (i) is shown in Section 8 of [4]. In the proof of (ii) we also follow the lines of Section 8 from [4] (the main ingredient here is the chain of pointwise inequalities  $0 \leq 1 - |a|^2e^\phi \leq 1 - |a|^2$  that follow by the maximum principle; details are left to the reader). ■

It follows from Lemma 3 that

$$F_\lambda[w^{(\xi)}, B^{(\xi)}] \leq \pi + F_\lambda[u, A] + \frac{1}{2} \int_G v(-\Delta(|a|^2e^\phi) + |u|^2(|a|^2e^\phi - 1)) dx + O(\delta^2|\log \delta|). \tag{4.11}$$

Observe that

$$\begin{aligned} & \Delta(|a|^2e^\phi) - |u|^2(|a|^2e^\phi - 1) \\ &= |u|^2(|a|^2e^\phi - 1)^2 + 4\left|\frac{\partial a}{\partial z}\right|^2 e^\phi + 4a\frac{\overline{\partial a}}{\partial z}e^\phi\frac{\partial \phi}{\partial z} + 4\overline{a}\frac{\partial a}{\partial z}e^\phi\frac{\partial \phi}{\partial \bar{z}} + |a|^2e^\phi|\nabla\phi|^2, \end{aligned}$$

where we have used (4.6). Then we make use of Lemma 3 and the fact that  $v = 0$  on  $\partial\Omega$  to get, after routine calculations,

$$\int_G v(\Delta(|a|^2e^\phi) - |u|^2(|a|^2e^\phi - 1)) dx = 4 \int_G v\left|\frac{\partial a}{\partial z}\right|^2 dx + O(\delta^2|\log \delta|). \tag{4.12}$$

Taking  $\tilde{v} = \frac{\partial v}{\partial \nu}(\tilde{\xi})\nu(\tilde{\xi})(x - \tilde{\xi})$  in place of  $v$  in the right-hand side of (4.12) (note that  $|v - \tilde{v}| \leq C|x - \tilde{\xi}|^2$ ), we obtain

$$4 \int_G v\left|\frac{\partial a}{\partial z}\right|^2 dx = 4 \int_\Omega \tilde{v}\left|\frac{\partial a}{\partial z}\right|^2 dx + O(\delta^2|\log \delta|). \tag{4.13}$$

Finally, since  $\Delta \log |a|^2 = 4\pi \delta_\xi(x)$  in  $\Omega$  (where  $\delta_\xi(x)$  stands for the Dirac delta centered at  $\xi$ ) and  $|a|^2 = 1$  on  $\partial\Omega$  while  $\Delta \tilde{v} = 0$  in  $\Omega$ , we have

$$4 \int_{\Omega} \tilde{v} \left| \frac{\partial a}{\partial z} \right|^2 dx = \int_{\Omega} \tilde{v} \Delta |a|^2 dx = \int_{\partial\Omega} \tilde{v} \frac{\partial |a|^2}{\partial \nu} ds = \int_{\partial\Omega} \tilde{v} \frac{\partial \log |a|^2}{\partial \nu} ds = 4\pi \tilde{v}(\xi). \tag{4.14}$$

We combine (4.11)–(4.14) to get

$$F_\lambda[w^{(\xi)}, B^{(\xi)}] \leq \pi + F_\lambda[u, A] + 2\pi \frac{\partial v}{\partial \nu}(\tilde{\xi})\delta + O(\delta^2 |\log \delta|), \tag{4.15}$$

where  $\tilde{\xi}$  is the nearest point projection of  $\xi$  on  $\partial\Omega$ . It is clear now that (4.15) yields the required inequality  $F_\lambda[w^{(\xi)}, B^{(\xi)}] < \pi + F_\lambda[u, A]$  when  $\delta$  is sufficiently small, provided that there is  $\tilde{\xi}$  on  $\partial\Omega$  where  $\frac{\partial v}{\partial \nu} < 0$ . The following result establishes the existence of the point  $\tilde{\xi}$  for large  $\lambda$ .

