

Algebro-Geometric Solutions to a New Hierarchy of Soliton Equations

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With the help of the Lenard recursion equations, we derive a new hierarchy of soliton equations associated with a 3×3 matrix spectral problem and establish Dubrovin type equations in terms of the introduced trigonal curve \mathcal{K}_{m-1} of arithmetic genus $m - 1$. Basing on the theory of algebraic curve, we construct the corresponding Baker–Akhiezer functions and meromorphic functions on \mathcal{K}_{m-1} . The known zeros and poles for the Baker–Akhiezer function and meromorphic functions allow us to find their theta function representations, from which algebro-geometric constructions and theta function representations of the entire hierarchy of soliton equations are obtained.

Key words: trigonal curve; Baker–Akhiezer function; algebro-geometric solutions.

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1. Introduction

The study of explicit solutions for various soliton equations has been very important in modern mathematics with ramifications to several areas of mathematics, physics and other sciences. As we all know, several systematic approaches have been developed to obtain explicit solutions of the soliton equations such as the inverse scattering transformation, the algebro-geometric method, the bilinear transformation methods of Hirota, the Bäcklund and Darboux transformations, and so on (see, e.g., [1–4] and references therein). Some interesting explicit solutions have been found, including N -soliton solutions, peaked soliton solutions, algebro-geometric (or finite-band, or quasi-periodic) solutions and others.

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Algebro-geometric solutions of soliton equations reveal inherent structure mechanism of solutions and describe the quasi-periodic behavior of nonlinear phenomenon or characteristic for the integrability of soliton equation [2, 5]. In recent years, based on the theory of the hyperelliptic curves, algebro-geometric solutions of many soliton equations associated with the 2×2 matrix spectral problems have been derived such as the KdV, nonlinear Schrödinger, mKdV, sine-Gordon, Camassa–Holm, Toda lattice, and Ablowitz–Ladik equations, etc. [2, 5–15]. However, there are very few studies on algebro-geometric solutions of soliton equations associated with the 3×3 matrix spectral problems, which are more difficult and complicated because the theory of non-hyperelliptic and trigonal curves [16–20] is involved. In [7, 8] and [21–27], certain algebro-geometric solutions of the Boussinesq equation related to a third-order differential operator were found as special solutions of the Kadomtsev–Peviashvili equation or by the reduction theory of Riemann theta functions. In [28], a unified framework was proposed which yields all algebro-geometric solutions of the entire Boussinesq hierarchy related to the third-order spectral problem. In [29, 30], we give a systematic method to define the trigonal curve by the characteristic polynomial of Lax matrix and develop the framework to deal with soliton equations associated with the 3×3 matrix spectral problems, from which the algebro-geometric solutions of some entire hierarchies are obtained, namely, the modified Boussinesq hierarchy and the Kaup–Kupershmidt hierarchy.

The principle aim of the present paper is to construct algebro-geometric solutions of a new hierarchy of soliton equations associated with a 3×3 matrix spectral problem on the basis of the approaches used in [28–30] but different from spectral problems considered in [28–30]. An obvious difference is that there is a potential on the main diagonal, which is a big trouble for us to construct proper meromorphic functions. However, we constructed the explicit theta function representations of the corresponding Baker–Akhiezer function, the meromorphic function, and the solutions of the associated nonlinear evolution equations.

The outline of the present paper is as follows. In Sec. 2, with the aid of the Lenard recursion equations we construct a new hierarchy of soliton equations associated with a 3×3 matrix spectral problem. In Sec. 3, a trigonal curve \mathcal{K}_{m-1} of arithmetic genus $m - 1$ is introduced in terms of characteristic polynomial of the Lax matrix, from which the stationary Baker–Akhiezer function and the associated meromorphic function are given on \mathcal{K}_{m-1} . In Sec. 4, we present the explicit theta function representations of the stationary Baker–Akhiezer function, the meromorphic function and, in particular, that of the potentials $u = (u_{12}, u_{13}, u_{23}, u_{31}, u_{32}, v)^T$ for the entire hierarchy of soliton equations. Sec. 5 then extends the analysis of Secs. 3 and 4 to the time-dependent case.

2. A New Hierarchy of Soliton Equations

In this section, we will derive a new hierarchy of soliton equations associated with a 3×3 matrix spectral problem

$$\psi_x = U(u, \lambda)\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad U = \begin{pmatrix} \lambda - v & u_{12} & u_{13} \\ 1 & -\lambda + v & u_{23} \\ u_{31} & u_{32} & 0 \end{pmatrix}, \quad (2.1)$$

where the potential $u = (u_{12}, u_{13}, u_{23}, u_{31}, u_{32}, v)^T$, λ is a constant spectral parameter. To this end, we introduce two sets of Lenard recursion equations

$$Kg_j = Jg_{j+1}, \quad g_{j+1}|_{u=0} = 0, \quad j \geq 0, \quad (2.2)$$

$$K\hat{g}_j = J\hat{g}_{j+1}, \quad \hat{g}_{j+1}|_{u=0} = 0, \quad j \geq 0 \quad (2.3)$$

with two starting points

$$g_0 = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0)^T, \quad \hat{g}_0 = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1)^T, \quad (2.4)$$

where the initial conditions are chosen so that the constants of integration are zeros, and two operators are defined by

$$K = \begin{pmatrix} \partial + 2v & u_{32} & u_{12}u_{31} & -u_{12}u_{23} & -u_{13} & \partial u_{12} + u_{12}\partial & 0 \\ u_{23} & \partial + v & \frac{1}{2}u_{13}u_{31} - u_{12} & -\frac{1}{2}u_{13}u_{23} & 0 & \partial u_{13} + \frac{1}{2}u_{13}\partial & -\frac{3}{2}u_{13} \\ 0 & -1 & \partial - v - \frac{1}{2}u_{23}u_{31} & -\frac{1}{2}u_{23}^2 & 0 & \partial u_{23} - \frac{1}{2}u_{23}\partial & -\frac{3}{2}u_{23} \\ 0 & 0 & -\frac{1}{2}u_{31}^2 & \partial - v + \frac{1}{2}u_{23}u_{31} & 1 & \partial u_{31} - \frac{1}{2}u_{31}\partial & \frac{3}{2}u_{31} \\ -u_{31} & 0 & \frac{1}{2}u_{31}u_{32} & u_{12} - \frac{1}{2}u_{23}u_{32} & \partial + v & \partial u_{32} + \frac{1}{2}u_{32}\partial & \frac{3}{2}u_{32} \\ -1 & 0 & u_{32} - \frac{1}{2}\partial u_{31} & \frac{1}{2}\partial u_{23} & -u_{23} & -\frac{1}{2}(\partial^2 - 2\partial v) & -\frac{1}{2}\partial \\ 0 & -u_{31} & -u_{32} & u_{13} & u_{23} & 0 & \partial \end{pmatrix},$$

$$J = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & 0 \\ 0 & -u_{31} & -u_{32} & u_{13} & u_{23} & 0 & \partial \end{pmatrix}.$$

Hence g_j and \hat{g}_j are uniquely determined, for example, the first several members read as

$$g_1 = \left(\frac{1}{2}u_{12,x}, \quad u_{13,x}, \quad -u_{23,x}, \quad -u_{31,x}, \quad u_{32,x}, \quad v, \quad u_{13}u_{31} - u_{23}u_{32} \right)^T,$$

$$\hat{g}_1 = (0, -\frac{3}{2}u_{13}, \frac{3}{2}u_{23}, -\frac{3}{2}u_{31}, \frac{3}{2}u_{32}, 0, 0)^T,$$

$$g_2 =$$

$$\begin{pmatrix} \frac{1}{4}u_{12,xx} + (vu_{12})_x + \frac{1}{2}(u_{13,x}u_{32} - u_{13}u_{32,x}) - \frac{1}{2}u_{12}(u_{23,x}u_{31} - u_{23}u_{31,x}) \\ u_{13,xx} + \frac{1}{2}u_{12,x}u_{23} + u_{12}u_{23,x} + 2vu_{13,x} + \frac{3}{2}u_{13}v_x + \frac{1}{2}u_{13}(u_{23}u_{31,x} - u_{23,x}u_{31} - 3u_{13}u_{31} + 3u_{23}u_{32}) \\ u_{23,xx} + u_{13,x} - \frac{1}{2}u_{23}v_x - 2vu_{23,x} - \frac{1}{2}u_{23}(u_{23,x}u_{31} - u_{23}u_{31,x} - 3u_{13}u_{31} + 3u_{23}u_{32}) \\ u_{31,xx} - u_{32,x} - \frac{1}{2}u_{31}v_x - 2vu_{31,x} - \frac{1}{2}u_{31}(u_{23,x}u_{31} - u_{23}u_{31,x} + 3u_{13}u_{31} - 3u_{23}u_{32}) \\ u_{32,xx} - \frac{1}{2}u_{12,x}u_{31} - u_{12}u_{31,x} + 2vu_{32,x} + \frac{3}{2}v_xu_{32} - \frac{1}{2}u_{32}(u_{23,x}u_{31} - u_{23}u_{31,x} - 3u_{13}u_{31} + 3u_{23}u_{32}) \\ -\frac{1}{2}(u_{12} + v_x - 2v^2 + u_{13}u_{31} + u_{23}u_{32} + u_{23}u_{31,x} - u_{23,x}u_{31}) \\ u_{13,x}u_{31} - u_{13}u_{31,x} + u_{23,x}u_{32} - u_{23}u_{32,x} + 2v(u_{13}u_{31} - u_{23}u_{32}) + u_{13}u_{32} + u_{12}u_{23}u_{31} \end{pmatrix},$$

$$\hat{g}_2 = \begin{pmatrix} \frac{3}{2}(u_{12}u_{23}u_{31} - u_{13}u_{32}) \\ -\frac{3}{2}(u_{13,x} + vu_{13} + u_{12}u_{23} - u_{13}u_{23}u_{31}) \\ -\frac{3}{2}(u_{23,x} - vu_{23} + u_{13} - u_{23}^2u_{31}) \\ \frac{3}{2}(u_{31,x} - vu_{31} - u_{32} + u_{23}u_{31}^2) \\ \frac{3}{2}(u_{32,x} + vu_{32} - u_{12}u_{31} + u_{23}u_{31}u_{32}) \\ -\frac{3}{2}u_{23}u_{31} \\ -\frac{3}{2}(u_{13}u_{31} + u_{23}u_{32}) \end{pmatrix}.$$

It is easy to see that

$$\ker J = \{\alpha_0 g_0 + \beta_0 \hat{g}_0 \mid \forall \alpha_0, \beta_0 \in \mathbb{R}\}.$$

In order to generate a hierarchy of evolution equations associated with spectral problem (2.1), we have to solve the stationary zero-curvature equation

$$V_x - [U, V] = 0, \quad V = (V_{ij})_{3 \times 3}. \tag{2.5}$$

Equation (2.5) can be written in the form

$$\begin{aligned} V_{11,x} + V_{12} + u_{31}V_{13} - u_{12}V_{21} - u_{13}V_{31} &= 0, \\ V_{22,x} - V_{12} + u_{12}V_{21} + u_{32}V_{23} - u_{23}V_{32} &= 0, \\ V_{33,x} - u_{31}V_{13} - u_{32}V_{23} + u_{13}V_{31} + u_{23}V_{32} &= 0, \\ V_{12,x} - 2\lambda V_{12} + 2vV_{12} + u_{12}(V_{11} - V_{22}) + u_{32}V_{13} - u_{13}V_{32} &= 0, \\ V_{13,x} - \lambda V_{13} + vV_{13} + u_{13}(V_{11} - V_{33}) + u_{23}V_{12} - u_{12}V_{23} &= 0, \\ V_{21,x} + 2\lambda V_{21} - 2vV_{21} + V_{22} - V_{11} + u_{31}V_{23} - u_{23}V_{31} &= 0, \\ V_{23,x} + \lambda V_{23} - vV_{23} + u_{23}(V_{22} - V_{33}) - V_{13} + u_{13}V_{21} &= 0, \\ V_{31,x} + \lambda V_{31} - vV_{31} + u_{31}(V_{33} - V_{11}) - u_{32}V_{21} + V_{32} &= 0, \\ V_{32,x} - \lambda V_{32} + vV_{32} + u_{32}(V_{33} - V_{22}) - u_{31}V_{12} + u_{12}V_{31} &= 0, \end{aligned} \tag{2.6}$$

where each entry $V_{ij} = V_{ij}(a, b, c, d, e, f, h)$ is a Laurent expansion in λ :

$$\begin{aligned} V_{11} &= \lambda f + \frac{1}{2}((\partial - 2v)f + u_{31}c - u_{23}d - h), & V_{12} &= a + u_{12}f, & V_{13} &= b + u_{13}f, \\ V_{21} &= f, & V_{22} &= -\lambda f - \frac{1}{2}((\partial - 2v)f + u_{31}c - u_{23}d + h), & V_{23} &= c + u_{23}f, \\ V_{31} &= d + u_{31}f, & V_{32} &= e + u_{32}f, & V_{33} &= h, \end{aligned} \tag{2.7}$$

with

$$(a, b, c, d, e, f, h) = \sum_{j \geq 0} (a_j, b_j, c_j, d_j, e_j, f_j, h_j) \lambda^{-j}. \quad (2.8)$$

A direct calculation shows that (2.6) and (2.7) imply the Lenard equation

$$KG = \lambda JG, \quad G = (a, b, c, d, e, f, h)^T. \quad (2.9)$$

Substituting (2.7), (2.8) into (2.6) and collecting the terms with the same powers of λ , we arrive at the recursion relation

$$KG_j = JG_{j+1}, \quad JG_0 = 0, \quad (2.10)$$

where $G_j = (a_j, b_j, c_j, d_j, e_j, f_j, h_j)^T$. It is easy to see that g_j, \hat{g}_j determined by (2.2), (2.3) and G_j satisfy (2.10). Then G_j can be expressed as

$$G_j = \alpha_0 g_j + \beta_0 \hat{g}_j + \dots + \alpha_j g_0 + \beta_j \hat{g}_0, \quad j \geq 0, \quad (2.11)$$

where α_j and β_j are arbitrary constants.

Let ψ satisfy the spectral problem (2.1) and the auxiliary problem

$$\psi_{t_r} = \tilde{V}^{(r)} \psi, \quad \tilde{V}^{(r)} = (\tilde{V}_{ij}^{(r)})_{3 \times 3}, \quad (2.12)$$

where each entry $\tilde{V}_{ij}^{(r)} = \tilde{V}_{ij}(\tilde{a}^{(r)}, \tilde{b}^{(r)}, \tilde{c}^{(r)}, \tilde{d}^{(r)}, \tilde{e}^{(r)}, \tilde{f}^{(r)}, \tilde{h}^{(r)})$,

$$(\tilde{a}^{(r)}, \tilde{b}^{(r)}, \tilde{c}^{(r)}, \tilde{d}^{(r)}, \tilde{e}^{(r)}, \tilde{f}^{(r)}, \tilde{h}^{(r)}) = \sum_{j=0}^r (\tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j, \tilde{e}_j, \tilde{f}_j, \tilde{h}_j) \lambda^{(r-j)}, \quad (2.13)$$

and $\tilde{G}_j = (\tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j, \tilde{e}_j, \tilde{f}_j, \tilde{h}_j)^T$ is determined by

$$\tilde{G}_j = \tilde{\alpha}_0 g_j + \tilde{\beta}_0 \hat{g}_j + \dots + \tilde{\alpha}_j g_0 + \tilde{\beta}_j \hat{g}_0, \quad j \geq 0.$$

Then the compatibility condition of (2.1) and (2.12) yields the zero-curvature equation $U_{t_r} - \tilde{V}_x^{(r)} + [U, \tilde{V}^{(r)}] = 0$, which is equivalent to the hierarchy of nonlinear evolution equations

$$u_{t_r} = X_r, \quad r \geq 0, \quad (2.14)$$

where the vector fields $X_j = X(u; \underline{\tilde{\alpha}}^{(j)}, \underline{\tilde{\beta}}^{(j)}) = P(K\tilde{G}_j) = P(J\tilde{G}_{j+1})$, P is the projective map $\chi = (\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \chi^{(4)}, \chi^{(5)}, \chi^{(6)}, \chi^{(7)})^T \rightarrow (\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \chi^{(4)}, \chi^{(5)}, \chi^{(6)})^T$, and $\underline{\tilde{\alpha}}^{(j)} = (\tilde{\alpha}_0, \dots, \tilde{\alpha}_j)$, $\underline{\tilde{\beta}}^{(j)} = (\tilde{\beta}_0, \dots, \tilde{\beta}_j)$, which are constant vectors different from $\underline{\alpha}^{(j)} = (\alpha_0, \dots, \alpha_j)$, $\underline{\beta}^{(j)} = (\beta_0, \dots, \beta_j)$.

The first nontrivial equation in the above hierarchy under condition (i): $\tilde{\beta}_0 = \tilde{\beta}_1 = 0$ is that

$$\begin{aligned}
 u_{12,t} &= \tilde{\alpha}_0 \left(\frac{1}{2} u_{12,xx} + 2(vu_{12})_x + u_{13,x}u_{32} - u_{13}u_{32,x} - u_{12}(u_{23,x}u_{31} - u_{23}u_{31,x}) \right) \\
 &\quad + \tilde{\alpha}_1 u_{12,x}, \\
 u_{13,t} &= \tilde{\alpha}_0 (u_{13,xx} + \frac{1}{2} u_{12,x}u_{23} + u_{12}u_{23,x} + 2vu_{13,x} + \frac{3}{2} u_{13}v_x + \frac{1}{2} u_{13}(u_{23}u_{31,x} \\
 &\quad - u_{23,x}u_{31} - 3u_{13}u_{31} + 3u_{23}u_{32})) + \tilde{\alpha}_1 u_{13,x}, \\
 u_{23,t} &= -\tilde{\alpha}_0 (u_{23,xx} + u_{13,x} - \frac{1}{2} u_{23}v_x - 2vu_{23,x} - \frac{1}{2} u_{23}(u_{23,x}u_{31} - u_{23}u_{31,x} \\
 &\quad - 3u_{13}u_{31} + 3u_{23}u_{32})) + \tilde{\alpha}_1 u_{23,x}, \\
 u_{31,t} &= -\tilde{\alpha}_0 (u_{31,xx} - u_{32,x} - \frac{1}{2} u_{31}v_x - 2vu_{31,x} - \frac{1}{2} u_{31}(u_{23,x}u_{31} - u_{23}u_{31,x} \\
 &\quad + 3u_{13}u_{31} - 3u_{23}u_{32})) + \tilde{\alpha}_1 u_{31,x}, \\
 u_{32,t} &= \tilde{\alpha}_0 (u_{32,xx} - \frac{1}{2} u_{12,x}u_{31} - u_{12}u_{31,x} + 2vu_{32,x} + \frac{3}{2} v_x u_{32} - \frac{1}{2} u_{32}(u_{23,x}u_{31} \\
 &\quad - u_{23}u_{31,x} - 3u_{13}u_{31} + 3u_{23}u_{32})) + \tilde{\alpha}_1 u_{32,x}, \\
 v_t &= -\frac{1}{2} \tilde{\alpha}_0 (u_{12} + v_x - 2v^2 + u_{13}u_{31} + u_{23}u_{32} + u_{23}u_{31,x} - u_{23,x}u_{31})_x + \tilde{\alpha}_1 v_x,
 \end{aligned} \tag{2.15}$$

and under condition (ii): $\tilde{\alpha}_0 = 0$ is that

$$\begin{aligned}
 u_{12,t} &= \tilde{\alpha}_1 u_{12,x} + 3\tilde{\beta}_0 (u_{12}u_{23}u_{31} - u_{13}u_{32}), \\
 u_{13,t} &= (\tilde{\alpha}_1 - \frac{3}{2} \tilde{\beta}_0) u_{13,x} - \frac{3}{2} \tilde{\beta}_0 (vu_{13} + u_{12}u_{23} - u_{13}u_{23}u_{31}) - \frac{3}{2} \tilde{\beta}_1 u_{13}, \\
 u_{23,t} &= (\tilde{\alpha}_1 + \frac{3}{2} \tilde{\beta}_0) u_{23,x} - \frac{3}{2} \tilde{\beta}_0 (vu_{23} - u_{13} + u_{23}^2 u_{31}) - \frac{3}{2} \tilde{\beta}_1 u_{23}, \\
 u_{31,t} &= (\tilde{\alpha}_1 - \frac{3}{2} \tilde{\beta}_0) u_{31,x} + \frac{3}{2} \tilde{\beta}_0 (vu_{31} + u_{32} - u_{23}u_{31}^2) + \frac{3}{2} \tilde{\beta}_1 u_{31}, \\
 u_{32,t} &= (\tilde{\alpha}_1 + \frac{3}{2} \tilde{\beta}_0) u_{32,x} + \frac{3}{2} \tilde{\beta}_0 (vu_{32} - u_{12}u_{31} + u_{23}u_{31}u_{32}) + \frac{3}{2} \tilde{\beta}_1 u_{32}, \\
 v_t &= \tilde{\alpha}_1 v_x - \frac{3}{2} \tilde{\beta}_0 (u_{23}u_{31})_x.
 \end{aligned} \tag{2.16}$$

