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On the Characteristic Operator of an Integral Equation with a Nevanlinna Measure in the Infinite-Dimensional Case

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We define the families of maximal and minimal relations generated by an integral equation with a Nevanlinna operator measure in the infinite-dimensional case and prove their holomorphic property. We show that if the restrictions of maximal relations are continuously invertible, then the operators inverse to these restrictions are integral. By using these results, we prove the existence of the characteristic operator and describe the families of linear relations generating the characteristic operator.

Key words: Hilbert space, linear relation, integral equation, characteristic operator, Nevanlinna measure.

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1. Introduction

Integral equations with a Nevanlinna operator measure are sufficiently general equations. For example, they include differential equations whose coefficients are generalized functions [1], differential equations with holomorphic (with respect to the spectral parameter) coefficients and with the Dirichlet integral whose imaginary part is nonpositive [2], integro-differential equations with the Stieltjes integrals [3] (see also references therein).

On a finite or infinite interval (a, b), we consider the integral equation

$$y(t) = y(t_0) - iJ\int_{t_0}^{t}(d\widetilde{Z}_\lambda)y(s) - iJ\int_{t_0}^{t}(d\widetilde{\mathrm{V}})f(s), \qquad (1)$$

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where y is the desired function. Here y, f are the functions with values belonging to a separable Hilbert space H; J is an operator in H such that $J^* = J$, $J^2 = E$ (E is the identity operator); the function $\Delta \to \widetilde{Z}_\lambda(\Delta)$ is an operator measure on (a,b) such that the measures $\mathrm{Re}\widetilde{Z}_i$, $\widetilde{\mathrm{V}} = (\mathrm{Im}\lambda_0)^{-1}\mathrm{Im}\widetilde{Z}_{\lambda_0}$ have locally bounded variations on (a,b), Δ is a Borel set; the function $\lambda \to \widetilde{Z}_\lambda(\Delta)$ is the Nevanlinna function for any fixed Δ, i.e., this function is holomorphic if $\mathrm{Im}\lambda \neq 0$, $\widetilde{Z}^*_\lambda(\Delta) = \widetilde{Z}_{\bar{\lambda}}(\Delta)$, and $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}\widetilde{Z}_\lambda(\Delta) \geqslant 0$.

Equation (1) generates holomorphic families of maximal and minimal relations. If the measure $\widetilde{Z}_\lambda$ is absolutely continuous, then (1) can be reduced to a differential equation with a Nevanlinna operator function. The linear relations generated by this differential equation were studied in [4–6]. For the case where H is a finite-dimensional space, equation (1) is considered in [7, 8]. The infinite-dimensional case differs essentially from the finite-dimensional case. It can be explained by the fact that the space $\mathfrak{H} = L_2(H, d\widetilde{\mathrm{V}}; a, b)$ is rather sophisticated ($\mathfrak{H}$ is the space in which the minimal and maximal relations are considered). The elements of $\mathfrak{H}$ are not necessarily the functions with values in H.

In the present paper, we define the families of maximal and minimal relations generated by (1) in the infinite-dimensional case, study the properties of these relations and describe the continuously invertible restrictions of maximal relations. Thereby we generalize some assertions from [7, 8] to the infinite-dimensional case. We apply the obtained results to prove the existence of the characteristic operator and to describe the families of linear relations generating the characteristic operator.

We note that the definition of the characteristic operator for a differential equation with a Nevanlinna operator function is given in [4, 5]. In these papers, the existence of the characteristic operator is established by studying special boundary value problems with a spectral parameter in the boundary condition.

2. Main Assumptions and Notations

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Denote by $\mathcal{B}$ a set of bounded Borel subsets Δ such that $\overline{\Delta} \subset (a,b)$. We consider the function $\Delta \to \widetilde{\mathcal{X}}(\Delta)$ defined on $\mathcal{B}$ and ranging over a set of bounded linear operators acting in H. The function $\widetilde{\mathcal{X}}$ is called the operator measure on (a,b) (see, e.g., [9, ch. 5]) if $\widetilde{\mathcal{X}}$ is equal to zero on the empty set and the equality

$$\widetilde{\mathcal{X}}\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} \widetilde{\mathcal{X}}(\Delta_n)$$

holds for disjoint Borel sets $\Delta_n \in \mathcal{B}$, where the series converges weakly. By

$\mathbf{V}_\Delta(\widetilde{\mathcal{X}})$, denote

$$\mathbf{V}_\Delta(\widetilde{\mathcal{X}}) = \widetilde{\rho}(\Delta) = \sup \sum_j \left\| \widetilde{\mathcal{X}}(\Delta_j) \right\|,$$

where "sup" can be applied to the finite sums of disjoint Borel sets $\Delta_j \subset \Delta$. The number $\mathbf{V}_\Delta(\widetilde{\mathcal{X}})$ is called the variation of the measure $\widetilde{\mathcal{X}}$ on the Borel set Δ. Suppose that the measure $\widetilde{\mathcal{X}}$ has a locally bounded variation, i.e., $\mathbf{V}_{[a_1,b_1]}(\widetilde{\mathcal{X}}) < \infty$ for any segment $[a_1, b_1] \subset (a, b)$. The function $\Delta \to \widetilde{\rho}(\Delta)$ is a nonnegative measure on (a, b). We assign $\widetilde{\rho}((a, b_1]) = \lim_{n\to\infty} \widetilde{\rho}((a_n, b_1]) \leqslant \infty$, where $a_n \to a$ as $n \to \infty$, $a_n > a$, $b_1 < b$ (similarly for the endpoint b).

The following statement can be found in [9, ch. 5].

Statement 1. *Suppose that the measure $\widetilde{\mathcal{X}}$ has a locally bounded variation on (a, b). Then for $\widetilde{\rho}$-almost all $\xi \in (a, b)$ there exists an operator function $\xi \to \Psi(\xi)$ such that Ψ has values in the set of bounded linear operators acting in H, $\|\Psi(\xi)\| = 1$, and the equality*

$$\widetilde{\mathcal{X}}(\Delta) = \int_\Delta \Psi(\xi) d\widetilde{\rho} \tag{2}$$

holds for all Borel sets $\Delta \in \mathcal{B}$. The function Ψ is uniquely determined up to a set of ρ-measure zero. In the case $\widetilde{\rho}(\Delta) < \infty$, the integral sums for (2) *converge with respect to the usual norm of operators.*

Let $\{\Delta, \lambda\} \to \widetilde{Z}_\lambda(\Delta)$ be a function with values in the set of linear bounded operators acting in H, where $\Delta \in \mathcal{B}$, $\lambda \in \mathbb{C}_0$, $\mathbb{C}_0 \supset \mathbb{C}\backslash\mathbb{R}$, the symbol $\{x_1, x_2\}$ denotes an ordered pair consisting of x_1, x_2. We assume that this function is the Nevanlinna function for any fixed Δ, i.e., the following conditions hold: (a) each point from $\mathbb{C}_0$ has a neighborhood (independent of Δ) such that the function $\lambda \to \widetilde{Z}_\lambda(\Delta)$ is holomorphic in this neighborhood; (b) $\widetilde{Z}^*_\lambda(\Delta) = \widetilde{Z}_{\bar\lambda}(\Delta)$; (c) $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}\widetilde{Z}_\lambda(\Delta) \geqslant 0$ for all $\Delta \in \mathcal{B}$ and all λ such that $\mathrm{Im}\lambda \neq 0$. Moreover, this function satisfies condition (d). Before formulating condition (d), we introduce some notations. We put $\widetilde{\mathrm{V}}_\lambda(\Delta) = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}\widetilde{Z}_\lambda(\Delta)$. Then for all $\nu \in \mathbb{C}_0 \cap \mathbb{R}$ there exists (at least in the weak sense) $\lim_{\lambda\to\nu\pm i0} \widetilde{\mathrm{V}}_\lambda(\Delta) = \widetilde{\mathrm{V}}_\nu(\Delta)$.

In [4], it was shown that conditions (a)–(c) imply

$$k_1(\widetilde{\mathrm{V}}_\lambda(\Delta)g, g) \leqslant (\widetilde{\mathrm{V}}_\mu(\Delta)g, g) \leqslant k_2(\widetilde{\mathrm{V}}_\lambda(\Delta)g, g). \tag{3}$$

It follows from [4, 6] that

$$\left|(\lambda - \mu)^{-1}((\widetilde{Z}_\lambda(\Delta) - \mathrm{Re}\widetilde{Z}_\mu(\Delta))g, h)\right| \leqslant k \left\|\widetilde{\mathrm{V}}^{1/2}_\zeta(\Delta)g\right\| \left\|\widetilde{\mathrm{V}}^{1/2}_\eta(\Delta)h\right\|. \tag{4}$$

In inequalities (3), (4), $g, h \in H$, $\lambda, \mu, \zeta, \eta \in \mathbb{C}_0$, the constants $k, k_1, k_2 > 0$ are independent of $\Delta \in \mathcal{B}$, $\lambda, \mu, \zeta, \eta \in \mathrm{K}$ (K is an arbitrary fixed compact, $\mathrm{K} \subset \mathbb{C}_0$).

Condition (d): the function $\Delta \to \widetilde{Z}_\lambda(\Delta)$ is an operator measure on (a,b) for all $\lambda \in \mathbb{C}_0$, and the measures $\mathrm{Re}\widetilde{Z}_i$, $\widetilde{\mathrm{V}}_{\lambda_0}$ (for some point $\lambda_0 \in \mathbb{C}_0$) have locally bounded variations on (a,b).

It follows from condition (d) and (3), (4) that the measures $\widetilde{\mathrm{V}}_\lambda$, $\widetilde{Z}_\lambda - \mathrm{Re}\widetilde{Z}_\mu$, $\widetilde{Z}_\lambda$ $(\lambda,\mu \in \mathbb{C}_0)$ have locally bounded variations on (a,b). We fix some $\lambda_0 \in \mathbb{C}$ and put $\widetilde{\mathrm{V}} = \widetilde{\mathrm{V}}_{\lambda_0}$.

Suppose that the measure $\widetilde{\mathcal{X}}$ has a locally bounded variation on (a,b). We call the endpoint a *regular* for the measure $\widetilde{\mathcal{X}}$ if $a > -\infty$ and the measure $\widetilde{\mathcal{X}}$ has a locally bounded variation on each segment $[a,b_1]$ $(b_1 < b)$. The endpoint a is said to be *singular* if it is not regular. We define the regularity and singularity of the endpoint b in a similar way.

In the case of regular endpoint, we extend the considered measures on a larger interval as follows. We put $a_0 = a$ if the endpoint a is singular; if it is regular, we fix a certain point a_0 such that $a_0 < a$, and we put $\widetilde{X}(\Delta) = 0$ for all Borel sets $\Delta \subset [a_0,a)$. Analogously, we define a value b_0 for the endpoint b. We also put $\widetilde{X}(\Delta) = 0$ for all Borel sets $\Delta \subset (b,b_0]$ in the case when the endpoint b is regular. We use the previous notations for the extended measures. Note that equality (2) remains valid for all Borel sets Δ such that $\overline{\Delta} \subset (a_0,b_0)$.

It follows from (3), (4) that the endpoint a is regular for the measure $\widetilde{Z}_\lambda$ if and only if it is regular for the measures $\mathrm{Re}\widetilde{Z}_i$ and $\widetilde{\mathrm{V}} = \widetilde{\mathrm{V}}_{\lambda_0}$. The similar assertion is valid for the endpoint b.

The integral $\int_{t_0}^{t}$ is understood as the integral $\int_{[t_0,t)}$ if $t > t_0$ or 0, if $t = t_0$ or $-\int_{[t,t_0)}$, if $t < t_0$. In the integral $\int_{t_0}^{t}$, we assume that $a_0 < t_0 < b_0$, $a_0 < t < b_0$. If the endpoint a (or b) is regular, then it is possible that $t_0 = a_0$, $t = a_0$ (or $t_0 = b_0$, $t = b_0$, respectively).

3. Solution of Integral Equations

A function f with values in H is called a step function on a Borel set Δ if Δ can be represented as the union of a finite number of disjoint Borel sets Δ_j such that f is constant on each Δ_j. Obviously, the integral $\int_\Delta (d\widetilde{\mathcal{X}})f(t)$ exists for any operator measure $\widetilde{\mathcal{X}}$ on (a,b) and for any step function f on $\Delta \in \mathcal{B}$. Suppose the measure $\widetilde{\mathcal{X}}$ has a locally bounded variation on (a,b), and Δ is a Borel set with the property $\overline{\Delta} \subset (a_0,b_0)$. Then the above integral is defined by the equality

$$\int_\Delta (d\widetilde{\mathcal{X}})f(t) = \int_\Delta \Psi(t)f(t)d\widetilde{\rho} \tag{5}$$

for a function f such that the Bochner integral exists in the right-hand side of (5), where Ψ is the function from (2). Let $\{f_n\}$ be a sequence of step functions on Δ. If $\int_\Delta \|f(s) - f_n(s)\| \, d\widetilde{\rho} \to 0$ as $n \to \infty$, then

$$\int_\Delta (d\widetilde{\mathcal{X}}) f(t) = \lim_{n\to\infty} \int_\Delta (d\widetilde{\mathcal{X}}) f_n(t).$$

The following integral is defined by the equality

$$\int_\Delta ((d\widetilde{\mathcal{X}}) f(t), g(t)) = \int_\Delta (\Psi(t) f(t), g(t)) d\widetilde{\rho}$$

under the condition that the integral exists in the right-hand side of this equality. If $\widetilde{\rho}(\Delta) = \infty$, then the above integrals can be defined by the standard method.