**Lemma 4.** *Let  $(u^{(\lambda)}, A^{(\lambda)})$  be a local minimizer of  $F_\lambda[u, A]$  in  $\mathcal{J}$ . Assume that  $(u^{(\lambda)}, A^{(\lambda)}) \in \mathcal{J}^{(d)} = \bigcup_{p,q \in \mathbb{Z}} \mathcal{J}_{p,q}^{(d)}$  (where  $d$  is a fixed positive integer number) and  $F_\lambda[u^{(\lambda)}, A^{(\lambda)}] \leq \Lambda$  for some fixed  $\Lambda > 0$ . Then  $h^{(\lambda)} = \text{curl } A^{(\lambda)}$  satisfies*

$$\frac{\partial h^{(\lambda)}}{\partial \nu}(\xi^{(\lambda)}) < 0 \text{ for some } \xi^{(\lambda)} \in \partial\Omega$$

when  $\lambda$  is sufficiently large,  $\lambda \geq \lambda_2(\Lambda)$ .

*P r o o f.* Assume by contradiction that  $\frac{\partial h^{(\lambda)}}{\partial \nu} \geq 0$  on  $\partial\Omega$  for a sequence  $\lambda = \lambda_k, \lambda_k \rightarrow \infty$ .

Since  $\text{div } A^{(\lambda)} = 0$  in  $\Omega$  and  $A^{(\lambda)} \cdot \nu = 0$  on  $\partial\Omega$ , we have  $\|A^{(\lambda)}\|_{H^1(\Omega; \mathbb{R}^2)} \leq C \|h\|_{L^2(\Omega)}$ . It follows that  $\|A^{(\lambda)}\|_{H^1(\Omega; \mathbb{R}^2)} \leq C$  thanks to the bound  $F_\lambda[u^{(\lambda)}, A^{(\lambda)}] \leq \Lambda$ . These two bounds imply that  $\|u^{(\lambda)}\|_{H^1(G; \mathbb{C})} \leq C$ , where  $C$  is independent of  $\lambda$ . Therefore, up to extracting a subsequence,  $(u^{(\lambda)}, A^{(\lambda)}) \rightarrow (u, A)$  weakly in  $H^1(G; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$  as  $\lambda \rightarrow \infty$ , and  $|u| = 1$  a.e. in  $G$  (since  $\| |u^{(\lambda)}|^2 - 1 \|_{L^2(G)}^2 \leq 8\Lambda/\lambda \rightarrow 0$ ).

It follows from (2.3) that

$$-\Delta h^{(\lambda)} + h^{(\lambda)} = 2 \frac{\partial u^{(\lambda)}}{\partial x_1} \wedge \frac{\partial u^{(\lambda)}}{\partial x_2} + \text{curl}((1 - |u^{(\lambda)}|^2)A^{(\lambda)}) \text{ in } G. \tag{4.16}$$

We also have  $h^{(\lambda)} = 0$  on  $\partial\Omega$  and  $h^{(\lambda)} = \text{const}$  in  $\bar{\omega}$ . Let  $V \in C^1(\bar{\Omega})$  be the unique solution of the equation

$$\begin{cases} -\Delta V + V = 0 \text{ in } G, \\ V = g \text{ on } \partial\Omega, V = 0 \text{ on } \partial\omega, \end{cases} \tag{4.17}$$

where  $g \in C^1(\partial\Omega)$  is some nonnegative function. Multiply (4.16) by  $V$  and integrate on  $G$  to get, after integrating by parts,

$$-\int_{\partial\Omega} \frac{\partial h^{(\lambda)}}{\partial\nu} g \, ds = \int_{\partial\omega} \frac{\partial V}{\partial\nu} h^{(\lambda)} \, ds + \int_{\partial\Omega} u^{(\lambda)} \wedge \frac{\partial u^{(\lambda)}}{\partial\tau} g \, ds - 2\pi\Phi(u^{(\lambda)}, A^{(\lambda)}, V).$$

