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Optimal Control Method for Solving the Cauchy–Neumann Problem for the Poisson Equation

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In the paper, the ill-posed Cauchy–Neumann problem is considered for the Poisson equation. The problem is reduced to the optimal control problem that is regularized. Optimization methods are applied to the solution of the obtained problem.

Key words: Ill-posed problem, Cauchy–Neumann problem, optimal control method, Poisson equation.

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1. Introduction

Optimal control methods have a wide field of applications such as modeling of various practical systems, studying of some classes of inverse and ill-posed problems [2,3], etc. It is known that the Cauchy–Neumann problem for elliptic equations, particularly for the Poisson equation, is an ill-posed problem (see [6, 11]). The necessity to study the ill-posed problems of mathematical physics and their well-posed formulation was first noted by A.N. Tikhonov. It was stimulated by the fact that some physical processes mathematically are described by these problems. Systematic studying of these problems has begun since the 50-th of the last century, and various methods have been developed for their investigation.

In [1], optimal control methods are applied to the solution of the ill-posed Cauchy–Dirichlet problem for the Poisson equation. Here we consider the ill-posed Cauchy–Neumann problem for the Poisson equation that reduces to the solution of the optimal control problem with specially constructed functional. It takes us to the adjoint problem with naturally simple form.

Note that the interest to the studying of this problem is encouraged with the strong relations of the Poisson equation with several applications as well as problems of electrostatics, mathematical engineering, and theoretical physics.

2. Problem Statement

In the domain $Q = \{(x,\, t) \mid 0 < x < \pi,\ -1 < t < 1\}$, consider the boundary problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = f(x,t), \quad (x,t) \in Q, \tag{2.1}$$

$$\left.\frac{\partial u(x,\, t\,)}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u(x,\, t)}{\partial x}\right|_{x=\pi} = 0, \quad t \in (-1,\, 1), \tag{2.2}$$

$$u(x,\, t)|_{t=-1} = \varphi_0(x), \quad \left.\frac{\partial u(x,\, t)}{\partial t}\right|_{t=-1} = \varphi_1(x),\ x \in (0,\pi). \tag{2.3}$$

It is supposed that $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=1} \in U_\partial$, where U_∂ is a convex closed set in $L_2(0,\pi)$, $0 \in U_\partial$ and $f \in L_2(Q),\ \varphi_0 \in W_2^1(0,\pi),\ \varphi_1 \in L_2(0,\pi)$ are given functions.

It is known that (2.1)–(2.3) is an ill-posed problem [4, 7].

Let us introduce an optimal control problem in correspondence with the problem above. For this purpose, we replace the conditions (2.3) by the following ones:

$$\left.\frac{\partial u(x,\, t)}{\partial t}\right|_{t=-1} = \varphi_1(x), \quad \left.\frac{\partial u(x,\, t)}{\partial t}\right|_{t=1} = v(x), \quad x \in (0,\pi) \tag{2.4}$$

and consider the problem on finding in U_∂ the minimum of the functional

$$J(v) = \int_0^\pi \left[u(x,\ -1) - \varphi_0(x)\right]^2 dx \tag{2.5}$$

subject to (2.1), (2.2), (2.4).

As is known from the general optimal control theory, (2.1), (2.2), (2.4), (2.5) is also an ill-posed problem. Note that if $f \in L_2(Q),\ \varphi_1 \in L_2(0,\pi)$, $v \in L_2(0,\pi)$, then the boundary problem (2.1), (2.2), (2.4) has the unique solution from $W_2^1(Q)$ [5].

3. Regularization of the Optimal Control Problem (2.1), (2.2), (2.4), (2.5)

The regularization method is one of important techniques used for solving ill-posed problems [5, 10]. We apply this method to solve the problem (2.1), (2.2), (2.4), (2.5). We take the functional $\varepsilon \int_0^\pi |v(x)|^2 \, dx$ $(\varepsilon > 0)$ as a stabilizer in the considering problem. Thus, for the minimization, we obtain the functional

$$J_\varepsilon(v) = J(v) + \varepsilon \int_0^\pi |v(x)|^2 \, dx = \int_0^\pi [u(x, -1) - \varphi_0(x)]^2 \, dx + \varepsilon \int_0^\pi |v(x)|^2 \, dx \quad (3.1)$$

in the class U_∂ subject to (2.1), (2.2), (2.5).

