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# A Study on the $\phi$-Symmetric K-Contact Manifold Admitting Quarter-Symmetric Metric Connection

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The local $\phi$-symmetry and $\phi$-symmetry of a K-contact manifold with respect to the quarter-symmetric metric connection are studied and the results concerning the $\phi$-symmetry, scalar curvature with respect to the quarter-symmetric and the Levi–Civita connection are obtained. Further, the locally C-Bochner $\phi$-symmetric and the locally $\phi$-symmetric K-contact manifolds with respect to the quarter-symmetric metric connection are studied and some results are obtained. The results are assisted by the examples.

*Key words*: K-contact manifold, connection, $\phi$-symmetry.

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## 1. Introduction

In 1924, A. Friedman and J.A. Schouten [9] introduced the notion of a semi-symmetric linear connection on a differentiable manifold. In 1932, H.A. Hayden [11] introduced the idea of metric connection with torsion on a Riemannian manifold. In 1970, K. Yano [19] studied some curvature and derivational conditions for semi-symmetric connections in Riemannian manifolds. Later on, some interesting results on semi-symmetric metric connection were obtained by K.S. Amur and S.S. Pujar [1], C.S. Bagewadi [2], U.C. De [8], Mukut Mani Tripathi [13], T.Q. Binh [4], A.A. Shaikh et. al. [17].

In 1975, S. Golab [10] introduced the idea of quarter-symmetric metric connections and studied their properties. In 1980, R.S. Mishra and S.N. Pandey [12] studied quarter-symmetric metric F-connections in Riemanniann Kaehlerian and Sasakian manifolds. Later on, K. Yano and T. Imai [20], S.C. Rastogi [15], S. Mukhopadhyay, A.K. Roy and B. Barua [14], C.S. Bagewadi, D.G. Prakasha and

Venkatesha [3] studied some properties of quarter-symmetric metric connection on different manifolds. Note that a quarter-symmetric metric connection is a Hayden connection with the torsion tensor of the form $(1, 2)$.

The notion of local symmetry of Riemannian manifolds has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [18] introduced the notion of local $\phi$-symmetry on Sasakian manifolds. In the context of contact Geometry, the notion of $\phi$-symmetry is introduced and studied by E. Boeckx, P. Buecken and L. Vanhecke [7] with several examples. The paper is organized as follows: Section 3 is concerned with the relation between the Levi–Civita connection and the quarter-symmetric metric connection in a K-contact manifold. Section 4 deals with the locally $\phi$-symmetric K-contact manifold with respect to the quarter-symmetric metric connection. In Section 5, we study the $\phi$-symmetric K-contact manifold with respect to the quarter-symmetric metric connection. Section 6 is devoted to the study of the locally C-Bochner $\phi$-symmetric K-contact manifold with respect to the quarter-symmetric metric connection. Finally, we construct an example.

A linear connection $\tilde{\nabla}$ in an $n$-dimensional differentiable manifold is said to be a quarter-symmetric connection [10] if its torsion tensor $T$ is of the form

$$\begin{aligned} T(X,Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \\ &= \pi(Y)FX - \pi(X)FY, \end{aligned} \tag{1.1}$$

where $\pi$ is a 1-form and $F$ is a tensor of type (1.1). A quarter-symmetric linear connection $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection if $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0$$

for all $X, Y, Z \in \mathcal{X}(M)$, where $\mathcal{X}(M)$ is the Lie algebra of vector fields of the manifold $M$. For the contact manifold admitting quarter-symmetric connection, we take $\pi = \eta$ and $F = \phi$ in (1.1). Then it can be written as

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{1.2}$$

## 2. Preliminaries

An $n$-dimensional differentiable manifold $M$ is said to have an almost contact structure $(\phi, \xi, \eta)$ if it carries a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ on $M$ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi \circ \xi = 0. \tag{2.1}$$

Thus, the manifold $M$ equipped with the structure $(\phi, \xi, \eta)$ is called an almost contact manifold and is denoted by $(M, \phi, \xi, \eta)$. If $g$ is a Riemannian metric on an almost contact manifold $M$ such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

where $X$ and $Y$ are the vector fields defined on $M$, then it is said to have an almost contact metric structure $(\phi, \xi, \eta, g)$ and the manifold $M$ with the structure $(\phi, \xi, \eta, g)$ is called an almost contact metric manifold and is denoted by $(M, \phi, \xi, \eta, g)$.