We know that  $u^{(\lambda)} \rightarrow u$  weakly in  $H^{1/2}(\partial\Omega; S^1)$ . Due to the result from [17] (see Lemma 3.2 therein) there is a subsequence such that for every  $g \in C^1(\partial\Omega)$ ,

$$\int_{\partial\Omega} u^{(\lambda)} \wedge \frac{\partial u^{(\lambda)}}{\partial\tau} g \, ds \rightarrow \int_{\partial\Omega} u \wedge \frac{\partial u}{\partial\tau} g \, ds + 2\pi \sum_{\text{finite}} D_k g(\alpha_k),$$

where the points  $\alpha_k \in \partial\Omega$  and integers  $D_k$  are independent of  $g$ . Now choose a nonnegative function  $g \not\equiv 0$  such that  $g(\alpha_k) = 0$  for every  $\alpha_k$ . Then we have

$$\begin{aligned} \int_{\partial\Omega} u^{(\lambda)} \wedge \frac{\partial u^{(\lambda)}}{\partial\tau} g \, ds - 2\pi\Phi(u^{(\lambda)}, A^{(\lambda)}, V) &\rightarrow \int_{\partial\Omega} u \wedge \frac{\partial u}{\partial\tau} g \, ds - 2\pi\Phi(u, A, V) \\ &= 2 \int_G \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2} V \, dx = 0, \end{aligned}$$

where we have used the continuity of  $\Phi(\cdot, V)$  with respect to the weak convergence in  $H^1(G; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$  and the pointwise equality  $\frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2} = 0$  a.e. in  $G$  (which holds for every  $u \in H^1(G; S^1)$ ). Thus,

$$0 \leq \int_{\partial\Omega} \frac{\partial h^{(\lambda)}}{\partial\nu} g \, ds = - \int_{\partial\omega} \frac{\partial V}{\partial\nu} h^{(\lambda)} \, ds + o(1) \quad \text{when } \lambda \rightarrow \infty.$$

On the other hand, since  $g \geq 0$  and  $g \not\equiv 0$ , by the strong maximum principle and Hopf's boundary lemma applied to (4.17) we have  $\frac{\partial V}{\partial\nu} > 0$  on  $\partial\omega$ . Therefore we will have a contradiction, if we show that  $\int_{\partial\omega} h^{(\lambda)} \, ds \geq c > 0$  for sufficiently large  $\lambda$ .

To this end, note that in view of (2.3),

$$\begin{aligned} 2\pi\Phi(u^{(\lambda)}, A^{(\lambda)}, V_0) &= - \int_G \nabla h^{(\lambda)} \cdot \nabla V_0 \, dx + \int_G A^{(\lambda)} \cdot \nabla^\perp V_0 \, dx \\ &= \int_{\partial\omega} \frac{\partial V_0}{\partial\nu} h^{(\lambda)} \, ds + \int_G h^{(\lambda)} \Delta V_0 \, dx + \int_G A^{(\lambda)} \cdot \nabla^\perp V_0 \, dx. \end{aligned}$$

Since  $V_0$  solves (1.6), we have

$$\begin{aligned} 2\pi\Phi(u^{(\lambda)}, A^{(\lambda)}, V_0) &= \int_{\partial\omega} \frac{\partial V_0}{\partial\nu} h^{(\lambda)} ds + \int_G h^{(\lambda)}(V_0 - 1) dx + \int_G A^{(\lambda)} \cdot \nabla^\perp V_0 dx \\ &= \int_{\partial\omega} \frac{\partial V_0}{\partial\nu} h^{(\lambda)} ds + \int_G \operatorname{curl}((V_0 - 1)A^{(\lambda)}) dx. \end{aligned} \quad (4.18)$$