Let $u(x, t; v)$ be a solution of the problem (2.1), (2.2), (2.4) corresponding to the given control $v \in U_\partial$; $u(x, t; 0)$ be a solution of the problem (2.1), (2.2), (2.4) for $v(x) \equiv 0$.

Let us define

$$a(v_1, v_2) = \int_0^\pi [u(x, -1; v_1) - u(x, -1; 0)] \, [u(x, -1; v_2) - u(x, -1; 0)] \, dx$$

$$+\varepsilon \int_0^\pi v_1(x) v_2(x) \, dx,$$

$$L(v) = \int_0^\pi [\varphi_0(x) - u(x, -1; 0)] \, [u(x, -1; v) - u(x, -1; 0)] \, dx,$$

where $a(v_1, v_2)$ is a bilinear continuous symmetric form on U_∂; $L(v)$ is a linear form on U_∂.

Using these definitions and taking $v_1 = v_2 = v$ in the expression for $a(v_1, v_2)$, the functional (3.1) can be rewritten in the form

$$J_\varepsilon(v) = a(v, v) - 2\, L(v) + \int_0^\pi [u(x, -1; 0) - \varphi_0(x)]^2 \, dx.$$

Since $a(v_1, v_2)$ is a bilinear continuous symmetric form and it satisfies the condition (see definition for $a(v_1, v_2)$)

$$a(v, v) \geq \varepsilon \, \|v\|_{L_2}^2 ,$$

it follows from the well-known theorem from [8, page 13] that the theorem below is valid.

Theorem 3.1. *For the optimal control problem* (2.1), (2.2), (2.4), (3.1) *there exists the element* $\bar{v} \in U_\partial$ *such that* $J_\varepsilon(\bar{v}) = \inf_{v \in U_\partial} J_\varepsilon(v)$ *and this element is unique.*

Basing on the theorem from [8, page 18], we can easily prove the theorem below.

Theorem 3.2. *For* $\bar{v} \in U_\partial$ *to be an optimal control, it is necessary and sufficient to fulfill the inequality*

$$J'_{\varepsilon v}(\bar{v})(v-\bar{v}) \geq 0 \quad \forall v \in U_\partial,$$

which is equivalent to

$$\int_0^\pi [u(x,\,-1;\bar{v}) - \varphi_0(x)]\, u_v(x,\,-1;\bar{v})\,[v(x)-\bar{v}(x)]\,dx$$

$$+\varepsilon\int_0^\pi \bar{v}(x)\,[v(x)-\bar{v}(x)]\,dx \geq 0 \;\; \forall v \in U_\partial, \tag{3.2}$$

where $J'_{\varepsilon v}$ *is a Gateaux derivative with respect to* v*;* $u_v(x,t;v)$ *is a derivative of the solution of the problem* (2.1), (2.2), (2.5) *with respect to* v*.*

Let us transform inequality (3.2). The linear boundary problem (2.1), (2.2), (2.4) can be rewritten in an operator form

$$Au = F \equiv \{f, \varphi_1, v\},$$

where A is an unbounded linear operator from the space $L_2(Q)$ to the Hilbert space $L_2(Q) \times L_2(0,\,\pi) \times L_2(0,\,\pi)$,

$$A:\; u(x,t) \mapsto \left\{ \Delta u(x,\,t),\; \left.\frac{\partial u(x,\,t)}{\partial t}\right|_{t=-1},\; \left.\frac{\partial u(x,\,t)}{\partial t}\right|_{t=1} \right\}.$$

As the domain of A, we take a set of functions from $W_2^2(Q)$ satisfying the conditions

$$\left.\frac{\partial u(x,\,t\,)}{\partial x}\right|_{x=0} = 0,\quad \left.\frac{\partial u(x,\,t)}{\partial x}\right|_{x=\pi} = 0,\quad t \in (-1,\,1).$$

Then the operator A admits the closure $\bar{A}$ that has an inverse. It means that the operator equation above has a generalized solution $u = \bar{A}^{-1}F$ belonging to $W_2^1(Q)$ [5,10].