If on $(M, \phi, \xi, \eta, g)$ the exterior derivative of 1-form $\eta$ satisfies

$$d\eta(X,Y) = g(X, \phi Y), \tag{2.4}$$

then $(\phi, \xi, \eta, g)$ is said to be a contact metric structure and $M$ equipped with a contact metric structure is called a contact metric manifold.

If, moreover, $\xi$ is a Killing vector field, then $M$ is called a K-contact manifold [6, 16]. In a K-contact manifold $M$ the following relations holds:

$$\eta(R(\xi, X)Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.5}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.6}$$

$$\nabla_X \xi = -\phi X, \tag{2.7}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{2.8}$$

for any vector fields $X$, $Y$, and $Z$, where $R$ and $S$ are the Riemannian curvature tensor and the Ricci tensor of $M$, respectively.

**Definition 2.1.** *A K-contact manifold $M$ is said to be locally $\phi$-symmetric if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0 \tag{2.9}$$

*for all vector fields $X$, $Y$, $Z$, and $W$ orthogonal to $\xi$. This notion was introduced by T. Takahashi* [18] *for Sasakian manifolds.*

**Definition 2.2.** *A K-contact manifold $M$ is said to be $\phi$-symmetric if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0 \tag{2.10}$$

*for the arbitrary vector fields $X$, $Y$, $Z$, and $W$.*

**Definition 2.3.** *A K-contact manifold $M$ is said to be locally C-Bochner $\phi$-symmetric if*

$$\phi^2((\nabla_W B)(X,Y)Z) = 0 \tag{2.11}$$

*for all vector fields X, Y, Z and W orthogonal to $\xi$, where B is the C-Bochner curvature tensor given by*

$$\begin{aligned} B(X,Y)Z = R(X,Y)Z + \frac{1}{n+3}[g(X,Z)QY - S(Y,Z)X - g(Y,Z)QX + S(X,Z)Y \\ + g(\phi X,Z)Q\phi Y - S(\phi Y,Z)\phi X - g(\phi Y,Z)Q\phi X + S(\phi X,Z)\phi Y \\ + 2S(\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z + \eta(Y)\eta(Z)QX - \eta(Y)S(X,Z)\xi \\ + \eta(X)S(Y,Z)\xi - \eta(X)\eta(Z)QY] - \frac{D+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ + 2g(\phi X,Y)\phi Z] + \frac{D}{n+3}[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y \\ - \eta(X)g(Y,Z)\xi] - \frac{D-4}{n+3}[g(X,Z)Y - g(Y,Z)X], \end{aligned} \quad (2.12)$$

*where* $D = \frac{r+n-1}{n+1}$.

## 3. Relation between Levi–Civita Connection and the Quarter-Symmetric Metric Connection in a K-Contact Manifold

Let $\tilde{\nabla}$ be a linear connection and $\nabla$ be a Riemannian connection of an almost contact metric manifold $M$ such that

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X,Y), \quad (3.1)$$

where $H$ is a tensor of type $(1,1)$. If $\tilde{\nabla}$ is a quarter-symmetric metric connection in $M$, then we have [10]

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)] \quad (3.2)$$

and

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \quad (3.3)$$

From (1.2) and (3.3), we get

$$T'(X,Y) = g(X,\phi Y)\xi - \eta(X)\phi Y. \quad (3.4)$$

Using (1.2) and (3.4) in (3.2), we obtain

$$H(X,Y) = -\eta(X)\phi Y.$$

Hence, a quarter-symmetric metric connection $\tilde{\nabla}$ in a K-contact manifold is given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \quad (3.5)$$

Therefore, (3.5) is the relation between the Levi–Civita connection and the quarter-symmetric metric connection on a K-contact manifold.