Using Stokes' theorem twice, we get

$$\int_G \operatorname{curl}((V_0 - 1)A^{(\lambda)}) dx = \int_{\partial\omega} A^{(\lambda)} \cdot \tau ds = \int_\omega h^{(\lambda)} dx = \frac{|\omega|}{|\partial\omega|} \int_{\partial\omega} h^{(\lambda)} ds,$$

while

$$\int_{\partial\omega} \frac{\partial V_0}{\partial\nu} h^{(\lambda)} ds = \int_{\partial\omega} \frac{\partial V_0}{\partial\nu} ds \int_{\partial\omega} h^{(\lambda)} \frac{ds}{|\partial\omega|} = \int_G (|\nabla V_0|^2 + (V_0 - 1)^2) dx \int_{\partial\omega} h^{(\lambda)} \frac{ds}{|\partial\omega|}.$$

Thus, passing to the limit in (4.18) yields

$$\lim_{\lambda \rightarrow \infty} \int_{\partial\omega} h^{(\lambda)} ds = 2\pi d |\partial\omega| / \left( |\omega| + \int_G (|\nabla V_0|^2 + (V_0 - 1)^2) dx \right) > 0,$$

where we have used the fact that  $\Phi(u^{(\lambda)}, A^{(\lambda)}, V_0) \rightarrow d$  as  $\lambda \rightarrow \infty$ . ■

**R e m a r k 5.** In exactly the same way as in Lemma 4, one can show that  $\frac{\partial h^{(\lambda)}}{\partial\nu} > 0$  at a point on  $\partial\omega$  for sufficiently large  $\lambda$ .

**Corollary 6.** *If  $\xi^{(\lambda)}$  is as in Lemma 4, then  $\frac{\partial v^{(\lambda)}}{\partial\nu}(\xi^{(\lambda)}) < 0$ , where  $v^{(\lambda)} = \operatorname{curl} A^{(\lambda)} - (|u^{(\lambda)}|^2 - 1)/2$ .*

**P r o o f.** Since  $|u^{(\lambda)}| = 1$  on  $\partial G$  and  $|u^{(\lambda)}| \leq 1$  in  $G$ , we have  $\frac{\partial |u^{(\lambda)}|^2}{\partial\nu}(\xi^{(\lambda)}) \geq 0$ . ■

Now, if we take  $\xi \in G$  sufficiently close to the point  $\xi^{(\lambda)}$ , where  $\frac{\partial v^{(\lambda)}}{\partial\nu}(\xi^{(\lambda)}) < 0$ , by (4.15) and Corollary 6, we have  $F_\lambda[w^{(\xi)}, B^{(\xi)}] < \pi + F_\lambda[u, A]$ . On the other hand,  $(w^{(\xi)}, B^{(\xi)}) \in \mathcal{J}_{p(q-1)}$  and  $(w^{(\xi)}, B^{(\xi)})$  converges weakly to  $(\gamma u, A)$  ( $\gamma = \operatorname{const} \in S^1$ ) as  $\xi \rightarrow \partial\Omega$ , up to a subsequence. Consequently,  $\Phi(w^{(\xi)}, B^{(\xi)}, V_0) \rightarrow \Phi(u, A, V_0)$ . Thus, if  $d - 1/2 < \Phi(u, A, V_0) < d + 1/2$ , then  $(w^{(\xi)}, B^{(\xi)}) \in \mathcal{J}_{p(q-1)}^{(d)}$ , when  $\xi$  is sufficiently close to  $\xi^{(\lambda)}$ . Quite similarly, we can show that there exists a testing pair from  $\mathcal{J}_{(p-1)q}^{(d)}$  whose Ginzburg–Landau energy is strictly less than  $\pi + F_\lambda[u, A]$ . These results are summarized in

**Lemma 7.** *Given integers  $d > 0$ ,  $p$  and  $q$ , if  $m_\lambda(p, q, d)$  is attained and  $\lambda$  is sufficiently large,  $\lambda \geq \lambda_3(p, q, d)$ , then  $m_\lambda(p - 1, q, d) < m_\lambda(p, q, d) + \pi$  and  $m_\lambda(p, q - 1, d) < m_\lambda(p, q, d) + \pi$ .*

**P r o o f.** It is not hard to prove that the bound  $m_\lambda(p, q, d) \leq \Lambda(p, q, d)$  holds for some  $\Lambda(p, q, d)$  independent of  $\lambda$  (see, e.g., [6]). Then the above results in conjunction with Proposition 1 yield the statement of the lemma. ■

## 5. Existence of Minimizers

To begin with, let us quote the following result which is an important tool in the proof of Theorem 2.