Let us take a derivative of this solution in the direction $v-\bar{v} : u_v(x,t;\bar{v})\;[v-\bar{v}] = u(x,t\,;v) - u\,(x,t\,;\bar{v})$.

Then inequality (3.2) takes the form

$$\int_0^\pi [u(x,\,-1;\bar{v}) - \varphi_0(x)]\,[u(x,\,-1;\,v) - \;u(x,\,-1;\,\bar{v})]\,dx$$

$$+\,\varepsilon\int_0^\pi \bar{v}(x)\,[v(x)-\bar{v}(x)]\,dx \ge 0\;\;\forall v \in U_\partial. \tag{3.3}$$

4. Optimality Condition

Let us introduce the adjoint boundary problem

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial t^2} = 0, \quad (x,t)\in Q, \tag{4.1}$$

$$\left.\frac{\partial\psi(x,\,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial u(x,\,t)}{\partial x}\right|_{x=\pi} = 0,\;\; t\in(-1,\,1), \tag{4.2}$$

$$\left.\frac{\partial\psi(x,\,t)}{\partial t}\right|_{t=-1} = u(x,-1;\bar{v}) - \varphi_0(x),\; \left.\frac{\partial\psi(x,\,t)}{\partial t}\right|_{t=1} = 0,\;\; x\in(0,\pi). \tag{4.3}$$

Note that the problem (4.1)–(4.3) has the unique solution from $W_2^2(Q)$ [5].

Using the boundary problem (4.1)–(4.3), we can transform the first term of inequality (3.3). If to take $\tilde{u}(x,t) = u(x,t;v) - u(x,t;\bar{v})$, then it is clear that

$$\iint_Q \left[\frac{\partial^2\tilde{u}}{\partial x^2} + \frac{\partial^2\tilde{u}}{\partial t^2}\right]\,\psi(x,\,t;\,\bar{v})\,dxdt = 0.$$

Integrating by parts in the above equation and taking into account (2.2), (2.4), (4.1)–(4.3), we have

$$\int\!\!\int_Q \left[\frac{\partial^2\tilde{u}}{\partial x^2} + \frac{\partial^2\tilde{u}}{\partial t^2}\right]\,\psi(x,t;\bar{v})dxdt = \int_0^\pi [v(x)-\bar{v}(x)]\;\psi(x,1;\bar{v})dx$$

$$+\int_0^\pi [u(x,-1;v) - u(x,-1;\bar{v})]\,\left.\frac{\partial\psi(x,\,t;\bar{v}\,)}{\partial t}\right|_{t=-1} dx = 0, \tag{4.4}$$

where $\psi(x,\,t;\,\bar{v})$ is a solution of the problem (4.1)–(4.3) corresponding to the control $\bar{v}\in U_\partial$.

Taking into account the first condition of (4.3) from (4.4), we can obtain

$$\int_0^{\pi} \left[u\left(x,-1;\,\bar{v}\right)-\varphi_0(x)\right]\,\left[u\left(x,-1;\,\,v\right)-u\left(x,-1;\,\bar{v}\right)\right]\,dx$$

$$= -\int_0^{\pi} \psi(x,1;\,\bar{v})\,\left[v\left(x\right)-\bar{v}\left(x\right)\right]\,dx. \tag{4.5}$$

Then from (3.3) and (4.5) it follows that

$$\int_0^{\pi} \left[-\psi(x,1;\bar{v})+\varepsilon\bar{v}(x)\right]\,\left[v(x)-\bar{v}(x)\right]dx \geq 0 \quad \forall v \in U_{\partial}. \tag{4.6}$$

This proves the optimality condition in the form of the following theorem.

