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The Plasticity of Some Fittable Surfaces on a Given Quadruple of Points in the Three-Dimensional Euclidean Space

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We construct a two-dimensional sphere in the three-dimensional Euclidean space which intersects a circular cylinder in three given points and the corresponding weighted Fermat–Torricelli point for a geodesic triangle such that these three points and the corresponding weighted Fermat–Torricelli point remain the same on the sphere for a different triad of weights which correspond to the vertices on the surface of the sphere. We derive a circular cone which passes from the same points that a circular cylinder passes. By applying the inverse weighted Fermat–Torricelli problem for different weights, we obtain the plasticity equations which provide the new weights of the weighted Fermat–Torricelli point for fixed geodesic triangles on the surface of a fittable sphere and a fittable circular cone with respect to the given quadruple of points on a circular cylinder, which inherits the curvature of the corresponding fittable surfaces.

Key words: weighted Fermat–Torricelli point, sphere, circular cylinder, circular cone, fittable surfaces.

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1. Introduction

The weighted Fermat–Torricelli problem states that:

Given are three points A_1, A_2, A_3 in the Euclidean plane, three positive real numbers w_i (weight) which correspond to the vertex A_i, find a point X in the Euclidean plane that minimizes the sum of the weighted Euclidean distances $f(X) = w_1\|A_1X\| + w_2\|A_2X\| + w_3\|A_3X\|$.

The solution of the weighted Fermat–Torricelli problem is named as the weighted Fermat–Torricelli point F. E. Torricelli was the first to discover the isogonal property (or 120° property) of the weighted Fermat–Torricelli point $\angle A_1FA_2 = \angle A_2FA_3 = \angle A_3FA_1 = 120°$ for equal weights. B. Cavalieri was the first who stated that if at most one angle $\angle A_iA_jA_k \geq 120°$, then $F = A_j$ for $w_1 = w_2 = w_3$, $i, j, k = 1, 2, 3, i \neq j \neq k$ (see [2, 4]).

The isogonal property of the equally weighted Fermat–Torricelli point holds in Riemmanian manifolds ([3]) and in an Alexandrov surface of the bounded curvature ([5], in the surface of polyhedra).

We introduce a problem of the (curvature) plasticity of a surface which passes from four given points in $\mathbb{R}^3$:

Problem 1 (Problem of plasticity of fittable surfaces in $\mathbb{R}^3$)**.** *Suppose that F is the corresponding weighted Fermat–Torricelli point of a geodesic triangle $\triangle A_1A_2A_3$ on a C^2 complete surface M with weights w_1, w_2, and w_3. Find a fittable Alexandrov surface M' of the bounded curvature which passes from A_1, A_2, A_3, and F such that F is the corresponding weighted Fermat–Torricelli point of $\triangle A_1A_2A_3$ on M' with weights w'_1, w'_2, and w'_3.*

In this paper, we apply the weighted Fermat-Torricelli problem for geodesic triangles on certain surfaces in the three- dimensional Euclidean space, the inverse weighted Fermat-Torricelli problem, in order to derive the equations which allow us to compute the weights corresponding to the fittable surfaces for three fixed points and a fixed fourth point (weighted Fermat-Torricelli point) located at the interior of the geodesic triangle for the case of a two-dimensional sphere in the three-dimensional Euclidean space which intersects a circular cylinder in three given points and the corresponding weighted Fermat–Torricelli point for a geodesic triangle and a fittable circular cone which passes from the same points that a circular cylinder passes.

2. Plasticity of a Sphere and Circular Cone with Respect to a Circular Cylinder in the Three-Dimensional Euclidean Space

Let $\triangle (A_1A_2A_3)_C$ be a geodesic triangle, for instance, on a circular cylinder $x^2 + y^2 = 1$ for $z_1 \leq z \leq z_2$ and $F_C \equiv A_0$ ia the corresponding weighted Fermat–Torricelli point for given weights w_1, w_2, and w_3.