It follows from (5) that if a Borel measurable function f is bounded and $\widetilde{\rho}(\Delta) < \infty$, then

$$\left\| \int_\Delta (d\widetilde{\mathcal{X}}) f(t) \right\| \leqslant \sup_{t\in\Delta} \|f(t)\| \, \widetilde{\rho}(\Delta). \tag{6}$$

Let the function f be locally integrable with respect to the measure $\widetilde{\mathcal{X}}$ on (a_0, b_0). We claim that the function $y(t) = \int_{t_0}^t (d\widetilde{\mathcal{X}}) f(s)$ is continuous from the left (here $t_0, t \in (a_0, b_0)$). Indeed, if $t < t_1$, then $y(t_1) - y(t) = \int_{[t,t_1)} (d\widetilde{\mathcal{X}}) f(s)$. Using (2), we get

$$\|y(t_1) - y(t)\| \leqslant \int_{[t,t_1)} \|\Psi(\xi) f(\xi)\| \, d\widetilde{\rho}.$$

Now the desired assertion follows from the equality $\bigcap_t [t, t_1) = \varnothing$.

Suppose a segment $[l_1, l_2] \subset (a_0, b_0)$. We consider a set of functions bounded on $[l_1, l_2]$, continuous from the left (in the strong sense) on $(l_1, l_2]$, and ranging over H. We introduce the norm $\|u\|_{[l_1,l_2]} = \sup_{t\in[l_1,l_2]} \|u(t)\|$ on this set and obtain a Banach space denoted by $C_-[l_1, l_2]$.

Theorem 1. *Suppose that the measure $\widetilde{\mathcal{X}}$ has a locally bounded variation on (a, b), and the function $h \in C_-[a_1, b_1]$, where $[a_1, b_1] \subset (a_0, b_0)$. Then for all $y_0 \in H$ there exists a unique solution of the equation*

$$y(t) = y_0 - iJ \int_{t_0}^t (d\widetilde{\mathcal{X}}) y(s) - iJh(t) \quad (a_1 \leqslant t_0 \leqslant b_1, \;\; a_1 \leqslant t \leqslant b_1) \tag{7}$$

belonging to the space $C_-([a_1, b_1])$.

P r o o f. By the definition of the interval (a_0, b_0), the measure $\widetilde{\mathcal{X}}$ has the bounded variation on the segment $[a_1, b_1]$. First, we will show that there exists a segment $\mathcal{I}_\delta = [t_0 - \delta, t_0 + \delta]$ such that equation (7) has a unique solution in the space $C_-(\mathcal{I}_\delta)$ $(\delta > 0)$ for each $y_0 \in H$. (If $t_0 = a_1$, we set $\mathcal{I}_\delta = [t_0, t_0 + \delta]$, and if $t_0 = b_1$, we set $\mathcal{I}_\delta = [t_0 - \delta, t_0]$.)

Let $t \to \rho(t)$ be a continuous from the left function generating the measure $\widetilde{\rho}$. By $\rho_s(t_0)$, we denote the jump of the function ρ at the point t_0 (it is possible that $\rho_s(t_0) = 0$). We set $r(t) = \rho(t)$ for $t \leqslant t_0$ and $r(t) = \rho(t) - \rho_s(t_0)$ for $t > t_0$. The function r is continuous at t_0. Let $\widetilde{r}$ denote the measure generated by the function r. We introduce the operator measure

$$\widetilde{\mathcal{Y}}(\Delta) = \int_\Delta \Psi(\xi) d\widetilde{r}. \tag{8}$$

Under this notation, equation (7) has the form $y = Ay + z$, where

$$(Ay)(t) = -iJ \int_{t_0}^{t} (d\widetilde{\mathcal{Y}}) y(s), \quad z = y_0 - iJ\widetilde{\mathcal{X}}(\{t_0\}) y_0 - iJh(t). \tag{9}$$

Taking into account (6), (8), (9), and the continuity of r, we obtain

$$\|(Ay)(t)\| \leqslant \sup_{t \in \mathcal{I}_\delta} \|y(t)\| \, |r(t) - r(t_0)| < \varepsilon \sup_{t \in \mathcal{I}_\delta} \|y(t)\| .$$

Consequently, $\sup_{t \in \mathcal{I}_\delta} \|(Ay)(t)\| \leqslant \varepsilon \sup_{t \in \mathcal{I}_\delta} \|y(t)\|$. Using the continuity of r, we take $\delta > 0$ under which $\varepsilon < 1$. Then $\|A\|_{C_-(\mathcal{I}_\delta)} < 1$. Hence the operator $E - A$ has the bounded everywhere defined inverse operator in the space $C_-(\mathcal{I}_\delta)$. Thus there exists a unique solution of equation (7) on the interval $\mathcal{I}_\delta$. The solution is found by using the formula $y = (E - A)^{-1} z$. Now the desired statement can be obtained in the standard way. The theorem is proved.

R e m a r k 1. If y is the solution of equation (7), then $\lim_{t \to t_0 - 0} y(t) = y_0$.

Theorem 2. *Assume that the operator measures* $\widetilde{\mathcal{X}}$, $\widetilde{\mathcal{X}}_n$ *have locally bounded variations on* (a, b), $[a_1, b_1] \subset (a_0, b_0)$, $y_{n,0} \in H$, $h_n \in C_-[a_1, b_1]$, *and assume that the sequence* $\{h_n\}$ *converges uniformly to* h *on* $[a_1, b_1]$, *and the sequence* $\{y_{n,0}\}$ *converges to* y_0 *in* H. *Let* y_n *be the solution of the equation*

$$y_n(t) = y_{n,0} - iJ \int_{t_0}^{t} (d\widetilde{\mathcal{X}}_n) y_n(s) - iJh_n(t), \quad a_1 \leqslant t_0 \leqslant b_1, \quad a_1 \leqslant t \leqslant b_1. \tag{10}$$

If $\lim_{n \to \infty} \mathbf{V}_{[a_1, b_1]}(\widetilde{\mathcal{X}}_n - \widetilde{\mathcal{X}}) = 0$, *then the sequence* $\{y_n\}$ *converges uniformly to the solution* y *of equation* (7) *on* $[a_1, b_1]$.

P r o o f. It follows from the definition of the interval (a_0, b_0) that the measures $\widetilde{\mathcal{X}}$, $\widetilde{\mathcal{X}}_n$ have bounded variations on the segment $[a_1, b_1]$. We construct the measure $\widetilde{\mathcal{Y}}_n$ by the measure $\widetilde{\mathcal{X}}_n$ in the same way as the measure $\widetilde{\mathcal{Y}}$ is constructed by the measure $\widetilde{\mathcal{X}}$ in the proof of Theorem 1. Then equation (10) can be written as $y_n = A_n y_n + z_n$, where

$$(A_n u)(t) = -iJ\int_{t_0}^{t}(d\widetilde{\mathcal{Y}}_n)u(\tau), \;\; z_n = y_{n,0} - iJ\widetilde{\mathcal{X}}_n(\{t_0\})y_{n,0} - iJh_n(t). \tag{11}$$

We claim that the sequence $\{A_n\}$ converges to A in the uniform operator topology of the operators acting in $C_-(\mathcal{I}_\delta)$, where A, δ are the same as in the proof of Theorem 1. Indeed, we denote $\widetilde{v}_n = \mathbf{V}_{[a_1,b_1]}(\widetilde{\mathcal{Y}}_n - \widetilde{\mathcal{Y}})$. Then $\lim\limits_{n\to\infty} \widetilde{v}_n = 0$. From this, Statement 1, and inequality (6) applied to the measure $\widetilde{\mathcal{Y}}_n - \widetilde{\mathcal{Y}}$, we obtain that for all function $x \in C_-(\mathcal{I}_\delta)$, for all $\varepsilon > 0$, the inequality

$$\|(A_n - A)x\|_{C_-(\mathcal{I}_\delta)} \leqslant \sup_{t\in\mathcal{I}_\delta}\|x(t)\|\,\widetilde{v}_n(\mathcal{I}_\delta) < \varepsilon \sup_{t\in\mathcal{I}_\delta}\|x(t)\| = \varepsilon\,\|x\|_{C_-(\mathcal{I}_\delta)}$$

holds for large enough n.

Thus the sequence $\{A_n\}$ converges to A in the uniform operator topology. Therefore, for large enough n, the operator $E - A_n$ has the bounded everywhere defined inverse operator, and the sequence $\{(E - A_n)^{-1}\}$ converges to $(E - A)^{-1}$ in the uniform operator topology of the operators acting in $C_-(\mathcal{I}_\delta)$. Using (9), (11), we get $y = (E - A)^{-1}z$, $y_n = (E - A_n)^{-1}z_n$. Hence the sequence $\{y_n\}$ converges to y in $C_-(\mathcal{I}_\delta)$ since $\{z_n\}$ converges to z in $C_-(\mathcal{I}_\delta)$. Now the desired statement can be obtained in the standard way. The theorem is proved.

Lemma 1. *Suppose the measures $\widetilde{\mathcal{X}}_\lambda$ satisfy the conditions: $\widetilde{\mathcal{X}}_\lambda$ have locally bounded variations on (a, b); the function $\lambda \to \widetilde{\mathcal{X}}_\lambda(\Delta)$ is holomorphic in some neighborhood of the point λ_1 for any fixed Borel set $\Delta \subseteq [a_1, b_1] \subset (a_0, b_0)$ and this neighborhood is independent of Δ; $\lim\limits_{\lambda\to\lambda_1} \mathbf{V}_{[a_1,b_1]}(\widetilde{\mathcal{X}}_\lambda - \widetilde{\mathcal{X}}_{\lambda_1}) = 0$. If the function $t \to y_\lambda(t)$ is the solution of equation (7), in which $\widetilde{\mathcal{X}}$ is changed to $\widetilde{\mathcal{X}}_\lambda$, then the point λ_1 has a neighborhood independent of $t \in [a_1, b_1]$ such that the function $\lambda \to y_\lambda(t)$ is holomorphic in this neighborhood for all $t \in [a_1, b_1]$.*

P r o o f. We construct the operator A_λ by the measure $\widetilde{\mathcal{X}}_\lambda$ in the same way as the operator A_n is constructed by the measure $\widetilde{\mathcal{X}}_n$ in the proof of Theorem 2. Then the function $\lambda \to A_\lambda$ is holomorphic in the neighborhood of λ_1, and

$$\lim_{\lambda\to\lambda_1}(E - A_\lambda)^{-1} = (E - A_{\lambda_1})^{-1}.$$

Consequently, the equality $\lim\limits_{\lambda\to\lambda_1} y_\lambda = y_{\lambda_1}$ holds in the space $C_-(\mathcal{I}_\delta)$. Hence the function $\lambda\to y_\lambda$ is holomorphic in the neighborhood of λ_1 in the space $C_-(\mathcal{I}_\delta)$. This implies the desired assertion. The lemma is proved.

Lemma 2. *Let $\widetilde{\mathcal{X}}_1$, $\widetilde{\mathcal{X}}_2$, $\widetilde{\mathcal{V}}$ be operator measures having locally bounded variations on (a,b), ranging over the set of bounded linear operators acting in H, and $\widetilde{\mathcal{V}}^*(\Delta) = \widetilde{\mathcal{V}}(\Delta)$ for all Δ such that $\overline{\Delta} \subset (a_0,b_0)$; let the functions f, g be locally integrable on (a_0,b_0) with respect to the measure $\widetilde{\mathcal{V}}$; $y_0, z_0 \in H$. Then for all functions y, z, having the form*

$$y(t)=y_0 - iJ\int_{t_0}^{t}(d\widetilde{\mathcal{X}}_1)y(s) - iJ\int_{t_0}^{t}(d\widetilde{\mathcal{V}})f(s),\ z(t)=z_0 - iJ\int_{t_0}^{t}(d\widetilde{\mathcal{X}}_2)z(s) - iJ\int_{t_0}^{t}(d\widetilde{\mathcal{V}})g(s),$$

the equality (the Lagrange formula)

$$\int_{c_1}^{c_2}((d\widetilde{\mathcal{V}})f(t), z(t)) - \int_{c_1}^{c_2}(y(t), (d\widetilde{\mathcal{V}})g(t)) = (iJy(c_2), z(c_2)) - (iJy(c_1), z(c_1))$$
$$+ \int_{c_1}^{c_2}(y(t), (d\widetilde{\mathcal{X}}_2)z(t)) - \int_{c_1}^{c_2}((d\widetilde{\mathcal{X}}_1)y(t), z(t)) \quad (12)$$

holds, where $t_0, t, c_1, c_2 \in (a_0,b_0)$.