**Lemma 8.** ([2]) *Let  $(u^{(n)}, A^{(n)}) \in \mathcal{J}_{pq}$  be a sequence converging weakly to  $(u, A)$  in  $H^1(G; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ . Then*

$$\liminf \frac{1}{2} \int_G |\nabla u^{(n)} - iA^{(n)}u^{(n)}|^2 dx \geq \pi(|p - \deg(u, \partial\omega)| + |q - \deg(u, \partial\Omega)|) + \frac{1}{2} \int_G |\nabla u - iAu|^2 dx.$$

Now consider the auxiliary minimization problem

$$M_\lambda(d) := \inf\{F_\lambda[u, A]; (u, A) \in \mathcal{J}^{(d)}\}, \tag{5.1}$$

where  $\mathcal{J}^{(d)} = \bigcup_{p,q \in \mathbb{Z}} \mathcal{J}_{pq}^{(d)} = \{(u, A) \in \mathcal{J}; d - 1/2 \leq \Phi(u, A, V_0) \leq d + 1/2\}$ . Note that  $M_\lambda(d)$  is always attained. Indeed,  $\Phi(\cdot, V_0)$  is continuous with respect to the weak convergence (in  $H^1(G; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ ), and  $\mathcal{J}^{(d)} \neq \emptyset$  ( $\mathcal{J}^{(d)}$  contains, in particular, pairs  $(u, 0)$  with  $u \in H^1(G; S^1)$  and  $\deg(u, \partial\Omega) = d$ ). Therefore every minimizing sequence  $(u^{(n)}, A^{(n)})$  contains a subsequence converging weakly in  $H^1(G; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$  to a minimizer  $(u, A) \in \mathcal{J}^{(d)}$ .

**Lemma 9.** *For sufficiently large  $\lambda$ ,  $\lambda \geq \lambda_4(d)$ ,  $M_\lambda(d) = m_\lambda(d, d, d)$ , and minimizers of (1.3) (with  $p = q = d$ ) and (5.1) coincide.*

**P r o o f.** Clearly,  $M_\lambda(d) \leq m_\lambda(d, d, d)$ . Now assume by contradiction that for a sequence  $\lambda = \lambda_k$ ,  $\lambda_k \rightarrow \infty$ , we have

$$M_\lambda(d) = F_\lambda[u^{(\lambda)}, A^{(\lambda)}] \text{ and } (u^{(\lambda)}, A^{(\lambda)}) \in \mathcal{J}^{(d)} \setminus \mathcal{J}_{dd}^{(d)}. \tag{5.2}$$

In other words, either  $\deg(u^{(\lambda)}, \partial\Omega) \neq d$  or  $\deg(u^{(\lambda)}, \partial\omega) \neq d$ . We assume that  $\operatorname{div} A^{(\lambda)} = 0$  in  $\Omega$  and  $A^{(\lambda)} \cdot \nu = 0$  on  $\partial\Omega$  (Coulomb gauge). Thanks to the obvious

bound

$$M_\lambda(d) \leq M_\infty(d) := \inf \left\{ \frac{1}{2} \int_G |\nabla u - iAu|^2 dx + \frac{1}{2} \int_\Omega (\operatorname{curl} A)^2 dx; \right. \\ \left. (u, A) \in \mathcal{J}^{(d)}, u \in H^1(G; S^1) \right\} < \infty \quad (5.3)$$

we can extract a subsequence such that  $u^{(\lambda)} \rightarrow u$  weakly in  $H^1(G; \mathbb{C})$ ,  $A^{(\lambda)} \rightarrow A$  weakly in  $H^1(\Omega; \mathbb{R}^2)$  as  $\lambda \rightarrow \infty$ , and  $\deg(u^{(\lambda)}, \partial\Omega) = q$ ,  $\deg(u^{(\lambda)}, \partial\omega) = p$  with integers  $p, q$  independent of  $\lambda$ . Note that  $(u, A) \in \mathcal{J}^{(d)}$  and  $u \in H^1(G; S^1)$ . Therefore,