3. The Stationary Baker–Akhiezer Function

In this section, we introduce the stationary Baker–Akhiezer function and the associated meromorphic function. Then we derive the system of Dubrovin-type differential equations. Let us consider the stationary equations $X_n = 0$, $n \geq 0$, which are equivalent to the stationary zero-curvature equations $V_x^{(n)} = [U, V^{(n)}]$, $V^{(n)} = (\lambda^n V)_+ = (V_{ij}^{(n)})_{3 \times 3}$, $V_{ij}^{(n)} = \sum_{k=0}^n V_{ij,k} \lambda^{n-k}$ where $V_{ij,k}$ are determined by (2.7) and (2.11). A direct calculation shows that the matrix $yI - V^{(n)}$ also satisfies the stationary zero-curvature equation. Then the characteristic polynomials of the Lax matrix $V^{(n)}$, $\mathcal{F}_m(\lambda, y) = \det(yI - V^{(n)})$, are independent constants of x and have the expansion

$$\det(yI - V^{(n)}) = y^3 + yS_m(\lambda) - T_m(\lambda), \tag{3.1}$$

where $S_m(\lambda)$ and $T_m(\lambda)$ are the polynomials of λ with the constant coefficients:

$$\begin{aligned}
 S_m(\lambda) &= \begin{vmatrix} V_{11}^{(n)} & V_{12}^{(n)} \\ V_{21}^{(n)} & V_{22}^{(n)} \end{vmatrix} + \begin{vmatrix} V_{11}^{(n)} & V_{13}^{(n)} \\ V_{31}^{(n)} & V_{33}^{(n)} \end{vmatrix} + \begin{vmatrix} V_{22}^{(n)} & V_{23}^{(n)} \\ V_{32}^{(n)} & V_{33}^{(n)} \end{vmatrix} \\
 &= -\alpha_0^2 \lambda^{2n+2} - 2\alpha_0 \alpha_1 \lambda^{2n+1} + \dots, \\
 T_m(\lambda) &= \begin{vmatrix} V_{11}^{(n)} & V_{12}^{(n)} & V_{13}^{(n)} \\ V_{21}^{(n)} & V_{22}^{(n)} & V_{23}^{(n)} \\ V_{31}^{(n)} & V_{32}^{(n)} & V_{33}^{(n)} \end{vmatrix} \\
 &= -\alpha_0^2 \beta_0 \lambda^{3n+2} - (\alpha_0^2 \beta_1 + 2\alpha_0 \alpha_1 \beta_0) \lambda^{3n+1} \\
 &\quad - (\alpha_0^2 \beta_2 + 2\alpha_0 \alpha_1 \beta_1 + 2\alpha_0 \alpha_2 \beta_0 + \alpha_1^2 \beta_0 - \frac{1}{4} \beta_0^3) \lambda^{3n} + \dots
 \end{aligned} \tag{3.2}$$

It is easy to see the following fact.

$T_m(\lambda)$ is a polynomial of degree $3n + 2$ with respect to λ as $\alpha_0 \beta_0 \neq 0$. Then $\mathcal{F}_m(\lambda, y) = 0$ naturally leads to a trigonal curve

$$\mathcal{K}_{m-1}: \quad \mathcal{F}_m(\lambda, y) = y^3 + yS_m(\lambda) - T_m(\lambda) = 0 \tag{3.3}$$

with $m = 3n + 2$. For the sake of convenience, we denote the compactification of the curve \mathcal{K}_{m-1} by the same symbol \mathcal{K}_{m-1} . Thus, \mathcal{K}_{m-1} becomes a three-sheeted Riemann surface of arithmetic genus $m - 1$ if it is nonsingular or smooth. Here nonsingular means that at each point $P' = (\lambda', y') \in \mathcal{K}_{m-1}$, $(\frac{\partial \mathcal{F}_m(\lambda, y)}{\partial \lambda}, \frac{\partial \mathcal{F}_m(\lambda, y)}{\partial y})|_{(\lambda, y)=(\lambda', y')} \neq (0, 0)$ holds. For $m \geq 4$, these curves are typically non-hyperelliptic. The point P on \mathcal{K}_{m-1} is represented as the pair $P = (\lambda, y(P))$ satisfying (3.3), particularly, $P_{\infty_j} = (\infty_j, \infty_j)$, $j = 1, 2, 3$, are the points at infinity. The complex structure on \mathcal{K}_{m-1} is defined in the usual way by introducing the local coordinates $\zeta_Q = \lambda - \lambda_Q$ near the points $Q = (\lambda_Q, y(Q)) \in \mathcal{K}_{m-1}$ which are neither branch nor singular points of \mathcal{K}_{m-1} . However, a trigonal curve of degree $m = 3n + 2$ has three infinity points $P_{\infty_1}, P_{\infty_2}, P_{\infty_3}$ with local coordinates $\lambda = \zeta^{-1}$; similarly at other branch and singular points of \mathcal{K}_{m-1} .

We now introduce the stationary Baker–Akhiezer function

$$\begin{aligned}
 \psi_x(P, x, x_0) &= U(u(x); \lambda(P))\psi(P, x, x_0), \\
 V^{(n)}(u(x); \lambda(P))\psi(P, x, x_0) &= y(P)\psi(P, x, x_0), \\
 \psi_3(P, x_0, x_0) &= 1, \quad P = (\lambda, y) \in \mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}, \quad x \in \mathbb{C}.
 \end{aligned} \tag{3.4}$$

Closely related to $\psi(P, x, x_0)$ are the two meromorphic functions $\phi_2(P, x)$ and $\phi_3(P, x)$ on \mathcal{K}_{m-1} defined by

$$\phi_2(P, x) = \frac{\psi_1(P, x, x_0)}{\psi_3(P, x, x_0)}, \quad P \in \mathcal{K}_{m-1}, \quad x \in \mathbb{C}, \tag{3.5}$$

$$\phi_3(P, x) = \frac{\psi_2(P, x, x_0)}{\psi_3(P, x, x_0)}, \quad P \in \mathcal{K}_{m-1}, \quad x \in \mathbb{C}. \tag{3.6}$$

Using (3.4), a direct calculation shows that

$$\phi_2 = \frac{yV_{12}^{(n)} + C_m}{yV_{32}^{(n)} + A_m} = \frac{F}{y^2V_{12}^{(n)} - yC_m + D_m} = \frac{y^2V_{32}^{(n)} - yA_m + B_m}{E}, \quad (3.7)$$

$$\phi_3 = \frac{yV_{21}^{(n)} + C_m}{yV_{31}^{(n)} + A_m} = \frac{\mathcal{F}}{y^2V_{21}^{(n)} - yC_m + D_m} = \frac{y^2V_{31}^{(n)} - yA_m + B_m}{-E}, \quad (3.8)$$

where

$$\begin{aligned} A_m &= V_{12}^{(n)}V_{31}^{(n)} - V_{11}^{(n)}V_{32}^{(n)}, \\ B_m &= V_{32}^{(n)}(V_{22}^{(n)}V_{33}^{(n)} - V_{23}^{(n)}V_{32}^{(n)}) + V_{31}^{(n)}(V_{12}^{(n)}V_{33}^{(n)} - V_{13}^{(n)}V_{32}^{(n)}), \\ C_m &= V_{13}^{(n)}V_{32}^{(n)} - V_{12}^{(n)}V_{33}^{(n)}, \\ D_m &= V_{12}^{(n)}(V_{11}^{(n)}V_{22}^{(n)} - V_{12}^{(n)}V_{21}^{(n)}) + V_{13}^{(n)}(V_{11}^{(n)}V_{32}^{(n)} - V_{12}^{(n)}V_{31}^{(n)}), \end{aligned} \quad (3.9)$$

$$\begin{aligned} \mathcal{A}_m &= V_{21}^{(n)}V_{32}^{(n)} - V_{22}^{(n)}V_{31}^{(n)}, \\ \mathcal{B}_m &= V_{31}^{(n)}(V_{11}^{(n)}V_{33}^{(n)} - V_{13}^{(n)}V_{31}^{(n)}) + V_{32}^{(n)}(V_{21}^{(n)}V_{33}^{(n)} - V_{23}^{(n)}V_{31}^{(n)}), \\ \mathcal{C}_m &= V_{23}^{(n)}V_{31}^{(n)} - V_{21}^{(n)}V_{33}^{(n)}, \\ \mathcal{D}_m &= V_{21}^{(n)}(V_{11}^{(n)}V_{22}^{(n)} - V_{12}^{(n)}V_{21}^{(n)}) + V_{23}^{(n)}(V_{22}^{(n)}V_{31}^{(n)} - V_{21}^{(n)}V_{32}^{(n)}), \end{aligned} \quad (3.10)$$

$$\begin{aligned} E &= V_{32}^{(n)}(V_{21}^{(n)}V_{32}^{(n)} - V_{22}^{(n)}V_{31}^{(n)}) + V_{31}^{(n)}(V_{11}^{(n)}V_{32}^{(n)} - V_{12}^{(n)}V_{31}^{(n)}), \\ F &= V_{12}^{(n)}(V_{12}^{(n)}V_{23}^{(n)} - V_{13}^{(n)}V_{22}^{(n)}) + V_{13}^{(n)}(V_{12}^{(n)}V_{33}^{(n)} - V_{13}^{(n)}V_{32}^{(n)}), \\ \mathcal{F} &= V_{23}^{(n)}(V_{21}^{(n)}V_{33}^{(n)} - V_{23}^{(n)}V_{31}^{(n)}) + V_{21}^{(n)}(V_{13}^{(n)}V_{21}^{(n)} - V_{11}^{(n)}V_{23}^{(n)}). \end{aligned} \quad (3.11)$$

We can easily show that among the polynomials $A_m, B_m, C_m, D_m, \mathcal{A}_m, \mathcal{B}_m, \mathcal{C}_m, \mathcal{D}_m, E, F, \mathcal{F}, S_m, T_m$ there exist various interrelationships, some of which are listed below:

$$\begin{aligned} V_{32}^{(n)}F &= V_{12}^{(n)}D_m - (V_{12}^{(n)})^2S_m - C_m^2, \\ A_mF &= (V_{12}^{(n)})^2T_m + C_mD_m, \end{aligned} \quad (3.12)$$

$$\begin{aligned} V_{12}^{(n)}E &= V_{32}^{(n)}B_m - (V_{32}^{(n)})^2S_m - A_m^2, \\ C_mE &= (V_{32}^{(n)})^2T_m + A_mB_m, \end{aligned} \quad (3.13)$$

$$\begin{aligned} V_{12}^{(n)}B_m + V_{32}^{(n)}D_m - V_{32}^{(n)}V_{12}^{(n)}S_m + A_mC_m &= 0, \\ V_{32}^{(n)}V_{12}^{(n)}T_m + V_{12}^{(n)}A_mS_m + V_{32}^{(n)}C_mS_m - B_mC_m - A_mD_m &= 0, \\ V_{12}^{(n)}A_mT_m + V_{32}^{(n)}C_mT_m + EF - B_mD_m &= 0, \end{aligned} \quad (3.14)$$

$$\begin{aligned} V_{31}^{(n)}\mathcal{F} &= V_{21}^{(n)}\mathcal{D}_m - (V_{21}^{(n)})^2S_m - \mathcal{C}_m^2, \\ \mathcal{A}_m\mathcal{F} &= (V_{21}^{(n)})^2T_m + \mathcal{C}_m\mathcal{D}_m, \end{aligned} \quad (3.15)$$

$$\begin{aligned} -V_{21}^{(n)}E &= V_{31}^{(n)}\mathcal{B}_m - (V_{31}^{(n)})^2S_m - \mathcal{A}_m^2, \\ -\mathcal{C}_mE &= (V_{31}^{(n)})^2T_m + \mathcal{A}_m\mathcal{B}_m, \end{aligned} \quad (3.16)$$

$$\begin{aligned} V_{21}^{(n)}\mathcal{B}_m + V_{31}^{(n)}\mathcal{D}_m - V_{31}^{(n)}V_{21}^{(n)}S_m + \mathcal{A}_m\mathcal{C}_m &= 0, \\ V_{31}^{(n)}V_{21}^{(n)}T_m + V_{21}^{(n)}\mathcal{A}_mS_m + V_{31}^{(n)}\mathcal{C}_mS_m - \mathcal{B}_m\mathcal{C}_m - \mathcal{A}_m\mathcal{D}_m &= 0, \\ V_{21}^{(n)}\mathcal{A}_mT_m + V_{31}^{(n)}\mathcal{C}_mT_m - E\mathcal{F} - \mathcal{B}_m\mathcal{D}_m &= 0, \end{aligned} \quad (3.17)$$

$$\begin{aligned} E_x &= -u_{31}(2V_{32}^{(n)}S_m - 3\mathcal{B}_m) + u_{32}(2V_{31}^{(n)}S_m - 3\mathcal{B}_m), \\ V_{12}^{(n)}F_x &= [3(\lambda - v)V_{12}^{(n)} - 3u_{12}V_{11}^{(n)}]F + (u_{12}V_{13}^{(n)} - u_{13}V_{12}^{(n)})(2V_{12}^{(n)}S_m - 3\mathcal{D}_m), \\ V_{21}^{(n)}\mathcal{F}_x &= [-3(\lambda - v)V_{21}^{(n)} - 3V_{22}^{(n)}]\mathcal{F} + (V_{23}^{(n)} - u_{23}V_{21}^{(n)})(2V_{21}^{(n)}S_m - 3\mathcal{D}_m). \end{aligned} \quad (3.18)$$

Next we introduce the holomorphic map $*$, changing sheets, which is defined by

$$\begin{aligned} * : \begin{cases} \mathcal{K}_{m-1} \rightarrow \mathcal{K}_{m-1} \\ P = (\lambda, y_i(\lambda)) \rightarrow P^* = (\lambda, y_{i+1(\text{mod}3)}(\lambda)), \quad i = 0, 1, 2, \\ P^{**} := (P^*)^*, \quad \text{etc.}, \end{cases} \end{aligned} \quad (3.19)$$

where $y_i(\lambda), i = 0, 1, 2$, denote the three branches of $y(P)$ satisfying $\mathcal{F}_m(\lambda, y) = 0$, namely,

$$(y - y_0(\lambda))(y - y_1(\lambda))(y - y_2(\lambda)) = y^3 + yS_m(\lambda) - T_m(\lambda) = 0. \quad (3.20)$$

From (3.20), we can easily get

$$\begin{aligned} y_0 + y_1 + y_2 &= 0, \\ y_0y_1 + y_0y_2 + y_1y_2 &= S_m(\lambda), \\ y_0y_1y_2 &= T_m(\lambda), \\ y_0^2 + y_1^2 + y_2^2 &= -2S_m(\lambda), \\ y_0^3 + y_1^3 + y_2^3 &= 3T_m(\lambda), \\ y_0^2y_1^2 + y_0^2y_2^2 + y_1^2y_2^2 &= S_m^2(\lambda). \end{aligned} \quad (3.21)$$

Then the further properties of $\phi_2(P, x)$, $\phi_3(P, x)$ and $\psi_3(P, x, x_0)$ are summarized in:

Lemma 3.1. Assume (3.4)–(3.6), $P = (\lambda, y(P)) \in \mathcal{K}_{m-1} \setminus \{P_{\infty 1}, P_{\infty 2}, P_{\infty 3}\}$, and let $(\lambda, x, x_0) \in \mathbb{C}^3$. Then

$$\begin{aligned} \phi_{2,x}(P, x) + u_{31}(x)\phi_2^2(P, x) + u_{32}(x)\phi_2(P, x)\phi_3(P, x) \\ - (\lambda - v)\phi_2(P, x) - u_{12}(x)\phi_3(P, x) - u_{13} = 0, \end{aligned} \quad (3.22)$$

$$\begin{aligned} \phi_{3,x}(P, x) + u_{32}(x)\phi_3^2(P, x) + u_{31}(x)\phi_2(P, x)\phi_3(P, x) \\ + (\lambda - v)\phi_2(P, x) - \phi_2(P, x) - u_{23}(x) = 0, \end{aligned} \quad (3.23)$$

$$\phi_2(P, x)\phi_2(P^*, x)\phi_2(P^{**}, x) = -\frac{F(\lambda, x)}{E(\lambda, x)}, \quad (3.24)$$

$$\phi_3(P, x)\phi_3(P^*, x)\phi_3(P^{**}, x) = \frac{\mathcal{F}(\lambda, x)}{E(\lambda, x)}, \quad (3.25)$$

$$\begin{aligned} \phi_2(P, x) + \phi_2(P^*, x) + \phi_2(P^{**}, x) &= -\frac{2V_{32}^{(n)}(\lambda, x)S_m(\lambda) - 3B_m(\lambda, x)}{E(\lambda, x)} \\ &= \left[\frac{E_x(\lambda, x)}{E(\lambda, x)} + 3u_{32}(x) \frac{V_{33}^{(n)}(\lambda, x)}{V_{32}^{(n)}(\lambda, x)} \right] \frac{V_{32}^{(n)}(\lambda, x)}{u_{31}(x)V_{32}^{(n)}(\lambda, x) - u_{32}(x)V_{31}^{(n)}(\lambda, x)}, \end{aligned} \tag{3.26}$$

$$\begin{aligned} \phi_3(P, x) + \phi_3(P^*, x) + \phi_3(P^{**}, x) &= \frac{2V_{31}^{(n)}(\lambda, x)S_m(\lambda) - 3B_m(\lambda, x)}{E(\lambda, x)} \\ &= -\left[\frac{E_x(\lambda, x)}{E(\lambda, x)} + 3u_{31}(x) \frac{V_{33}^{(n)}(\lambda, x)}{V_{31}^{(n)}(\lambda, x)} \right] \frac{V_{31}^{(n)}(\lambda, x)}{u_{31}(x)V_{32}^{(n)}(\lambda, x) - u_{32}(x)V_{31}^{(n)}(\lambda, x)}, \end{aligned} \tag{3.27}$$

$$\begin{aligned} u_{31}(x)[\phi_2(P, x) + \phi_2(P^*, x) + \phi_2(P^{**}, x)] + u_{32}(x)[\phi_3(P, x) + \phi_3(P^*, x) \\ + \phi_3(P^{**}, x)] &= \frac{E_x(\lambda, x)}{E(\lambda, x)}, \end{aligned} \tag{3.28}$$

$$\begin{aligned} \frac{1}{\phi_2(P, x)} + \frac{1}{\phi_2(P^*, x)} + \frac{1}{\phi_2(P^{**}, x)} \\ &= \frac{[3(\lambda - v)V_{12}^{(n)}(\lambda, x) - 3u_{12}(x)V_{11}^{(n)}(\lambda, x)]F(\lambda, x) - V_{12}^{(n)}(\lambda, x)F_x(\lambda, x)}{[u_{12}(x)V_{13}^{(n)}(\lambda, x) - u_{13}(x)V_{12}^{(n)}(\lambda, x)]F(\lambda, x)}, \end{aligned} \tag{3.29}$$

$$\begin{aligned} \frac{1}{\phi_3(P, x)} + \frac{1}{\phi_3(P^*, x)} + \frac{1}{\phi_3(P^{**}, x)} \\ &= -\frac{[3(\lambda - v)V_{21}^{(n)}(\lambda, x) + 3V_{22}^{(n)}(\lambda, x)]\mathcal{F}(\lambda, x) + V_{21}^{(n)}(\lambda, x)\mathcal{F}_x(\lambda, x)}{[V_{23}^{(n)}(\lambda, x) - u_{23}(x)V_{21}^{(n)}(\lambda, x)]\mathcal{F}(\lambda, x)}, \end{aligned} \tag{3.30}$$

$$\begin{aligned} \frac{\phi_2(P, x)}{\phi_3(P, x)} + \frac{\phi_2(P^*, x)}{\phi_3(P^*, x)} + \frac{\phi_2(P^{**}, x)}{\phi_3(P^{**}, x)} \\ &= \frac{V_{23}^{(n)}(\lambda, x) V_{21}^{(n)}(\lambda, x)\mathcal{F}_x(\lambda, x) + 3[(\lambda - v)V_{21}^{(n)}(\lambda, x) + V_{22}^{(n)}(\lambda, x)]\mathcal{F}(\lambda, x)}{V_{21}^{(n)}(\lambda, x) [V_{23}^{(n)}(\lambda, x) - u_{23}(x)V_{21}^{(n)}(\lambda, x)]\mathcal{F}(\lambda, x)} \\ &\quad - 3\frac{V_{22}^{(n)}(\lambda, x)}{V_{21}^{(n)}(\lambda, x)}, \end{aligned} \tag{3.31}$$

$$\begin{aligned} \frac{\phi_3(P, x)}{\phi_2(P, x)} + \frac{\phi_3(P^*, x)}{\phi_2(P^*, x)} + \frac{\phi_3(P^{**}, x)}{\phi_2(P^{**}, x)} \\ &= \frac{V_{13}^{(n)}(\lambda, x) V_{12}^{(n)}(\lambda, x)F_x(\lambda, x) - 3[(\lambda - v)V_{12}^{(n)}(\lambda, x) - u_{12}(x)V_{11}^{(n)}(\lambda, x)]F(\lambda, x)}{V_{12}^{(n)}(\lambda, x) [u_{12}(x)V_{13}^{(n)}(\lambda, x) - u_{13}(x)V_{12}^{(n)}(\lambda, x)]F(\lambda, x)} \\ &\quad - 3\frac{V_{11}^{(n)}(\lambda, x)}{V_{12}^{(n)}(\lambda, x)}, \end{aligned} \tag{3.32}$$

$$\psi_3(P, x, x_0) = \exp \left(\int_{x_0}^x [u_{31}(x')\phi_2(P, x') + u_{32}(x')\phi_3(P, x')]dx' \right), \quad (3.33)$$

$$\psi_3(P, x, x_0)\psi_3(P^*, x, x_0)\psi_3(P^{**}, x, x_0) = \frac{E(\lambda, x)}{E(\lambda, x_0)}. \quad (3.34)$$