The proof of the lemma is done by the routine transformations of the left-hand side of equality (12). The transformations are carried out in the same way as in [7], where the finite-dimensional case is considered. In these transformations, the interchange of the order of integration is of great importance. It follows from (2) that the interchange is possible in the infinite-dimensional case.

R e m a r k 2. In Lemma 2, if the endpoint a (or b) is regular, then for t, t_0, c_1, c_2 one can take a_0 (or b_0, respectively).

Let $W_\lambda(t)$ denote the operator solution of the equation

$$W_\lambda(t)x_0 = x_0 - iJ\int_{t_0}^{t}(d\widetilde{Z}_\lambda)W_\lambda(s)x_0 \quad (x_0\in H,\ \lambda\in\mathbb{C}_0,\ t_0, t \in (a_0,b_0)).$$

It follows from inequalities (3), (4) and Lemma 1 that the function $\lambda\to W_\lambda(t)$ is holomorphic for all fixed $t\in(a_0,b_0)$. The equality

$$W^*_{\bar\lambda}(t)JW_\lambda(t) = J \quad (13)$$

can be proved analogously to the corresponding equality in [7].

Lemma 3. *The function y is the solution of the equation*

$$y(t) = x_0 - iJ\int_{t_0}^{t}(d\widetilde{Z}_\lambda)y(s) - iJ\int_{t_0}^{t}(d\widetilde{\mathrm{V}})f(s) \tag{14}$$

if and only if y has the form

$$y(t) = W_\lambda(t)x_0 - W_\lambda(t)iJ\int_{t_0}^{t}W^*_{\bar{\lambda}}(s)(d\widetilde{\mathrm{V}})f(s), \tag{15}$$

where f is locally integrable with respect to the measure $\widetilde{V}$, $x_0 \in H$, $\lambda \in \mathbb{C}_0$, $a_0 < t_0 < b_0$, $a_0 < t < b_0$.

P r o o f. It follows from Theorem 1 that there exists a unique solution of equation (14). The limit as $t \to t_0 - 0$ of the right-hand sides of (14), (15) is equal to x_0. Suffice it to prove that we will obtain the identity if we substitute the right-hand side of equality (15) for y in (14). The substitution generates transformations carried out in the same way as the corresponding transformations in [7]. The lemma is proved.

We denote $\widetilde{U}(\Delta) = \mathrm{Re}\widetilde{Z}_i(\Delta)$ and consider the special case where $\widetilde{Z}_\lambda(\Delta) = \widetilde{U}(\Delta) + \lambda\widetilde{\mathrm{V}}(\Delta)$ ($\lambda \in \mathbb{C}$). Obviously, conditions (a)–(d) hold for this measure, and $\mathrm{Im}\widetilde{Z}_\lambda(\Delta) = (\mathrm{Im}\lambda)\widetilde{\mathrm{V}}(\Delta)$, $\mathrm{Re}\widetilde{Z}_i(\Delta) = \widetilde{U}(\Delta)$. Therefore all preceding results remain valid for this case. In particular, Lemma 3 implies the following statement.

Corollary 1. *The function y is the solution of the equation*

$$y(t) = x_0 - iJ\int_{t_0}^{t}(d\widetilde{U})y(s) - i\lambda J\int_{t_0}^{t}(d\widetilde{\mathrm{V}})y(s) - iJ\int_{t_0}^{t}(d\widetilde{\mathrm{V}})f(s) \tag{16}$$

if and only if y has the form

$$y(t) = \mathcal{W}_\lambda(t)x_0 - \mathcal{W}_\lambda(t)iJ\int_{t_0}^{t}\mathcal{W}^*_{\bar{\lambda}}(s)(d\widetilde{\mathrm{V}})f(s), \tag{17}$$

where $x_0 \in H$, $\lambda \in \mathbb{C}$, $t_0, t \in (a_0, b_0)$, the function $t \to \mathcal{W}_\lambda(t)$ is the operator solution of the equation

$$\mathcal{W}_\lambda(t)x_0 = x_0 - iJ\int_{t_0}^{t}(d\widetilde{U})\mathcal{W}_\lambda(s)x_0 - i\lambda J\int_{t_0}^{t}(d\widetilde{\mathrm{V}})\mathcal{W}_\lambda(s)x_0.$$

4. The Space $L_2(H, d\widetilde{\mathrm{V}}; a, b)$ Generated by the Nevanlinna Measure

On a set of step functions finite on the interval (a_0, b_0) and ranging over H, we introduce the quasi-scalar product

$$(x, y)_{\mathrm{V}} = \int_{a_0}^{b_0} ((d\widetilde{\mathrm{V}})x(t), y(t)). \tag{18}$$

We identify the functions y such that $(y, y)_{\mathrm{V}} = 0$ with zero and perform the completion. Then we obtain a Hilbert space denoted by $\mathfrak{H} = L_2(H, d\widetilde{\mathrm{V}}; a, b)$. The elements of $\mathfrak{H}$ are the classes of functions identical to the norm $\|y\|_{\mathrm{V}} = (y, y)_{\mathrm{V}}^{1/2}$. Here, to avoid complicating terminology, we denote the class of functions with the representative y by the same symbol. We will also say that the function y belongs to $\mathfrak{H}$. We treat the equalities between the functions belonging to $\mathfrak{H}$ as the equalities between the corresponding equivalence classes. The space $\mathfrak{H}$ does not change whenever we replace the interval (a_0, b_0) by (a_0', b_0'), where the points a_0', b_0' are introduced in the same way as the points a_0, b_0.

R e m a r k 3. It follows from (3) that the space $\mathfrak{H}$ does not depend on the choice of the point $\lambda_0 \in \mathbb{C}_0$ in the following sense. If we change the measure $\widetilde{\mathrm{V}} = \widetilde{\mathrm{V}}_{\lambda_0}$ to $\widetilde{\mathrm{V}}_\lambda$ $(\lambda \in \mathbb{C}_0)$ in (18), then we obtain the same set $\mathfrak{H}$ supplied with an equivalent norm.

We denote $\widetilde{\mathcal{Z}}_\lambda(\Delta) = \widetilde{Z}_\lambda(\Delta) - \mathrm{Re}\widetilde{Z}_i(\Delta)$.

Lemma 4. *The inequality*

$$\left| \int_{a_0}^{b_0} ((d\widetilde{\mathcal{Z}}_\lambda)y(t), x(t)) \right| \leqslant k \|y\|_{\mathrm{V}} \|x\|_{\mathrm{V}} \tag{19}$$

holds for all functions $y, x \in \mathfrak{H}$, where $k > 0$ is independent of $\lambda \in \mathrm{K}$ (K is an arbitrary fixed compact, $\mathrm{K} \subset \mathbb{C}_0$).

P r o o f. Using (4), we obtain that inequality (19) holds for step functions. Now the desired statement follows from the definition of the space $\mathfrak{H}$. The lemma is proved.

It follows from Lemma 4 that the sesquilinear form in the left-hand side of (19) is continuous on $\mathfrak{H} \times \mathfrak{H}$. Hence there exists a bounded operator $\mathbf{Z}_\lambda : \mathfrak{H} \to \mathfrak{H}$ such that

$$\int_{a_0}^{b_0} ((d\widetilde{\mathcal{Z}}_\lambda)y(t), x(t)) = (\mathbf{Z}_\lambda y, x)_{\mathrm{V}}. \tag{20}$$

In (20), we take $g = \mathbf{Z}_\lambda y$, $x(t) = \chi(t)x_0$, where $x_0 \in H$, $\chi(t)$ is the characteristic function of the Borel set Δ. Then we obtain $\int_\Delta((d\widetilde{\mathcal{Z}}_\lambda)y(t), x_0) = \int_\Delta((d\widetilde{\mathrm{V}})g(t), x_0)$. Consequently,

$$\int\limits_\Delta (d\widetilde{\mathcal{Z}}_\lambda)y(t) = \int\limits_\Delta (d\widetilde{\mathrm{V}})(\mathbf{Z}_\lambda y)(t) \qquad (21)$$

for any Borel set Δ and any function $y \in \mathfrak{H}$.

Theorem 3. *The operator function $\lambda \to \mathbf{Z}_\lambda$ is holomorphic on $\mathbb{C}_0$.*

P r o o f. Taking into account condition (a) and equality (20), we obtain that the function $\lambda \to \mathbf{Z}_\lambda y$ is holomorphic for any finite step-function y. Now the desired statement follows from Lemma 4 and the density of the set of finite step-functions in $\mathfrak{H}$. The theorem is proved.

We study the structure of the space $\mathfrak{H}$ in detail. It follows from condition (d) that there exists an operator function Ψ satisfying all conditions of Statement 1 in which the measure $\widetilde{\mathcal{X}}$ is changed to the measure $\widetilde{\mathrm{V}}$. In particular, the equality $\widetilde{\mathrm{V}}(\Delta) = \int_\Delta \Psi(\xi) d\widetilde{\rho}$ holds.

The inequality $\widetilde{\mathrm{V}}(\Delta) \geqslant 0$ implies $\Psi(\xi) \geqslant 0$ for ρ-almost all $\xi \in (a_0, b_0)$. We use some constructions from [10]. We denote $G(t) = \ker \Psi(t)$; $H(t) = H \ominus G(t)$; $\Psi_0(t)$ is the restriction of $\Psi(t)$ to $H(t)$. Then the operator $\Psi_0(t)$ acting in $H(t)$ has the inverse $\Psi_0^{-1}(t)$ (which, in general, is unbounded). Let $\{H_\tau(t)\}$ $(-\infty < \tau < \infty)$ be a Hilbert scale of spaces generated by the operator $\Psi_0^{-1}(t)$ [9, ch. 3], [11, ch. 2]. It follows from the definition of the Hilbert scale that the operator $\Psi_0(t)$ can be extended to the operator $\widehat{\Psi}_0(t)$, which maps $H_{-\alpha}(t)$ continuously and bijectively onto $H_{1-\alpha}(t)$ $(\alpha \geqslant 0)$. Let $\widehat{\Psi}(t)$ denote the operator defined on $H_{-\alpha}(t) \oplus G(t)$ and coinciding with $\widehat{\Psi}_0(t)$ on $H_{-\alpha}(t)$ and with zero on $G(t)$. The operator $\widehat{\Psi}(t)$ is the extension of $\Psi(t)$. Below the case $\alpha = 1/2$ is considered. Thus, the operator $\widehat{\Psi}(t)$ maps $H_{-1/2}(t) \oplus G(t)$ continuously onto $H_{1/2}(t)$.

In [10], the case, where the measure $\widetilde{\rho}$ is the usual Lebesgue measure, i.e., $\widetilde{\rho}([a_1, b_1)) = b_1 - a_1$, is considered. By literally repeating argumentation from [10], it is proved that the spaces $H_{-1/2}(t)$ are $\widetilde{\rho}$-measurable with respect to the parameter t [12, ch. 1] whenever for measurable functions the functions of the form $t \to \widehat{\Psi}_0^{-1}(t)\Psi^{1/2}(t)h(t)$ are taken, where h is an arbitrary $\widetilde{\rho}$-measurable function ranging over H. The space $\mathfrak{H}$ is a measurable sum of the spaces $H_{-1/2}(t)$ with respect to the measure $\widetilde{\rho}$, and $\mathfrak{H}$ consists of all functions of the form $t \to \widehat{\Psi}_0^{-1}(t)\Psi^{1/2}(t)g(t)$, where g is an arbitrary $\widetilde{\rho}$-measurable function ranging over H such that $\int_{a_0}^{b_0} \|g(t)\|^2 \, d\widetilde{\rho} < \infty$. We note that the above description of the space $\mathfrak{H}$ follows also from [13]. Thus we obtain that the equality

$$(x,y)_{\mathrm{V}} = \int_{a_0}^{b_0} (\widehat{\Psi}(t)x(t),y(t))d\widetilde{\rho} = \int_{a_0}^{b_0} (\Psi^{1/2}(t)x(t),\Psi^{1/2}(t)y(t))d\widetilde{\rho} \qquad (22)$$

holds for all functions $x,y \in \mathfrak{H}$. It follows from (2), (22) that $\int_{\Delta}(d\widetilde{\mathrm{V}})y(t) \in H$ for all functions $y \in \mathfrak{H}$ and for all Borel sets Δ with the property $\overline{\Delta} \subset (a_0,b_0)$.