A relation between the curvature tensor of $M$ with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi–Civita connection $\nabla$ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + 2g(\phi X,Y)\phi Z + [\eta(X)g(Y,Z) \\ &- \eta(Y)g(X,Z)]\xi + [\eta(Y)X - \eta(X)Y]\eta(Z),\end{aligned} \tag{3.6}$$

where $\tilde{R}$ and $R$ are the Riemannian curvatures of the connections $\tilde{\nabla}$ and $\nabla$, respectively.
From (3.6), it follows that

$$\tilde{S}(Y,Z) = S(Y,Z) - g(Y,Z) + n\eta(Y)\eta(Z), \tag{3.7}$$

where $\tilde{S}$ and $S$ are the Ricci tensors of the connections $\tilde{\nabla}$ and $\nabla$, respectively.
Contracting (3.7), we get

$$\tilde{r} = r, \tag{3.8}$$

where $\tilde{r}$ and $r$ are the scalar curvatures of the connections $\tilde{\nabla}$ and $\nabla$, respectively.

## 4. Locally $\phi$-Symmetric K-Contact Manifold with Respect to the Quarter-Symmetric Metric Connection

In this section we define a locally $\phi$-symmetric K-contact manifold with respect to the quarter-symmetric metric connection by

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = 0 \tag{4.1}$$

for all vector fields $X$, $Y$, $Z$, and $W$ orthogonal to $\xi$.
Using (3.5), we can write

$$\begin{aligned}((\tilde{\nabla}_W \tilde{R})(X,Y)Z) &= (\nabla_W \tilde{R})(X,Y)Z - \eta(W)\phi\tilde{R}(X,Y)Z + \eta(W)\tilde{R}(\phi X,Y)Z \\ &+ \eta(W)\tilde{R}(X,\phi Y)Z + \eta(W)\tilde{R}(X,Y)\phi Z.\end{aligned} \tag{4.2}$$

Now, differentiating (3.6) with respect to $W$ and using (2.6), we obtain

$$\begin{aligned}(\nabla_W \tilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z + 2[\eta(Y)g(X,W) - \eta(X)g(W,Y)]\phi Z \\ &+[g(W,\phi X)g(Y,Z) - 2g(X,\phi Y)g(W,Z) - g(W,\phi Y)g(X,Z)]\xi \\ &+[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\phi W - [g(Y,\phi W)\eta(Z) + g(Z,\phi W)\eta(Y)]X \\ &+[g(X,\phi W)\eta(Z) + g(Z,\phi W)\eta(X)]Y - 2g(\phi X,Y)\eta(Z)W.\end{aligned} \tag{4.3}$$

Using (2.1) and (4.3) in (4.2), we get

$$\begin{aligned}\phi^2(\tilde{\nabla}_W\tilde{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z + 2[\eta(Y)g(X,W) - \eta(X)g(W,Y)]\phi^2(\phi Z)\\ +[g(W,\phi X)g(Y,Z) - 2g(X,\phi Y)g(W,Z) - g(W,\phi Y)g(X,Z)]\phi^2\xi\\ +[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\phi^2(\phi W) - [g(Y,\phi W)\eta(Z)\\ +g(Z,\phi W)\eta(Y)]\phi^2 X + [g(X,\phi W)\eta(Z) + g(Z,\phi W)\eta(X)]\phi^2 Y\\ -2g(\phi X,Y)\eta(Z)\phi^2 W - \eta(W)\phi^2(\phi R(X,Y)Z) + \eta(W)[\phi^2 R(\phi X,Y)Z\\ +\phi^2 R(X,\phi Y)Z + \phi^2 R(X,Y)\phi Z]. \qquad (4.4)\end{aligned}$$

If we consider $X$, $Y$, $Z$ and $W$ orthogonal to $\xi$, then (4.4) reduces to

$$\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \qquad (4.5)$$

Hence we can state the following:

**Theorem 4.1.** *A K-contact manifold is locally $\phi$-symmetric with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is locally $\phi$-symmetric with respect to the Levi–Civita connection.*

## 5. A $\phi$-Symmetric K-Contact Manifold with Respect to the Quarter-Symmetric Metric Connection

A K-contact manifold $M$ is said to be $\phi$-symmetric with respect to the quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) = 0 \qquad (5.1)$$

for the arbitrary vector fields $X$, $Y$, $Z$, and $W$.