Due to the observation of (2.11) and (3.11), one infers that E , F and \mathcal{F} are polynomials with respect to λ of degree $m - 1$. Hence we can write them in the form

$$E(\lambda, x) = 2\alpha_0^3 u_{31}(x)u_{32}(x) \prod_{j=1}^{m-1} (\lambda - \mu_j(x)), \quad (3.35)$$

$$F(\lambda, x) = \alpha_0^3 u_{12}(x)u_{13}(x) \prod_{j=1}^{m-1} (\lambda - \nu_j(x)), \quad (3.36)$$

$$\mathcal{F}(\lambda, x) = -\alpha_0^3 u_{23}(x) \prod_{j=1}^{m-1} (\lambda - \xi_j(x)), \quad (3.37)$$

where $\alpha_0 \neq 0$.

Defining

$$\begin{aligned} \hat{\mu}_j(x) = \left(\mu_j(x), y(\hat{\mu}_j(x)) \right) &= \left(\mu_j(x), -\frac{A_m(\mu_j(x), x)}{V_{32}^{(n)}(\mu_j(x), x)} \right) \\ &= \left(\mu_j(x), -\frac{\mathcal{A}_m(\mu_j(x), x)}{V_{31}^{(n)}(\mu_j(x), x)} \right) \in \mathcal{K}_{m-1}, \end{aligned} \quad (3.38)$$

$$\hat{\nu}_j(x) = \left(\nu_j(x), y(\hat{\nu}_j(x)) \right) = \left(\nu_j(x), -\frac{C_m(\nu_j(x), x)}{V_{12}^{(n)}(\nu_j(x), x)} \right) \in \mathcal{K}_{m-1}, \quad (3.39)$$

$$\begin{aligned} \hat{\xi}_j(x) = \left(\xi_j(x), y(\hat{\xi}_j(x)) \right) &= \left(\xi_j(x), -\frac{C_m(\xi_j(x), x)}{V_{21}^{(n)}(\xi_j(x), x)} \right) \in \mathcal{K}_{m-1}, \\ 1 \leq j \leq m - 1, x \in \mathbb{C}, \end{aligned} \quad (3.40)$$

it is easy to see that the two representations of $\hat{\mu}_j(x)$ are equivalent. In fact,

$$\begin{aligned} E|_{\lambda=\mu_j(x)} &= [(V_{32}^{(n)})^2 V_{21}^{(n)} + V_{31}^{(n)} V_{32}^{(n)} (V_{11}^{(n)} - V_{22}^{(n)}) - (V_{31}^{(n)})^2 V_{12}^{(n)}]|_{\lambda=\mu_j(x)} \\ &= [V_{32}^{(n)}(\mu_j(x), x)\mathcal{A}_m(\mu_j(x), x) - V_{31}^{(n)}(\mu_j(x), x)A_m(\mu_j(x), x)] = 0, \end{aligned}$$

which means

$$\frac{A_m(\mu_j(x), x)}{V_{31}^{(n)}(\mu_j(x), x)} = \frac{A_m(\mu_j(x), x)}{V_{32}^{(n)}(\mu_j(x), x)}.$$

The dynamics of the zeros $\{\mu_j(x)\}_{j=1,\dots,m-1}$, $\{\nu_j(x)\}_{j=1,\dots,m-1}$ and $\{\xi_j(x)\}_{j=1,\dots,m-1}$ of E , F and \mathcal{F} with respect to x can be described by the first-order system of nonlinear differential equations also called Dubrovin-type equations.

Lemma 3.2. *Suppose the zeros $\{\mu_j(x)\}_{j=1,\dots,m-1}$, $\{\nu_j(x)\}_{j=1,\dots,m-1}$ and $\{\xi_j(x)\}_{j=1,\dots,m-1}$ of $E(\lambda, x)$, $F(\lambda, x)$ and $\mathcal{F}(\lambda, x)$ remain distinct.*

Then $\{\mu_j(x)\}_{j=1,\dots,m-1}$, $\{\nu_j(x)\}_{j=1,\dots,m-1}$ and $\{\xi_j(x)\}_{j=1,\dots,m-1}$ satisfy the system of differential equations

$$\begin{aligned} \mu_{j,x}(x) &= \frac{u_{32}(x)V_{31}^{(n)}(\mu_j(x), x) - u_{31}(x)V_{32}^{(n)}(\mu_j(x), x)}{2\alpha_0^3 u_{31}(x)u_{32}(x)} \\ &\times \frac{[3y^2(\hat{\mu}_j(x)) + S_m(\mu_j(x))]}{\prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\mu_j(x) - \mu_k(x))}, \quad 1 \leq j \leq m-1, \end{aligned} \quad (3.41)$$

$$\begin{aligned} \nu_{j,x}(x) &= \frac{u_{12}(x)V_{13}^{(n)}(\nu_j(x), x) - u_{13}(x)V_{12}^{(n)}(\nu_j(x), x)}{\alpha_0^3 u_{12}(x)u_{13}(x)} \\ &\times \frac{[3y^2(\hat{\nu}_j(x)) + S_m(\nu_j(x))]}{\prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\nu_j(x) - \nu_k(x))}, \quad 1 \leq j \leq m-1, \end{aligned} \quad (3.42)$$

$$\begin{aligned} \xi_{j,x}(x) &= \frac{u_{23}(x)V_{21}^{(n)}(\xi_j(x), x) - V_{23}^{(n)}(\xi_j(x), x)}{\alpha_0^3 u_{23}(x)} \\ &\times \frac{[3y^2(\hat{\xi}_j(x)) + S_m(\xi_j(x))]}{\prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\xi_j(x) - \xi_k(x))}, \quad 1 \leq j \leq m-1. \end{aligned} \quad (3.43)$$

P r o o f. Substituting $\lambda = \mu_j(x)$ into (3.13) and (3.16), we arrive at

$$\begin{aligned} (V_{32}^{(n)}(\mu_j(x), x))^2 S_m(\mu_j(x)) - V_{32}^{(n)}(\mu_j(x), x)B_m(\mu_j(x), x) + A_m^2(\mu_j(x), x) &= 0, \\ (V_{31}^{(n)}(\mu_j(x), x))^2 S_m(\mu_j(x)) - V_{31}^{(n)}(\mu_j(x), x)\mathcal{B}_m(\mu_j(x), x) + \mathcal{A}_m^2(\mu_j(x), x) &= 0. \end{aligned}$$

Then we have

$$\begin{aligned} B_m(\mu_j(x), x) &= V_{32}^{(n)}(\mu_j(x), x)S_m(\mu_j(x)) + \frac{A_m^2(\mu_j(x), x)}{V_{32}^{(n)}(\mu_j(x), x)} \\ &= V_{32}^{(n)}(\mu_j(x), x)[S_m(\mu_j(x)) + y^2(\hat{\mu}_j(x))], \\ \mathcal{B}_m(\mu_j(x), x) &= V_{31}^{(n)}(\mu_j(x), x)S_m(\mu_j(x)) + \frac{\mathcal{A}_m^2(\mu_j(x), x)}{V_{31}^{(n)}(\mu_j(x), x)} \\ &= V_{31}^{(n)}(\mu_j(x), x)[S_m(\mu_j(x)) + y^2(\hat{\mu}_j(x))]. \end{aligned} \quad (3.44)$$

After inserting $\lambda = \mu_j(x)$ into the first equation of (3.18) and in view of (3.44), we obtain

$$E_x(\mu_j(x), x) = [u_{31}(x)V_{32}^{(n)}(\mu_j(x), x) - u_{32}(x)V_{31}^{(n)}(\mu_j(x), x)][3y^2(\hat{\mu}_j(x)) + S_m(\mu_j(x))]. \quad (3.45)$$

On the other hand, differentiating (3.35) with respect to x , we find

$$E_x(\mu_j(x), x) = -2\alpha_0^3 u_{31}(x)u_{32}(x)\mu_{j,x}(x) \prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\mu_j(x) - \mu_k(x)). \quad (3.46)$$

Comparing (3.45) and (3.46), we derive (3.41). In a similar way, we can prove (3.42) and (3.43).

4. Algebro-Geometric Solutions to the Stationary Hierarchy

In this section, we will obtain the explicit Riemann theta function representations for the two meromorphic functions $\phi_2(P, x)$, $\phi_3(P, x)$, the Baker–Akhiezer function $\psi_3(P, x, x_0)$, and, finally, for each of the potential u of the stationary hierarchy.

To study the asymptotic expansions of $\phi_2(P, x)$, $\phi_3(P, x)$ and $\psi_3(P, x, x_0)$ near $P_{\infty_1}, P_{\infty_2}, P_{\infty_3} \in \mathcal{K}_{m-1}$, we choose the local coordinate $\zeta = \lambda^{-1}$, then we have the following Lemma.

Lemma 4.1. *Suppose that u satisfies the n th stationary system $X_n = 0$. Moreover, let $P \in \mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$, $x, x_0 \in \mathbb{C}$. Then*

$$\phi_2(P, x) \underset{\zeta \rightarrow 0}{=} \begin{cases} \frac{1}{u_{31}(x)}\zeta^{-1} + \frac{u_{31,x}(x)}{u_{31}^2(x)} - v(x)\frac{1}{u_{31}(x)} - \frac{1}{2}\frac{u_{32}(x)}{u_{31}^2(x)} + O(\zeta), \\ \text{as } P \rightarrow P_{\infty_1}, \\ \frac{1}{2}\frac{u_{12}(x)}{u_{32}(x)} + O(\zeta), & \text{as } P \rightarrow P_{\infty_2}, \\ -u_{13}(x)\zeta - [u_{12}(x)u_{23}(x) + v(x)u_{13}(x) + u_{13,x}(x)]\zeta^2 + O(\zeta^3), \\ \text{as } P \rightarrow P_{\infty_3}, \end{cases} \quad (4.1)$$

$$\phi_3(P, x) \underset{\zeta \rightarrow 0}{=} \begin{cases} \frac{1}{2}\frac{1}{u_{31}(x)} + O(\zeta), & \text{as } P \rightarrow P_{\infty_1}, \\ -\frac{1}{u_{32}(x)}\zeta^{-1} + \frac{u_{32,x}(x)}{u_{32}^2(x)} + v(x)\frac{1}{u_{32}(x)} - \frac{1}{2}\frac{u_{12}(x)u_{31}(x)}{u_{32}^2(x)} + O(\zeta), \\ \text{as } P \rightarrow P_{\infty_2}, \\ u_{23}(x)\zeta + [-u_{13}(x) + v(x)u_{23}(x) - u_{23,x}(x)]\zeta^2 + O(\zeta^3), \\ \text{as } P \rightarrow P_{\infty_3}, \end{cases} \quad (4.2)$$

$$\psi_3(P, x, x_0) \underset{\zeta \rightarrow 0}{=} \begin{cases} \frac{u_{31}(x)}{u_{31}(x_0)} c^{-1}(x, x_0) \exp\left(\zeta^{-1}(x - x_0) + O(\zeta)\right), & \text{as } P \rightarrow P_{\infty_1}, \\ \frac{u_{32}(x)}{u_{32}(x_0)} c(x, x_0) \exp\left(-\zeta^{-1}(x - x_0) + O(\zeta)\right), & \text{as } P \rightarrow P_{\infty_2}, \\ \exp\left(O(\zeta)\right), & \text{as } P \rightarrow P_{\infty_3}, \end{cases} \quad (4.3)$$

where $c(x, x_0) = \exp(\int_{x_0}^x v(x')dx')$.

P r o o f. Inserting the three sets of ansatzes

$$\begin{aligned} (1) \quad \phi_2 \underset{\zeta \rightarrow 0}{=} \kappa_{1,-1}\zeta^{-1} + \kappa_{1,0} + O(\zeta), & \quad \phi_3 \underset{\zeta \rightarrow 0}{=} \chi_{1,0} + O(\zeta); \\ (2) \quad \phi_2 \underset{\zeta \rightarrow 0}{=} \kappa_{2,0} + O(\zeta), & \quad \phi_3 \underset{\zeta \rightarrow 0}{=} \chi_{2,-1}\zeta^{-1} + \chi_{2,0} + O(\zeta); \\ (3) \quad \phi_2 \underset{\zeta \rightarrow 0}{=} \kappa_{3,1}\zeta + \kappa_{3,2}\zeta^2 + O(\zeta^3), & \quad \phi_3 \underset{\zeta \rightarrow 0}{=} \chi_{3,1}\zeta + \chi_{3,2}\zeta^2 + O(\zeta^3); \end{aligned}$$

into the Riccati-type equations (3.22), (3.23) and comparing the coefficients of the same powers of ζ , we obtain (4.1) and (4.2). Equation (4.3) then follows after substituting (4.1) and (4.2) into (3.33).

One infers from (3.7), (3.8), (4.1) and (4.2) that the divisors $(\phi_2(P, x))$ and $(\phi_3(P, x))$ of $\phi_2(P, x)$ and $\phi_3(P, x)$, respectively, are given by

$$(\phi_2(P, x)) = \mathcal{D}_{P_{\infty_3}, \hat{\nu}_1(x), \dots, \hat{\nu}_{m-1}(x)}(P) - \mathcal{D}_{P_{\infty_1}, \hat{\mu}_1(x), \dots, \hat{\mu}_{m-1}(x)}(P), \quad (4.4)$$

$$(\phi_3(P, x)) = \mathcal{D}_{P_{\infty_3}, \hat{\xi}_1(x), \dots, \hat{\xi}_{m-1}(x)}(P) - \mathcal{D}_{P_{\infty_2}, \hat{\mu}_1(x), \dots, \hat{\mu}_{m-1}(x)}(P). \quad (4.5)$$

That is, $P_{\infty_3}, \hat{\nu}_1(x), \dots, \hat{\nu}_{m-1}(x)$ are m zeros of $\phi_2(P, x)$ and $P_{\infty_1}, \hat{\mu}_1(x), \dots, \hat{\mu}_{m-1}(x)$ are m poles; $P_{\infty_3}, \hat{\xi}_1(x), \dots, \hat{\xi}_{m-1}(x)$ are m zeros of $\phi_3(P, x)$ and $P_{\infty_2}, \hat{\mu}_1(x), \dots, \hat{\mu}_{m-1}(x)$ are m poles.

A tedious calculation reveals that the asymptotic behaviors of $y(P)$ and S_m near $P_{\infty_1}, P_{\infty_2}, P_{\infty_3}$ are as follows:

$$y(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \zeta^{-n-1}[\alpha_0 + (\alpha_1 - \frac{1}{2}\beta_0)\zeta + O(\zeta^2)], & \text{as } P \rightarrow P_{\infty_1}, \\ -\zeta^{-n-1}[\alpha_0 + (\alpha_1 + \frac{1}{2}\beta_0)\zeta + O(\zeta^2)], & \text{as } P \rightarrow P_{\infty_2}, \\ \zeta^{-n}[\beta_0 + O(\zeta)], & \text{as } P \rightarrow P_{\infty_3}, \end{cases} \quad (4.6)$$

$$S_m \underset{\zeta \rightarrow 0}{=} -\zeta^{-2n-2}\{\alpha_0^2 + O(\zeta)\}, \quad (4.7)$$

as $P \rightarrow P_{\infty_j}, \quad j = 1, 2, 3.$

Equip the Riemann surface \mathcal{K}_{m-1} with homology basis $\{a_j, b_j\}_{j=1}^{m-1}$, whose elements are independent and have the intersection numbers

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1, \dots, m - 1.$$

For the present, we will choose as our basis the following set:

$$\varpi_l(P) = \frac{1}{3y^2(P) + S_m} \begin{cases} \lambda^{l-1}d\lambda, & 1 \leq l \leq 2n + 1, \\ y(P)\lambda^{l+n-m}d\lambda, & 2n + 2 \leq l \leq m - 1 \end{cases}, \quad (4.8)$$

the elements of which are $m - 1$ linearly independent holomorphic differentials on \mathcal{K}_{m-1} . By using the basis $\{a_j\}_{j=1}^{m-1}$ and $\{b_j\}_{j=1}^{m-1}$, the period matrices $A = (A_{jk})$ and $B = (B_{jk})$ can be constructed from

$$A_{jk} = \int_{a_k} \varpi_j, \quad B_{jk} = \int_{b_k} \varpi_j. \quad (4.9)$$

It is possible to show that the matrices A and B are invertible. Now we define the matrices C and τ by $C = A^{-1}$, $\tau = A^{-1}B$. Then the matrix τ can be shown to be symmetric ($\tau_{jk} = \tau_{kj}$) and to have a positive-definite imaginary part ($\text{Im } \tau > 0$) [31, 32]. If we normalize $\varpi_l(P)$ into new basis ω_j ,

$$\omega_j = \sum_{l=1}^{m-1} C_{jl} \varpi_l, \quad (4.10)$$

then we have $\int_{a_k} \omega_j = \delta_{jk}$, $\int_{b_k} \omega_j = \tau_{jk}$, $j, k = 1, \dots, m - 1$.