Theorem 4.1. *Let the function $\bar{v}(x) \in U_{\partial}$ be an optimal control for the problem* (2.1), (2.2), (2.4), (3.1). *Then it is necessary and sufficient that this function satisfy the boundary problems* (2.1), (2.2), (2.4), (4.1)–(4.3) *and variational inequality* (4.6).

5. Application of the Fourier Method

Now we analyze the boundary problems (2.1), (2.2), (2.4) and (4.1)–(4.3) by using the Fourier method. The applicability of this method to the considered problems is shown in [5]. We will look for the solutions of the boundary problems in the forms

$$u(x,t)=u_0(t)+\sum_{k=1}^{\infty}u_k(t)X_k(x), \quad \psi(x,t)=\psi_0(t)+\sum_{k=1}^{\infty}\psi_k(t)X_k(x),$$

where

$$X_0(x)=\frac{1}{\sqrt{\pi}},\lambda_o=0,\;X_k(x)=\sqrt{\frac{2}{\pi}}\,\cos k\,x\,;\;\;\lambda_k=\,-k^2,\;\;k=1,2,\ldots \tag{5.1}$$

are the systems of orthonormal eigenfunctions and eigenvalues of the spectral problem

$$X''(x)=\lambda X(x),\;\;X'(0)=X'(\pi)=0.$$

From (2.1), (2.2), (2.4), (4.1)–(4.3) and (4.6) we obtain

$$\begin{cases} \ddot{u}_k(t)-k^2u_k(t)=f_k(t), & t\in(-1,\,1), \\ \dot{u}_k(-1)=\varphi_{1k},\;\;\dot{u}_k(1)=\bar{v}_k, & k=0,1,2,\ldots, \end{cases} \tag{5.2}$$

$$\begin{cases} \ddot{\psi}_k(t) - k^2\psi_k(t) = 0, & t \in (-1,\,1), \\ \dot{\psi}_k(-1) = u_k(-1) - \varphi_{0k},\ \dot{\psi}_k(1) = 0, & k = 0,1,2,\ldots, \end{cases} \tag{5.3}$$

$$[-\psi_k(1) + \varepsilon\bar{v}_k]\ [v_k - \bar{v}_k] \geq 0 \quad \forall v_k,\ \ k = 0,1,2,\ldots, \tag{5.4}$$

where $f_k(t), \varphi_{0k}, \varphi_{1k},\ \bar{v}_k,\ v_k,\ k = 0,1,2,\ldots$ are Fourier coefficients of the functions $f(x,\,t),\ \varphi_0(x),\ \varphi_1(x),\ \bar{v}(x),\ v(x)$ with respect to the system (5.1).

From the general theory of the boundary problems for the ordinary differential equations [9], we can conclude that the solutions of the boundary problem (5.2) for $k = 1,2,\ldots$ can be written in the form

$$u_k(t) = \bar{v}_k \frac{\operatorname{ch} k(1+t)}{k \operatorname{sh} 2k} - \varphi_{1k} \frac{\operatorname{ch} k(1-t)}{k \operatorname{sh} 2k} + \int\limits_{-1}^{1} G_k(t;\tau) f_k(\tau) d\tau, \tag{5.5}$$

where

$$G_k(t,\tau) = \begin{cases} -\dfrac{\operatorname{ch} k(1-\tau) \operatorname{ch} k(1+t)}{k \operatorname{sh} 2k}, & t \in [-1,\tau], \\ \dfrac{1}{k} \operatorname{sh} k(t-\tau) - \dfrac{\operatorname{ch} k(1-\tau) \operatorname{ch} k(1+t)}{k \operatorname{sh} 2k}, & t \in [\tau,\,1] \end{cases}$$

is a Green function for the problem (5.2).