Let us consider a $\phi$-symmetric K-contact manifold with respect to the quarter-symmetric metric connection. Then by virtue of (2.1) and (5.1), we have

$$-((\tilde{\nabla}_W\tilde{R})(X,Y)Z) + \eta((\tilde{\nabla}_W\tilde{R})(X,Y)Z)\xi = 0, \qquad (5.2)$$

from which it follows that

$$-g((\tilde{\nabla}_W\tilde{R})(X,Y)Z,U) + \eta((\tilde{\nabla}_W\tilde{R})(X,Y)Z)g(\xi,U) = 0. \qquad (5.3)$$

Let $\{e_i : i = 1,2,\ldots,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then, putting $X = U = e_i$ in (5.3) and taking summation over $i, 1 \le i \le n$, we get

$$-(\tilde{\nabla}_W\tilde{S})(Y,Z) + \sum_{i=1}^{n}\eta((\tilde{\nabla}_W\tilde{R})(e_i,Y)Z)\eta(e_i) = 0. \qquad (5.4)$$

By putting $Z = \xi$, the second term of (5.4) takes the form

$$\eta((\tilde{\nabla}_W \tilde{R})(e_i, Y)Z)\eta(e_i) = g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi)g(e_i, \xi). \tag{5.5}$$

Thus, by using (3.5) and (4.2), we can write

$$\begin{aligned} g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) &= g(\tilde{\nabla}_W \tilde{R}(e_i, Y)\xi, \xi) - g(\tilde{R}(\tilde{\nabla}_W e_i, Y)\xi, \xi) \\ &- g(\tilde{R}(e_i, \tilde{\nabla}_W Y)\xi, \xi) - g(\tilde{R}(e_i, Y)\tilde{\nabla}_W \xi, \xi). \end{aligned} \tag{5.6}$$

By simplifying (5.6), we obtain

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = g((\nabla_W R)(e_i, Y)\xi, \xi). \tag{5.7}$$

In the K-contact manifold $M$, we have $g((\nabla_W R)(e_i, Y)\xi, \xi) = 0$ and thus from (5.7) we get

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = 0. \tag{5.8}$$

By replacing $Z = \xi$ in (5.4) and using (5.8), we get

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = 0. \tag{5.9}$$

We know that

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \tilde{\nabla}_W \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_W Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi). \tag{5.10}$$

Using (2.7), (2.8), (3.5) and (3.7) in (5.10), we obtain

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = S(Y, \phi W) - (2n-1)g(Y, \phi W). \tag{5.11}$$

Using (5.11) in (5.9) and simplifying it, we have

$$S(Y, W) = (2n-1)g(Y, W) - n\eta(Y)\eta(W). \tag{5.12}$$

Then, after contracting the last equation, we get

$$r = 2n(n-1). \tag{5.13}$$

This leads to the following:

**Theorem 5.2.** *Let $M$ be a $\phi$-symmetric K-contact manifold with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Then the manifold has a scalar curvature $r$ with respect to the Levi–Civita connection $\nabla$ of $M$ given by* (5.13)*.*

## 6. Locally C-Bochner $\phi$-Symmetric K-Contact Manifold with Respect to the Quarter-Symmetric Metric Connection

A K-contact manifold $M$ is said to be locally C-Bochner $\phi$-symmetric with respect to the quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_W \tilde{B})(X,Y)Z) = 0 \tag{6.1}$$

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$, where $\tilde{B}$ is the C-Bochner curvature tensor with respect to the quarter-symmetric metric connection. It is given by

$$\begin{aligned}\tilde{B}(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{n+3}[g(X,Z)\tilde{Q}Y - \tilde{S}(Y,Z)X - g(Y,Z)\tilde{Q}X + \tilde{S}(X,Z)Y \\ +g(\phi X,Z)\tilde{Q}\phi Y - \tilde{S}(\phi Y,Z)\phi X - g(\phi Y,Z)\tilde{Q}\phi X + \tilde{S}(\phi X,Z)\phi Y \\ +2\tilde{S}(\phi X,Y)\phi Z + 2g(\phi X,Y)\tilde{Q}\phi Z + \eta(Y)\eta(Z)\tilde{Q}X - \eta(Y)\tilde{S}(X,Z)\xi \\ +\eta(X)\tilde{S}(Y,Z)\xi - \eta(X)\eta(Z)\tilde{Q}Y] - \frac{\tilde{D}+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ +2g(\phi X,Y)\phi Z] + \frac{\tilde{D}}{n+3}[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y \\ -\eta(X)g(Y,Z)\xi] - \frac{\tilde{D}-4}{n+3}[g(X,Z)Y - g(Y,Z)X].\end{aligned} \tag{6.2}$$