A cumbersome calculation yields the following Laurent expansions of (4.10) near $\{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$:

$$\omega = (\omega_1, \dots, \omega_j, \dots, \omega_{m-1}),$$

$$\omega_j \underset{\zeta \rightarrow 0}{=} \begin{cases} \left(\frac{-C_{j,2n+1} + \alpha_0 C_{j,m-1}}{2\alpha_0^2} + O(\zeta) \right) d\zeta, & P \rightarrow P_{\infty_1}, \\ \left(\frac{-C_{j,2n+1} + \alpha_0 C_{j,m-1}}{2\alpha_0^2} + O(\zeta) \right) d\zeta, & P \rightarrow P_{\infty_2}, \\ \left(\frac{C_{j,2n+1}}{\alpha_0^2} + O(\zeta) \right) d\zeta, & P \rightarrow P_{\infty_3}. \end{cases} \quad (4.11)$$

Let $\omega_{P_{\infty_s},2}^{(2)}(P)$, $s = 1, 2, 3$ denote the normalized Abelian differential of the second kind satisfying

$$\int_{a_k} \omega_{P_{\infty_s},2}^{(2)}(P) = 0, \quad k = 1, \dots, m - 1, \quad (4.12)$$

$$\omega_{P_{\infty_s},2}^{(2)}(P) \underset{\zeta \rightarrow 0}{=} (\zeta^{-2} + O(1))d\zeta, \quad \text{as } P \rightarrow P_{\infty_s}. \quad (4.13)$$

We introduce

$$\Omega^{(2)}(P) = \omega_{P_{\infty_1},2}^{(2)}(P) - \omega_{P_{\infty_2},2}^{(2)}(P), \quad (4.14)$$

then we have

$$\int_{Q_0}^P \Omega^{(2)}(P) = \begin{cases} -\zeta^{-1} + e_1^{(2)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_1}, \\ \zeta^{-1} + e_2^{(2)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_2}, \\ e_3^{(2)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_3}, \end{cases} \quad (4.15)$$

where $e_1^{(2)}(Q_0), e_2^{(2)}(Q_0), e_3^{(2)}(Q_0)$ are integral constants, and Q_0 is an appropriately chosen base point on $\mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$. The b -periods of the differential $\Omega^{(2)}(P)$ are denoted by

$$U_2^{(2)} = (U_{2,1}^{(2)}, \dots, U_{2,m-1}^{(2)}), \quad U_{2,k}^{(2)} = \frac{1}{2\pi i} \int_{b_k} \Omega^{(2)}(P), \quad k = 1, \dots, m-1. \quad (4.16)$$

Thus, from (4.11) and (4.14), we have

$$\begin{aligned} U_{2,k}^{(2)} &= \frac{1}{2\pi i} \left[\int_{b_k} \omega_{P_{\infty_1},2}^{(2)}(P) - \int_{b_k} \omega_{P_{\infty_2},2}^{(2)}(P) \right] \\ &= -\frac{C_{k,2n+1} + \alpha_0 C_{k,m-1}}{2\alpha_0^2} + \frac{C_{k,2n+1} - \alpha_0 C_{k,m-1}}{2\alpha_0^2} = -\frac{C_{k,m-1}}{\alpha_0}, \end{aligned}$$

in which we used the result: $\frac{1}{2\pi i} \int_{b_k} \omega_{Q_1,2}^{(2)}(P) = \varrho_{k,0}$, if $\omega_k = \sum_{l=0}^{\infty} \varrho_{k,l} \zeta_{Q_1}^l d\zeta_{Q_1}$.

Furthermore, the normalized Abelian differentials of the third kind $\omega_{P_{\infty_3}, P_{\infty_j}}^{(3)}(P)$, $j = 1, 2$ are holomorphic on $\mathcal{K}_{m-1} \setminus \{P_{\infty_3}, P_{\infty_j}\}$ with simple poles at P_{∞_3} and P_{∞_j} with residues ± 1 , respectively, that is,

$$\begin{aligned} \omega_{P_{\infty_3}, P_{\infty_1}}^{(3)}(P) &\underset{\zeta \rightarrow 0}{=} \begin{cases} (-\zeta^{-1} + O(1))d\zeta, & \text{as } P \rightarrow P_{\infty_1}, \\ O(1)d\zeta, & \text{as } P \rightarrow P_{\infty_2}, \\ (\zeta^{-1} + O(1))d\zeta, & \text{as } P \rightarrow P_{\infty_3}, \end{cases} \\ \omega_{P_{\infty_3}, P_{\infty_2}}^{(3)}(P) &\underset{\zeta \rightarrow 0}{=} \begin{cases} O(1)d\zeta, & \text{as } P \rightarrow P_{\infty_1}, \\ (-\zeta^{-1} + O(1))d\zeta, & \text{as } P \rightarrow P_{\infty_2}, \\ (\zeta^{-1} + O(1))d\zeta, & \text{as } P \rightarrow P_{\infty_3}, \end{cases} \end{aligned} \quad (4.17)$$

then

$$\begin{aligned} \int_{Q_0}^P \omega_{P_{\infty_3}, P_{\infty_1}}^{(3)}(P) &= \begin{cases} -\ln \zeta + e_{1,\infty_1}^{(3)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_1}, \\ e_{1,\infty_2}^{(3)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_2}, \\ \ln \zeta + e_{1,\infty_3}^{(3)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_3}, \end{cases} \\ \int_{Q_0}^P \omega_{P_{\infty_3}, P_{\infty_2}}^{(3)}(P) &= \begin{cases} e_{2,\infty_1}^{(3)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_1}, \\ -\ln \zeta + e_{2,\infty_2}^{(3)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_2}, \\ \ln \zeta + e_{2,\infty_3}^{(3)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_3}, \end{cases} \end{aligned} \quad (4.18)$$

where $e_{k,\infty_j}^{(3)}(Q_0), k = 1, 2, j = 1, 2, 3$, are integration constants.

Let \mathcal{T}_{m-1} be the period lattice $\{z \in \mathbb{C}^{m-1} | z = \underline{N} + \tau \underline{L}, \underline{N}, \underline{L} \in \mathbb{Z}^{m-1}\}$. The complex torus $\mathcal{J}_{m-1} = \mathbb{C}^{m-1} / \mathcal{T}_{m-1}$ is called the Jacobian variety of \mathcal{K}_{m-1} . An Abel map $\mathcal{A}, \mathcal{K}_{m-1} \rightarrow \mathcal{J}_{m-1}$, is defined by

$$\mathcal{A}(P) = \left(\mathcal{A}_1(P), \dots, \mathcal{A}_{m-1}(P) \right) = \left(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_{m-1} \right) \pmod{\mathcal{T}_{m-1}}$$

with the natural linear extension to the factor group $\text{Div}(\mathcal{K}_{m-1})$

$$\mathcal{A}\left(\sum n_k P_k\right) = \sum n_k \mathcal{A}(P_k).$$

Considering the nonspecial divisors $\mathcal{D}_{\hat{\mu}(x)} = \sum_{k=1}^{m-1} \hat{\mu}_k(x)$, $\mathcal{D}_{\hat{\nu}(x)} = \sum_{k=1}^{m-1} \hat{\nu}_k(x)$ and $\mathcal{D}_{\hat{\xi}(x)} = \sum_{k=1}^{m-1} \hat{\xi}_k(x)$, we define

$$\mathcal{A}\left(\sum_{k=1}^{m-1} P_k^{(j)}(x)\right) = \sum_{k=1}^{m-1} \mathcal{A}(P_k^{(j)}(x)) = \sum_{k=1}^{m-1} \int_{Q_0}^{P_k^{(j)}(x)} \omega = \rho^{(j)}(x), \quad j = 1, 2, 3, \tag{4.19}$$

where $P_k^{(1)}(x) = \hat{\mu}_k(x)$, $P_k^{(2)}(x) = \hat{\nu}_k(x)$, $P_k^{(3)}(x) = \hat{\xi}_k(x)$, $\hat{\mu}(x) = (\hat{\mu}_1(x), \dots, \hat{\mu}_{m-1}(x)) \in \sigma^{m-1} \mathcal{K}_{m-1}$, $\hat{\nu}(x) = (\hat{\nu}_1(x), \dots, \hat{\nu}_{m-1}(x)) \in \sigma^{m-1} \mathcal{K}_{m-1}$, $\hat{\xi}(x) = (\hat{\xi}_1(x), \dots, \hat{\xi}_{m-1}(x)) \in \sigma^{m-1} \mathcal{K}_{m-1}$, and $\sigma^n \mathcal{K}_{m-1} = \{\{Q_1, \dots, Q_n\} | Q_j \in \mathcal{K}_{m-1}, j = 1, \dots, n\}$ denotes the n th symmetric power of \mathcal{K}_{m-1} .

Theorem 4.2. Assume that the curve \mathcal{K}_{m-1} is nonsingular and let $x, x_0 \in \mathbb{C}$. Then

$$\begin{aligned} \rho^{(1)}(x) &= \rho^{(1)}(x_0) + U_2^{(2)}(x - x_0), \\ \rho^{(2)}(x) &= \rho^{(2)}(x_0) + U_2^{(2)}(x - x_0), \\ \rho^{(3)}(x) &= \rho^{(3)}(x_0) + U_2^{(2)}(x - x_0). \end{aligned} \tag{4.20}$$

P r o o f. We prove only the first linearity of the Abel map with respect to x in (4.20). Assume that

$$\mu_j(x) \neq \mu_{j'}(x) \quad \text{for } j \neq j'.$$

Then one computes

$$\frac{d}{dx} \rho_l^{(1)}(x) = \frac{d}{dx} \sum_{j=1}^{m-1} \int_{Q_0}^{\hat{\mu}_j} \omega_l = \sum_{j=1}^{m-1} \mu_{j,x} \omega_l(\hat{\mu}_j) = \sum_{j=1}^{m-1} \mu_{j,x} \sum_{k=1}^{m-1} C_{lk} \varpi_k$$

$$\begin{aligned}
 &= \sum_{j=1}^{m-1} \frac{[u_{32}(x)V_{31}^{(n)}(\mu_j, x) - u_{31}(x)V_{32}^{(n)}(\mu_j, x)]}{2\alpha_0^3 u_{31}(x)u_{32}(x)} \\
 &\quad \times \frac{[3y^2(\hat{\mu}_j) + S_m(\mu_j)]}{\prod_{\substack{k=1 \\ r \neq j}}^{m-1} (\mu_j(x) - \mu_r(x))} \times \left(\sum_{k=1}^{2n+1} C_{lk} \frac{\mu_j^{k-1}}{3y^2(\hat{\mu}_j) + S_m(\mu_j)} \right. \\
 &\quad \left. + \sum_{k=2n+2}^{m-1} C_{lk} \frac{y(\hat{\mu}_j)\mu_j^{k+n-m}}{3y^2(\hat{\mu}_j) + S_m(\mu_j)} \right) \\
 &= \frac{1}{2\alpha_0^3} \left\{ \sum_{j=1}^{m-1} \sum_{k=1}^{2n+1} C_{lk} \frac{[\frac{V_{31}^{(n)}(\mu_j, x)}{u_{31}(x)} - \frac{V_{32}^{(n)}(\mu_j, x)}{u_{32}(x)}] \mu_j^{k-1}}{\prod_{\substack{r=1 \\ r \neq j}}^{m-1} (\mu_j - \mu_r)} \right. \\
 &\quad \left. + \sum_{j=1}^{m-1} \sum_{k=2n+2}^{m-1} C_{lk} \frac{[\frac{V_{31}^{(n)}(\mu_j, x)}{u_{31}(x)} - \frac{V_{32}^{(n)}(\mu_j, x)}{u_{32}(x)}] y(\hat{\mu}_j) \mu_j^{k+n-m}}{\prod_{\substack{r=1 \\ r \neq j}}^{m-1} (\mu_j - \mu_r)} \right\} \\
 &= \frac{1}{2\alpha_0^3} \left\{ \sum_{k=1}^{2n+1} C_{lk} \sum_{j=1}^{m-1} \frac{[\alpha_0 - \alpha_0] \mu_j^{k+n-1}}{\prod_{\substack{r=1 \\ r \neq j}}^{m-1} (\mu_j - \mu_r)} \right. \\
 &\quad \left. + \sum_{k=2n+2}^{m-1} C_{lk} \sum_{j=1}^{m-1} \frac{[-\alpha_0^2 - \alpha_0^2] \mu_j^{k+3n+1-m}}{\prod_{\substack{r=1 \\ r \neq j}}^{m-1} (\mu_j - \mu_r)} \right\} \\
 &= -\frac{C_{k,m-1}}{\alpha_0},
 \end{aligned}$$

where we can use the standard Lagrange interpolation argument. Obviously, we have

$$\frac{d}{dx} \rho_l^{(1)}(x) = U_{2,l}^{(2)},$$

from which the first representation of (4.20) holds. The second and third equalities in (4.20) follow from the same calculations.

Let $\theta(z)$ denote the Riemann theta function associated with \mathcal{K}_{m-1} and an appropriately fixed homology basis. For brevity, define the function $\underline{z} : \mathcal{K}_{m-1} \times \sigma^{m-1} \mathcal{K}_{m-1} \rightarrow \mathbb{C}$ by

$$\begin{aligned}
 \underline{z}(P, \underline{Q}) &= M^{(j)} - \mathcal{A}(P) + \sum_{Q \in \underline{Q}} \mathcal{D}(Q) \mathcal{A}(Q), \quad P \in \mathcal{K}_{m-1}, \\
 \underline{Q} &= (Q_1, \dots, Q_{m-1}) \in \sigma^{m-1} \mathcal{K}_{m-1}, \quad j = 1, 2, 3,
 \end{aligned} \tag{4.21}$$

where $M^{(j)}$ are the vectors of Riemann constants.

Given these preparations, the theta function representations of $\phi_2(P, x)$, $\phi_3(P, x)$, $\psi_3(P, x, x_0)$, and the algebra-geometric solutions of the hierarchy read as follows.

Theorem 4.3. Assume that the curve \mathcal{K}_{m-1} is nonsingular. Let $P = (\lambda, y) \in \mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$ and let $(x, x_0) \in \mathbb{C}^2$. Suppose that $\mathcal{D}_{\hat{\mu}(x)}$ or $\mathcal{D}_{\hat{\nu}(x)}$ or $\mathcal{D}_{\hat{\xi}(x)}$ is nonspecial. Then

$$\phi_2(P, x) = N_1(x, x_0) \frac{\theta(\underline{z}(P, \hat{\nu}(x)))}{\theta(\underline{z}(P, \hat{\mu}(x)))} \exp\left(\int_{Q_0}^P \omega_{P_{\infty_3}, P_{\infty_1}}^{(3)}\right), \quad (4.22)$$

$$\phi_3(P, x) = N_2(x, x_0) \frac{\theta(\underline{z}(P, \hat{\xi}(x)))}{\theta(\underline{z}(P, \hat{\mu}(x)))} \exp\left(\int_{Q_0}^P \omega_{P_{\infty_3}, P_{\infty_2}}^{(3)}\right), \quad (4.23)$$

$$\psi_3(P, x, x_0) = \frac{\theta(\underline{z}(P, \hat{\mu}(x)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0)))}{\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x)))\theta(\underline{z}(P, \hat{\mu}(x_0)))} \exp\left((e_3^{(2)}(Q_0) - \int_{Q_0}^P \Omega^{(2)})(x - x_0)\right), \quad (4.24)$$

where $c(x, x_0) = \exp(\int_{x_0}^x v(x')dx')$

$$c^2(x, x_0) = -\frac{1}{2} \frac{u_{32}(x_0) \theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x_0)))\theta(\underline{z}(P_{\infty_2}, \hat{\xi}(x)))}{u_{31}(x_0) \theta(\underline{z}(P_{\infty_2}, \hat{\mu}(x_0)))\theta(\underline{z}(P_{\infty_1}, \hat{\xi}(x)))} \\ \times \exp\left((e_1^{(2)}(Q_0) - e_2^{(2)}(Q_0))(x - x_0) + e_{2, \infty_2}^{(3)}(Q_0) - e_{2, \infty_1}^{(3)}(Q_0)\right), \quad (4.25)$$

$$N_1(x, x_0) = \frac{1}{u_{31}(x_0)c(x, x_0)} \frac{\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x_0)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x)))}{\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0)))\theta(\underline{z}(P_{\infty_1}, \hat{\nu}(x)))} \\ \times \exp\left((e_1^{(2)}(Q_0) - e_3^{(2)}(Q_0))(x - x_0) - e_{1, \infty_1}^{(3)}(Q_0)\right), \quad (4.26)$$

$$N_2(x, x_0) = \frac{1}{2u_{31}(x_0)c(x, x_0)} \frac{\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x_0)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x)))}{\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0)))\theta(\underline{z}(P_{\infty_1}, \hat{\xi}(x)))} \\ \times \exp\left((e_1^{(2)}(Q_0) - e_3^{(2)}(Q_0))(x - x_0) - e_{2, \infty_1}^{(3)}(Q_0)\right). \quad (4.27)$$

Finally, $u_{12}(x), u_{13}(x), u_{23}(x), u_{31}(x), u_{32}(x), v(x)$ are of the form

$$u_{12}(x) = -4 \frac{\theta(\underline{z}(P_{\infty_2}, \hat{\nu}(x)))\theta(\underline{z}(P_{\infty_1}, \hat{\xi}(x)))}{\theta(\underline{z}(P_{\infty_1}, \hat{\nu}(x)))\theta(\underline{z}(P_{\infty_2}, \hat{\xi}(x)))} \\ \times \exp\left(e_{1, \infty_2}^{(3)}(Q_0) - e_{1, \infty_1}^{(3)}(Q_0) + e_{2, \infty_1}^{(3)}(Q_0) - e_{2, \infty_2}^{(3)}(Q_0)\right), \quad (4.28)$$

$$\begin{aligned}
 u_{13}(x) &= -\frac{1}{u_{31}(x_0)c(x, x_0)} \frac{\theta(\underline{z}(P_{\infty_3}, \hat{\nu}(x)))\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x_0)))}{\theta(\underline{z}(P_{\infty_1}, \hat{\nu}(x)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0)))} \\
 &\quad \times \exp\left((e_1^{(2)}(Q_0) - e_3^{(2)}(Q_0))(x - x_0) + e_{1, \infty_3}^{(3)}(Q_0) - e_{1, \infty_1}^{(3)}(Q_0)\right), \tag{4.29}
 \end{aligned}$$

$$\begin{aligned}
 u_{23}(x) &= \frac{1}{2} \frac{1}{u_{31}(x_0)c(x, x_0)} \frac{\theta(\underline{z}(P_{\infty_3}, \hat{\xi}(x)))\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x_0)))}{\theta(\underline{z}(P_{\infty_1}, \hat{\xi}(x)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0)))} \\
 &\quad \times \exp\left(e_{2, \infty_3}^{(3)}(Q_0) - e_{2, \infty_1}^{(3)}(Q_0) + (e_1^{(2)}(Q_0) - e_3^{(2)}(Q_0))(x - x_0)\right), \tag{4.30}
 \end{aligned}$$

$$\begin{aligned}
 u_{31}(x) &= u_{31}(x_0)c(x, x_0) \frac{\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0)))}{\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x)))\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x_0)))} \\
 &\quad \times \exp\left((e_3^{(2)}(Q_0) - e_1^{(2)}(Q_0))(x - x_0)\right), \tag{4.31}
 \end{aligned}$$

$$\begin{aligned}
 u_{32}(x) &= \frac{u_{32}(x_0)}{c(x, x_0)} \frac{\theta(\underline{z}(P_{\infty_2}, \hat{\mu}(x)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0)))}{\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x)))\theta(\underline{z}(P_{\infty_2}, \hat{\mu}(x_0)))} \\
 &\quad \times \exp\left((e_3^{(2)}(Q_0) - e_2^{(2)}(Q_0))(x - x_0)\right), \tag{4.32}
 \end{aligned}$$

$$v(x) = \partial_x \ln c(x, x_0). \tag{4.33}$$

P r o o f. Let Ψ_3 be defined by the right-hand side of (4.24). We intend to prove that $\psi_3 = \Psi_3$ with ψ_3 given by (3.33). For this purpose, we first investigate the zeros and poles of ψ_3 . Since they can only come from zeros of ϕ_2 and ϕ_3 , one computes using (3.7) and (3.8),