By $A_i = (\cos\varphi_i, \sin\varphi_i, z_i)$, we denote the points located on the circular cylinder $x^2 + y^2 = 1$, by $(a_{ij})_C$, the length of the geodesic arc A_iA_j, by $\vec{r}_{ij} = (\cos t, \sin t, b_{ij}t)$, the circular helix from A_i to A_j, by $(\alpha_{ijk})_C$, the angle formed by A_iA_j and A_jA_k, by A_{ip}, the projection of A_i to the circle of the cylinder which passes from $A_1 = (1, 0, 0)$, by ω_0, the angle $\angle A_{0p}A_1A_0$, by z_0, the linear segment A_0A_{0p}, and by L_0, the linear segment A_1A_{0p} for $i, j, k = 0, 1, 2, 3$, $i \neq j$ and $j \neq k$.

We set $b_{12} \equiv \frac{z_2}{\varphi_2}$ and $b_{13} =\equiv \frac{z_3}{\varphi_3}$, where $0 < \varphi_i < \pi$, for $i, j = 1, 2, 3$ and $i \neq j$.

We need the following two lemmata proved in [12] and [11] (see also in [12]):

Lemma 1. [12, Theorem 1, p. 173]. *The exact location of the weighted Fermat–Torricelli point $A_0 = A_0(x_0, y_0, z_0)$ of $\triangle(A_1A_2A_3)_C$, composed of three circular helixes on the circular cylinder, is given by the following three equations:*

$$\omega_0 = \arctan b_{12} + \arccos\left(\frac{1+b_{12}b_{13}}{\sqrt{1+b_{12}^2}\sqrt{1+b_{13}^2}}\right)$$
$$-\operatorname{arccot}\left[\left(\sqrt{1-\left(\frac{1+b_{12}b_{13}}{\sqrt{1+b_{12}^2}\sqrt{1+b_{13}^2}}\right)^2}\right.\right.$$
$$-\frac{1+b_{12}b_{13}}{\sqrt{1+b_{12}^2}\sqrt{1+b_{13}^2}}\cot\left(\arccos\frac{w_3^2-w_1^2-w_2^2}{2w_1w_2}\right)$$
$$\left.-\frac{\sqrt{1+b_{13}^2}\varphi_3}{\sqrt{1+b_{12}^2}\varphi_2}\cot\left(\arccos\frac{w_2^2-w_1^2-w_3^2}{2w_1w_3}\right)\right)\Big/$$
$$\left(-\frac{1+b_{12}b_{13}}{\sqrt{1+b_{12}^2}\sqrt{1+b_{13}^2}}-\sqrt{1-\left(\frac{1+b_{12}b_{13}}{\sqrt{1+b_{12}^2}\sqrt{1+b_{13}^2}}\right)^2}\cot\left(\arccos\frac{w_3^2-w_1^2-w_2^2}{2w_1w_2}\right)\right.$$
$$\left.\left.+\frac{\sqrt{1+b_{13}^2}\varphi_3}{\sqrt{1+b_{12}^2}\varphi_2}\right)\right] \tag{2.1}$$

$$z_0 = \frac{\sin\left(\arctan b_{13} - \omega_0 + \arccos\frac{w_2^2-w_1^2-w_3^2}{2w_1w_3}\right)\sqrt{1+b_{13}^2}\varphi_3}{\sin\left(\arccos\frac{w_2^2-w_1^2-w_3^2}{2w_1w_3}\right)}\sin\omega_0 \tag{2.2}$$

and

$$L_0 = \frac{\sin\left(\arctan b_{13} - \omega_0 + \arccos\frac{w_2^2-w_1^2-w_3^2}{2w_1w_3}\right)\sqrt{1+b_{13}^2}\varphi_3}{\sin\left(\arccos\frac{w_2^2-w_1^2-w_3^2}{2w_1w_3}\right)}\cos\omega_0. \tag{2.3}$$

We consider the same points A_1, A_2, A_3, and A_0 on a sphere $S(A_0, R)$ and we denote by $\triangle(A_1A_2A_3)_S$ the geodesic triangle on $S(A_0, R)$, by $(a_{ij})_S$, the length of the geodesic arc A_iA_j, by $(\alpha_{ijk})_S$, the angle formed by A_iA_j and A_jA_k and w_i', the weight which corresponds to A_i and minimizes the objective function $w_1'(a_{01})_S + w_2'(a_{02})_S + w_3'(a_{03})_S$ for $i, j, k = 0, 1, 2, 3$ and $i \neq j \neq k$.