5. Maximal and Minimal Relations in the Regular Case

Let $\mathbf{B}_1$, $\mathbf{B}_2$ be Banach spaces. A linear relation T is understood as a linear manifold $\mathrm{T} \subset \mathbf{B}_1 \times \mathbf{B}_2$. The terminology of the linear relations can be found, for example, in [14]. From now onwards, the following notations are used: $\ker \mathrm{T}$ is a set of the elements $x \in \mathbf{B}_1$ such that $\{x,0\} \in \mathrm{T}$; KerT is a set of ordered pairs of the form $\{x,0\} \in \mathrm{T}$; $\mathcal{D}(\mathrm{T})$ is a domain of T; $\mathcal{R}(\mathrm{T})$ is a range of T. The relation T is called continuously invertible if T^{-1} is a bounded everywhere defined operator. Linear operators are treated as linear relations. Since all relations considered further are linear, the word "linear" will often be omitted.

A family of linear relations is understood as a function $\lambda \to \mathrm{T}(\lambda)$ $(\lambda \in \mathcal{D} \subset \mathbb{C})$, where $\mathrm{T}(\lambda)$ is a linear relation, $\mathrm{T}(\lambda) \subset \mathbf{B}_1 \times \mathbf{B}_2$. A family of closed relations $\mathrm{T}(\lambda)$ is called holomorphic at a point $\lambda_1 \in \mathbb{C}$ if there exists a Banach space $\mathbf{B}_0$ and a family of bounded linear operators $\mathcal{K}(\lambda) : \mathbf{B}_0 \to \mathbf{B}_1 \times \mathbf{B}_2$ such that the operator $\mathcal{K}(\lambda)$ maps $\mathbf{B}_0$ bijectively onto $\mathrm{T}(\lambda)$ for any fixed λ, and the family $\lambda \to \mathcal{K}(\lambda)$ is holomorphic in some neighborhood of λ_1. A family of relations is called holomorphic on the domain $\mathcal{D}$ if it is holomorphic at all points belonging to $\mathcal{D}$. These definitions generalize the corresponding definitions of holomorphic families of closed operators [15, ch. 7].

Now we introduce the linear relations generated by equations (14), (16) for the case of regular endpoints. We represent integral equation (14) as

$$y(t) = x_0 - iJ\int_{t_0}^{t}(d\widetilde{U})y(s) - iJ\int_{t_0}^{t}(d\widetilde{\mathcal{Z}}_{\lambda})y(s) - iJ\int_{t_0}^{t}(d\widetilde{\mathrm{V}})f(s). \qquad (23)$$

Let $L'(\lambda)$ ($\mathcal{L}'$, respectively) be the relation consisting of the pairs $\{\tilde{y},\tilde{f}\} \in \mathfrak{H}\times\mathfrak{H}$ satisfying the condition: for each pair $\{\tilde{y},\tilde{f}\}$ there exists a pair $\{y,f\}$ such that the pairs $\{\tilde{y},\tilde{f}\}$, $\{y,f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$, and equality (14) (equality (16) for $\lambda = 0$, respectively) holds on (a_0,b_0). By $L(\lambda)$ (by $\mathcal{L}$), denote the closure of $L'(\lambda)$ (of $\mathcal{L}'$) and call $L(\lambda)$ ($\mathcal{L}$) *the maximal relation* generated by integral equation (14) (equation (16) for $\lambda = 0$, respectively). We define *the minimal* *relation* $L_0(\lambda)$ ($\mathcal{L}_0$) as the restriction of $L'(\lambda)$ ($\mathcal{L}'$) to the set of functions y such that $y(a_0)=y(b_0)=0$, where y is a solution of (14) ((16) for $\lambda=0$, respectively).

R e m a r k 4. It follows from the definition of the points a_0, b_0 that $y(a_0)=\lim\limits_{t\to a-0} y(t)$, $y(b_0)=\lim\limits_{t\to b+0} y(t)$. The maximal and minimal relations do not change if we replace the interval (a_0,b_0) by (a_0',b_0'), where the points a_0', b_0' are defined in the same way as a_0, b_0. Thus the minimal relations $L_0(\lambda)$, $\mathcal{L}_0$ can be defined as the restrictions of $L'(\lambda)$, $\mathcal{L}'$ to a set of the functions y finite on (a_0,b_0), where y is a solution of (14) or (16) for $\lambda=0$, respectively. We note that in [7], [8] the minimal relations should be defined in the same way, i.e., as the restrictions of $L'(\lambda)$, $\mathcal{L}'$ to a set of the functions y finite on (a_0,b_0) (in the finite-dimensional case, $L'(\lambda)=L(\lambda)$, $\mathcal{L}'=\mathcal{L}$).

Using (21) and (23), we obtain

$$L(\lambda)=\mathcal{L}-\mathbf{Z}_\lambda, \quad L_0(\lambda)=\mathcal{L}_0-\mathbf{Z}_\lambda. \tag{24}$$

Let Q_0 be a set of the elements $x\in H$ such that the function $t\to W_\mu(t)x$ ($\mu\in\mathbb{C}_0$) is identical to zero in $\mathfrak{H}$. We denote $Q=H\ominus Q_0$. We claim that the set Q_0 (and therefore Q) does not change if we substitute W_λ ($\lambda\in\mathbb{C}_0$) for W_μ or $\mathcal{W}_\lambda$ ($\lambda\in\mathbb{C}$) for W_μ. Indeed, let $u_\mu=\mathbf{Z}_\mu(W_\mu(\cdot)x)$, $v_\mu=\mathbf{Z}_\lambda(W_\mu(\cdot)x)$, $w_\lambda=\mathbf{Z}_\mu(\mathcal{W}_\lambda(\cdot)x)$. Using Lemma 3, Corollary 1 and equality (21), we get

$$W_\mu(t)x=W_\lambda(t)x-W_\lambda(t)iJ\int_{t_0}^{t} W^*_{\bar\lambda}(s)(d\widetilde{\mathrm{V}})(u_\mu(s)-v_\mu(s)), \tag{25}$$

$$W_\mu(t)x=\mathcal{W}_\lambda(t)x-\mathcal{W}_\lambda(t)iJ\int_{t_0}^{t} \mathcal{W}^*_{\bar\lambda}(s)(d\widetilde{\mathrm{V}})(u_\mu(s)-\lambda W_\mu(s)x), \tag{26}$$

$$\mathcal{W}_\lambda(t)x=W_\mu(t)x-W_\mu(t)iJ\int_{t_0}^{t} W^*_{\bar\mu}(s)(d\widetilde{\mathrm{V}})(\lambda\mathcal{W}_\lambda(s)x-w_\lambda(s)) \tag{27}$$

for all $x\in H$. It follows from (25) that $W_\lambda(t)x$ is identical to zero in $\mathfrak{H}$ if $x\in Q_0$. Substituting λ for μ and μ for λ in (25), we obtain the converse assertion. By (26), (27), it follows that Q_0 does not change if we substitute $\mathcal{W}_\lambda$ for W_μ.

In Q, we introduce the norm

$$\|c\|_- = \left(\int_{a_0}^{b_0}((d\widetilde{\mathrm{V}})W_\mu(s)c, W_\mu(s)c)\right)^{1/2}$$

$$=\left(\int_{a_0}^{b_0}\left\|\Psi^{1/2}(s)W_\mu(s)c\right\|^2 d\widetilde{\rho}\right)^{1/2} \leqslant \gamma\,\|c\|, \quad \mu\in\mathbb{C}_0,\ c\in Q,\ \gamma>0. \tag{28}$$

We denote the completion of Q with respect to this norm by Q_-. It follows from (25)–(27) that the replacement of $W_\mu(s)$ by $W_\lambda(s)$ ($\lambda \in \mathbb{C}_0$) or of $W_\mu(s)$ by $\mathcal{W}_\lambda$ ($\lambda \in \mathbb{C}$) in (28) leads to the same set Q_- with the equivalent norm. The space Q_- can be treated as a space with negative norm with respect to Q [9, ch. 1], [11, ch. 2]. By Q_+, denote the corresponding space with positive norm.

Suppose the sequence $\{c_n\}$ ($c_n \in Q$) converges to $c_0 \in Q_-$ in Q_-. Then the sequences $\{W_\lambda(\cdot)c_n\}$, $\{\mathcal{W}_\lambda(\cdot)c_n\}$ are fundamental in $\mathfrak{H}$ and hence converge in $\mathfrak{H}$ to some elements from $\mathfrak{H}$. Denote these elements by $W_\lambda(\cdot)c_0$ and $\mathcal{W}_\lambda(\cdot)c_0$. Let $\mathbf{W}(\lambda)$ and $\mathbf{w}(\lambda)$ denote the operators $c \to W_\lambda(\cdot)c$ and $c \to \mathcal{W}_\lambda(\cdot)c$, respectively, where $c \in Q_-$. The operators $\mathbf{W}(\lambda)$, $\mathbf{w}(\lambda)$ are continuous one-to-one mappings of Q_- into $\mathfrak{H}$ and their ranges are closed. Hence the adjoint operators $\mathbf{W}^*(\lambda)$, $\mathbf{w}^*(\lambda)$ map $\mathfrak{H}$ continuously onto Q_+. For all $x \in Q$, $f \in \mathfrak{H}$, we have

$$(f, \mathbf{W}(\lambda)x)_{\mathrm{V}} = \int_{a_0}^{b_0} ((d\widetilde{\mathrm{V}})f(s), W_\lambda(s)x) = \int_{a_0}^{b_0} (W_\lambda^*(s)(d\widetilde{\mathrm{V}})f(s), x) = (\mathbf{W}^*(\lambda)f, x).$$

The analogous equality holds for the operator $\mathbf{w}(\lambda)$. Hence, taking into account that Q is densely embedded in Q_-, we obtain

$$\mathbf{W}^*(\lambda)f = \int_{a_0}^{b_0} W_\lambda^*(s)(d\widetilde{\mathrm{V}})f(s), \quad \mathbf{w}^*(\lambda)f = \int_{a_0}^{b_0} \mathcal{W}_\lambda^*(s)(d\widetilde{\mathrm{V}})f(s). \tag{29}$$

Thus the following statement is obtained.

Lemma 5. *The operators* $\mathbf{W}^*(\lambda)$, $\mathbf{w}^*(\lambda)$ *map* $\mathfrak{H}$ *continuously onto* Q_+ *and have the form* (29).

The following lemma and corollaries can be proved in the same way as the corresponding assertions in [6–8].

Lemma 6. *The pair* $\{\tilde{y}, \tilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ *belongs to the relation* $L(\lambda)$ (*the relation* $\mathcal{L} - \lambda E$) *if and only if there exists a pair* $\{y, f\}$ *such that the pairs* $\{\tilde{y}, \tilde{f}\}$, $\{y, f\}$ *are identical in* $\mathfrak{H} \times \mathfrak{H}$, *and equality* (15) (*equality* (17), *respectively*) *holds, where* $x_0 \in Q_-$, $f \in \mathfrak{H}$.

Corollary 2. *The relations* $L_0(\lambda)$, $\mathcal{L}_0$ *are closed.*

Corollary 3. *The range of the relation* $L_0(\lambda)$ (*relation* $\mathcal{L}_0 - \lambda E$) *consists of all elements* $f \in \mathfrak{H}$ *such that the following equalities hold, respectively:*

$$\mathbf{W}^*(\lambda)f = \int_{a_0}^{b_0} W_{\bar{\lambda}}^*(s)(d\widetilde{\mathrm{V}})f(s) = 0, \quad \mathbf{w}^*(\lambda)f = \int_{a_0}^{b_0} \mathcal{W}_{\bar{\lambda}}^*(s)(d\widetilde{\mathrm{V}})f(s) = 0.$$

Corollary 4. *The operators* $\mathbf{W}(\lambda)$ *and* $\mathbf{w}(\lambda)$ *are continuous one-to-one mappings of* Q_- *onto* $\ker L(\lambda)$ *and onto* $\ker(\mathcal{L} - \lambda E)$*, respectively.*

Theorem 3, Corollary 2 and equalities (24) imply the following statement.

Corollary 5. *The families of the relations* $L(\lambda), L_0(\lambda)$ *are holomorphic on* $\mathbb{C}_0$.

Lemma 7. $\mathcal{L}_0$ *is a symmetric relation and* $\mathcal{L} = \mathcal{L}_0^*$.