For the problem (5.3), we have

$$\psi_k(t) = -\frac{u_k(-1) - \varphi_{0k}}{k \operatorname{sh} 2k} \operatorname{ch} k(1-t). \tag{5.6}$$

For $k = 0$, the solutions of the problems (5.2), (5.3) are in the forms

$$u_0(t) = \int\limits_{-1}^{t} (t-s) f_0(s) ds + \varphi_{10}(t+1) + \varphi_{00},$$

$$\psi_0(t) = c,$$

where c is a random constant and $\int_{-1}^{1} f_0(t)dt + \varphi_{10} = \bar{v}_0$, moreover, $\varphi_{00} = \frac{1}{\sqrt{\pi}} \int_0^\pi \varphi_0(x)dx$, $\varphi_{10} = \frac{1}{\sqrt{\pi}} \int_0^\pi \varphi_1(x)dx$, $\bar{v}_0 = \frac{1}{\sqrt{\pi}} \int_0^\pi v(x)dx$, $f_0(t) = \frac{1}{\sqrt{\pi}} \int_0^\pi f(x,t)dx$.

From (5.5), (5.6) and (5.4), we get

$$u_k(-1) = \frac{\bar{v}_k}{k \operatorname{sh} 2k} - \varphi_{1k} \frac{\operatorname{cth} 2k}{k} + \int\limits_{-1}^{1} G_k(-1;\tau) f_k(\tau) d\tau, \quad k = 1,2,\ldots,$$

$$-\psi_k(1) = \frac{u_k(-1) - \varphi_{0k}}{k \operatorname{sh} 2k},\ \ k = 1,2,\ldots,$$

$$[u_k(-1) - \varphi_{0k} + \varepsilon k \operatorname{sh} 2k \cdot \bar{v}_k]\ [v_k - \bar{v}_k] \geq 0 \ \ \forall v_k, \quad k = 1, 2, \dots . \tag{5.7}$$

The condition (5.7) can be transformed as

$$\left[-\varphi_{1k} \frac{\operatorname{cth} 2k}{k} + \bar{v}_k \left(\frac{1}{k \operatorname{sh}\, 2k} + \varepsilon k \operatorname{sh} 2k \right) - \varphi_{ok} + \int_{-1}^{1} G_k(-1;\tau) f_k(\tau) d\tau \right]$$
$$\times (v_k - \bar{v}_k) \geq 0 \ \ \forall v_k, \ \ k = 1, 2, \dots . \tag{5.7'}$$

Now let us consider the case $U_\partial = L_2(0, \pi)$. Then from (5.7′) we obtain

$$\bar{v}_k = \beta_{k\varepsilon}^{-1} \left[\varphi_{0k} + \varphi_{1k} \frac{\operatorname{cth} 2k}{k} - \int_{-1}^{1} G_k(-1;\tau) f_k(\tau) d\tau \right], \tag{5.8}$$

where

$$\beta_{k\varepsilon} = \frac{1 + \varepsilon k^2 \operatorname{sh}^2 2k}{k \operatorname{sh} 2k}, \quad k = 1, 2, \dots .$$

As follows from (5.4), $\bar{v}_0 = \frac{\psi_0(1)}{\varepsilon}$ for $k = 0$.

Since we are interested in the bounded value of $\bar{v}_0$ for $\varepsilon \to 0$, the value $\psi_0(1)$ should be equal to zero. Then $\bar{v}_0 = 0$, and the solution of the problem (5.3) for $k = 0$ is equal to zero.

Since the solution of the problem (5.2) for $k = 0$ has the form

$$u_0(t) = \int_{-1}^{t} (t - s) f_0(s) ds + \varphi_{10}(t + 1) + \varphi_{00},$$

it is necessary to fulfill the condition $\int_{-1}^{1} f_0(t) dt + \varphi_{10} = 0$.