We set

$$c_i \equiv \frac{\sin(\kappa(a_{jk})_S)}{\sin((\alpha_{j0k})_S)}$$

for $i,j,k=1,2,3$ and $i\neq j\neq k$, where

$$\kappa=\begin{cases}\sqrt{K} & \text{if } K=\frac{1}{R^2}>0,\\ i\sqrt{-K} & \text{if } K<0.\end{cases}$$

Lemma 2. [13, Theorem 2.4, p. 115]. *A finite set of solutions of the weighted Fermat–Torricelli problem on the K-plane(two-dimensional sphere, hyperbolic plane), which yields the global minimum point A_0 (weighted Fermat–Torricelli point), is given by the following equation with respect to the variable $z=\sin(\alpha_{013})_S$:*

$$\frac{c_3}{c_2}\left(\pm\sin(\alpha_{123})_S\sqrt{1-\left(\frac{c_2 z}{c_1}\right)^2}-\cos(\alpha_{123})_S\left(\frac{c_2 z}{c_1}\right)\right)$$
$$=-\frac{c_3}{c_1}\sin(\alpha_{213})_S\cos(\alpha_{132})_S\sqrt{1-z^2}+\frac{c_3}{c_1}\cos(\alpha_{213})_S\cos(\alpha_{132})_S z$$
$$\pm(\sin\alpha_{132})_S\sqrt{1-\left(\frac{c_3}{c_1}\right)^2\left[-\sin(2(\alpha_{213})_S)z\sqrt{1-z^2}+\cos(2(\alpha_{213})_S)z^2+\sin^2(\alpha_{213})_S\right]}. \tag{2.4}$$

We recall the inverse weighted Fermat–Torricelli problem on a C^2 surface in $\mathbb{R}^3$ first stated by S. Gueron and R. Tessler in $\mathbb{R}^2$ ([2, 8, 9, 10]):

Problem 2. [2, p. 449], [8, Problem 3.2, p. 61] [9, Problem 2, p. 52], [10]. *Given is a point $A_0\in\triangle A_1A_2A_3$ on a C^2 surface in $\mathbb{R}^3$. Does there exist a unique set of positive weights w_i, normalized by $w_1+w_2+w_3=1$, for which A_0 minimizes*

$$w_1(a_{01})_g+w_2(a_{02})_g+w_3(a_{03})_g,$$

where $(a_{0i})_g$ is the length of the geodesic arc A_0A_i?

Lemma 3. [2], [8, Proposition 3.2, Corollary 3.3, p. 61] [9, Proposition 5, p. 52], [10]. *The solution of the inverse weighted Fermat–Torricelli problem on a C^2 surface in $\mathbb{R}^3$ is given by*

$$w_i=\frac{1}{1+\frac{\sin\alpha_{i0j}}{\sin\alpha_{j0k}}+\frac{\sin\alpha_{i0k}}{\sin\alpha_{j0k}}} \tag{2.5}$$

for $i,j,k=1,2,3$ and $i\neq j\neq k$.

We assume that $A_i=(x_i,y_i,z_i)$ and $F_C=F_S\equiv A_0=(x_F,y_F,z_F)$ are located at the intersection of the circular cylinder C and the sphere $S(x_0,y_0,z_0;R)$ for $i=1,2,3$.