where

$$\tilde{D} = \frac{\tilde{r}+n-1}{n+1},$$

where $\tilde{R}$ and $\tilde{r}$ are the Riemannian curvature tensor and the scalar curvature with respect to the quarter-symmetric metric connection. Using (3.5), we can write

$$\begin{aligned}((\tilde{\nabla}_W \tilde{B})(X,Y)Z) &= (\nabla_W \tilde{B})(X,Y)Z - \eta(W)\phi\tilde{B}(X,Y)Z + \eta(W)\tilde{B}(\phi X,Y)Z \\ &+ \eta(W)\tilde{B}(X,\phi Y)Z + \eta(W)\tilde{B}(X,Y)\phi Z.\end{aligned} \tag{6.3}$$

Now differentiating (6.2) with respect to $W$ and by making use of (4.3), (3.8), (6.2) in (6.3), we get

$$\begin{aligned}(\tilde{\nabla}_W \tilde{B})(X,Y)Z = (\nabla_W R)(X,Y)Z + 2[g(W,X)\eta(Y) - g(W,Y)\eta(X)]\phi Z - \\ 2g(\phi X,Y)[\eta(Z)W - g(W,Z)\xi] + \frac{1}{n+3}[S(W,Z)[\eta(Y)\phi X - \eta(X)\phi Y] \\ -2\eta(X)S(W,Y)\phi Z + S(\phi Y,Z)[\eta(X)W - g(W,X)\xi] - S(\phi X,Z)[\eta(Y)W - g(W,Y)\xi] \\ -2S(\phi X,Y)[\eta(Z)W - g(W,Z)\xi] + [g(Y,\phi W)S(X,Z) - g(X,\phi W)S(Y,Z)]\xi \\ +[\eta(Y)S(X,Z) - \eta(X)S(Y,Z)]\phi W] + \frac{r-(n+3)}{(n+1)(n+3)}[\{\eta(X)g(Y,Z)\end{aligned}$$

$$
\begin{aligned}
&-\eta(Y)g(X,Z)\}\phi W + \{g(X,\phi W)g(Y,Z) - g(Y,\phi W)g(X,Z)\}\xi]\\
&-\frac{r+3n+1}{(n+1)(n+3)}[g(W,Z)\{\eta(Y)\phi X - \eta(X)\phi Y\} + g(\phi Y, Z)[\eta(X)W\\
&-g(W,X)\xi] - g(\phi X, Z)[\eta(Y)W - g(W,Y)\xi] - 2g(\phi X, Y)[\eta(Z)W\\
&-g(W,Z)\xi] - 2\eta(X)g(W,Y)\phi Z] + \frac{n^2-3n-2-r}{(n+1)(n+3)}[g(W,X)\eta(Z)\phi Y\\
&-g(W,Y)\eta(Z)\phi X + 2g(W,X)\eta(Y)\phi Z] + \frac{\nabla_W r}{(n+1)(n+3)}[g(\phi Y, Z)\phi X\\
&-g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z + g(\phi Y, \phi Z)X - g(\phi X, \phi Z)Y + \{\eta(Y)g(X,Z)\\
&-\eta(X)g(Y,Z)\}\xi] + \frac{r-(n^2+n+2)}{(n+1)(n+3)}[\eta(Z)\{g(Y,\phi W)X - g(X,\phi W)Y\}\\
&+g(Z,\phi W)\{\eta(Y)X - \eta(X)Y\}] + \eta(W)[R(\phi X, Y)Z + R(X,\phi Y)Z\\
&+R(X,Y)\phi Z] - \eta(W)\phi R(X,Y)Z. \qquad (6.4)
\end{aligned}
$$