P r o o f. In Lemma 2, we take $\widetilde{\mathcal{X}}_1 = \widetilde{\mathcal{X}}_2 = \widetilde{U}$, $\widetilde{\mathcal{V}} = \widetilde{\mathrm{V}}$, $c_1 = a_0$, $c_2 = b_0$. It follows from (12) that $\mathcal{L}_0$ is the symmetric relation and $\mathcal{L}' \subset \mathcal{L}_0^*$. Consequently, $\mathcal{L} \subset \mathcal{L}_0^*$. We prove the converse inclusion. Using Corollaries 3, 4, we get $\mathfrak{N}_\lambda = \ker(\mathcal{L} - \bar{\lambda} E)$, where $\mathfrak{N}_\lambda$ is the defect subspace of the relation $\mathcal{L}_0$, i.e., $\mathfrak{N}_\lambda$ is the orthogonal complement of the range $\mathcal{R}(\mathcal{L}_0 - \lambda E)$. By $\widehat{\mathfrak{N}}_\lambda$, denote a set of all pairs of the form $\{z, \bar{\lambda} z\}$, where $z \in \mathfrak{N}_\lambda$ ($\mathrm{Im}\lambda \neq 0$). It follows from Lemma 6 and the equality $\mathcal{L}_0^* = \mathcal{L}_0 \dotplus \widehat{\mathfrak{N}}_\lambda \dotplus \widehat{\mathfrak{N}}_{\bar{\lambda}}$ that $\mathcal{L}_0^* \subset \mathcal{L}$. The lemma is proved.

Corollary 6. *The equation* $L_0^*(\lambda) = L(\bar{\lambda})$ $(\lambda \in \mathbb{C}_0)$ *holds.*

Theorem 4. *The equality*

$$\mathrm{Im}(f, y)_{\mathrm{V}} = -\int_{a_0}^{b_0} ((d\mathrm{Im}\widetilde{Z}_\lambda) y(t), y(t)) \tag{30}$$

holds for all $\lambda \in C_0$ *and for all pairs* $\{y, f\} \in L_0(\lambda)$.

P r o o f. In Lemma 2, we take $\widetilde{\mathcal{X}}_1 = \widetilde{\mathcal{X}}_2 = \widetilde{Z}_\lambda$, $\widetilde{\mathcal{V}} = \widetilde{\mathrm{V}}$, $z = y$, $g = f$, $c_1 = a_0$, $c_2 = b_0$. The desired statement follows from (12) and the definition of $L_0(\lambda)$.

Corollary 7. *The inequality* $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(f, y)_{\mathrm{V}} \leqslant -k\,\|y\|^2_{\mathrm{V}}$ *holds for any fixed* λ $(\mathrm{Im}\lambda \neq 0)$ *and for all pairs* $\{y, f\} \in L_0(\lambda)$*, where* $k > 0$.

We construct a space of boundary values for the relation $L(\lambda)$. According to Lemma 6, the pair $\{\tilde{y}, \tilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ belongs to the relation $L(\lambda)$ if and only if there exists a pair $\{y, f\}$ such that the pairs $\{\tilde{y}, \tilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$, and the equality

$$y(t) = W_\lambda(t) c_\lambda + F_\lambda(t) \tag{31}$$

holds, where $c_\lambda \in Q_-$,

$$F_\lambda(t) = -W_\lambda(t) iJ \int_{a_0}^{t} W_{\bar{\lambda}}^*(s)(d\widetilde{\mathrm{V}}) f(s) ds. \tag{32}$$

To each pair $\{y,f\}\in L(\lambda)$, we assign a pair of boundary values $\{\mathbf{y}_\lambda,\mathbf{y}'_\lambda\}\in Q_-\times Q_+$ by the formulas

$$\mathbf{y}'_\lambda=\mathbf{W}^*(\lambda)f=\int_{a_0}^{b_0}W^*_{\bar\lambda}(s)(d\widetilde{\mathrm{V}})f(s)\in Q_+,\quad \mathbf{y}_\lambda=c_\lambda-(1/2)iJ\mathbf{y}'_\lambda\in Q_-.$$

It follows from (31), (32) that if the pairs $\{\tilde y,\tilde f\}$, $\{y,f\}$ are identical in $\mathfrak{H}\times\mathfrak{H}$, then their boundary values coincide. Let $\gamma(\lambda)$, $\gamma_1(\lambda)$, $\gamma_2(\lambda)$ be the operators defined by the equalities $\gamma(\lambda)\{y,f\}=\{\mathbf{y}_\lambda,\mathbf{y}'_\lambda\}$, $\gamma_1(\lambda)\{y,f\}=\mathbf{y}_\lambda$, $\gamma_2(\lambda)\{y,f\}=\mathbf{y}'_\lambda$. It follows from Lemma 5 and Corollary 4 that $\gamma(\lambda)$ maps $L(\lambda)$ continuously onto $Q_-\times Q_+$, and the restriction of $\gamma_1(\lambda)$ to $\mathrm{Ker}L(\lambda)$ is the one-to-one mapping of $\mathrm{Ker}L(\lambda)$ onto Q_-. Hence the quadruple $(Q_-,Q_+,\gamma_1(\lambda),\gamma_2(\lambda))$ is the space of boundary values for the relation $L(\lambda)$ from the viewpoint of [16] (see also references in [6]).

For fixed $\lambda\in\mathbb{C}_0$, between the relations $\widehat{L}(\lambda)$ with the property $L_0(\lambda)\subset\widehat{L}(\lambda)\subset L(\lambda)$ and the relations $\theta(\lambda)\subset Q_-\times Q_+$ there exists a one-to-one correspondence determined by the equality $\gamma(\lambda)\widehat{L}(\lambda)=\theta(\lambda)$. In this case we denote $\widehat{L}(\lambda)=L_{\theta(\lambda)}(\lambda)$.

Lemma 8. *For all pairs* $\{y,f\}\in L(\lambda)$, $\{z,g\}\in L(\bar\lambda)$ $(\lambda\in\mathbb{C}_0)$*, "the Green formula"*

$$(f,z)_{\mathrm{V}}-(y,g)_{\mathrm{V}}=(\mathbf{y}'_\lambda,\mathbf{z}_{\bar\lambda})-(\mathbf{y}_\lambda,\mathbf{z}'_{\bar\lambda})\tag{33}$$

holds, where $\gamma(\lambda)\{y,f\}=\{\mathbf{y}_\lambda,\mathbf{y}'_\lambda\}$, $\gamma(\lambda)\{z,g\}=\{\mathbf{z}_\lambda,\mathbf{z}'_\lambda\}$.

P r o o f. According to Lemma 6, the function y has the form (31). Analogously, the function z can be represented as $z(t)=W_{\bar\lambda}(t)d_{\bar\lambda}+G_{\bar\lambda}(t)$, where $d_{\bar\lambda}\in Q_-$, $G_{\bar\lambda}(t)=-W_{\bar\lambda}(t)iJ\int_{a_0}^t W^*_\lambda(s)(d\widetilde{\mathrm{V}})g(s)ds$. In (12), we take $\widetilde{\mathcal{X}}_1=\widetilde{Z}_\lambda$, $\widetilde{\mathcal{X}}_2=\widetilde{Z}_{\bar\lambda}$, $z(t)=G_{\bar\lambda}(t)$, $y(t)=F_\lambda(t)$, $c_1=a_0$, $c_2=b_0$. Using (13) and Lemmas 2, 3, we get

$$(f,G_{\bar\lambda})_{\mathrm{V}}-(F_\lambda,g)_{\mathrm{V}}=(iJW_\lambda(b_0)iJ\mathbf{y}'_\lambda,W_{\bar\lambda}(b_0)iJ\mathbf{z}_{\bar\lambda})=(iJ\mathbf{y}'_\lambda,\mathbf{z}'_{\bar\lambda}).\tag{34}$$

We take two sequences $\{c_{\lambda,n}\}$, $\{d_{\bar\lambda,n}\}$ such that $c_{\lambda,n},d_{\bar\lambda,n}\in Q$ and $\{c_{\lambda,n}\}$, $\{d_{\bar\lambda,n}\}$ converge to $c_\lambda,d_{\bar\lambda}\in Q_-$ in Q_-, respectively. We denote $v_n(t)=W_\lambda(t)c_{\lambda,n}$, $u_n(t)=W_{\bar\lambda}(t)d_{\bar\lambda,n}$, $v(t)=W_\lambda(t)c_\lambda$, $u(t)=W_{\bar\lambda}(t)d_{\bar\lambda}$. Then the sequences $\{v_n\}$, $\{u_n\}$ converge to v, u in $\mathfrak{H}$, respectively. From (13) and Lemmas 2, 3, we obtain

$$(f,u_n)_{\mathrm{V}}=(f,u_n)_{\mathrm{V}}-(F_\lambda,0)_{\mathrm{V}}=-(iJW_\lambda(b_0)iJ\mathbf{y}'_\lambda,W_{\bar\lambda}(b_0)d_{\bar\lambda,n})=(\mathbf{y}'_\lambda,d_{\bar\lambda,n}),$$

$$-(v_n,g)_{\mathrm{V}}=(0,G_{\bar\lambda})_{\mathrm{V}}-(v_n,g)_{\mathrm{V}}=-(iJW_\lambda(b_0)c_{\lambda,n},W_{\bar\lambda}(b_0)iJ\mathbf{z}'_{\bar\lambda})=-(c_{\lambda,n},\mathbf{z}'_{\bar\lambda}).$$

In these equalities, we pass to the limit as $n\to\infty$. Then we get

$$(f,u)_{\mathrm{V}} = (\mathbf{y}'_\lambda, d_{\bar\lambda}), \quad (v,g)_{\mathrm{V}} = (c_\lambda, \mathbf{z}'_{\bar\lambda}). \tag{35}$$

Using (34), (35), we obtain

$$(f,z)_{\mathrm{V}} - (y,g)_{\mathrm{V}} = (\mathbf{y}'_\lambda, d_{\bar\lambda}) - (c_\lambda, \mathbf{z}'_{\bar\lambda}) + (iJ\mathbf{y}'_\lambda, \mathbf{z}'_{\bar\lambda}) =$$
$$= (\mathbf{y}'_\lambda, d_{\bar\lambda} - (1/2)iJ\mathbf{z}'_{\bar\lambda}) - (c_\lambda - (1/2)iJ\mathbf{y}'_\lambda, \mathbf{z}'_{\bar\lambda}) = (\mathbf{y}'_\lambda, \mathbf{z}_{\bar\lambda}) - (\mathbf{y}_\lambda, \mathbf{z}'_{\bar\lambda}).$$

The lemma is proved.

Corollary 8. *The equation $(L_{\theta(\lambda)}(\lambda))^* = L_{\theta^*(\lambda)}(\bar\lambda)$ holds.*

In view of Lemma 6 and Corollary 8, the following theorem can be proved analogously to the corresponding assertions in [6, 8].

Theorem 5. *For any fixed $\lambda \in \mathbb{C}_0$, the relation $\widehat{L}(\lambda) = L_{\theta(\lambda)}(\lambda)$ is continuously invertible if and only if the relation $\theta(\lambda)$ has the same property. In this case, the operator $R(\lambda) = \widehat{L}^{-1}(\lambda)$ is integral,*

$$R(\lambda)g = \int_{a_0}^{b_0} K(t,s,\lambda)(d\widetilde{\mathrm{V}})g(s) \quad (g \in \mathfrak{H}),$$

where $K(t,s,\lambda) = W_\lambda(t)(\theta^{-1}(\lambda) + (1/2)\mathrm{sgn}(s-t)iJ)W^*_{\bar\lambda}(s)$. The equalities $\theta^*(\lambda) = \theta(\bar\lambda)$ and $R^*(\lambda) = R(\bar\lambda)$ hold simultaneously. The function $\lambda \to R_\lambda$ ($\lambda \in \mathcal{D} \subset \mathbb{C}_0$) is holomorphic at a point $\lambda_1 \in \mathcal{D}$ if and only if the function $\lambda \to \theta^{-1}(\lambda)$ is holomorphic at the same point.

6. Maximal and Minimal Relations in the Singular Case

Let $L'_0(\lambda)$ ($\mathcal{L}'_0$, respectively) be the relation consisting of the pairs $\{\tilde y, \tilde f\} \in \mathfrak{H} \times \mathfrak{H}$ satisfying the conditions: for each pair $\{\tilde y, \tilde f\}$ there exists a pair $\{y, f\}$ such that the pairs $\{\tilde y, \tilde f\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$; the function y is finite on (a_0, b_0); and equality (14) (equality (16) for $\lambda = 0$, respectively) holds. We define *the minimal relations* $L_0(\lambda)$, $\mathcal{L}_0$ as the closures of the relations $L'_0(\lambda)$, $\mathcal{L}_0$, respectively. The relations $(L_0(\bar\lambda))^* = L^*_0(\bar\lambda)$, $\mathcal{L}^*_0$ are *maximal relations.*

Using (21), (23), we get $L'_0(\lambda) = \mathcal{L}'_0 - \mathbf{Z}_\lambda$. Consequently, $L_0(\lambda) = \mathcal{L}_0 - \mathbf{Z}_\lambda$. It follows from Theorem 3 that the family $\lambda \to L_0(\lambda)$ is holomorphic on $\mathbb{C}_0$. Therefore $\lambda \to L^*_0(\bar\lambda)$ is the holomorphic family (see [6]). Thus, the families of minimal and maximal relations are holomorphic on $\mathbb{C}_0$. Suppose $\{y, f\} \in L'_0(\lambda)$. In Lemma 2, we take $\widetilde{\mathcal{X}}_1 = \widetilde{\mathcal{X}}_2 = \widetilde{Z}_\lambda$, $\widetilde{\mathcal{V}} = \widetilde{\mathrm{V}}$, $z = y$, $g = f$ and choose c_1, c_2 such that $y(c_1)=y(c_2)=0$. Then equality (30) holds. Passing to the limit in (30), we can see that Theorem 4 remains valid for $L_0(\lambda)$ in the singular case.