$$\begin{aligned}
 &u_{31}(x)\phi_2(P, x) + u_{32}(x)\phi_3(P, x) \\
 &= u_{31}(x) \frac{y^2 V_{32}^{(n)} - yA_m + B_m}{E} - u_{32} \frac{y^2 V_{31}^{(n)} - yA_m + B_m}{E} \\
 &= \frac{1}{E} \{ [u_{31}V_{32}^{(n)} - u_{32}V_{31}^{(n)}]y^2 - [u_{31}A_m - u_{32}A_m]y \\
 &\quad + \frac{1}{3}E_x + \frac{2}{3}(u_{31}V_{32}^{(n)} - u_{32}V_{31}^{(n)})S_m \} \\
 &= \frac{1}{3} \frac{E_x}{E} + \frac{2}{3} \frac{[u_{31}V_{32}^{(n)} - u_{32}V_{31}^{(n)}](3y^2 + S_m)}{E} \\
 &\quad - \frac{u_{31}V_{32}^{(n)}y[y + \frac{A_m}{V_{32}^{(n)}}] - u_{32}V_{31}^{(n)}y[y + \frac{A_m}{V_{31}^{(n)}}]}{E} \\
 &\underset{\lambda \rightarrow \mu_j(x)}{=} -\frac{\mu_{j,x}(x)}{\lambda - \mu_j(x)} + O(1) \\
 &\underset{\lambda \rightarrow \mu_j(x)}{=} \partial_x \ln(\lambda - \mu_j(x)) + O(1).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \psi_3(P, x, x_0) &= \exp \left(\int_{x_0}^x [u_{31}(x')\phi_2(P, x') + u_{32}(x')\phi_3(P, x')] dx' \right) \\
 &= \frac{\lambda - \mu_j(x)}{\lambda - \mu_j(x_0)} O(1) \\
 &= \begin{cases} (\lambda - \mu_j(x))O(1) & \text{for } P \text{ near } \hat{\mu}_j(x) \neq \hat{\mu}_j(x_0), \\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(x) = \hat{\mu}_j(x_0), \\ (\lambda - \mu_j(x_0))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0) \neq \hat{\mu}_j(x), \end{cases} \tag{4.34}
 \end{aligned}$$

where $O(1) \neq 0$. Consequently, all zeros and poles of ψ_3 and Ψ_3 on $\mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$ are simple and coincident. It remains to identify the essential singularities of ψ_3 and Ψ_3 at $P_{\infty_1}, P_{\infty_2}, P_{\infty_3}$. Taking into account (3.33), (4.3), (4.14), the expression for Ψ_3 in (4.24), we observe that ψ_3 and Ψ_3 share the same singularities and zeros. The Riemann–Roch uniqueness results in that the holomorphic function $\frac{\Psi_3}{\psi_3} = \gamma$, where γ is a constant. By (4.3), (4.14) and the right-hand side of (4.24), we have

$$\frac{\Psi_3(P, x, x_0)}{\psi_3(P, x, x_0)} \underset{\zeta \rightarrow 0}{=} \frac{\exp(O(\zeta))(1 + O(\zeta))}{\exp(O(\zeta))} \underset{\zeta \rightarrow 0}{=} 1 + O(\zeta), \quad P \rightarrow P_{\infty_3}. \tag{4.35}$$

Then we conclude that $\gamma = 1$, which completes the proof of (4.24). With the help of the asymptotic properties of ψ_3 near $P_{\infty_1}, P_{\infty_2}$, we obtain (4.31) and (4.32). Equations (4.4), (4.5) and (4.18) immediately yield that ϕ_2 and ϕ_3 have the following forms:

$$\begin{aligned}
 \phi_2(P, x) &= N_1(x, x_0) \frac{\theta(\underline{z}(P, \hat{\nu}(x)))}{\theta(\underline{z}(P, \hat{\mu}(x)))} \exp \left(\int_{Q_0}^P \omega_{P_{\infty_3}, P_{\infty_1}}^{(3)} \right), \\
 \phi_3(P, x) &= N_2(x, x_0) \frac{\theta(\underline{z}(P, \hat{\xi}(x)))}{\theta(\underline{z}(P, \hat{\mu}(x)))} \exp \left(\int_{Q_0}^P \omega_{P_{\infty_3}, P_{\infty_2}}^{(3)} \right).
 \end{aligned}$$

Considering the asymptotic expansions of ϕ_2 and ϕ_3 near $P_{\infty_1}, P_{\infty_2}, P_{\infty_3}$, we have

$$\begin{aligned}
 \frac{1}{u_{31}(x)} &= N_1(x, x_0) \frac{\theta(\underline{z}(P_{\infty_1}, \hat{\nu}(x)))}{\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x)))} \exp \left(e_{1, \infty_1}^{(3)}(Q_0) \right), \\
 \frac{1}{2} \frac{1}{u_{31}(x)} &= N_2(x, x_0) \frac{\theta(\underline{z}(P_{\infty_1}, \hat{\xi}(x)))}{\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x)))} \exp \left(e_{2, \infty_1}^{(3)}(Q_0) \right),
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2} \frac{u_{12}(x)}{u_{32}(x)} &= N_1(x, x_0) \frac{\theta(\underline{z}(P_{\infty_2}, \hat{\underline{\nu}}(x)))}{\theta(\underline{z}(P_{\infty_2}, \hat{\underline{\mu}}(x)))} \exp\left(e_{1, \infty_2}^{(3)}(Q_0)\right), \\
 -\frac{1}{u_{32}(x)} &= N_2(x, x_0) \frac{\theta(\underline{z}(P_{\infty_2}, \hat{\underline{\xi}}(x)))}{\theta(\underline{z}(P_{\infty_2}, \hat{\underline{\mu}}(x)))} \exp\left(e_{2, \infty_2}^{(3)}(Q_0)\right), \\
 -u_{13}(x) &= N_1(x, x_0) \frac{\theta(\underline{z}(P_{\infty_3}, \hat{\underline{\nu}}(x)))}{\theta(\underline{z}(P_{\infty_3}, \hat{\underline{\mu}}(x)))} \exp\left(e_{1, \infty_3}^{(3)}(Q_0)\right), \\
 u_{23}(x) &= N_2(x, x_0) \frac{\theta(\underline{z}(P_{\infty_3}, \hat{\underline{\xi}}(x)))}{\theta(\underline{z}(P_{\infty_3}, \hat{\underline{\mu}}(x)))} \exp\left(e_{2, \infty_3}^{(3)}(Q_0)\right).
 \end{aligned} \tag{4.36}$$

which together with (4.31) and (4.32) show the expressions (4.25)–(4.33).

5. Algebro-Geometric Solutions of the Entire Hierarchy

In this section, we extend the stationary Baker–Akhiezer function to the time-dependent case, from which all the results obtained in Sections 3 and 4 are generalized to the time-dependent case. In particular, we obtain Riemann theta function representations for the time-dependent Baker–Akhiezer function, the meromorphic function, and algebro-geometric solutions.

In analogy to (3.4), we introduce the time-dependent Baker–Akhiezer function

$$\begin{aligned}
 \psi_x(P, x, x_0, t_r, t_{0,r}) &= U(u(x, t_r); \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}), \\
 \psi_{t_r}(P, x, x_0, t_r, t_{0,r}) &= \tilde{V}^{(r)}(u(x, t_r); \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}), \\
 V^{(n)}(u(x, t_r); \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}) &= y(P)\psi(P, x, x_0, t_r, t_{0,r}), \\
 \psi_3(P, x_0, x_0, t_{0,r}, t_{0,r}) &= 1, \quad P = (\lambda, y) \in \mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}, \quad x \in \mathbb{C},
 \end{aligned} \tag{5.1}$$

with the conditions $\alpha_0\beta_0 \neq 0$ and $\tilde{\alpha}_0\tilde{\beta}_0 \neq 0$ in the Lax matrices $V^{(r)}$ and $\tilde{V}^{(r)}$, respectively. The compatibility conditions of the first three equations in (5.1) yield that

$$U_{t_r} - \tilde{V}_x^{(r)} + [U, \tilde{V}^{(r)}] = 0, \tag{5.2}$$

$$-V_x^{(n)} + [U, V^{(n)}] = 0, \tag{5.3}$$

$$-V_{t_r}^{(n)} + [\tilde{V}^{(r)}, V^{(n)}] = 0. \tag{5.4}$$

A direct calculation shows that $yI - V^{(n)}$ satisfies (5.3) and (5.4). Then the characteristic polynomial of the Lax matrix $V^{(n)}$ for the hierarchy is a constant independent of the variables x and t_r with the expansion

$$\det(yI - V^{(n)}) = y^3 + yS_m(\lambda) - T_m(\lambda),$$

where $S_m(\lambda)$ and $T_m(\lambda)$ are defined as in (3.2). Then the curve \mathcal{K}_{m-1} is determined by

$$\mathcal{K}_{m-1} : \mathcal{F}_m(\lambda, y) = y^3 + yS_m(\lambda) - T_m(\lambda) = 0.$$

Closely related to $\psi(P, x, x_0, t_r, t_{0,r})$ are the two meromorphic functions $\phi_2(P, x, t_r)$ and $\phi_3(P, x, t_r)$ on \mathcal{K}_{m-1} defined by

$$\phi_2(P, x, t_r) = \frac{\psi_1(P, x, x_0, t_r, t_{0,r})}{\psi_3(P, x, x_0, t_r, t_{0,r})}, \quad P \in \mathcal{K}_{m-1}, \quad x \in \mathbb{C}, \quad (5.5)$$

$$\phi_3(P, x, t_r) = \frac{\psi_2(P, x, x_0, t_r, t_{0,r})}{\psi_3(P, x, x_0, t_r, t_{0,r})}, \quad P \in \mathcal{K}_{m-1}, \quad x \in \mathbb{C}, \quad (5.6)$$

which imply from (5.1) that

$$\begin{aligned} \phi_2(P, x, t_r) &= \frac{yV_{12}^{(n)}(\lambda, x, t_r) + C_m(\lambda, x, t_r)}{yV_{32}^{(n)}(\lambda, x, t_r) + A_m(\lambda, x, t_r)} \\ &= \frac{F(\lambda, x, t_r)}{E(\lambda, x, t_r)} \\ &= \frac{y^2V_{12}^{(n)}(\lambda, x, t_r) - yC_m(\lambda, x, t_r) + D_m(\lambda, x, t_r)}{y^2V_{32}^{(n)}(\lambda, x, t_r) - yA_m(\lambda, x, t_r) + B_m(\lambda, x, t_r)}, \end{aligned} \quad (5.7)$$

$$\begin{aligned} \phi_3(P, x, t_r) &= \frac{yV_{21}^{(n)}(\lambda, x, t_r) + \mathcal{C}_m(\lambda, x, t_r)}{yV_{31}^{(n)}(\lambda, x, t_r) + \mathcal{A}_m(\lambda, x, t_r)} \\ &= \frac{\mathcal{F}(\lambda, x, t_r)}{E(\lambda, x, t_r)} \\ &= \frac{y^2V_{21}^{(n)}(\lambda, x, t_r) - y\mathcal{C}_m(\lambda, x, t_r) + \mathcal{D}_m(\lambda, x, t_r)}{y^2V_{31}^{(n)}(\lambda, x, t_r) - y\mathcal{A}_m(\lambda, x, t_r) + \mathcal{B}_m(\lambda, x, t_r)}, \end{aligned} \quad (5.8)$$

where $P = (\lambda, y) \in \mathcal{K}_{m-1}$, $(x, t_r) \in \mathbb{C}^2$, and $A_m(\lambda, x, t_r)$, $B_m(\lambda, x, t_r)$, $C_m(\lambda, x, t_r)$, $D_m(\lambda, x, t_r)$, $\mathcal{A}_m(\lambda, x, t_r)$, $\mathcal{B}_m(\lambda, x, t_r)$, $\mathcal{C}_m(\lambda, x, t_r)$, $\mathcal{D}_m(\lambda, x, t_r)$, $E(\lambda, x, t_r)$, $F(\lambda, x, t_r)$, $\mathcal{F}(\lambda, x, t_r)$ are defined as in (3.9)–(3.11). Therefore, (3.12)–(3.18) also hold in the present context. Similarly, one can write E , F and \mathcal{F} in the following form:

$$E(\lambda, x, t_r) = 2\alpha_0^3 u_{31}(x, t_r) u_{32}(x, t_r) \prod_{j=1}^{m-1} (\lambda - \mu_j(x, t_r)), \quad (5.9)$$

$$F(\lambda, x, t_r) = \alpha_0^3 u_{12}(x, t_r) u_{13}(x, t_r) \prod_{j=1}^{m-1} (\lambda - \nu_j(x, t_r)), \quad (5.10)$$

$$\mathcal{F}(\lambda, x, t_r) = -\alpha_0^3 u_{23}(x, t_r) \prod_{j=1}^{m-1} (\lambda - \xi_j(x, t_r)). \quad (5.11)$$

Defining

$$\begin{aligned} \hat{\mu}_j(x, t_r) &= \left(\mu_j(x, t_r), y(\hat{\mu}_j(x, t_r)) \right) = \left(\mu_j(x, t_r), -\frac{A_m(\mu_j(x, t_r), x, t_r)}{V_{32}^{(n)}(\mu_j(x, t_r), x, t_r)} \right) \\ &= \left(\mu_j(x, t_r), -\frac{\mathcal{A}_m(\mu_j(x, t_r), x, t_r)}{V_{31}^{(n)}(\mu_j(x, t_r), x, t_r)} \right) \in \mathcal{K}_{m-1}, \end{aligned} \tag{5.12}$$

$$\hat{\nu}_j(x, t_r) = \left(\nu_j(x, t_r), y(\hat{\nu}_j(x, t_r)) \right) = \left(\nu_j(x, t_r), -\frac{C_m(\nu_j(x, t_r), x, t_r)}{V_{12}^{(n)}(\nu_j(x, t_r), x, t_r)} \right) \in \mathcal{K}_{m-1}, \tag{5.13}$$

$$\begin{aligned} \hat{\xi}_j(x, t_r) &= \left(\xi_j(x, t_r), y(\hat{\xi}_j(x, t_r)) \right) = \left(\xi_j(x, t_r), -\frac{C_m(\xi_j(x, t_r), x, t_r)}{V_{21}^{(n)}(\xi_j(x, t_r), x, t_r)} \right) \in \mathcal{K}_{m-1}, \\ &1 \leq j \leq m-1, (x, t_r) \in \mathbb{C}^2, \end{aligned} \tag{5.14}$$

one infers from (4.1), (4.2), (5.7)–(5.11) that the divisors $(\phi_2(P, x, t_r))$ and $(\phi_3(P, x, t_r))$ of $\phi_2(P, x, t_r)$ and $\phi_3(P, x, t_r)$ are given by

$$(\phi_2(P, x, t_r)) = \mathcal{D}_{P_{\infty_3}, \hat{\nu}_1(x, t_r), \dots, \hat{\nu}_{m-1}(x, t_r)}(P) - \mathcal{D}_{P_{\infty_1}, \hat{\mu}_1(x, t_r), \dots, \hat{\mu}_{m-1}(x, t_r)}(P), \tag{5.15}$$

$$(\phi_3(P, x, t_r)) = \mathcal{D}_{P_{\infty_3}, \hat{\xi}_1(x, t_r), \dots, \hat{\xi}_{m-1}(x, t_r)}(P) - \mathcal{D}_{P_{\infty_2}, \hat{\mu}_1(x, t_r), \dots, \hat{\mu}_{m-1}(x, t_r)}(P). \tag{5.16}$$

That is, $P_{\infty_3}, \hat{\nu}_1(x, t_r), \dots, \hat{\nu}_{m-1}(x, t_r)$ are m zeros of $\phi_2(P, x, t_r)$ and $P_{\infty_1}, \hat{\mu}_1(x, t_r), \dots, \hat{\mu}_{m-1}(x, t_r)$ are m poles; $P_{\infty_3}, \hat{\xi}_1(x, t_r), \dots, \hat{\xi}_{m-1}(x, t_r)$ are m zeros of $\phi_3(P, x, t_r)$ and $P_{\infty_2}, \hat{\mu}_1(x, t_r), \dots, \hat{\mu}_{m-1}(x, t_r)$ are m poles.

Differentiating (5.5) and (5.6) with respect to t_r and using (5.1), we have

$$\begin{aligned} \phi_{2,t_r} &= \left(\frac{\psi_1}{\psi_3} \right)_{t_r} \\ &= \frac{\tilde{V}_{11}^{(r)}\psi_1 + \tilde{V}_{12}^{(r)}\psi_2 + \tilde{V}_{13}^{(r)}\psi_3}{\tilde{V}_{31}^{(r)}\psi_1 + \tilde{V}_{32}^{(r)}\psi_2 + \tilde{V}_{33}^{(r)}\psi_3} - \phi_2 \frac{\psi_3}{\tilde{V}_{31}^{(r)}\psi_1 + \tilde{V}_{32}^{(r)}\psi_2 + \tilde{V}_{33}^{(r)}\psi_3} \\ &= \tilde{V}_{13}^{(r)} + (\tilde{V}_{11}^{(r)} - \tilde{V}_{33}^{(r)})\phi_2 + \tilde{V}_{12}^{(r)}\phi_3 - \tilde{V}_{31}^{(r)}\phi_2^2 - \tilde{V}_{32}^{(r)}\phi_2\phi_3, \\ \phi_{3,t_r} &= \left(\frac{\psi_2}{\psi_3} \right)_{t_r} \\ &= \frac{\tilde{V}_{21}^{(r)}\psi_1 + \tilde{V}_{22}^{(r)}\psi_2 + \tilde{V}_{23}^{(r)}\psi_3}{\tilde{V}_{31}^{(r)}\psi_1 + \tilde{V}_{32}^{(r)}\psi_2 + \tilde{V}_{33}^{(r)}\psi_3} - \phi_3 \frac{\psi_3}{\tilde{V}_{31}^{(r)}\psi_1 + \tilde{V}_{32}^{(r)}\psi_2 + \tilde{V}_{33}^{(r)}\psi_3} \\ &= \tilde{V}_{23}^{(r)} + (\tilde{V}_{22}^{(r)} - \tilde{V}_{33}^{(r)})\phi_3 + \tilde{V}_{21}^{(r)}\phi_2 - \tilde{V}_{32}^{(r)}\phi_3^2 - \tilde{V}_{31}^{(r)}\phi_2\phi_3. \end{aligned} \tag{5.17}$$