The above equation (6.4) can be written in the form

$$
\begin{aligned}
&(\tilde{\nabla}_W \tilde{B})(X,Y)Z = (\nabla_W B)(X,Y)Z + 2[g(W,X)\eta(Y) - g(W,Y)\eta(X)]\phi Z - g(\phi X, Y)\\
&\times[\eta(Z)W - g(W,Z)\xi] - \frac{2}{n+3}[\{g(Y,\phi W)\eta(Z) + g(Z,\phi W)\eta(Y)\}X - \{g(X,\phi W)\eta(Z)\\
&+g(Z,\phi W)\eta(X)\}Y + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi W + \{g(\phi Y, Z)\eta(X)\\
&-g(\phi X, Z)\eta(Y) - 2g(\phi X, Y)\eta(Z)\}W + \eta(Y)g(W,Z)\phi X - \eta(X)g(W,Z)\phi Y\\
&-2\eta(X)g(W,Y)\phi Z + \{g(X,\phi W)g(Y,Z) - g(Y,\phi W)g(X,Z)\\
&+g(\phi X, Z)g(W,Y) - g(\phi Y, Z)g(W,X) + 2g(\phi X, Y)g(W,Z)\}\xi]\\
&-\eta(W)\phi R(X,Y)Z + \eta(W)\{R(\phi X, Y)Z + R(X,\phi Y)Z + R(X,Y)\phi Z\}. \qquad (6.5)
\end{aligned}
$$

Applying (2.1) to (6.5), we get

$$
\begin{aligned}
&\phi^2(\tilde{\nabla}_W \tilde{B})(X,Y)Z = \phi^2(\nabla_W B)(X,Y)Z + 2[g(W,X)\eta(Y) - g(W,Y)\eta(X)]\phi^2\phi Z\\
&-g(\phi X, Y)[\eta(Z)\phi^2 W - g(W,Z)\phi^2\xi] - \frac{2}{n+3}[\{g(Y,\phi W)\eta(Z)\\
&+g(Z,\phi W)\eta(Y)\}\phi^2 X - \{g(X,\phi W)\eta(Z) + g(Z,\phi W)\eta(X)\}\phi^2 Y\\
&+\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\phi^2\phi W + \{g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)\\
&-2g(\phi X, Y)\eta(Z)\}\phi^2 W + \eta(Y)g(W,Z)\phi^2\phi X - \eta(X)g(W,Z)\phi^2\phi Y\\
&-2\eta(X)g(W,Y)\phi^2\phi Z + \{g(X,\phi W)g(Y,Z) - g(Y,\phi W)g(X,Z)\\
&+g(\phi X, Z)g(W,Y) - g(\phi Y, Z)g(W,X) + 2g(\phi X, Y)g(W,Z)\}\phi^2\xi]\\
&-\eta(W)\phi^2\phi R(X,Y)Z + \eta(W)\{\phi^2 R(\phi X, Y)Z + \phi^2 R(X,\phi Y)Z\\
&+\phi^2 R(X,Y)\phi Z\}. \qquad (6.6)
\end{aligned}
$$

If we are considering $X, Y, Z, W$ to be orthogonal to $\xi$, then we obtain

$$\phi^2((\tilde{\nabla}_W \tilde{B})(X,Y)Z) \;=\; \phi^2((\nabla_W B)(X,Y)Z). \tag{6.7}$$

**Theorem 6.3.** *A K-contact manifold is locally C-Bochner $\phi$-symmetric with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is locally C-Bochner $\phi$-symmetric with respect to the Levi–Civita connection $\nabla$.*

Applying (2.1) to (6.4) and again considering $X$, $Y$, $Z$, and $W$ to be orthogonal to $\xi$ and the scalar curvature $r$ with respect to the Levi–Civita connection be constant in (6.4), we can reduce (6.4) to

$$\phi^2((\tilde{\nabla}_W \tilde{B})(X,Y)Z) \;=\; \phi^2((\nabla_W R)(X,Y)Z). \tag{6.8}$$