Theorem 1. *The following equations provide the plasticity of a sphere derived by a circular cylinder with respect to the fixed points* $\{A_1A_2A_3A_0\}$ *for a different triad of weights* w_1, w_2, w_3, *and* w_1', w_2', w_3' *such that* $F_C = F_S \equiv A_0$:

$$w_i = \frac{1}{1 + \frac{\sin(\alpha_{i0j})_S}{\sin(\alpha_{j0k})_S} + \frac{\sin(\alpha_{i0k})_S}{\sin(\alpha_{j0k})_S}}\,, \tag{2.6}$$

for $i, j, k = 1, 2, 3$ *and* $i \neq j \neq k$, *the angles* $(\alpha_{i0j})_S$ *are determined by the equations*

$$x_0 = -\frac{-d_3g_2h_1 + d_2g_3h_1 + d_3g_1h_2 - d_1g_3, h_2 - d_2g_1h_3 + d_1g_2h_3}{f_3g_2h_1 - f_2g_3h_1 - f_3, g_1h_2 + f_1g_3h_2 + f_2g_1h_3 - f_1g_2h_3}\,, \tag{2.7}$$

$$y_0 = -\frac{d_3f_2h_1 - d_2f_3h_1 - d_3f_1h_2 + d_1f_3h_2 + d_2f_1h_3 - d_1f_2h_3}{f_3g_2h_1 - f_2g_3h_1 - f_3g_1h_2 + f_1g_3h_2 + f_2g_1h_3 - f_1g_2h_3}\,, \tag{2.8}$$

$$z_0 = \frac{d_3f_2g_1 - d_2f_3g_1 - d_3f_1g_2 + d_1f_3g_2 + d_2f_1g_3 - d_1f_2g_3}{f_3g_2h_1 - f_2g_3h_1 - f_3g_1h_2 + f_1g_3h_2 + f_2g_1h_3 - f_1g_2h_3}\,, \tag{2.9}$$

$$w_1' + w_2' + w_3' = 1, \tag{2.10}$$

where

$$f_1 = (x_i - x_F), \tag{2.11}$$

$$g_1 = (y_i - y_F), \tag{2.12}$$

$$h_1 = (z_i - z_F), \tag{2.13}$$

and

$$d_i = 0.5[(x_i - x_F)(x_i + x_F) + (y_i - y_F)(y_i + y_F) + (z_i - z_F)(z_i + z_F)]. \tag{2.14}$$

P r o o f. Let $\triangle(A_1A_2A_3)_C$ be a geodesic triangle which is composed of three circular helixes on a circular cylinder $x^2 + y^2 = 1$ for $z_1 \leq z \leq z_2$, and F_C be the corresponding weighted Fermat–Torricelli point. By unrolling the cylinder, we get an isometric mapping of $\triangle A_1A_2A_3$ to the Euclidean plane $\mathbb{R}^2$. From Lemma 1, we derive the exact location of $F_C = (x_F, y_F, z_F)$.

We construct a sphere $S(A_0(x_0, y_0, z_0), R)$ which passes from $A_1 = (x_1, y_1, z_1)$, $A_2 = (x_2, y_2, z_2)$, $A_3 = (x_3, y_3, z_3)$ and $F = (x_F, y_F, z_F)$.

The bisectors of the linear segments A_iF pass from $M_i = (\frac{x_i + x_F}{2}, \frac{y_i + y_F}{2}, \frac{z_i + z_F}{2})$ and intersect at $A_0 = (x_0, y_0, z_0)$, such that $\|A_iA_0\| = R$, for $i = 1, 2, 3$ (Fig. 1).