$$F_\lambda[u^{(\lambda)}, A^{(\lambda)}] \leq M_\lambda(d) \leq F_\lambda[u, A] = \frac{1}{2} \int_G (|\nabla u - iAu|^2 dx + \frac{1}{2} \int_\Omega (\operatorname{curl} A)^2 dx,$$

while

$$\liminf_{\lambda \rightarrow \infty} F_\lambda[u^{(\lambda)}, A^{(\lambda)}] \geq \pi(|d-p| + |d-q|) + \frac{1}{2} \int_G |\nabla u - iAu|^2 dx + \frac{1}{2} \int_\Omega (\operatorname{curl} A)^2 dx$$

by virtue of Lemma 8. Thus  $p = q = d$  for sufficiently large  $\lambda$ , i.e.,  $u^{(\lambda)} \in \mathcal{J}_{dd}^{(d)}$ , which contradicts (5.2). ■

Thus, Lemma 9 shows that  $m_\lambda(d, d, d)$  is always attained for  $\lambda \geq \lambda_4(d)$ . Moreover, thanks to Proposition 1 every minimizer of (1.3) with  $p = q = d$  is a local minimizer of  $F_\lambda[u, A]$  in  $\mathcal{J} \times H^1(\Omega; \mathbb{R}^2)$  when  $\lambda \geq \lambda_0(M_\infty(d) + 1)$ , where  $M_\infty(d)$  is defined in (5.3) (note that  $m_\lambda(d, d, d) < M_\infty(d) + 1$ ).

We next prove the attainability of the infimum (1.3) in

**Proposition 10.** *For every  $K = 0, 1, 2, \dots$  there is  $\lambda_5 = \lambda_5(K) > 0$  such that for all  $\lambda \geq \lambda_5$  and all integers  $p$  and  $q$  satisfying  $p \leq d, q \leq d$ , and  $|q - d| + |p - d| \leq K$ :*

- (i) *the infimum  $m_\lambda(p, q, d)$  is attained,*
- (ii) *if  $p \leq p' \leq d, q \leq q' \leq d$  and either  $p \neq p'$  or  $q \neq q'$ , then*

$$m_\lambda(p, q, d) < m_\lambda(p', q', d) + \pi(|p - p'| + |q - q'|).$$

*Proof.* By Lemma 9, Proposition 10 holds for  $K = 0$  (induction base). Now assume that (i) and (ii) hold for given  $K \geq 0$ . Then by Lemma 7, (ii) holds for  $K + 1$  in place of  $K$  when  $\lambda \geq \max\{\lambda_5(K), \max\{\lambda_3(p, q, d); |q - d| + |p - d| = K, p \leq d, q \leq d\}\}$ .

To show (i), we consider a minimizing sequence  $(u^{(n)}, A^{(n)})$  for problem (1.3). This minimizing sequence exists since  $m_\lambda(p, q, d) < m_\lambda(d, d, d) + \pi(|p - d| + |q - d|) \leq M_\infty(d) + \pi(K + 1)$ . Thanks to this bound, up to extracting a subsequence,  $u^{(n)} \rightharpoonup u \in \mathcal{J}$  weakly in  $H^1(G; \mathbb{C})$ , and  $A^{(n)} \rightharpoonup A$  weakly in  $H^1(\Omega; \mathbb{R}^2)$ . By virtue of Lemma 8, we have

$$m_\lambda(p, q, d) = \lim_{n \rightarrow \infty} F_\lambda[u^{(n)}, A^{(n)}] \geq F_\lambda[u, A] + \pi(|q - \deg(u, \partial\Omega)| + |p - \deg(u, \partial\omega)|). \quad (5.4)$$

Let us show that  $\deg(u, \partial\Omega) = q$  and  $\deg(u, \partial\omega) = p$ . To this end, we need the following.