Let Q_0 be a set of the elements $c \in H$ such that

$$\int_{a_0}^{b_0} ((d\widetilde{\mathrm{V}})W_\mu(s)c, W_\mu(s)c) = 0.$$

We denote $Q = H \ominus Q_0$. The sets Q_0, Q do not depend on $\mu \in \mathbb{C}_0$. (This assertion can be proved analogously to the corresponding assertion for the regular case.) Let $[\alpha_n, \beta_n]$ be a sequence of intervals such that $a_0 < \alpha_{n+1} < \alpha_n < \beta_n < \beta_{n+1} < b_0$ and $\alpha_n \to a_0$, $\beta_n \to b_0$ as $n \to \infty$. In Q, we introduce a system of semi-norms

$$p_n(x) = \left(\int_{[\alpha_n, \beta_n]} ((d\widetilde{\mathrm{V}})W_\mu(s)c, W_\mu(s)c) \right)^{1/2}, \quad \mu \in \mathbb{C}_0, \ \ c \in Q. \tag{36}$$

By Q_-, denote the completion of Q with respect to this system. The space Q_- is generated by the counting system of semi-norms. Hence Q_- is a Frechet space [17, ch. 2]. Arguing as above, we can see that the replacement of μ by $\lambda \in \mathbb{C}_0$ in (36) leads to an isomorphic space.

We define the function $t \to W_\lambda(t)c$ for all $c \in Q_-$. The reasonings are similar to those given in [18, 6] and thus are omitted.

By $\mathfrak{H}_n$, denote $\mathfrak{H}_n = L_2(H, d\widetilde{\mathrm{V}}; [\alpha_n, \beta_n])$. Let $Q_0(n)$ be a set of the elements $c \in Q$ such that the function $t \to W_\lambda(t)c$ is identical to zero in the space $\mathfrak{H}_n$, $Q(n) = Q \ominus Q_0(n)$. Obviously, $Q(n) \supset Q(m)$ for $n \geqslant m$. If $c \in Q(n)$, then $p_n(c) > 0$. Hence p_n is the norm on the set $Q(n)$. We denote it by $\|\cdot\|_-^{(n)}$. Let $Q_-(n)$ be the completion of $Q(n)$ with respect to the norm $\|\cdot\|_-^{(n)}$.

We define the mappings $h_{mn} : Q_-(n) \to Q_-(m)$ $(n \geqslant m)$ as follows. Let $Q(n,m) = Q(n) \ominus Q(m)$. For all elements $c_1 \in Q(m)$, $c_0 \in Q(n,m)$, we put $h_{mn}c_1 = c_1$, $h_{mn}c_0 = 0$. Using the equality $p_m(c_0) = 0$, we obtain

$$\|h_{mn}(c_1 + c_0)\|_-^{(m)} = \|c_1\|_-^{(m)} = p_m(c_1) = p_m(c_1 + c_0) \leqslant \|c_1 + c_0\|_-^{(n)}.$$

This implies that the mapping h_{mn} is extended by the continuity to the space $Q_-(n)$.

We consider the projective limit $\lim(pr)h_{mn}Q_-(n)$ of the family of the spaces $\{Q_-(n); n \in \mathbb{N}\}$ with respect to the mappings h_{mn} $(m, n \in \mathbb{N}, m \leqslant n)$ [17, ch. 2]. Repeating the corresponding arguments from [17, ch. 2, proof 5.4], one can show that this projective limit is isomorphic to the space Q_- introduced after formula (36). It follows from the definition of projective limit that Q_- is a subspace of the product $\prod_n Q_-(n)$, and Q_- consists of all elements $c = \{c_n\}$ such that $c_m = h_{mn}c_n$ for all m, n, where $m \leqslant n$.

The space $Q_-(n)$ can be treated as a negative one with respect to $Q(n)$ [9, ch. 1], [11, ch. 2]. By $Q_+(n)$, denote the corresponding space with positive norm. Then $Q_+(m) \subset Q_+(n)$ for $m \leqslant n$ and the inclusion map of $Q_+(m)$ into $Q_+(n)$ is continuous. This inclusion map coincides with h^+_{mn}, where $h^+_{mn} : Q_+(m) \to Q_+(n)$ is the adjoint mapping of h_{mn}. By Q_+, denote the inductive limit [17, ch. 2] of the family $\{Q_+(n); n \in \mathbb{N}\}$ with respect to the mappings h^+_{mn}, i.e., $Q_+ = \lim(ind)h^+_{mn}Q_+(n)$. According to [17, ch. 4], Q_+ is the adjoint space of Q_-. The space Q_+ can be treated as the union $Q_+ = \cup_n Q_+(n)$ with the strongest topology such that all inclusion maps of $Q_+(n)$ into Q_+ are continuous [17, ch. 2].

By Corollary 4, the operator $c_n \to W_\lambda(t)c_n$ is a continuous one-to-one mapping of $Q_-(n)$ into $\mathfrak{H}_n$ and it has the closed range. We denote this operator by $\mathbf{W}_n(\lambda)$. It follows from Lemma 5 that the adjoint operator $\mathbf{W}^*_n(\lambda)$ maps $\mathfrak{H}_n$ continuously onto $Q_+(n)$, and

$$\mathbf{W}^*_n(\lambda)f = \int_{\alpha_n}^{\beta_n} W^*_\lambda(s)(d\widetilde{\mathrm{V}})f(s).$$

Consequently,

$$\int_{\alpha}^{\beta} W^*_\lambda(s)(d\widetilde{\mathrm{V}})f(s) \in Q_+$$

for each function $f \in \mathfrak{H}$ and for all α, β such that $a_0 < \alpha < \beta < b_0$.

Suppose $c = \{c_n\} \in Q_-$. Then $c_m = h_{mn}c_n$ $(m \leqslant n)$. Hence the restriction of the function $\widehat{\Psi}^{1/2}(t)W_\lambda(t)c_n$ to the segment $[\alpha_m, \beta_m]$ coincides with the function $\widehat{\Psi}^{1/2}(t)W_\lambda(t)c_m$ in the space $L_2(H, d\widetilde{\rho}; [\alpha_m, \beta_m])$. By $\widehat{\Psi}^{1/2}(t)W_\lambda(t)c$, we denote the function equal to $\widehat{\Psi}^{1/2}(t)W_\lambda(t)c_n$ on each segment $[\alpha_n, \beta_n]$. Now by $W_\lambda(t)c$ denote the function ranging over $H_{-1/2}(t) \oplus G(t)$ and coinciding with $W_\lambda(t)c_n$ in the space $\mathfrak{H}_n$ for any $n \in \mathbb{N}$. For all m, n $(m \leqslant n)$, the equality $W_\lambda(t)c_n = W_\lambda(t)c_m$ holds in the space $\mathfrak{H}_m$.

From Lemma 6, we obtain the following statement.

Lemma 9. *If the pair $\{\tilde{y}, \tilde{f}\} \in L^*_0(\bar{\lambda})$, then there exists a pair $\{y, f\}$ such that the pairs $\{\tilde{y}, \tilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$, and the equality*

$$y(t) = W_\lambda(t)c - W_\lambda(t)iJ\int_{t_0}^{t} W^*_{\bar{\lambda}}(s)(d\widetilde{\mathrm{V}})f(s)ds, \quad c \in Q_-,\ t_0,\ t \in (a_0, b_0)$$

holds.

The following theorem can be proved in the similar way as the corresponding assertions in [6, 8, 18]. Note that in [6, 18], the linear relations satisfy the requirements under which the boundedness condition for the function $\lambda \to \|R(\lambda)\|$ on some neighborhood of the point λ_1 holds automatically.

Theorem 6. *Assume that the relation $\widehat{L}(\lambda)$ has the property $L_0(\lambda) \subset \widehat{L}(\lambda) \subset L_0^*(\bar{\lambda})$. If the relation $\widehat{L}(\lambda)$ is continuously invertible, then the operator $R(\lambda) = \widehat{L}^{-1}(\lambda)$ is integral,*

$$R(\lambda)g = \int_{a_0}^{b_0} K(t,s,\lambda)(d\widetilde{\mathrm{V}})g(s)ds, \quad g \in \mathfrak{H}, \tag{37}$$

where $K(t,s,\lambda) = W_\lambda(t)(M(\lambda) + (1/2)\mathrm{sgn}(s-t)iJ)W^*_{\bar{\lambda}}(s)$, $M(\lambda) : Q_+ \to Q_-$ is the continuous operator. The equalities $R^*(\lambda) = R(\bar{\lambda})$, $M^*(\lambda) = M(\bar{\lambda})$ hold simultaneously. The integral (37) converges at least weakly in the space $\mathfrak{H}$. If the function $\lambda \to R(\lambda)$ $(\lambda \in \mathcal{D} \subset \mathbb{C}_0)$ is holomorphic at a point $\lambda_1 \in \mathcal{D}$, then the function $\lambda \to M(\lambda)x$ is holomorphic at the same point for all $x \in Q_+$. The converse statement is valid whenever the function $\lambda \to \|R(\lambda)\|$ is bounded on some neighborhood of the point λ_1.

7. The Characteristic Operator

In this section, the endpoints of the interval (a,b) may be regular or singular. In the space $\mathfrak{H} = L_2(H, d\widetilde{\mathrm{V}}; a, b)$, the scalar product is defined by equality (18). According to Remark 3, the replacement of $\widetilde{\mathrm{V}} = \widetilde{\mathrm{V}}_{\lambda_0}$ by $\widetilde{\mathrm{V}}_\lambda$ $(\lambda \in \mathbb{C}_0)$ in (18) leads to the same set $\mathfrak{H}$ with an equivalent norm (the designation of $\widetilde{\mathrm{V}}_\lambda$ is given before formula (3)). By $\|\cdot\|_{\mathrm{V}_\lambda}$ (by $(\cdot,\cdot)_{\mathrm{V}_\lambda}$), we denote the norm (the scalar product, respectively) in the space $L_2(H, d\widetilde{\mathrm{V}}_\lambda; a, b)$. According to these notations, $\|\cdot\|_{\mathfrak{H}} = \|\cdot\|_{\mathrm{V}}$.

We replace $\widetilde{\mathrm{V}}$ by $\widetilde{\mathrm{V}}_\lambda$ $(\lambda \in \mathbb{C}_0)$ in (14) (or in (1)). Then we obtain the equation

$$y(t) = y(t_0) - iJ\int_{t_0}^{t}(d\widetilde{Z}_\lambda)y(s) - iJ\int_{t_0}^{t}(d\widetilde{\mathrm{V}}_\lambda)f(s). \tag{38}$$

The following definition of the characteristic operator for a differential equation with a Nevanlinna operator function is given in [4, 5].

Definition 1. *Let $\lambda \to M(\lambda) = M^*(\bar{\lambda})$ be a function holomorphic for $\mathrm{Im}\lambda \neq 0$ whose values are bounded linear operators and $\mathcal{D}(M(\lambda)) = Q_+$, $\mathcal{R}(M(\lambda)) \subset Q_-$. This function M is called a characteristic operator of equation (38) if the operator*

$\mathbf{R}(\lambda)$, *defined by the equality*

$$(\mathbf{R}(\lambda)f)(t) = \int_{a_0}^{b_0} W_\lambda(t)(M(\lambda) + (1/2)\mathrm{sgn}(s-t)iJ)W^*_{\bar{\lambda}}(s)(d\widetilde{\mathrm{V}}_\lambda)f(s), \tag{39}$$

satisfies the inequality

$$\|\mathbf{R}(\lambda)f\|^2_{\mathrm{V}_\lambda} \leqslant \mathrm{Im}(\mathbf{R}(\lambda)f, f)_{\mathrm{V}_\lambda}/\mathrm{Im}\lambda, \quad \mathrm{Im}\lambda \neq 0 \tag{40}$$

for all function $f \in \mathfrak{H}$ such that f is finite on (a_0, b_0).

Using (40), we get the inequality $\|\mathbf{R}(\lambda)f\|_{\mathrm{V}_\lambda} \leqslant (\mathrm{Im}\lambda)^{-1}\|f\|_{\mathrm{V}_\lambda}$ ($\mathrm{Im}\lambda \neq 0$) for all finite functions $f \in \mathfrak{H}$. Hence the operator $\mathbf{R}(\lambda)$ is bounded in $\mathfrak{H}$. Therefore $\mathbf{R}(\lambda)$ is extended by the continuity to the whole space $\mathfrak{H}$. We denote the extended operator by the same symbol $\mathbf{R}(\lambda)$.