Then we list the properties of $\phi_2(P, x, t_r)$ and $\phi_3(P, x, t_r)$ as follows:

$$\begin{aligned} \phi_{2,x}(P, x, t_r) + u_{31}(x, t_r)\phi_2^2(P, x, t_r) + u_{32}(x, t_r)\phi_2(P, x, t_r)\phi_3(P, x, t_r) \\ - (\lambda - v)\phi_2(P, x, t_r) - u_{12}(x, t_r)\phi_3(P, x, t_r) - u_{13}(x, t_r) = 0, \end{aligned} \quad (5.18)$$

$$\begin{aligned} \phi_{3,x}(P, x, t_r) + u_{32}(x, t_r)\phi_3^2(P, x, t_r) + u_{31}(x, t_r)\phi_2(P, x, t_r)\phi_3(P, x, t_r) \\ + (\lambda - v)\phi_3(P, x, t_r) - \phi_2(P, x, t_r) - u_{23}(x, t_r) = 0, \end{aligned} \quad (5.19)$$

$$\phi_{2,t_r}(P, x, t_r) = \tilde{V}_{13}^{(r)} + (\tilde{V}_{11}^{(r)} - \tilde{V}_{33}^{(r)})\phi_2 + \tilde{V}_{12}^{(r)}\phi_3 - \tilde{V}_{31}^{(r)}\phi_2^2 - \tilde{V}_{32}^{(r)}\phi_2\phi_3, \quad (5.20)$$

$$\phi_{3,t_r}(P, x, t_r) = \tilde{V}_{23}^{(r)} + (\tilde{V}_{22}^{(r)} - \tilde{V}_{33}^{(r)})\phi_3 + \tilde{V}_{21}^{(r)}\phi_2 - \tilde{V}_{32}^{(r)}\phi_3^2 - \tilde{V}_{31}^{(r)}\phi_2\phi_3, \quad (5.21)$$

$$\phi_2(P, x, t_r)\phi_2(P^*, x, t_r)\phi_2(P^{**}, x, t_r) = -\frac{F(\lambda, x, t_r)}{E(\lambda, x, t_r)}, \quad (5.22)$$

$$\phi_3(P, x, t_r)\phi_3(P^*, x, t_r)\phi_3(P^{**}, x, t_r) = \frac{\mathcal{F}(\lambda, x, t_r)}{E(\lambda, x, t_r)}, \quad (5.23)$$

$$\phi_2(P, x, t_r) + \phi_2(P^*, x, t_r) + \phi_2(P^{**}, x, t_r) = -\frac{2V_{32}^{(n)}(\lambda, x, t_r)S_m(\lambda) - 3B_m(\lambda, x, t_r)}{E(\lambda, x, t_r)}, \quad (5.24)$$

$$\phi_3(P, x, t_r) + \phi_3(P^*, x, t_r) + \phi_3(P^{**}, x, t_r) = \frac{2V_{31}^{(n)}(\lambda, x, t_r)S_m(\lambda) - 3\mathcal{B}_m(\lambda, x, t_r)}{E(\lambda, x, t_r)}, \quad (5.25)$$

$$\begin{aligned} u_{31}(x, t_r)[\phi_2(P, x, t_r) + \phi_2(P^*, x, t_r) + \phi_2(P^{**}, x, t_r)] + u_{32}(x, t_r)[\phi_3(P, x, t_r) \\ + \phi_3(P^*, x, t_r) + \phi_3(P^{**}, x, t_r)] = \frac{E_x(\lambda, x, t_r)}{E(\lambda, x, t_r)}, \end{aligned} \quad (5.26)$$

$$\begin{aligned} \frac{1}{\phi_2(P, x, t_r)} + \frac{1}{\phi_2(P^*, x, t_r)} + \frac{1}{\phi_2(P^{**}, x, t_r)} \\ = \frac{3(\lambda - v)V_{12}^{(n)}(\lambda, x, t_r) - 3u_{12}(x, t_r)V_{11}^{(n)}(\lambda, x, t_r)}{u_{12}(x, t_r)V_{13}^{(n)}(\lambda, x, t_r) - u_{13}(x, t_r)V_{12}^{(n)}(\lambda, x, t_r)} \\ - \frac{V_{12}^{(n)}(\lambda, x, t_r)F_x(\lambda, x, t_r)}{[u_{12}(x, t_r)V_{13}^{(n)}(\lambda, x, t_r) - u_{13}(x, t_r)V_{12}^{(n)}(\lambda, x, t_r)]F(\lambda, x, t_r)}, \end{aligned} \quad (5.27)$$

$$\begin{aligned} \frac{1}{\phi_3(P, x, t_r)} + \frac{1}{\phi_3(P^*, x, t_r)} + \frac{1}{\phi_3(P^{**}, x, t_r)} \\ = -\frac{3(\lambda - v)V_{21}^{(n)}(\lambda, x, t_r) + 3V_{22}^{(n)}(\lambda, x, t_r)}{V_{23}^{(n)}(\lambda, x, t_r) - u_{23}(x, t_r)V_{21}^{(n)}(\lambda, x, t_r)} \\ - \frac{V_{21}^{(n)}(\lambda, x, t_r)\mathcal{F}_x(\lambda, x, t_r)}{[V_{23}^{(n)}(\lambda, x, t_r) - u_{23}(x, t_r)V_{21}^{(n)}(\lambda, x, t_r)]\mathcal{F}(\lambda, x, t_r)}, \end{aligned} \quad (5.28)$$

$$\begin{aligned} & \frac{\phi_2(P, x, t_r)}{\phi_3(P, x, t_r)} + \frac{\phi_2(P^*, x, t_r)}{\phi_3(P^*, x, t_r)} + \frac{\phi_2(P^{**}, x, t_r)}{\phi_3(P^{**}, x, t_r)} = \frac{V_{23}^{(n)}(\lambda, x, t_r)}{V_{21}^{(n)}(\lambda, x, t_r)} \\ & \times \frac{V_{21}^{(n)}(\lambda, x, t_r)\mathcal{F}_x(\lambda, x, t_r) + 3[(\lambda - v)V_{21}^{(n)}(\lambda, x, t_r) + V_{22}^{(n)}(\lambda, x, t_r)]\mathcal{F}(\lambda, x, t_r)}{[V_{23}^{(n)}(\lambda, x, t_r) - u_{23}(x, t_r)V_{21}^{(n)}(\lambda, x, t_r)]\mathcal{F}(\lambda, x, t_r)} \\ & - 3\frac{V_{22}^{(n)}(\lambda, x, t_r)}{V_{21}^{(n)}(\lambda, x, t_r)}, \end{aligned} \tag{5.29}$$

$$\begin{aligned} & \frac{\phi_3(P, x, t_r)}{\phi_2(P, x, t_r)} + \frac{\phi_3(P^*, x, t_r)}{\phi_2(P^*, x, t_r)} + \frac{\phi_3(P^{**}, x, t_r)}{\phi_2(P^{**}, x, t_r)} = \frac{V_{13}^{(n)}(\lambda, x, t_r)}{V_{12}^{(n)}(\lambda, x, t_r)} \\ & \times \frac{V_{12}^{(n)}(\lambda, x, t_r)F_x(\lambda, x, t_r) - 3[(\lambda - v)V_{12}^{(n)}(\lambda, x, t_r) - u_{12}(x, t_r)V_{11}^{(n)}(\lambda, x, t_r)]F(\lambda, x, t_r)}{[u_{12}(x, t_r)V_{13}^{(n)}(\lambda, x, t_r) - u_{13}(x, t_r)V_{12}^{(n)}(\lambda, x, t_r)]F(\lambda, x, t_r)} \\ & - 3\frac{V_{11}^{(n)}(\lambda, x, t_r)}{V_{12}^{(n)}(\lambda, x, t_r)}. \end{aligned} \tag{5.30}$$

Lemma 5.1. *Let $(\lambda, x, t_r) \in \mathbb{C}^3$. Then*

$$\begin{aligned} E_{t_r}(\lambda, x, t_r) &= E_x(\lambda, x, t_r) \frac{V_{32}^{(n)}\tilde{V}_{31}^{(r)} - V_{31}^{(n)}\tilde{V}_{32}^{(r)}}{u_{31}V_{32}^{(n)} - u_{32}V_{31}^{(n)}} \\ &+ E \left\{ 3(\tilde{V}_{33}^{(r)} - \frac{u_{32}\tilde{V}_{31}^{(r)} - u_{31}\tilde{V}_{32}^{(r)}}{u_{32}V_{31}^{(n)} - u_{31}V_{32}^{(n)}}V_{33}^{(n)}) \right\}, \\ F_{t_r}(\lambda, x, t_r) &= F_x(\lambda, x, t_r) \frac{V_{13}^{(n)}\tilde{V}_{12}^{(r)} - V_{12}^{(n)}\tilde{V}_{13}^{(r)}}{u_{12}V_{13}^{(n)} - u_{13}V_{12}^{(n)}} \\ &+ F(\lambda, x, t_r) \left\{ 3(\tilde{V}_{11}^{(r)} - \frac{u_{13}\tilde{V}_{12}^{(r)} - u_{12}\tilde{V}_{13}^{(r)}}{u_{13}V_{12}^{(n)} - u_{12}V_{13}^{(n)}}V_{11}^{(n)}) \right. \\ &\left. - 3(\lambda - v)\frac{V_{13}^{(n)}\tilde{V}_{12}^{(r)} - V_{12}^{(n)}\tilde{V}_{13}^{(r)}}{u_{12}V_{13}^{(n)} - u_{13}V_{12}^{(n)}} \right\}, \\ \mathcal{F}_{t_r}(\lambda, x, t_r) &= \mathcal{F}_x(\lambda, x, t_r) \frac{V_{23}^{(n)}\tilde{V}_{21}^{(r)} - V_{21}^{(n)}\tilde{V}_{23}^{(r)}}{V_{23}^{(n)} - u_{23}V_{21}^{(n)}} \\ &+ \mathcal{F}(\lambda, x, t_r) \left\{ 3(\tilde{V}_{22}^{(r)} - \frac{u_{23}\tilde{V}_{21}^{(r)} - \tilde{V}_{23}^{(r)}}{V_{23}^{(n)} - u_{23}V_{21}^{(n)}}V_{22}^{(n)}) \right. \\ &\left. + 3(\lambda - v)\frac{V_{23}^{(n)}\tilde{V}_{21}^{(r)} - V_{21}^{(n)}\tilde{V}_{23}^{(r)}}{V_{23}^{(n)} - u_{23}V_{21}^{(n)}} \right\}. \end{aligned} \tag{5.31}$$

P r o o f. Differentiating (5.26) with respect to t_r , we have

$$\begin{aligned}
 \left(\frac{E_x}{E}\right)_{t_r} &= \partial_x \partial_{t_r} (\ln E) = [u_{31}(\phi_2 + \phi_2^* + \phi_2^{**}) + u_{32}(\phi_3 + \phi_3^* + \phi_3^{**})]_{t_r} \\
 &= [u_{31,t_r} + u_{31}(\tilde{V}_{11}^{(r)} - \tilde{V}_{33}^{(r)}) + u_{32}\tilde{V}_{21}^{(r)}](\phi_2 + \phi_2^* + \phi_2^{**}) \\
 &\quad + [u_{32,t_r} + u_{32}(\tilde{V}_{22}^{(r)} - \tilde{V}_{33}^{(r)}) + u_{31}\tilde{V}_{12}^{(r)}](\phi_3 + \phi_3^* + \phi_3^{**}) \\
 &\quad - u_{31}\tilde{V}_{31}^{(r)}[\phi_2^2 + (\phi_2^*)^2 + (\phi_2^{**})^2] - u_{32}\tilde{V}_{32}^{(r)}[\phi_3^2 + (\phi_3^*)^2 + (\phi_3^{**})^2] \\
 &\quad - (u_{31}\tilde{V}_{32}^{(r)} + u_{32}\tilde{V}_{31}^{(r)})(\phi_2\phi_3 + \phi_2^*\phi_3^* + \phi_2^{**}\phi_3^{**}) + 3(u_{31}\tilde{V}_{13}^{(r)} + u_{32}\tilde{V}_{23}^{(r)}) \\
 &= [\tilde{V}_{31}^{(r)}(\phi_2 + \phi_2^* + \phi_2^{**}) + \tilde{V}_{32}^{(r)}(\phi_3 + \phi_3^* + \phi_3^{**}) + 3\tilde{V}_{33}^{(r)}]_x.
 \end{aligned}$$

Thus, without loss of generality, we choose the integration constant as zero

$$\begin{aligned}
 \partial_{t_r} (\ln E(\lambda, x, t_r)) &= \tilde{V}_{31}^{(r)}(\phi_2 + \phi_2^* + \phi_2^{**}) + \tilde{V}_{32}^{(r)}(\phi_3 + \phi_3^* + \phi_3^{**}) + 3\tilde{V}_{33}^{(r)} \\
 &= \frac{E_x}{E}(\lambda, x, t_r) \frac{V_{32}^{(n)}\tilde{V}_{31}^{(r)} - V_{31}^{(n)}\tilde{V}_{32}^{(r)}}{u_{31}V_{32}^{(n)} - u_{32}V_{31}^{(n)}} \\
 &\quad + 3\left(\tilde{V}_{33}^{(r)} - \frac{u_{32}\tilde{V}_{31}^{(r)} - u_{31}\tilde{V}_{32}^{(r)}}{u_{32}V_{31}^{(n)} - u_{31}V_{32}^{(n)}}V_{33}^{(n)}\right), \tag{5.32}
 \end{aligned}$$

which implies the first equation in (5.31). Differentiating (5.22) with respect to t_r and using (5.22), (5.27), (5.30), we have

$$\begin{aligned}
 -\left(\frac{F}{E}\right)_{t_r} &= \phi_2\phi_2^*\phi_2^{**}\left(\frac{\phi_{2,t_r}}{\phi_2} + \frac{\phi_{2,t_r}^*}{\phi_2^*} + \frac{\phi_{2,t_r}^{**}}{\phi_2^{**}}\right) \\
 &= \phi_2\phi_2^*\phi_2^{**}\left[\tilde{V}_{13}^{(r)}\left(\frac{1}{\phi_2} + \frac{1}{\phi_2^*} + \frac{1}{\phi_2^{**}}\right) + \tilde{V}_{12}^{(r)}\left(\frac{\phi_3}{\phi_2} + \frac{\phi_3^*}{\phi_2^*} + \frac{\phi_3^{**}}{\phi_2^{**}}\right) + 3\tilde{V}_{11}^{(r)}\right. \\
 &\quad \left. - \tilde{V}_{31}^{(r)}(\phi_2 + \phi_2^* + \phi_2^{**}) - \tilde{V}_{32}^{(r)}(\phi_3 + \phi_3^* + \phi_3^{**}) - 3\tilde{V}_{33}^{(r)}\right] \\
 &= -\frac{F}{E}\left\{\frac{V_{13}^{(n)}\tilde{V}_{12}^{(r)} - V_{12}^{(n)}\tilde{V}_{13}^{(r)}}{u_{12}V_{13}^{(n)} - u_{13}V_{12}^{(n)}}\left[\frac{F_x}{F} - 3(\lambda - v)\right]\right. \\
 &\quad \left.+ 3\left(\tilde{V}_{11}^{(r)} - \frac{u_{13}\tilde{V}_{12}^{(r)} - u_{12}\tilde{V}_{13}^{(r)}}{u_{13}V_{12}^{(n)} - u_{12}V_{13}^{(n)}}V_{11}^{(n)}\right) - \frac{E_{t_r}}{E}\right\}. \tag{5.33}
 \end{aligned}$$

Thus we get the second expression in (5.31). The last expression can be proved in the same way.

Next, we will display the properties of $\psi_3(P, x, x_0, t_r, t_{0,r})$ in the following Lemma immediately.

Lemma 5.2. *Assume (5.1), (5.5), $P = (\lambda, y(P)) \in \mathcal{K}_{m-1} \setminus \{P_{\infty 1}, P_{\infty 2}, P_{\infty 3}\}$, and let $(\lambda, x, x_0, t_r, t_{0,r}) \in \mathbb{C}^5$. Then*

$$\begin{aligned} \psi_3(P, x, x_0, t_r, t_{0,r}) &= \exp \left(\int_{x_0}^x [u_{31}(x', t_r)\phi_2(P, x', t_r) + u_{32}(x', t_r)\phi_3(P, x', t_r)] dx' \right. \\ &\quad \left. + \int_{t_{0,r}}^{t_r} [\tilde{V}_{31}^{(r)}(\lambda, x_0, t')\phi_2(P, x_0, t') + \tilde{V}_{32}^{(r)}(\lambda, x_0, t')\phi_3(P, x_0, t') + \tilde{V}_{33}^{(r)}(\lambda, x_0, t')] dt' \right). \end{aligned} \tag{5.34}$$

In analogy to Lemma 3.1, we will show that the zeros $\{\mu_j(x, t_r)\}_{j=1, \dots, m-1}$ of $E(\lambda, x, t_r)$, $\{\nu_j(x, t_r)\}_{j=1, \dots, m-1}$ of $F(\lambda, x, t_r)$ and $\{\xi_j(x, t_r)\}_{j=1, \dots, m-1}$ of $\mathcal{F}(\lambda, x, t_r)$ obey the following Dubrovin-type equations.

Lemma 5.3.

(i) *Suppose the zeros $\{\mu_j(x, t_r)\}_{j=1, \dots, m-1}$ of $E(\lambda, x, t_r)$ remain distinct for $(x, t_r) \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}^2$ is open and connected. Then $\{\mu_j(x, t_r)\}_{j=1, \dots, m-1}$ satisfy the system of differential equations*

$$\begin{aligned} \mu_{j,x}(x, t_r) &= \frac{[u_{32}(x, t_r)V_{31}^{(n)}(\mu_j(x, t_r), x, t_r) - u_{31}(x, t_r)V_{32}^{(n)}(\mu_j(x, t_r), x, t_r)]}{2\alpha_0^3 u_{31}(x, t_r)u_{32}(x, t_r)} \\ &\quad \times \frac{[3y^2(\hat{\mu}_j(x, t_r)) + S_m(\mu_j(x, t_r))]}{\prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\mu_j(x, t_r) - \mu_k(x, t_r))}, \quad 1 \leq j \leq m-1, \end{aligned} \tag{5.35}$$

$$\begin{aligned} \mu_{j,t_r}(x, t_r) &= \frac{[\tilde{V}_{32}^{(r)}V_{31}^{(n)} - \tilde{V}_{31}^{(r)}V_{32}^{(n)}] \Big|_{\lambda=\mu_j(x, t_r)}}{2\alpha_0^3 u_{31}(x, t_r)u_{32}(x, t_r)} \\ &\quad \times \frac{[3y^2(\hat{\mu}_j(x, t_r)) + S_m(\mu_j(x, t_r))]}{\prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\mu_j(x, t_r) - \mu_k(x, t_r))}, \quad 1 \leq j \leq m-1; \end{aligned} \tag{5.36}$$

(ii) *Suppose the zeros $\{\nu_j(x, t_r)\}_{j=1, \dots, m-1}$ of $F(\lambda, x, t_r)$ remain distinct for $(x, t_r) \in \Omega_\nu$, where $\Omega_\nu \subseteq \mathbb{C}^2$ is open and connected. Then $\{\nu_j(x, t_r)\}_{j=1, \dots, m-1}$ satisfy*

the system of differential equations

$$\nu_{j,x}(x, t_r) = \frac{[u_{12}(x, t_r)V_{13}^{(n)}(\nu_j(x, t_r), x, t_r) - u_{13}(x, t_r)V_{12}^{(n)}(\nu_j(x, t_r), x, t_r)]}{\alpha_0^3 u_{12}(x, t_r) u_{13}(x, t_r)} \times \frac{[3y^2(\hat{\nu}_j(x, t_r)) + S_m(\nu_j(x, t_r))]}{\prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\nu_j(x, t_r) - \nu_k(x, t_r))}, \quad 1 \leq j \leq m-1, \quad (5.37)$$

$$\nu_{j,t_r}(x, t_r) = \frac{[\tilde{V}_{12}^{(r)} V_{13}^{(n)} - V_{12}^{(n)} \tilde{V}_{13}^{(r)}] \Big|_{\lambda=\nu_j(x, t_r)}}{\alpha_0^3 u_{12}(x, t_r) u_{13}(x, t_r)} \times \frac{[3y^2(\hat{\nu}_j(x, t_r)) + S_m(\nu_j(x, t_r))]}{\prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\nu_j(x, t_r) - \nu_k(x, t_r))}, \quad 1 \leq j \leq m-1; \quad (5.38)$$

(iii) Suppose the zeros $\{\xi_j(x, t_r)\}_{j=1, \dots, m-1}$ of $\mathcal{F}(\lambda, x, t_r)$ remain distinct for $(x, t_r) \in \Omega_\xi$, where $\Omega_\xi \subseteq \mathbb{C}^2$ is open and connected. Then $\{\xi_j(x, t_r)\}_{j=1, \dots, m-1}$ satisfy the system of differential equations

$$\xi_{j,x}(x, t_r) = \frac{[u_{23}(x, t_r)V_{21}^{(n)}(\xi_j(x, t_r), x, t_r) - V_{23}^{(n)}(\xi_j(x, t_r), x, t_r)]}{\alpha_0^3 u_{23}(x, t_r)} \times \frac{[3y^2(\hat{\xi}_j(x, t_r)) + S_m(\xi_j(x, t_r))]}{\prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\xi_j(x, t_r) - \xi_k(x, t_r))}, \quad 1 \leq j \leq m-1, \quad (5.39)$$

$$\xi_{j,t_r}(x, t_r) = \frac{[\tilde{V}_{23}^{(r)} V_{21}^{(n)} - V_{23}^{(n)} \tilde{V}_{21}^{(r)}] \Big|_{\lambda=\xi_j(x, t_r)}}{\alpha_0^3 u_{23}(x, t_r)} \times \frac{[3y^2(\hat{\xi}_j(x, t_r)) + S_m(\xi_j(x, t_r))]}{\prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\xi_j(x, t_r) - \xi_k(x, t_r))}, \quad 1 \leq j \leq m-1. \quad (5.40)$$