**Lemma 11.** *For every  $\Lambda > 0$  there is  $\lambda_6 = \lambda_6(\Lambda)$  such that  $m_\lambda(p, q, d) \geq M_\infty(d) + \pi(|p - d| + |q - d| - 1/2)$  when  $\lambda \geq \lambda_6$  and  $M_\infty(d) + \pi(|p - d| + |q - d|) \leq \Lambda$ .*

This lemma and (5.4) imply that if  $\lambda \geq \lambda_6$ , then

$$\begin{aligned} m_\lambda(p, q, d) &\geq m_\lambda(\deg(u, \partial\omega), \deg(u, \partial\Omega), d) + \pi(|q - \deg(u, \partial\Omega)| + |p - \deg(u, \partial\omega)|) \\ &\geq M_\infty(d) + \pi(|p - \deg(u, \partial\omega)| + |\deg(u, \partial\omega) - d|) \\ &\quad + \pi(|q - \deg(u, \partial\Omega)| + |\deg(u, \partial\Omega) - d|) - \pi/2. \end{aligned}$$

On the other hand, (ii) guarantees that  $m_\lambda(p, q, d) < M_\infty(d) + \pi(|p - d| + |q - d|)$ , therefore  $p \leq \deg(u, \partial\omega) \leq d$  and  $q \leq \deg(u, \partial\Omega) \leq d$ . Furthermore, if we assume that either  $\deg(u, \partial\Omega) \neq q$  or  $\deg(u, \partial\omega) \neq p$ , then (5.4) implies that  $m_\lambda(p, q, d) \geq m_\lambda(\deg(u, \partial\omega), \deg(u, \partial\Omega), d) + \pi(|q - \deg(u, \partial\Omega)| + |p - \deg(u, \partial\omega)|)$ , but this contradicts (ii). Thus  $m_\lambda(p, q, d)$  is always attained for sufficiently large  $\lambda$  when  $p \leq d$ ,  $q \leq d$ , and  $|q - d| + |p - d| \leq K + 1$ . Proposition 10 is proved. ■

*P r o o f* of Lemma 11. Similarly to Lemma 9, we argue by contradiction. Namely, assume that  $m_\lambda(p, q, d) < M_\infty(d) + \pi(|p - d| + |q - d| - 1/2)$  for some integers  $p, q, d$  and a sequence  $\lambda = \lambda_k, \lambda_k \rightarrow \infty$ . In other words, there are  $(u^{(\lambda)}, A^{(\lambda)}) \in \mathcal{J}_{pq}^{(d)}$  such that  $F_\lambda[u^{(\lambda)}, A^{(\lambda)}] < M_\infty(d) + \pi(|p - d| + |q - d| - 1/2)$  for  $\lambda = \lambda_k$ . Since  $F_\lambda[u^{(\lambda)}, A^{(\lambda)}]$  is bounded, up to extracting a subsequence,  $u^{(\lambda)} \rightharpoonup u$  weakly in  $H^1(G; \mathbb{C})$ ,  $A^{(\lambda)} \rightharpoonup A$  weakly in  $H^1(\Omega; \mathbb{R}^2)$ . Besides,  $u \in \mathcal{J}^{(d)} \cap H^1(G; S^1)$ . Therefore,

$$\frac{1}{2} \int_G |\nabla u - iAu|^2 dx + \frac{1}{2} \int_\Omega (\operatorname{curl} A)^2 dx \geq M_\infty(d).$$

On the other hand, by virtue of Lemma 8, we have

$$\liminf_{\lambda \rightarrow \infty} F_\lambda[u^{(\lambda)}, A^{(\lambda)}] \geq \frac{1}{2} \int_G |\nabla u - iAu|^2 dx + \frac{1}{2} \int_\Omega (\operatorname{curl} A)^2 dx + \pi(|p - d| + |q - d|),$$

and thus we arrive at a contradiction with the bound  $F_\lambda[u^{(\lambda)}, A^{(\lambda)}] < M_\infty(d) + \pi(|p - d| + |q - d| - 1/2)$ . ■

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