We claim that for any function $f \in \mathfrak{H}$ and for any fixed $\lambda \in \mathbb{C}_0$ there exists a unique function $g \in \mathfrak{H}$ such that the equality

$$\int_{t_0}^{t} (d\widetilde{\mathrm{V}}_\lambda)f(s) = \int_{t_0}^{t} (d\widetilde{\mathrm{V}})g(s) \tag{41}$$

holds for all $t \in (a_0, b_0)$. Indeed, we consider the sesquilinear form $(f, u)_{\mathrm{V}_\lambda}$, where u is an arbitrary function belonging to $\mathfrak{H}$. This form is continuous on $\mathfrak{H} \times \mathfrak{H}$. Consequently, there exists a unique function $g \in \mathfrak{H}$ such that $(f, u)_{\mathrm{V}_\lambda} = (g, u)_{\mathrm{V}}$. From this equality, we obtain the desired assertion.

We consider the operator $R(\lambda)$ defined by the equality

$$(R(\lambda)g)(t) = \int_{a_0}^{b_0} W_\lambda(t)(M(\lambda) + (1/2)\mathrm{sgn}(s-t)iJ)W^*_{\bar{\lambda}}(s)(d\widetilde{\mathrm{V}})g(s), \tag{42}$$

where $g \in \mathfrak{H}$, and g is finite on (a_0, b_0). Suppose that the functions $f, g \in \mathfrak{H}$ are connected by equality (41). Using (39), (42), we get $R(\lambda)g = \mathbf{R}(\lambda)f$. Therefore inequality (40) is equivalent to the following inequality:

$$\|R(\lambda)g\|^2_{\mathrm{V}_\lambda} \leqslant (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)g, g)_{\mathrm{V}}, \quad \mathrm{Im}\lambda \neq 0. \tag{43}$$

Then, since the norms $\|\cdot\|_{\mathrm{V}}$ and $\|\cdot\|_{\mathrm{V}_\lambda}$ are equivalent, the operator $R(\lambda)$ is bounded in the space $\mathfrak{H}$. Consequently $R(\lambda)$ is extended by the continuity to the whole space $\mathfrak{H}$. We denote the extended operator by the same symbol $R(\lambda)$. It follows from (43) that the function $\lambda \to \|R(\lambda)\|$ is bounded in some neighborhood of the arbitrary point λ_1 ($\mathrm{Im}\lambda_1 \neq 0$). Theorems 5, 6 imply that the functions $\lambda \to R(\lambda)$ and $\lambda \to M(\lambda)$ are simultaneously holomorphic for $\mathrm{Im}\lambda \neq 0$.

Suppose that a family of closed relations $\lambda \to \mathfrak{L}(\lambda)$ satisfies the condition $L_0(\lambda) \subset \mathfrak{L}(\lambda) \subset L_0^*(\bar{\lambda})$ for $\mathrm{Im}\lambda \neq 0$. We say that this family *generates a characteristic operator* $M(\lambda)$ if the equality $\mathfrak{L}^{-1}(\lambda) = R(\lambda)$ holds.

Now we need the description of the families generating the characteristic operator. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces. By $\mathcal{F}(\mathfrak{H}_1, \mathfrak{H}_2)$, denote a set of holomorphic functions $\lambda \to F(\lambda)$ ($\mathrm{Im}\lambda \neq 0$) whose values are bounded operators satisfying the conditions: (i) $\mathcal{D}(F(\lambda)) = \mathfrak{H}_1$, $\mathcal{R}(F(\lambda)) \subset \mathfrak{H}_2$ for $\mathrm{Im}\lambda > 0$; (ii) $F(\bar{\lambda}) = F^*(\lambda)$; (iii) $\|F(\lambda)\| \leqslant 1$. Let $\mathfrak{N}_i = \mathfrak{H} \ominus \mathcal{R}(\mathcal{L}_0 - iE) = \ker(\mathcal{L}_0^* + iE)$ be the defect subspace of the relation $\mathcal{L}_0$. We consider the holomorphic operator function $\lambda \to F(\lambda)$ belonging to $\mathcal{F}(\mathfrak{N}_i, \mathfrak{N}_{-i})$. By $\mathcal{L}_{F(\lambda)}$, denote the linear relation consisting of all pairs of the form $\{y_0 + F(\lambda)z_0 - z_0, y_1 + iF(\lambda)z_0 + iz_0\}$, where $\{y_0, y_1\} \in \mathcal{L}_0$, $z_0 \in \mathfrak{N}_i$. Then $(\mathcal{L}_{F(\lambda)})^* = \mathcal{L}_{F(\bar{\lambda})}$. The family $\lambda \to \mathcal{L}_{F(\lambda)}$ is holomorphic for $\mathrm{Im}\lambda \neq 0$. The relations $\mathcal{L}_{F(\lambda)}$ are maximal accumulative for $\mathrm{Im}\lambda > 0$ and maximal dissipative for $\mathrm{Im}\lambda < 0$. Conversely, let $\lambda \to \widehat{\mathcal{L}}(\lambda)$ ($\mathrm{Im}\lambda \neq 0$) be a holomorphic family of linear relations with the properties: $\mathcal{L}_0 \subset \widehat{\mathcal{L}}(\lambda) \subset \mathcal{L}_0^*$; $\widehat{\mathcal{L}}^*(\lambda) = \widehat{\mathcal{L}}(\bar{\lambda})$; each relation $\widehat{\mathcal{L}}(\lambda)$ is maximal accumulative for $\mathrm{Im}\lambda > 0$. Then there exists an operator function $\lambda \to F(\lambda)$ belonging to $\mathcal{F}(\mathfrak{N}_i, \mathfrak{N}_{-i})$, and $\widehat{\mathcal{L}}(\lambda) = \mathcal{L}_{F(\lambda)}$ (see [19, 20]).

Theorem 7. *If the function* $\lambda \to F(\lambda)$ *belongs to* $\mathcal{F}(\mathfrak{N}_i, \mathfrak{N}_{-i})$*, then the family of relations* $\mathfrak{L}(\lambda) = \mathcal{L}_{F(\lambda)} - \mathbf{Z}_\lambda$ *generates a characteristic operator of equation* (38) *and, conversely, if the family* $\lambda \to \mathfrak{L}(\lambda)$ ($\mathrm{Im}\lambda \neq 0$) *generates a characteristic operator, then there exists an operator function* $\lambda \to F(\lambda)$ *belonging to* $\mathcal{F}(\mathfrak{N}_i, \mathfrak{N}_{-i})$ *such that* $\mathfrak{L}(\lambda) = \mathcal{L}_{F(\lambda)} - \mathbf{Z}_\lambda$.

P r o o f. First notice that the inclusions $L_0(\lambda) \subset \mathfrak{L}(\lambda) \subset L_0^*(\bar{\lambda})$ and $\mathcal{L}_0 \subset \mathcal{L}_{F(\lambda)} \subset \mathcal{L}_0^*$ hold simultaneously. It follows from Theorem 3 that the family $\lambda \to \mathfrak{L}(\lambda)$ is holomorphic if and only if the family $\lambda \to \mathcal{L}_{F(\lambda)}$ is the same. The equalities $\mathfrak{L}(\bar{\lambda}) = \mathfrak{L}^*(\lambda)$ and $(\mathcal{L}_{F(\lambda)})^* = \mathcal{L}_{F(\bar{\lambda})}$ hold simultaneously since $\mathbf{Z}_\lambda^* = \mathbf{Z}_{\bar{\lambda}}$. Using (20), we get

$$\mathrm{Im}(\mathbf{Z}_\lambda f_0, f_0)_{\mathrm{V}} = (\mathrm{Im}\lambda) \int_{a_0}^{b_0} ((d\widetilde{\mathrm{V}}_\lambda) f_0(t), f_0(t))$$

for all functions $f_0 \in \mathfrak{H}$. Therefore, if $\mathfrak{L}(\lambda)$ is continuously invertible, then inequality

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(g, z)_{\mathrm{V}} \leqslant -(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(\mathbf{Z}_\lambda z, z)_{\mathrm{V}}, \quad \{z, g\} \in \mathfrak{L}(\lambda), \ \mathrm{Im}\lambda \neq 0 \qquad (44)$$

is equivalent to (43). By (44) and the equality $\mathfrak{L}(\bar{\lambda}) = \mathfrak{L}^*(\lambda)$, it follows that $\mathfrak{L}(\lambda)$ is continuously invertible. The pair $\{z, g\} \in \mathfrak{L}(\lambda)$ if and only if the pair $\{z, h\} \in \mathcal{L}_{F(\lambda)}$, where $g = h - \mathbf{Z}_\lambda z$. Hence the relation $\mathcal{L}_{F(\lambda)}$ is maximal accumulative for $\mathrm{Im}\lambda > 0$ if and only if (44) holds and $(\mathcal{L}_{F(\lambda)})^* = \mathcal{L}_{F(\bar{\lambda})}$.

Thus, if the function $\lambda \to F(\lambda)$ belongs to $\mathcal{F}(\mathfrak{N}_i, \mathfrak{N}_{-i})$, then the family $\lambda \to \mathfrak{L}(\lambda) = \mathcal{L}_{F(\lambda)} - \mathbf{Z}_\lambda$ has the following properties for $\mathrm{Im}\lambda \neq 0$: this family is holomorphic; $\mathfrak{L}(\bar{\lambda}) = \mathfrak{L}^*(\lambda)$; the inequality (44) holds. From Theorems 5, 6, it follows that the family $\lambda \to \mathfrak{L}(\lambda)$ generates a characteristic operator. The converse statement is valid since the above argument is reversible. The theorem is proved.

Corollary 9. *Let $\mathfrak{L}(\lambda)$ be linear relations with the property $L_0(\lambda) \subset \mathfrak{L}(\lambda) \subset L_0^*(\bar{\lambda})$, $\mathrm{Im}\lambda \neq 0$. The family $\lambda \to \mathfrak{L}(\lambda)$ generates a characteristic operator of equation* (38) *if and only if this family satisfies the following conditions:* 1) $\mathfrak{L}(\bar{\lambda}) = \mathfrak{L}^*(\lambda)$; 2) *the family $\lambda \to \mathfrak{L}(\lambda)$ is holomorphic for* $\mathrm{Im}\lambda \neq 0$; 3) *the inequality*

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(g, z)_{\mathrm{V}} \leqslant -(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(\mathbf{Z}_\lambda z, z)_{\mathrm{V}} = -\int_{a_0}^{b_0}((d\widetilde{\mathrm{V}}_\lambda)z(t), z(t))dt$$

holds for $\mathrm{Im}\lambda \neq 0$ *and for all pairs* $\{z, g\} \in \mathfrak{L}(\lambda)$.

We notice that the statements, similar to Theorem 7 and Corollary 9, for a differential equation with the Nevanlinna operator function are obtained in [4, 5] in another way.

For the regular case, in terms of boundary values we will describe the families generating the characteristic operator. According to Lemma 6, the pair $\{\tilde{y}, \tilde{f}\} \in \mathfrak{H} \times \mathfrak{H}$ belongs to the relation $\mathcal{L} - \lambda E$ if and only if there exists a pair $\{y, f\}$ such that the pairs $\{\tilde{y}, \tilde{f}\}$, $\{y, f\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$, and there holds the equality

$$y(t) = \mathcal{W}_\lambda(t)\widehat{c}_\lambda - \mathcal{W}_\lambda(t)iJ\int_{a_0}^{t}\mathcal{W}^*_{\bar{\lambda}}(s)(d\widetilde{\mathrm{V}})f(s)ds,$$

where $\widehat{c}_\lambda \in Q_-$.

We take here $\lambda = 0$. To each pair $\{y, f\} \in \mathcal{L}$, we assign a pair of boundary values $\{\mathrm{Y}, \mathrm{Y}'\} \in Q_- \times Q_+$ by the formulas

$$\mathrm{Y}' = \mathbf{w}^*(0)f = \int_{a_0}^{b_0}\mathcal{W}^*_0(s)(d\widetilde{\mathrm{V}})f(s) \in Q_+, \quad \mathrm{Y} = \widehat{c}_0 - (1/2)iJ\mathrm{Y}' \in Q_-.$$

Let Γ, Γ_1, Γ_2 be the operators defined by the equalities $\Gamma\{y, f\} = \{\mathrm{Y}, \mathrm{Y}'\}$, $\Gamma_1\{y, f\} = \mathrm{Y}$, $\Gamma_2\{y, f\} = \mathrm{Y}'$. We apply Lemma 8 to the relation $\mathcal{L}$. Then, using (33), we get

$$(f, z)_{\mathrm{V}} - (y, g)_{\mathrm{V}} = (\mathrm{Y}', \mathrm{Z}) - (\mathrm{Y}, \mathrm{Z}'), \tag{45}$$

where $\{y, f\}, \{z, g\} \in \mathcal{L}$, $\Gamma\{y, f\} = \{\mathrm{Y}, \mathrm{Y}'\}$, $\Gamma\{z, g\} = \{\mathrm{Z}, \mathrm{Z}'\}$. It follows from Lemma 5 and Corollary 4 that the mapping $\Gamma : \mathcal{L} \to Q_- \times Q_+$ is surjective. Thus the quadruple $(Q_-, Q_+, \Gamma_1, \Gamma_2)$ is the space of boundary values for the relation $\mathcal{L}$ in the sense of [21, 22]. (In [21, 22], the case, where the relation is an operator and $Q_- = Q_+$, is considered.)