For the sake of convenience, we introduce the notations

$$\tilde{V}_{3j}^{(r)} = \begin{cases} \tilde{V}_{3j}^{(r,1,0)} = \sum_{k=0}^r \tilde{V}_{3j,k}^{(1,0)} \lambda^{r-k}, & j = 1, 2, 3, \\ \tilde{V}_{3j}^{(r,0,1)} = \sum_{k=0}^r \tilde{V}_{3j,k}^{(0,1)} \lambda^{r-k}, & j = 1, 2, 3, \end{cases},$$

where

$$\begin{aligned} \tilde{V}_{3j,k}^{(1,0)} &= \tilde{V}_{3j,k} |_{\tilde{\alpha}_0=1, \tilde{\beta}_0=0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_r, \tilde{\beta}_1, \dots, \tilde{\beta}_r \in \mathbb{R}}, & j = 1, 2, 3, \\ \tilde{V}_{3j,k}^{(0,1)} &= \tilde{V}_{3j,k} |_{\tilde{\alpha}_0=0, \tilde{\beta}_0=1, \tilde{\alpha}_1, \dots, \tilde{\alpha}_r, \tilde{\beta}_1, \dots, \tilde{\beta}_r \in \mathbb{R}}, & j = 1, 2, 3, \end{aligned}$$

and the corresponding homogeneous cases

$$\tilde{\tilde{V}}_{3j}^{(r)} = \begin{cases} \tilde{\tilde{V}}_{3j}^{(r,1,0)} = \sum_{k=0}^r \tilde{\tilde{V}}_{3j,k}^{(1,0)} \lambda^{r-k}, & j = 1, 2, 3, \\ \tilde{\tilde{V}}_{3j}^{(r,0,1)} = \sum_{k=0}^r \tilde{\tilde{V}}_{3j,k}^{(0,1)} \lambda^{r-k}, & j = 1, 2, 3, \end{cases}$$

with

$$\begin{aligned} \tilde{\tilde{V}}_{3j,k}^{(1,0)} &= \tilde{\tilde{V}}_{3j,k} |_{\tilde{\alpha}_0=1, \tilde{\beta}_0=0, \tilde{\alpha}_1=\dots=\tilde{\alpha}_r=\tilde{\beta}_1=\dots=\tilde{\beta}_r=0}, & j = 1, 2, 3, \\ \tilde{\tilde{V}}_{3j,k}^{(0,1)} &= \tilde{\tilde{V}}_{3j,k} |_{\tilde{\alpha}_0=0, \tilde{\beta}_0=1, \tilde{\alpha}_1=\dots=\tilde{\alpha}_r=\tilde{\beta}_1=\dots=\tilde{\beta}_r=0}, & j = 1, 2, 3. \end{aligned}$$

From (5.34), one defines a function $I_r(P, x, t_r)$ by

$$I_r(P, x, t_r) = \tilde{V}_{31}^{(r)}(\lambda, x, t_r) \phi_2(P, x, t_r) + \tilde{V}_{32}^{(r)}(\lambda, x, t_r) \phi_3(P, x, t_r) + \tilde{V}_{33}^{(r)}(\lambda, x, t_r), \tag{5.41}$$

and the associated homogeneous function $\bar{I}_r(P, x, t_r)$ by replacing $\tilde{V}_{31}^{(r)}, \tilde{V}_{32}^{(r)}, \tilde{V}_{33}^{(r)}$ with the corresponding homogeneous polynomials $\tilde{\tilde{V}}_{31}^{(r)}, \tilde{\tilde{V}}_{32}^{(r)}, \tilde{\tilde{V}}_{33}^{(r)}$, that is,

$$\begin{aligned} \bar{I}_r(P, x, t_r) &= \tilde{\tilde{V}}_{31}^{(r)}(\lambda, x, t_r) \phi_2(P, x, t_r) + \tilde{\tilde{V}}_{32}^{(r)}(\lambda, x, t_r) \phi_3(P, x, t_r) \\ &\quad + \tilde{\tilde{V}}_{33}^{(r)}(\lambda, x, t_r). \end{aligned} \tag{5.42}$$

Lemma 5.4. Assume $(x, t_r) \in \mathbb{C}^2$, $\lambda = \zeta^{-1}$ denotes the local coordinate near P_{∞_j} , $j = 1, 2, 3$. Then

$$\bar{I}_r^{(1,0)} = \begin{cases} \zeta^{-r-1} - \frac{1}{u_{31}} \tilde{\tilde{V}}_{31,r+1}^{(1,0)} + O(\zeta), & P \rightarrow P_{\infty_1}, \\ -\zeta^{-r-1} + \frac{1}{u_{32}} \tilde{\tilde{V}}_{32,r+1}^{(1,0)} + O(\zeta), & P \rightarrow P_{\infty_2}, \\ O(\zeta), & P \rightarrow P_{\infty_3}, \end{cases} \tag{5.43}$$

$$\bar{I}_r^{(0,1)} = \begin{cases} -\frac{1}{2} \zeta^{-r} - \frac{1}{u_{31}} \tilde{\tilde{V}}_{31,r+1}^{(0,1)} + O(\zeta), & P \rightarrow P_{\infty_1}, \\ -\frac{1}{2} \zeta^{-r} + \frac{1}{u_{32}} \tilde{\tilde{V}}_{32,r+1}^{(0,1)} + O(\zeta), & P \rightarrow P_{\infty_2}, \\ \zeta^{-r} + O(\zeta), & P \rightarrow P_{\infty_3}. \end{cases} \tag{5.44}$$

P r o o f. We just prove (5.43) and accordingly obtain (5.44). By (5.42), it is easy to see that

$$\begin{aligned} \bar{I}_1^{(1,0)} &= u_{31}\phi_2 + u_{32}\phi_3 \\ &= \begin{cases} \zeta^{-1} - \frac{1}{u_{31}}\tilde{V}_{31,1}^{(1,0)} + O(\zeta), & P \rightarrow P_{\infty_1}, \\ -\zeta^{-1} + \frac{1}{u_{32}}\tilde{V}_{32,1}^{(1,0)} + O(\zeta), & P \rightarrow P_{\infty_2}, \\ -\tilde{V}_{33,1}^{(1,0)}\zeta + O(\zeta^2), & P \rightarrow P_{\infty_3}. \end{cases} \end{aligned} \quad (5.45)$$

So (5.43) holds for $r = 0$. With the aid of (5.1), (5.2), (5.18)-(5.21), we arrive at

$$\begin{aligned} &\partial_x \bar{I}_r^{(1,0)}(P, x, t_r) \\ &= \partial_x [\tilde{V}_{31}^{(r,1,0)}(\lambda, x, t_r)\phi_2(P, x, t_r) + \tilde{V}_{32}^{(r,1,0)}(\lambda, x, t_r)\phi_3(P, x, t_r) + \tilde{V}_{33}^{(r,1,0)}(\lambda, x, t_r)] \\ &= \partial_{t_r} [u_{31}(x, t_r)\phi_2(P, x, t_r) + u_{32}(x, t_r)\phi_3(P, x, t_r)]. \end{aligned} \quad (5.46)$$

I. When $P \rightarrow P_{\infty_1}$, by investigating (5.45), one can assume that $\bar{I}_r^{(1,0)}$ has the expansion

$$\bar{I}_r^{(1,0)}(P, x, t_r) \underset{\zeta \rightarrow 0}{=} \zeta^{-r-1} + \sum_{j=0}^{\infty} \varsigma_{1j}(x, t_r)\zeta^j, \text{ as } P \rightarrow P_{\infty_1}, \quad (5.47)$$

for some coefficients $\{\varsigma_{1j}(x, t_r)\}_{j \in \mathbb{N}_0}$. Suppose that (as $P \rightarrow P_{\infty_1}$)

$$\phi_2 = \sum_{j=0}^{\infty} \kappa_{1,j-1}\zeta^{j-1}, \quad \phi_3 = \sum_{j=0}^{\infty} \chi_{1,j}\zeta^j, \quad (5.48)$$

where $\kappa_{1,j-1}$ and $\chi_{1,j}$ are defined in (4.1) and (4.2). Inserting (5.47), (5.48) into (5.46) and comparing the same powers of ζ imply

$$\begin{aligned} \varsigma_{1j,x} &= (u_{31}\kappa_{1,j} + u_{32}\chi_{1,j})_{t_r}, \quad j = 0, 1, 2, 3, \dots, \\ \varsigma_{10,x} &= (u_{31}\kappa_{1,0} + u_{32}\chi_{1,0})_{t_r} = \left[-\frac{1}{u_{31}}\tilde{V}_{31,r+1}^{(1,0)}\right]_x, \\ \varsigma_{11,x} &= (u_{31}\kappa_{1,1} + u_{32}\chi_{1,1})_{t_r} \\ &= -\left[\tilde{V}_{33,r+1}^{(1,0)} + \frac{2u_{31,x} - 2vu_{31} - u_{32}}{2u_{31}^2}\tilde{V}_{31,r+1}^{(1,0)} + \frac{1}{2u_{31}}(\tilde{V}_{32,r+1}^{(1,0)} + 2\tilde{V}_{31,r+2}^{(1,0)})\right]_x, \end{aligned} \quad (5.49)$$

from which one infers

$$\begin{aligned} \varsigma_{10}(x, t_r) &= \epsilon_{1,0}(t_r) - \frac{1}{u_{31}}\tilde{V}_{31,r+1}^{(1,0)}, \\ \varsigma_{11}(x, t_r) &= \epsilon_{1,1}(t_r) - \left[\tilde{V}_{33,r+1}^{(1,0)} + \frac{2u_{31,x} - 2vu_{31} - u_{32}}{2u_{31}^2}\tilde{V}_{31,r+1}^{(1,0)} + \frac{1}{2u_{31}}(\tilde{V}_{32,r+1}^{(1,0)} + 2\tilde{V}_{31,r+2}^{(1,0)})\right], \end{aligned} \quad (5.50)$$

where $\epsilon_{1,0}(t_r), \epsilon_{1,1}(t_r)$ are integration constants. Noting that the coefficients of the power series for $\phi_2(P, x, t_r)$ and $\phi_3(P, x, t_r)$ near P_{∞_1} , and the coefficients of the homogeneous polynomials $\tilde{V}_{3j,r+1}^{(1,0)}(\zeta, x, t_r)$ are differential polynomials in u , with no arbitrary integration constants in their construction, and the definition of $\tilde{I}_r^{(1,0)}$, it follows that it also can have no arbitrary integration constants and must consist purely of differential polynomials in u . All the above considerations mean $\epsilon_{1,0}(t_r) = \epsilon_{1,1}(t_r) = 0$. Hence one concludes

$$\begin{aligned} \tilde{I}_r^{(1,0)}(P, x, t_r) &= \zeta^{-r-1} - \frac{1}{u_{31}} \tilde{V}_{31,r+1}^{(1,0)} - [\tilde{V}_{33,r+1}^{(1,0)} + \frac{2u_{31,x} - 2vu_{31} - u_{32}}{2u_{31}^2} \tilde{V}_{31,r+1}^{(1,0)} \\ &\quad + \frac{1}{2u_{31}} (\tilde{V}_{32,r+1}^{(1,0)} + 2\tilde{V}_{31,r+2}^{(1,0)})] \zeta + O(\zeta^2), \quad \text{as } P \rightarrow P_{\infty_1}. \end{aligned} \tag{5.51}$$

On the other hand, we have

$$\begin{aligned} \tilde{I}_{r+1}^{(1,0)}(P, x, t_r) &= \tilde{V}_{31}^{(r+1,1,0)} \phi_2 + \tilde{V}_{32}^{(r+1,1,0)} \phi_3 + \tilde{V}_{33}^{(r+1,1,0)} \\ &= \zeta^{-1} \tilde{I}_r^{(1,0)} + \tilde{V}_{31,r+1}^{(1,0)} \phi_2 + \tilde{V}_{32,r+1}^{(1,0)} \phi_3 + \tilde{V}_{33,r+1}^{(1,0)} \\ &= \zeta^{-r-2} - \frac{1}{u_{31}} \tilde{V}_{31,r+2}^{(1,0)} + O(\zeta), \quad \text{as } P \rightarrow P_{\infty_1}. \end{aligned} \tag{5.52}$$

II. When $P \rightarrow P_{\infty_2}$, by investigating (5.45), one can assume that $\tilde{I}_r^{(1,0)}$ has the expansion

$$\tilde{I}_r^{(1,0)}(P, x, t_r) \underset{\zeta \rightarrow 0}{=} -\zeta^{-r-1} + \sum_{j=0}^{\infty} \varsigma_{2j}(x, t_r) \zeta^j, \quad \text{as } P \rightarrow P_{\infty_2}, \tag{5.53}$$

for some coefficients $\{\varsigma_{2j}(x, t_r)\}_{j \in \mathbb{N}_0}$. Suppose that (as $P \rightarrow P_{\infty_2}$)

$$\phi_2 = \sum_{j=0}^{\infty} \kappa_{2,j} \zeta^j, \quad \phi_3 = \sum_{j=0}^{\infty} \chi_{2,j-1} \zeta^{j-1}, \tag{5.54}$$

where $\kappa_{2,j}$ and $\chi_{2,j-1}$ are defined in (4.1) and (4.2). Substituting (5.53), (5.54) into (5.46) and comparing the same powers of ζ imply

$$\begin{aligned} \varsigma_{2j,x} &= (u_{31} \kappa_{2,j} + u_{32} \chi_{2,j})_{t_r}, \quad j = 0, 1, 2, 3, \dots, \\ \varsigma_{20,x} &= (u_{31} \kappa_{2,0} + u_{32} \chi_{2,0})_{t_r} = \left[\frac{1}{u_{32}} \tilde{V}_{32,r+1}^{(1,0)} \right]_x, \\ \varsigma_{21,x} &= (u_{31} \kappa_{2,1} + u_{32} \chi_{2,1})_{t_r} \\ &= -\left[\tilde{V}_{33,r+1}^{(1,0)} + \frac{2u_{32,x} + 2vu_{32} - u_{12}u_{31}}{2u_{32}^2} \tilde{V}_{32,r+1}^{(1,0)} + \frac{1}{2u_{32}} (u_{12} \tilde{V}_{31,r+1}^{(1,0)} \right. \\ &\quad \left. - 2\tilde{V}_{32,r+2}^{(1,0)}) \right]_x, \end{aligned} \tag{5.55}$$

from which one infers

$$\begin{aligned} \varsigma_{20}(x, t_r) &= \epsilon_{2,0}(t_r) + \frac{1}{u_{32}} \tilde{V}_{32,r+1}^{(1,0)}, \\ \varsigma_{21}(x, t_r) &= \epsilon_{2,1}(t_r) - \left[\tilde{V}_{33,r+1}^{(1,0)} + \frac{2u_{32,x} + 2vu_{32} - u_{12}u_{31}}{2u_{32}^2} \tilde{V}_{32,r+1}^{(1,0)} + \frac{1}{2u_{32}} (u_{12} \tilde{V}_{31,r+1}^{(1,0)} - 2\tilde{V}_{32,r+2}^{(1,0)}) \right], \end{aligned} \tag{5.56}$$

where $\epsilon_{2,0}(t_r), \epsilon_{2,1}(t_r)$ are integration constants. Manage together, we find that $\epsilon_{2,0}(t_r) = \epsilon_{2,1}(t_r) = 0$. Therefore,

$$\begin{aligned} \bar{I}_r^{(1,0)}(P, x, t_r) &= \frac{1}{u_{13}} \tilde{V}_{13,r+1}^{(1,0)} - \left[\tilde{V}_{33,r+1}^{(1,0)} + \frac{2u_{32,x} + 2vu_{32} - u_{12}u_{31}}{2u_{32}^2} \tilde{V}_{32,r+1}^{(1,0)} + \frac{1}{2u_{32}} (u_{12} \tilde{V}_{31,r+1}^{(1,0)} - 2\tilde{V}_{32,r+2}^{(1,0)}) \right] \zeta + O(\zeta^2), \quad \text{as } P \rightarrow P_{\infty_2}. \end{aligned} \tag{5.57}$$

On the other hand,

$$\begin{aligned} \bar{I}_{r+1}^{(1,0)}(P, x, t_r) &= \tilde{V}_{31}^{(r+1,1,0)} \phi_2 + \tilde{V}_{32}^{(r+1,1,0)} \phi_3 + \tilde{V}_{33}^{(r+1,1,0)} \\ &= \zeta^{-1} \bar{I}_r^{(1,0)} + \tilde{V}_{31,r+1}^{(1,0)} \phi_2 + \tilde{V}_{32,r+1}^{(1,0)} \phi_3 + \tilde{V}_{33,r+1}^{(1,0)} \\ &= -\zeta^{-r-2} + \frac{1}{u_{32}} \tilde{V}_{32,r+2}^{(1,0)} + O(\zeta), \quad \text{as } P \rightarrow P_{\infty_2}. \end{aligned} \tag{5.58}$$

III. When $P \rightarrow P_{\infty_3}$, by (5.45), one can assume that $\bar{I}_r^{(1,0)}$ has the expansion

$$\bar{I}_r^{(1,0)}(P, x, t_r) \underset{\zeta \rightarrow 0}{=} \sum_{j=1}^{\infty} \varsigma_{3j}(x, t_r) \zeta^j, \quad \text{as } P \rightarrow P_{\infty_3}, \tag{5.59}$$

for some coefficients $\{\varsigma_{3j}(x, t_r)\}_{j \in \mathbb{N}}$. Assume that (as $P \rightarrow P_{\infty_3}$)

$$\phi_2 = \sum_{j=1}^{\infty} \kappa_{3,j} \zeta^j, \quad \phi_3 = \sum_{j=1}^{\infty} \chi_{3,j} \zeta^j, \tag{5.60}$$

where $\kappa_{3,j}$ and $\chi_{3,j}$ are defined in (4.1) and (4.2). Inserting (5.59), (5.60) into (5.46) and comparing the same powers of ζ imply

$$\begin{aligned} \varsigma_{3j,x} &= (u_{31} \kappa_{3,j} + u_{32} \chi_{3,j})_{t_r}, \quad j = 1, 2, 3, \dots, \\ \varsigma_{31,x} &= (u_{31} \kappa_{3,1} + u_{32} \chi_{3,1})_{t_r} = [-\tilde{V}_{33,r+1}^{(1,0)}]_x, \end{aligned} \tag{5.61}$$

from which one infers

$$\varsigma_{31}(x, t_r) = \epsilon_{3,1}(t_r) - \tilde{V}_{33,r+1}^{(1,0)}, \tag{5.62}$$

where $\epsilon_{3,1}(t_r)$ are integration constants. In the same way, we conclude that $\epsilon_{3,1}(t_r) = 0$. Thus,

$$\bar{I}_r^{(1,0)}(P, x, t_r) = -\bar{V}_{33,r+1}^{(1,0)}\zeta + O(\zeta^2), \text{ as } P \rightarrow P_{\infty_3}. \tag{5.63}$$

On the other hand, we have

$$\begin{aligned} \bar{I}_{r+1}^{(1,0)}(P, x, t_r) &= \bar{V}_{31}^{(r+1,1,0)}\phi_2 + \bar{V}_{32}^{(r+1,1,0)}\phi_3 + \bar{V}_{33}^{(r+1,1,0)} \\ &= \zeta^{-1}\bar{I}_r^{(1,0)} + \bar{V}_{31,r+1}^{(1,0)}\phi_2 + \bar{V}_{32,r+1}^{(1,0)}\phi_3 + \bar{V}_{33,r+1}^{(1,0)} \\ &= O(\zeta), \text{ as } P \rightarrow P_{\infty_3}. \end{aligned} \tag{5.64}$$

To sum up, we complete the proof of (5.43). Similarly, we can show that (5.44) holds.