**Theorem 6.4.** *A K-contact manifold is locally C-Bochner $\phi$-symmetric with respect to the quarter-symmetric metric connection if and only if it is locally $\phi$-symmetric with respect to the Levi–Civita connection provided the scalar curvature $r$ is constant with respect to the Levi–Civita connection.*

## 7. Example

Consider a 3-dimensional manifold $C^* \times R$. Let $(r, \theta, z)$ be standard coordinates in $C^* \times R$. Let $\{E_1, E_2, E_3\}$ be linearly independent:

$$E_1 = \frac{1}{r}\frac{\partial}{\partial\theta} + r\frac{\partial}{\partial z}, \qquad E_2 = \frac{\partial}{\partial r}, \qquad E_3 = \xi = \frac{\partial}{\partial z}.$$

Let $g$ be a Riemannian metric defined by

$$g(E_1,E_1) = g(E_2,E_2) = g(E_3,E_3) = 1,$$
$$g(E_1,E_2) = g(E_2,E_3) = g(E_3,E_1) = 0.$$

Then $(\phi, \xi, \eta)$ is given by

$$\xi = \frac{\partial}{\partial z}, \qquad \eta = dz - r^2 d\theta,$$
$$\phi E_1 = -E_2, \qquad \phi E_2 = E_1, \qquad \phi E_3 = 0.$$

The linearity of $\phi$ and $g$ yields

$$\eta(E_3) = 1, \qquad \phi^2 U = -U + \eta(U)E_3,$$
$$g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W)$$

for any vector fields $U, W$ on $M$. By the definition of Lie bracket, we have

$$[E_1, E_2] = \frac{1}{r}E_1 - 2E_3, \;\; [E_1,E_3] = [E_2,E_3] = 0.$$

Let $\nabla$ be a Levi–Civita connection with respect to the above metric $g$ given by the Koszula formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &- g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{7.1}$$

Then

$$\begin{aligned} \nabla_{E_1}E_1 &= \frac{-E_2}{r}, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_3}E_3 = 0, \\ \nabla_{E_1}E_2 &= \frac{E_1}{r} - E_3, \quad \nabla_{E_2}E_1 = E_3, \\ \nabla_{E_1}E_3 &= E_2, \quad \nabla_{E_3}E_1 = E_2, \\ \nabla_{E_2}E_3 &= -E_1, \quad \nabla_{E_3}E_2 = -E_1. \end{aligned} \tag{7.2}$$

The tangent vectors $X$, $Y$, $Z$ and $W$ to $C^* \times R$ are expressed as the linear combination of $\{E_1, E_2, E_3\}$, that is, $X = \sum_{i=1}^{3} a_i E_i$, $Y = \sum_{j=1}^{3} b_j E_j$, $Z = \sum_{k=1}^{3} c_k E_k$, and $W = \sum_{l=1}^{3} d_l E_l$, where $a_i$, $b_j$, $c_k$, and $d_l$ are scalars. Clearly, $(\phi, \xi, \eta, g)$ satisfy the equations of the K-contact manifold. Thus, $C^* \times R$ is a K-contact.

The non-zero terms $g(R(X, E_i)E_i, Y)$, $i = 1, 2, 3$, by virtue of (7.2), are given by

$$\begin{aligned} R(E_2,E_1)E_1 &= -3E_2, \quad R(E_3,E_1)E_1 = E_3, \\ R(E_1,E_2)E_2 &= -3E_1, \quad R(E_3,E_2)E_2 = E_3, \\ R(E_1,E_3)E_3 &= E_1, \quad R(E_2,E_3)E_3 = E_2. \end{aligned} \tag{7.3}$$

Using expressions (7.2) and (7.3), by virtue of the definition of the K-contact manifold and $\phi^2 E_3 = 0$, one can see that Theorems 4.1, 6.3 and 6.4 are verified as seen below:

$$\phi^2(\tilde{\nabla}_W \tilde{R})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z. \tag{7.4}$$

$$\phi^2(\tilde{\nabla}_W \tilde{B})(X,Y)Z = \phi^2(\nabla_W B)(X,Y)Z. \tag{7.5}$$

$$\phi^2(\tilde{\nabla}_W \tilde{B})(X,Y)Z = \phi^2(\nabla_W R)(X,Y)Z. \tag{7.6}$$

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