Thus, we have the optimal values of the Fourier coefficients $\bar{v}_k$ for the function $\bar{v}(x)$. Then, for $\varepsilon \to 0$ from (5.5), (5.8), we have

$$u_{k0}(t) = \lim_{\varepsilon \to 0} u_k(t) = \varphi_{0k} \operatorname{ch} k(1 + t) + \frac{\varphi_{1k}}{k \operatorname{sh} 2k} [\operatorname{ch} 2k \operatorname{ch} k(1 + t) - \operatorname{ch} k(1 - t)]$$
$$- \operatorname{ch} k(1 + t) \int_{-1}^{1} G_k(-1;\tau) f_k(\tau) d\tau + \int_{-1}^{1} G_k(t;\tau) f_k(\tau) d\tau, \ \ k = 1, 2, \dots . \tag{5.9}$$

$$\bar{v}_{k0} = \lim_{\varepsilon \to 0} \bar{v}_k = \varphi_{0k} k \operatorname{sh} 2k + \varphi_{1k} \operatorname{ch} 2k - k \operatorname{sh} 2k \int_{-1}^{1} G_k(-1;\tau) f_k(\tau) d\tau, \ \ k = 1, 2, \dots . \tag{5.10}$$

Note that the solution $u_k(t)$ given by (5.5) and corresponding to the optimal Fourier coefficients $\bar{v}_k,\ k = 1, 2, \ldots$ given by (5.8), satisfies the limit relations $\lim\limits_{\varepsilon \to 0} u_k(-1) = \varphi_{0k}$. This is consistent with the condition $u(x, -1) = \varphi_0(x)$ from (2.3).

Thus the exact solution of the problem (2.1), (2.2), (2.4), (2.5) has the form

$$\bar{v}(x) = \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} \operatorname{sh} 2k \left[k\varphi_{0k} + \varphi_{1k} \operatorname{cth} 2k - k \int_{-1}^{1} G_k(-1; \tau) f_k(\tau) d\tau \right] \cos kx.$$

Then the solution of the initial problem (2.1)–(2.3) has the form

$$u(x, t) = \frac{1}{\sqrt{\pi}} u_0(t)$$

$$+ \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} \Bigg\{ \varphi_{0k} \operatorname{ch} k(1 + t) + \frac{\varphi_{1k}}{k \operatorname{sh} 2k} \left[\operatorname{ch} 2k \ \operatorname{ch} k \left(1 + t\right) - \operatorname{ch} k \left(1 - t\right) \right]$$

$$- \operatorname{ch} k \left(1 + t\right) \int_{-1}^{1} G_k \left(-1; \tau\right) f_k \left(\tau\right) d\tau + \int_{-1}^{1} G_k \left(t; \tau\right) f_k \left(\tau\right) d\tau \Bigg\} \cos kx.$$

Now let us consider the analogue of the Hadamard example for the problem (2.1)–(2.3). For this purpose we take

$$f(x, t) = 0, \ \varphi_0(x) = 0, \ \varphi_1(x) = \frac{1}{k} \exp\left\{ -\sqrt{k} \right\} \cos kx, \ \ k \in N.$$

In this case, the solution of the Cauchy–Neumann problem for the Laplace equation is unique and has the form

$$u(x, t) = \frac{1}{k^2} \exp\left\{ -\sqrt{k} \right\} \cos kx \operatorname{sh} k(1 + t), \ \ k \in N. \tag{5.11}$$

At the same time, for $k \to \infty$, the function $\varphi_1(x)$ tents to zero uniformly and all its derivatives belong to $L_2(0, \pi)$. However, the solution (5.11) for any $t > -1$ has a cosinusodial form with arbitrary large amplitude not belonging to $L_2(Q)$.

For the function $\varphi_1(x)$ to satisfy $\{\exp\{2k\} \varphi_{1k}\}_{k=1}^{\infty} \in l_2$, it is necessary and sufficient that the Fourier coefficients φ_{1k} have asymptotics of order $\exp\{-(2 + \varepsilon)k\}$ for sufficiently large k, where $\varepsilon > 0$. In our case, we have only asymptotics of order $\exp\left\{ -\sqrt{k} \right\}$ that is not enough to provide the well-posedness of the Cauchy–Neumann problem for the Laplace equation.

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