From (5.1) one gets

$$I_r(P, x, t_r) = \sum_{l=0}^r \tilde{\alpha}_{r-l} \bar{I}_l^{(1,0)}(P, x, t_r) + \sum_{l=0}^r \tilde{\beta}_{r-l} \bar{I}_l^{(0,1)}(P, x, t_r). \tag{5.65}$$

Thus,

$$\int_{t_{0,r}}^{t_r} I_r(P, x, t') dt' \underset{\zeta \rightarrow 0}{=} \begin{cases} (t_r - t_{0,r}) \left(\sum_{l=0}^r \tilde{\alpha}_{r-l} \zeta^{-l-1} - \frac{1}{2} \sum_{l=0}^r \tilde{\beta}_{r-l} \zeta^{-l} \right) + \ln \frac{u_{31}(x, t_r)}{u_{31}(x, t_{0,r})} \\ \quad - \int_{x_0}^x v(x', t_r) dx' + O(\zeta), \text{ as } P \rightarrow P_{\infty_1}, \\ (t_r - t_{0,r}) \left(\sum_{l=0}^r \tilde{\alpha}_{r-l} \zeta^{-l-1} + \frac{1}{2} \sum_{l=0}^r \tilde{\beta}_{r-l} \zeta^{-l} \right) + \ln \frac{u_{32}(x, t_r)}{u_{32}(x, t_{0,r})} \\ \quad + \int_{x_0}^x v(x', t_r) dx' + O(\zeta), \text{ as } P \rightarrow P_{\infty_2}, \\ (t_r - t_{0,r}) \sum_{l=0}^r \tilde{\beta}_{r-l} \zeta^{-l} + O(\zeta), \text{ as } P \rightarrow P_{\infty_3}. \end{cases} \tag{5.66}$$

Let $\omega_{P_{\infty_l},j}^{(2)}, j \in \mathbb{N}, l = 1, 2, 3$, be the normalized differential of the second kind holomorphic on $\mathcal{K}_{m-1} \setminus \{P_{\infty_l}\}$ with a pole of order j at P_{∞_l} ,

$$\omega_{P_{\infty_l},j}^{(2)}(P) \underset{\zeta \rightarrow 0}{=} (\zeta^{-j} + O(1))d\zeta, \text{ as } P \rightarrow P_{\infty_l}. \tag{5.67}$$

Furthermore, we introduce the normalized differential of the second kind

$$\begin{aligned} \tilde{\Omega}_{r+1}^{(2)}(P) &= \sum_{l=0}^r \tilde{\alpha}_{r-l}(l+1)(\omega_{P_{\infty_1},l+2}^{(2)} - \omega_{P_{\infty_2},l+2}^{(2)}) - \frac{1}{2} \sum_{l=0}^r \tilde{\beta}_{r-l}l(\omega_{P_{\infty_1},l+1}^{(2)} \\ &\quad + \omega_{P_{\infty_2},l+1}^{(2)} - 2\omega_{P_{\infty_3},l+1}^{(2)}). \end{aligned} \tag{5.68}$$

In addition, we define the vector of b -periods of the differential of the second kind $\tilde{\Omega}_{r+1}^{(2)}$,

$$\tilde{U}_{r+1}^{(2)} = (\tilde{U}_{r+1,1}^{(2)}, \dots, \tilde{U}_{r+1,m-1}^{(2)}), \quad \tilde{U}_{r+1,k}^{(2)} = \frac{1}{2\pi i} \int_{b_k} \tilde{\Omega}_{r+1}^{(2)}, \quad k = 1, \dots, m-1. \tag{5.69}$$

Integrating (5.68), we have

$$\int_{Q_0}^P \tilde{\Omega}_{r+1}^{(2)} = \begin{cases} -\left(\sum_{l=0}^r \tilde{\alpha}_{r-l} \zeta^{-l-1} - \frac{1}{2} \sum_{l=0}^r \tilde{\beta}_{r-l} \zeta^{-l}\right) + \tilde{e}_1^{(2)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_1}, \\ \left(\sum_{l=0}^r \tilde{\alpha}_{r-l} \zeta^{-l-1} + \frac{1}{2} \sum_{l=0}^r \tilde{\beta}_{r-l} \zeta^{-l}\right) + \tilde{e}_2^{(2)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_2}, \\ -\sum_{l=0}^r \tilde{\beta}_{r-l} \zeta^{-l} + \tilde{e}_3^{(2)}(Q_0) + O(\zeta), & \text{as } P \rightarrow P_{\infty_3}, \end{cases} \tag{5.70}$$

where $\tilde{e}_1^{(2)}(Q_0), \tilde{e}_2^{(2)}(Q_0), \tilde{e}_3^{(2)}(Q_0)$ are constants.

Given these preparations, the theta function representations of $\phi_2(P, x, t_r), \phi_3(P, x, t_r), \psi_3(P, x, x_0, t_r, t_{0,r})$, and the algebro-geometric solutions of the entire hierarchy read as follows.

Theorem 5.5. *Assume that the curve \mathcal{K}_{m-1} is nonsingular. Let $P = (\lambda, y) \in \mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$ and let $(x, x_0, t_r, t_{0,r}) \in \mathbb{C}^4$. Suppose that $\mathcal{D}_{\hat{\mu}(x,t_r)}$ or $\mathcal{D}_{\hat{\nu}(x,t_r)}$ or $\mathcal{D}_{\hat{\xi}(x,t_r)}$ is nonspecial. Then*

$$\phi_2(P, x, t_r) = N_1(x, x_0, t_r, t_{0,r}) \frac{\theta(\underline{z}(P, \hat{\nu}(x, t_r)))}{\theta(\underline{z}(P, \hat{\mu}(x, t_r)))} \exp\left(\int_{Q_0}^P \omega_{P_{\infty_3}, P_{\infty_1}}^{(3)}\right), \tag{5.71}$$

$$\phi_3(P, x, t_r) = N_2(x, x_0, t_r, t_{0,r}) \frac{\theta(\underline{z}(P, \hat{\xi}(x, t_r)))}{\theta(\underline{z}(P, \hat{\mu}(x, t_r)))} \exp\left(\int_{Q_0}^P \omega_{P_{\infty_3}, P_{\infty_2}}^{(3)}\right), \tag{5.72}$$

$$\begin{aligned} \psi_3(P, x, x_0, t_r, t_{0,r}) &= \frac{\theta(\underline{z}(P, \hat{\mu}(x, t_r)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x, t_r)))\theta(\underline{z}(P, \hat{\mu}(x_0, t_{0,r})))} \\ &\times \exp\left(\left(e_3^{(2)}(Q_0) - \int_{Q_0}^P \Omega^{(2)}\right)(x - x_0) + \left(\tilde{e}_3^{(2)}(Q_0) - \int_{Q_0}^P \tilde{\Omega}_{r+1}^{(2)}\right)(t_r - t_{0,r})\right), \end{aligned} \tag{5.73}$$

where $c(x, x_0, t_r) = \exp(\int_{x_0}^x v(x', t_r) dx')$,

$$\begin{aligned} c^2(x, x_0, t_r) &= -\frac{u_{32}(x_0, t_{0,r})}{2u_{31}(x_0, t_{0,r})} \frac{\theta(\underline{z}(P_{\infty_1}, \hat{\underline{\mu}}(x_0, t_{0,r})))\theta(\underline{z}(P_{\infty_2}, \hat{\underline{\xi}}(x, t_r)))}{\theta(\underline{z}(P_{\infty_2}, \hat{\underline{\mu}}(x_0, t_{0,r})))\theta(\underline{z}(P_{\infty_1}, \hat{\underline{\xi}}(x, t_r)))} \\ &\times \exp\left((e_1^{(2)}(Q_0) - e_2^{(2)}(Q_0))(x - x_0) + (\tilde{e}_1^{(2)}(Q_0) - \tilde{e}_2^{(2)}(Q_0))(t_r - t_{0,r})\right) \\ &\times \exp\left(e_{2,\infty_2}^{(3)}(Q_0) - e_{2,\infty_1}^{(3)}(Q_0)\right), \end{aligned} \tag{5.74}$$

$$\begin{aligned} N_1(x, x_0, t_r, t_{0,r}) &= \frac{1}{u_{31}(x_0)c(x, x_0, t_r)} \frac{\theta(\underline{z}(P_{\infty_1}, \hat{\underline{\mu}}(x_0, t_{0,r})))\theta(\underline{z}(P_{\infty_3}, \hat{\underline{\mu}}(x, t_r)))}{\theta(\underline{z}(P_{\infty_3}, \hat{\underline{\mu}}(x_0, t_{0,r})))\theta(\underline{z}(P_{\infty_1}, \hat{\underline{\nu}}(x, t_r)))} \\ &\times \exp\left((e_1^{(2)}(Q_0) - e_3^{(2)}(Q_0))(x - x_0) + (\tilde{e}_1^{(2)}(Q_0) - \tilde{e}_3^{(2)}(Q_0))(t_r - t_{0,r})\right) \\ &\times \exp\left(-e_{1,\infty_1}^{(3)}(Q_0)\right), \end{aligned} \tag{5.75}$$

$$\begin{aligned} N_2(x, x_0, t_r, t_{0,r}) &= \frac{1}{2u_{31}(x_0)c(x, x_0, t_r)} \frac{\theta(\underline{z}(P_{\infty_1}, \hat{\underline{\mu}}(x_0, t_{0,r})))\theta(\underline{z}(P_{\infty_3}, \hat{\underline{\mu}}(x, t_r)))}{\theta(\underline{z}(P_{\infty_3}, \hat{\underline{\mu}}(x_0, t_{0,r})))\theta(\underline{z}(P_{\infty_1}, \hat{\underline{\xi}}(x, t_r)))} \\ &\times \exp\left((e_1^{(2)}(Q_0) - e_3^{(2)}(Q_0))(x - x_0) + (\tilde{e}_1^{(2)}(Q_0) - \tilde{e}_3^{(2)}(Q_0))(t_r - t_{0,r})\right) \\ &\times \exp\left(-e_{2,\infty_1}^{(3)}(Q_0)\right). \end{aligned} \tag{5.76}$$

Finally, $u_{12}(x, t_r), u_{13}(x, t_r), u_{23}(x, t_r), u_{31}(x, t_r), u_{32}(x, t_r), v(x, t_r)$ are of the form

$$\begin{aligned} u_{12}(x, t_r) &= -4 \frac{\theta(\underline{z}(P_{\infty_2}, \hat{\underline{\nu}}(x, t_r)))\theta(\underline{z}(P_{\infty_1}, \hat{\underline{\xi}}(x, t_r)))}{\theta(\underline{z}(P_{\infty_1}, \hat{\underline{\nu}}(x, t_r)))\theta(\underline{z}(P_{\infty_2}, \hat{\underline{\xi}}(x, t_r)))} \\ &\times \exp\left(e_{1,\infty_2}^{(3)}(Q_0) - e_{1,\infty_1}^{(3)}(Q_0) + e_{2,\infty_1}^{(3)}(Q_0) - e_{2,\infty_2}^{(3)}(Q_0)\right), \end{aligned} \tag{5.77}$$

$$\begin{aligned} u_{13}(x, t_r) &= -\frac{1}{u_{31}(x_0, t_{0,r})c(x, x_0, t_r)} \frac{\theta(\underline{z}(P_{\infty_3}, \hat{\underline{\nu}}(x, t_r)))\theta(\underline{z}(P_{\infty_1}, \hat{\underline{\mu}}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_1}, \hat{\underline{\nu}}(x, t_r)))\theta(\underline{z}(P_{\infty_3}, \hat{\underline{\mu}}(x_0, t_{0,r})))} \\ &\times \exp\left((e_1^{(2)}(Q_0) - e_3^{(2)}(Q_0))(x - x_0) + (\tilde{e}_1^{(2)}(Q_0) - \tilde{e}_3^{(2)}(Q_0))(t_r - t_{0,r})\right) \\ &\times \exp\left(e_{1,\infty_3}^{(3)}(Q_0) - e_{1,\infty_1}^{(3)}(Q_0)\right), \end{aligned} \tag{5.78}$$

$$\begin{aligned}
 u_{23}(x, t_r) &= \frac{1}{2u_{31}(x_0, t_{0,r})c(x, x_0, t_r)} \frac{\theta(\underline{z}(P_{\infty_3}, \hat{\xi}(x, t_r)))\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_1}, \hat{\xi}(x, t_r)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0, t_{0,r})))} \\
 &\times \exp\left((e_1^{(2)}(Q_0) - e_3^{(2)}(Q_0))(x - x_0) + (\tilde{e}_1^{(2)}(Q_0) - \tilde{e}_3^{(2)}(Q_0))(t_r - t_{0,r})\right) \\
 &\times \exp\left(e_{2,\infty_3}^{(3)}(Q_0) - e_{2,\infty_1}^{(3)}(Q_0)\right), \tag{5.79}
 \end{aligned}$$

$$\begin{aligned}
 u_{31}(x, t_r) &= u_{31}(x_0, t_{0,r})c(x, x_0, t_r) \frac{\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x, t_r)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x, t_r)))\theta(\underline{z}(P_{\infty_1}, \hat{\mu}(x_0, t_{0,r})))} \\
 &\times \exp\left((e_3^{(2)}(Q_0) - e_1^{(2)}(Q_0))(x - x_0) + (\tilde{e}_3^{(2)}(Q_0) - \tilde{e}_1^{(2)}(Q_0))(t_r - t_{0,r})\right), \tag{5.80}
 \end{aligned}$$

$$\begin{aligned}
 u_{32}(x, t_r) &= \frac{u_{32}(x_0, t_{0,r})}{c(x, x_0, t_r)} \frac{\theta(\underline{z}(P_{\infty_2}, \hat{\mu}(x, t_r)))\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_3}, \hat{\mu}(x, t_r)))\theta(\underline{z}(P_{\infty_2}, \hat{\mu}(x_0, t_{0,r})))} \\
 &\times \exp\left((e_3^{(2)}(Q_0) - e_2^{(2)}(Q_0))(x - x_0) + (\tilde{e}_3^{(2)}(Q_0) - \tilde{e}_2^{(2)}(Q_0))(t_r - t_{0,r})\right), \tag{5.81}
 \end{aligned}$$

$$v(x, t_r) = \partial_x \ln c(x, x_0, t_r). \tag{5.82}$$

P r o o f. The proofs of (5.71), (5.72) and (5.74)–(5.81) are similar to those of Theorem 4.3, so we just need to prove (5.73). Let Ψ_3 be defined by the right-hand side of (5.73). We intend to prove that $\psi_3 = \Psi_3$ with ψ_3 given by (5.34). For this purpose, one computes by using (5.7) and (5.8) that

$$\begin{aligned}
 &u_{31}(x, t_r)\phi_2(P, x, t_r) + u_{32}(x, t_r)\phi_3(P, x, t_r) \\
 &= u_{31} \frac{y^2 V_{32}^{(n)} - yA_m + B_m}{E} - u_{32} \frac{y^2 V_{31}^{(n)} - yA_m + B_m}{E}, \\
 &= \frac{1}{E} \{ [u_{31} V_{32}^{(n)} - u_{32} V_{31}^{(n)}] y^2 - [u_{31} A_m - u_{32} A_m] y \\
 &\quad + \frac{1}{3} E_{m-2,x} + \frac{2}{3} (u_{31} V_{32}^{(n)} - u_{32} V_{31}^{(n)}) S_m \}, \\
 &= \frac{1}{3} \frac{E_x}{E} + \frac{2}{3} \frac{[u_{31} V_{32}^{(n)} - u_{32} V_{31}^{(n)}] (3y^2 + S_m)}{E} \\
 &\quad - \frac{u_{31} V_{32}^{(n)} y [y + \frac{A_m}{V_{32}^{(n)}}] - u_{32} V_{31}^{(n)} y [y + \frac{A_m}{V_{31}^{(n)}}]}{E} \\
 &= \frac{\mu_{j,x}(x, t_r)}{\lambda - \mu_j(x, t_r)} + O(1) \\
 &\stackrel{\lambda \rightarrow \mu_j(x, t_r)}{=} \partial_x \ln(\lambda - \mu_j(x, t_r)) + O(1),
 \end{aligned}$$

and

$$\begin{aligned}
 & \tilde{V}_{31}^{(r)}(\lambda, x, t_r)\phi_2(P, x, t_r) + \tilde{V}_{32}^{(r)}(\lambda, x, t_r)\phi_3(P, x, t_r) + \tilde{V}_{33}^{(r)}(\lambda, x, t_r) \\
 &= \tilde{V}_{31}^{(r)}\frac{y^2V_{32}^{(n)} - yA_m + B_m}{E} - \tilde{V}_{32}^{(r)}\frac{y^2V_{31}^{(n)} - yA_m + B_m}{E} + \tilde{V}_{33}^{(r)} \\
 &= \frac{y^2(\tilde{V}_{31}^{(r)}V_{32}^{(n)} - \tilde{V}_{32}^{(r)}V_{31}^{(n)}) - y(\tilde{V}_{31}^{(r)}A_m - \tilde{V}_{32}^{(r)}A_m) + \frac{1}{3}E_{t_r} - \frac{2}{3}(\tilde{V}_{32}^{(r)}V_{31}^{(n)} - \tilde{V}_{31}^{(r)}V_{32}^{(n)})S_m}{E} \\
 &= \frac{1}{3}\frac{E_{t_r}}{E} + \frac{2}{3}\frac{[\tilde{V}_{31}^{(r)}V_{32}^{(n)} - \tilde{V}_{32}^{(r)}V_{31}^{(n)}](3y^2 + S_m)}{E} \\
 &\quad - \frac{\tilde{V}_{31}^{(r)}V_{32}^{(n)}y[y + \frac{A_m}{V_{32}^{(n)}}] - \tilde{V}_{32}^{(r)}V_{31}^{(n)}y[y + \frac{A_m}{V_{31}^{(n)}}]}{E} \\
 &= \frac{\mu_{j,t_r}(x, t_r)}{\lambda - \mu_j(x, t_r)} + O(1) \\
 &= \frac{\partial_{t_r} \ln(\lambda - \mu_j(x, t_r))}{\lambda - \mu_j(x, t_r)} + O(1).
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 & \psi_3(P, x, x_0, t_r, t_{0,r}) \\
 &= \exp\left(\int_{x_0}^x [u_{31}(x', t_r)\phi_2(P, x', t_r) + u_{32}(x', t_r)\phi_3(P, x', t_r)]dx'\right) \\
 &+ \int_{t_{0,r}}^{t_r} (\tilde{V}_{31}^{(r)}(\lambda, x_0, t')\phi_2(P, x_0, t') + \tilde{V}_{32}^{(r)}(\lambda, x_0, t')\phi_3(P, x_0, t') + \tilde{V}_{33}^{(r)}(\lambda, x_0, t'))dt' \\
 &= \frac{\lambda - \mu_j(x, t_r)}{\lambda - \mu_j(x_0, t_r)} \frac{\lambda - \mu_j(x_0, t_r)}{\lambda - \mu_j(x_0, t_{0,r})} O(1) \\
 &= \frac{\lambda - \mu_j(x, t_r)}{\lambda - \mu_j(x_0, t_{0,r})} O(1) \\
 &= \begin{cases} (\lambda - \mu_j(x, t_r))O(1) & \text{for } P \text{ near } \hat{\mu}_j(x, t_r) \neq \hat{\mu}_j(x_0, t_{0,r}), \\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(x, t_r) = \hat{\mu}_j(x_0, t_{0,r}), \\ (\lambda - \mu_j(x_0, t_{0,r}))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0, t_{0,r}) \neq \hat{\mu}_j(x, t_r), \end{cases} \tag{5.83}
 \end{aligned}$$

where $O(1) \neq 0$. Therefore, all zeros and poles of ψ_3 and Ψ_3 on $\mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}, P_{\infty_3}\}$ are simple and coincident. In the same way as in Theorem 4.3, we can show that ψ_3 and Ψ_3 have the same essential singularities at $P_{\infty_1}, P_{\infty_2}, P_{\infty_3}$. Then the Riemann–Roch uniqueness implies $\Psi_3 = \psi_3$.

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