Thus, we get

$$\begin{aligned}&(x_i - x_F)x + (y_i - y_F)y + (z_i - z_F)z \\ &= 0.5[(x_i - x_F)(x_i + x_F) + (y_i - y_F)(y_i + y_F) + (z_i - z_F)(z_i + z_F)]\end{aligned} \tag{2.15}$$

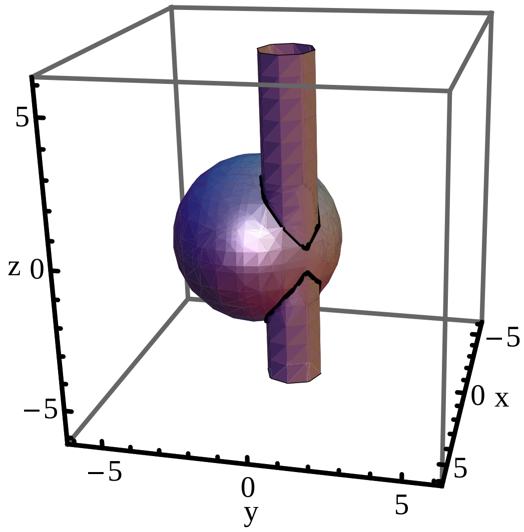

Fig. 1.

for $i = 1, 2, 3$. The intersection of the three planes (2.15) gives (2.7), (2.8), and (2.9).

Thus, we get

$$R = \frac{1}{\sqrt{K}} = \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2 + (z_i - z_0)^2}, \tag{2.16}$$

$$\cos\theta_{iF} = 1 - \frac{1}{2}\left(\frac{(x_i - x_F)^2 + (y_i - y_F)^2 + (z_i - z_F)^2}{R}\right)^2, \tag{2.17}$$

$$(a_{iF})_S = R\theta_{iF}, \tag{2.18}$$

and

$$(a_{ij})_S = R\theta_{ij} \tag{2.19}$$

for $i, j = 1, 2, 3$, $i \neq j$.

Therefore, the angles $(\alpha_{i0j})_S$ are determined by the spherical cosine law in $\triangle A_i A_j A_k$:

$$(\alpha_{i0j})_S = \arccos \frac{\cos\kappa(a_{ij})_S - \cos\kappa(a_{0i})_S \cos\kappa(a_{0j})_S}{\sin\kappa(a_{0i})_S \sin\kappa(a_{0j})_S}. \tag{2.20}$$

Then, by applying Lemma 3, we obtain (2.6). ∎

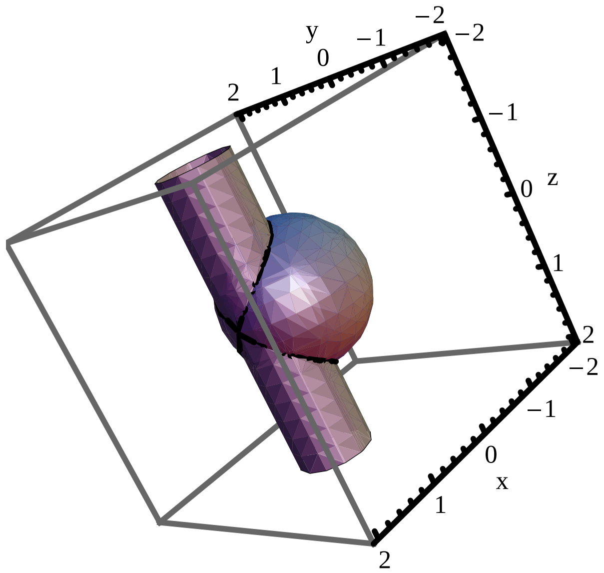

Fig. 2.

R e m a r k 1. We note that there is a particular case where A_1, A_2, A_3, and F can be located on a circular cylinder $x^2 + y^2 = Rx$ with the radius $\frac{R}{2}$ and on a sphere $x^2 + y^2 + z^2 = R^2$ with the radius R. The intersection of this circular cylinder and the sphere is called a Viviani curve with one point of self-intersection (Fig.2, [6, Example 1.2.4 (a), p. 5]).

We consider the intersection of a circular cylinder $C : x^2 + y^2 = 1$ for $z_1 \leq Z \leq z_2$ and a circular cone $Co : (x - x_0)^2 + (z - z_0)^2 = \left(\frac{r_1}{H}\right)^2 (z - H)^2$. By H, we denote the height of the circular cylinder and by r_1, the radius of the circle which corresponds to the basis of the circular cone.