R e m a r k 5. For the pairs $\{y,f\} \in \mathcal{L}'$ by straightforward calculations, we get

$$\mathrm{Y} = 2^{-1}(\mathcal{W}_0^{-1}(b_0)y(b_0) + \mathcal{W}_0^{-1}(a_0)y(a_0)),\ \mathrm{Y}' = iJ(\mathcal{W}_0^{-1}(b_0)y(b_0) - \mathcal{W}_0^{-1}(a_0)y(a_0)).$$

It follows from (45) and equality $\mathcal{R}(\Gamma) = Q_- \times Q_+$ that the pair $\{y, f\}$ belongs to $\mathcal{L}_0$ if and only if $\Gamma\{y, f\} = 0$. Consequently, between the relations $\widehat{\mathcal{L}}$ with the property $\mathcal{L}_0 \subset \widehat{\mathcal{L}} \subset \mathcal{L}$ and the relations $\vartheta \subset Q_- \times Q_+$ there exists a one-to-one correspondence determined by the equality $\Gamma\widehat{\mathcal{L}} = \vartheta$.

Denote by $\mathcal{T}(Q_-, Q_+)$ a set of holomorphic families of the linear relations $\lambda \to \vartheta(\lambda)$ ($\mathrm{Im}\lambda \neq 0$) whose values are the relations $\vartheta(\lambda) \subset Q_- \times Q_+$ such that $\vartheta(\bar{\lambda}) = \vartheta^*(\lambda)$ and $\vartheta(\lambda)$ is maximal accumulative for each λ with $\mathrm{Im}\lambda > 0$.

Theorem 8. *Between the functions $\lambda \to F(\lambda)$ belonging to $\mathcal{F}(\mathfrak{N}_i, \mathfrak{N}_{-i})$ and the families $\lambda \to \vartheta(\lambda)$ belonging to $\mathcal{T}(Q_-, Q_+)$ there exists a one-to-one correspondence determined by the equality $\Gamma\mathcal{L}_{F(\lambda)} = \vartheta(\lambda)$, i.e., the pair $\{y, f\} \in \mathcal{L}$ belongs to the relation $\mathcal{L}_{F(\lambda)}$ if and only if the pair $\Gamma\{y, f\} = \{\mathrm{Y}, \mathrm{Y}'\}$ belongs to $\vartheta(\lambda)$.*

P r o o f. It follows from the definitions of positive and negative spaces (see [9, ch. 1], [11, ch. 2]) that there exist the operators I_1, I_2 such that I_1, I_2 are isometrics Q_+ onto Q and Q onto Q_-, respectively, and the equality $(\mathrm{Y}, \mathrm{Y}') = (I_2^{-1}\mathrm{Y}, I_1\mathrm{Y}')$ holds for all elements $\mathrm{Y} \in Q_-$, $\mathrm{Y}' \in Q_+$. We define the operator $\widehat{\Gamma} : \mathcal{L} \to Q \times Q$ by the equality $\widehat{\Gamma}\{y, f\} = \{\widehat{\mathrm{Y}}, \widehat{\mathrm{Y}}'\}$, where $\{y, f\} \in \mathcal{L}$, $\widehat{\mathrm{Y}} = I_2^{-1}\mathrm{Y}$, $\widehat{\mathrm{Y}}' = I_1\mathrm{Y}'$, $\{\mathrm{Y}, \mathrm{Y}'\} = \Gamma\{y, f\}$. Then the operator $\widehat{\Gamma}$ maps $\mathcal{L}$ onto $Q \times Q$ and equality (45) holds when Y, Y', Z, Z' is replaced by $\widehat{\mathrm{Y}}$, $\widehat{\mathrm{Y}}'$, $\widehat{\mathrm{Z}}$, $\widehat{\mathrm{Z}}'$, respectively. According to [22], a pair $\{y, f\} \in \mathcal{L}$ belongs to $\mathcal{L}_{F(\lambda)}$ if and only if

$$(K(\lambda) - E)\widehat{\mathrm{Y}}' - i(K(\lambda) + E)\widehat{\mathrm{Y}} = 0,$$

where $\lambda \to K(\lambda)$ is a holomorphic function for $\mathrm{Im}\lambda > 0$ whose values are the operators in Q with the norm $\|K(\lambda)\| \leqslant 1$. The functions F and K determine uniquely each other.

By $\widehat{\vartheta}(\lambda)$, denote the Cayley transformation of the operator $K(\lambda)$, i.e., $\widehat{\vartheta}(\lambda) = i(K(\lambda) + E)(K(\lambda) - E)^{-1}$. We set $\widehat{\vartheta}(\bar{\lambda}) = \widehat{\vartheta}^*(\lambda)$. The function $\lambda \to K(\lambda)$ has the properties listed above if and only if the family of relations $\lambda \to \widehat{\vartheta}(\lambda)$ belongs $\mathcal{T}(Q, Q)$. On the other hand, the family of relations $\lambda \to \widehat{\vartheta}(\lambda)$ belongs $\mathcal{T}(Q, Q)$ if and only if the family $\lambda \to \vartheta(\lambda) = I_1^{-1}\widehat{\vartheta}(\lambda)I_2^{-1}$ belongs $\mathcal{T}(Q_-, Q_+)$. The theorem is proved.

It follows from (24) that $\{y, g\} \in L(\lambda)$ if and only if $\{y, g + \mathbf{Z}_\lambda y\} \in \mathcal{L}$. We set $\Gamma(\lambda)\{y, g\} = \Gamma\{y, g + \mathbf{Z}_\lambda y\}$ for each pair $\{y, g\} \in L(\lambda)$.

Theorem 8 implies the following statement.

Corollary 10. *The family of the relations* $\lambda \to \mathfrak{L}(\lambda)$, $\mathrm{Im}\lambda \neq 0$, *generates some characteristic operator if and only if the family* $\lambda \to \Gamma(\lambda)\mathfrak{L}(\lambda) = \vartheta(\lambda)$ *belongs to* $\mathcal{T}(Q_-, Q_+)$.

Notice that for the first time the linear relations were used to describe self-adjoint extensions of differential operators in [23] (further bibliography can be found in [11, 24]).

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References

[1] *A.M. Savchuk and A.A. Shkalikov*, Sturm–Liouville Operators with Singular Potentials. — *Mat. Zametki* **66** (1999), No. 6, 897–912. (Russian) (Engl. transl.: *Math. Notes* **66** (1999), No. 6, 741–753.)

[2] *F.S. Rofe-Beketov*, Square-Integrable Solutions, Self-Adjoint Extensions and Spectrum of Differential Systems. *Differential Equations. Proc. from the Uppsala 1977 Intern. Conf. on Differ. Equations.* Uppsala, 1977, 169–178.

[3] *Yu.V. Pokornyi, M.B. Zvereva, and S.A. Shabrov*, Sturm–Liouville Oscillation Theory for Impulsive Problems. — *Usp. Mat. Nauk* **68** (2008), No. 1, 111–154. (Russian) (Engl. transl.: *Russian Mathem. Surv.* **63** (2008), No. 1, 109–153.)

[4] *V.I. Khrabustovsky*, On the Characteristic Operators and Projections and on the Solutions of Weyl Type of Dissipative and Accumulative Operator Systems. 1. General case. 2. Abstract Theory. 3. Separated Boundary Conditions. — *J. Math. Phys., Anal., Geom.* **2** (2006), No. 2, 149–175, No. 3, 299–317, No. 4, 449–473.

[5] *V.I. Khrabustovsky*, On Characteristic Matrix of Weil-Titchmarsh Type for Differential-Operator Equations which Contains the Spectral Parameter in Linearly or Nevanlinna's Manner. — *Mat. Fiz., Anal., Geom.* **10** (2003), No. 2, 205–227.

[6] *V.M. Bruk*, On Linear Relations Generated by a Differential Expression and by a Nevanlinna Operator Function. — *J. Math. Phys., Anal., Geom.* **7** (2011), No. 2, 115–140.

[7] *V.M. Bruk*, On Linear Relations Generated by an Integral Equation with a Nevanlinna Measure. — *Izv. VUZ. Mathem.*(2012), No. 10, 3–19. (Russian) (Engl. transl.: *Rus. Mathem.* **56** (2012), No. 10, 1–14.)

[8] *V.M. Bruk*, On Invertible Linear Relations Generated by an Integral Equation with a Nevanlinna Measure. — *Izv. VUZ. Mathem.*(2013), No. 2, 16–29. (Russian) (Engl. transl.: *Rus. Mathem.* **57** (2013), No. 2, 13–24.)

[9] *Yu.M. Berezanski*, Expansions in Eigenfunctions of Selfadjoint Operators. Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965.)

[10] *V.M. Bruk*, On Linear Relations in a Space of Vector Functions. — *Mat. Zametki* **24** (1978), No. 4, 499–511. (Russian) (Engl. transl.: *Math. Notes* **24** (1979), No. 4, 767–773.)

[11] *V.I. Gorbatchuk and M.L. Gorbatchuk,* Boundary Value Problems for Differential Operator Equations. Kluwer Acad. Publ., Dordrecht, Boston, London, 1991. (Russian edition: Naukova Dumka, Kiev, 1984.)

[12] *J.-L. Lions and E. Magenes*, Problemes aux Limites non Homogenes et Applications. Dunod, Paris, 1968.

[13] *M.M. Malamud and S.M. Malamud*, On the Spectral Theory of Operator Measures. — *Funk. Anal.* **36** (2002), No. 2, 83–89. (Russian) (Engl. transl.: *Funct. Anal. and Appl.* **36** (2002), No. 2, 154–158.)

[14] *A.G. Baskakov*, Spectral Analysis of Differential Operators with Unbounded Operator-valued Coefficients, Difference Relations and Semigroups of Difference Relation. — *Izv. RAN. Ser. Mat.* **73** (2009), No. 2, 3–68. (Russian) (Engl. transl.: *Izv. Math.* **73** (2009), No. 2, 215–278.)

[15] *T. Kato*, Perturbation Theory for Linear Operators. Springer–Verlag, Berlin, Heidelberg, New York, 1966.

[16] *V.M. Bruk*, On Invertible Restrictions of Closed Operators in Banach Spaces. — *Funct. Anal.*, Uljanovsk **28** (1988), 17–22. (Russian)

[17] *H. Schaefer*, Topological Vector Spaces. The Macmillan Company, New York; Collier–Macmillan Limited, London, 1966.

[18] *V.M. Bruk*, Generalized Resolvents of Symmetric Relations Generated on Semi-Axis by a Differential Expression and a Nonnegative Operator Function. — *J. Math. Phys., Anal., Geom.* **2** (2006), No. 4, 372–387.

[19] *A.V. Strauss*, Generalized Resolvents of Symmetric Operators. — *Izv. Akad. Nauk SSSR, Ser. Mat.* **18** (1954), 51–86. (Russian)

[20] *A. Dijksma and H.S.V. de Snoo*, Self-adjoint Extensions of Symmetric Subspaces. — *Pacific J. Math.* **54** (1974), No. 1, 71–100.

[21] *A.N. Kochubei*, On Extensions of Symmetric Operators and Symmetric Binary Relations. — *Mat. Zametki* **17** (1975), No. 1, 41–48. (Russian) (Engl. transl.: *Math. Notes* **17** (1975), No. 1, 25–28.)

[22] *V.M. Bruk*, On One Class of Boundary Value Problems with a Spectral Parameter in the Boundary Condition. — *Mat. Sb.* **100** (1976), No. 2, 210–216. (Russian) (Engl. transl.: *Math. USSR-Sb.* **29** (1976), No. 2, 186–192.)

[23] *F.S. Rofe-Beketov*, Self-adjoint Extensions of Differential Operators in a Space of Vector Functions. — *Dokl. Akad. Nauk SSSR* **184** (1969), No. 5, 1034–1037. (Russian) (Engl. transl.: *Soviet. Math. Dokl.* **10** (1969), No. 1, 188–192.)

[24] *F.S. Rofe-Beketov and A.M. Khol'kin*, Spectral Analysis of Differential Operators. Interplay between Spectral and Oscillatory Properties. World Sci. Monogr. Ser. Math. **7**, Singapore (2005).