Theorem 2. *The following equations provide the plasticity of a circular cone derived by a circular cylinder with respect to the fixed points* $\{A_1A_2A_3A_0\}$ *for a different triad of weights* w_1, w_2, w_3, *and* w'_1, w'_2, w'_3 *such that* $F_{Co} = F_C \equiv A_0$:

$$w_i = \frac{1}{1 + \frac{\sin(\alpha_{i0j})_{Co}}{\sin(\alpha_{j0k})_{Co}} + \frac{\sin(\alpha_{i0k})_{Co}}{\sin(\alpha_{j0k})_{Co}}}, \tag{2.21}$$

and the angles $(\alpha_{i0j})_S$ *are determined by the equations*

$$(x_1 - x_0)^2 + (z_1 - z_0)^2 = \left(\frac{r_1}{H}\right)^2 (z_1 - H)^2, \tag{2.22}$$

$$(x_2 - x_0)^2 + (z_2 - z_0)^2 = \left(\frac{r_1}{H}\right)^2 (z_2 - H)^2, \qquad (2.23)$$

$$(x_3 - x_0)^2 + (z_3 - z_0)^2 = \left(\frac{r_1}{H}\right)^2 (z_3 - H)^2, \qquad (2.24)$$

$$(x_F - x_0)^2 + (z_F - z_0)^2 = \left(\frac{r_1}{H}\right)^2 (z_F - H)^2, \qquad (2.25)$$

where

$$w'_1 + w'_2 + w'_3 = 1. \qquad (2.26)$$

P r o o f. By considering a fittable circular cone $Co : (x - x_0)^2 + (z - z_0)^2 = \left(\frac{r_1}{H}\right)^2 (z - H)^2$, which passes from the points A_1, A_2, A_3, and $F_{Co} \equiv F_C = A_0$, we get the system of equations (2.22), (2.23), (2.24), and (2.25) with respect to the four variables x_0, y_0, r_1, and $z_0 = H$, which can give numerically the vertex A of the circular cone.

Then, by unrolling the circular cone Co along A_1A, we derive an isometric mapping from Co to $\mathbb{R}^2$, which determines the angles $(\alpha_{ijk})_0 = (\alpha_{ijk})_{Co}$, and obtain (2.21). ■

Taking into account that A is the vertex of the circular cone, r_1 is the radius of the circle $c(P, r_1)$ at the basis of the cone, H is the height of the cone, we denote by φ_0 the angle $\angle A_1PA_{0p}$, where A_{0p} is the point of intersection of AA_0 and $c(P, r_1)$, and by x_{00}, the length of the linear segment A_0A, and we consider the lemma proved in ([12]).

Lemma 4. [12, Theorem 2, p. 177–178]. *The exact location of the weighted Fermat–Torricelli point F_{Co} of $\triangle A_1A_2A_3$ on Co is given by the following two equations:*

$$x_{00} = \sqrt{(1 + H^2) + (a_{01})_g^2 - 2\sqrt{1 + H^2}(a_{01})_g \cos(\alpha_{013} + \angle A_3A_1A)}, \qquad (2.27)$$

where

$$\alpha_{013} = \operatorname{arccot}\left(\frac{\sin(\alpha_{213}) - \cos(\alpha_{213}) \cot(\alpha_{102})_{Co} - \frac{(a_{13})_0}{(a_{12})_0} \cot(\alpha_{103})_{Co}}{-\cos(\alpha_{213}) - \sin(\alpha_{213}) \cot(\alpha_{102})_{Co} + \frac{(a_{13})_0}{(a_{12})_0}}\right),$$

$$(a_{10})_0 = \frac{\sin(\alpha_{013} + (\alpha_{103})_{Co})\,(a_{13})_0}{\sin(\alpha_{103})_{Co}},$$

$$(\alpha_{i0j})_{Co} = \arccos\left(\frac{(w'_k)^2 - (w'_i)^2 - (w'_j)^2}{2(w'_i)(w'_j)}\right),$$

for $i, j, k = 1, 2, 3$, $i \neq j \neq k$, and

$$\varphi_0 = \frac{\sqrt{1 + H^2}}{r_1} \angle A_1AA_0(x_{00}, (a_{10})_g). \qquad (2.28)$$

R e m a r k 2. For given x_{00}, α_{013} and $(a_{10})_0$, the system of two nonlinear equations (2.27) and (2.28) gives numerically w_1', w_2', taking into account that $w_3' = 1 - w_1' - w_2'$.

E x a m p l e 1. Given are $A_1 = (\cos 0, \sin 0, 0)$, $A_2 = (\cos \frac{\pi}{3}, \sin \frac{\pi}{3}, 0.8)$, $A_3 = (\cos \frac{\pi}{6}, \sin \frac{\pi}{6}, 2)$, on the circular cylinder $x^2 + y^2 = 1$.

The isometric mapping of the circular cylinder to $\mathbb{R}^2$ induced by (φ, z) yields the points $A_1' = (0,0)$, $A_2' = (\frac{\pi}{3}, 0.8)$, $A_3' = (\frac{\pi}{6}, 2)$. Thus, the corresponding Fermat–Torricelli point of $\triangle A_1'A_2'A_3'$ $F' = (0.8404027, 0.8536775)$ gives $F_C = (0.667163, 0.744912, 0.8536775)$ for $w_1 = w_2 = w_3 = \frac{1}{3}$.

Given A_1, A_2, A_3, F_C, we calculate the center of the fittable sphere x_0, y_0, z_0, R from the equations (2.7)–(2.14): $x_0 = -1.31848$, $y_0 = -1.60442$, $z_0 = 1.31278$, $R = 3.11013$.

From (2.17), (2.18), (2.19), (2.20), and (2.6), we derive that

$$(\alpha_{102})_S = 1.99478 \; rad,$$

$$(\alpha_{203})_S = 2.10237 \; rad,$$

$$(\alpha_{103})_S = 2.18604 \; rad,$$

which give

$$w_1' = 0.33281,$$

$$w_2' = 0.315291$$

and

$$w_3' = 0.3519$$

such that

$$w_1' + w_2' + w_3' = 1.$$

As a future work, we consider the following problem that may provide some perspectives on the plasticity of geodesic triangles on some C^2 complete surfaces in $\mathbb{R}^3$:

Problem 3. *Suppose that F is the corresponding Fermat–Torricelli point of a geodesic triangle $\triangle A_1A_2A_3$ on a C^2 complete surface M with positive weights w_i such that*

$$w_1 + w_2 + w_3 = 1.$$

Find a fittable Alexandrov surface M' of a bounded curvature which passes from A_1, A_2 A_3 and F such that F is the corresponding Fermat–Torricelli point of $\triangle A_1A_2A_3$ on M', with positive weights w_i', satisfying the equations

$$w_1' + w_2' + w_3' = 1$$

and $w_1 = w_1'$ or $w_1 = w_1'$ and $w_2 = w_2'$.

R e m a r k 3. If the points A_1, A_2, A_3 are not fixed and belong to the surface of a circular cylinder, then there is an isometric mapping which is deduced by unrolling the circular cylinder by the line (generator of cylinder) which passes from the weighted Fermat–Torricelli point A_0 of $\triangle(\triangle A_1A_2A_3)_C$ and the corresponding weighted Fermat–Torricelli point of $\triangle(\triangle A_1A_2A_3)_P$ on the Euclidean plane coincides with A_0, which yields $w_1 = w_1'$ and $w_2 = w_2'$.

We are interested in the derivation of non-isometric mappings of fittable surfaces which solve Problem 3, which could also lead to a new way of creating two-dimensional fittable hyperbolic spaces (Plasticity of hyperbolic spaces).

Finally, we note that Problems 1 and 3 may provide an alternative characterization of a Wald curvature ([7]) by placing geometric properties of the weighted Fermat–Torricelli problem for geodesic triangles into Wald's nonlinear quad.

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