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Noncommutative Space-Time of the Relativistic Equations with a Coulomb Potential Using Seiberg–Witten Map

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We present an important contribution to the noncommutative approach to the hydrogen atom to deal with Lamb shift corrections. This can be done by studying the Klein–Gordon and Dirac equations in a non-commutative space-time up to first-order of the noncommutativity parameter using the Seiberg–Witten maps. We thus find the noncommutative modification of the energy levels and by comparing with the current experimental results on the Lamb shift of the 2P level to extract a bound on the parameter of noncommutativity, we show that the fundamental length ($\sqrt{\Theta}$) is compatible with the value of the electroweak length scale (l). Phenomenologically, this effectively confirms the presence of gravity at this level.

Key words: non-commutative geometry methods, field theory, Klein–Gordon and Dirac equations.

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1. Introduction

The standard concept of space-time as a geometric manifold is based on the notion of a manifold whose points are locally labelled by a finite number of real coordinates. However, it is generally believed that this picture of space-time as a manifold should break down at very short distances of the order of the Planck length. This implies that the mathematical concepts of high energy physics has to be changed or, more precisely, our classical geometric concepts may not be well-suited for the description of physical phenomenon at short distances [1–3]. The connection between the string theory and the non-commutativity [4–7] motivated a large amount of work to study and understand many physical phenomena. The study of this geometry has raised new physical consequences and thus, recently, a noncommutative description of quantum mechanics has stimulated a large amount of research [8–15]. The non-commutative field theory is characterized by the commutation relations between the position coordinate operators themselves, namely,

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\Theta^{\mu\nu}, \tag{1}$$

and the star Moyal product $*$ is defined between two fields $\psi(x)$ and $\varphi(x)$ by

$$\psi(x) * \varphi(x) = \exp\left(\frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}}\right)\psi(x)\,\varphi(y)\,|_{y=x}, \tag{2}$$

where $\Theta^{\mu\nu}$ are the noncommutative constant parameters in the canonical noncommutative space-time.

The issue of time-space noncommutativity is worth pursuing on its own right because of its deep connection with such fundamental notions as unitarity and causality. Much attention has been devoted in recent times to circumvent these difficulties in formulating theories with $\Theta^{0i} \neq 0$ [1, 2, 16, 17]. There are similar examples of theories with time-space noncommutativity in the literature [18–20] where unitarity is preserved by a perturbative approach [21].

The most obvious natural phenomena to search for noncommutative effects are simple systems of quantum mechanics in the presence of a magnetic field, such as a hydrogen atom. In the noncommutative time-space one expects the degeneracy of the spectrum levels to be lifted, and therefore one can say that the noncommutativity plays the role of the magnetic field. The study of the exact and approximate solutions of the relativistic hydrogen atom has proved to be fruitful and many papers have been published [22–25]. In this work we present an important contribution to the noncommutative approach to the relativistic description of the hydrogen atom. Our goal is to solve the Klein–Gordan and Dirac equations for the Coulomb potential in a noncommutative space-time up to first-order of the noncommutativity parameter using the Seiberg–Witten maps and the Moyal product. We thus find the noncommutative modification of the energy levels of the hydrogen atom and we show that the noncommutativity is the source of a magnetic field resulting in the Lamb shift corrections. We also note that the effect of noncommutativity confirms the presence of gravity at the very short distances.

In a previous work [26, 27], by solving the deformed Klein–Gordon and Dirac equations in a canonical noncommutative space, we showed that the energy is shifted, where the correction is proportional to the magnetic quantum number, which behavior is similar to the Zeeman effect as applied to a system without spin in a magnetic field, thus we explicitly accounted for spin effects in this space.

The purpose of this paper is to study the extension of the Klein–Gordon and Dirac fields in canonical noncommutative time-space by applying the result obtained to a hydrogen atom.

The paper is organized as follows. In Sec. 2, we propose an invariant action of the noncommutative boson and fermion fields in the presence of an electromagnetic field. In Sec. 3, using the generalized Euler–Lagrange field equations, we derive the deformed Klein–Gordon (KG) and Dirac equations for the hydrogen atom. We solve these deformed equations and obtain the noncommutative modification of the energy levels. Furthermore, we derive the non-relativistic limit of the noncommutative KG equation for a hydrogen atom and solve it using the perturbation theory. Finally, in Sec. 4, we draw our conclusions.

2. Action

The canonical noncommutative space-time is characterized by the commutation relations of coordinate operators satisfying relation (1). In order to preserve this relation, the infinitesimal gauge transformation is generalized by the following relation:

$$\hat{\phi}^A(A) + \hat{\delta}_{\hat{\lambda}}\hat{\phi}^A(A) = \hat{\phi}^A(A + \delta_\lambda A), \tag{3}$$

where $\hat{\phi}^A = (\hat{A}_\mu, \hat{\psi})$ is a non-commutative generic field, $\hat{A}_\mu$ and $\hat{\psi}$ are the non-commutative gauge and matter fields, respectively, λ is the U(1) gauge Lie-valued infinitesimal transformation parameter, δ_λ is the ordinary gauge transformation and $\hat{\delta}_{\hat{\lambda}}$ is a non-commutative gauge transformation which are defined by:

$$\hat{\delta}_{\hat{\lambda}}\hat{\psi} = i\hat{\lambda} * \hat{\psi}, \qquad \delta_\lambda\psi = i\lambda\psi, \tag{4}$$

$$\hat{\delta}_{\hat{\lambda}}\hat{A}_\mu = \partial_\mu\hat{\lambda} + i\left[\hat{\lambda}, \hat{A}_\mu\right]_*, \qquad \delta_\lambda A_\mu = \partial_\mu\lambda. \tag{5}$$

Now using these transformations one can get at second order in the noncommutative parameter $\Theta^{\mu\nu}$ the following Seiberg–Witten maps [4]:

$$\hat{\psi} = \psi + \psi^1 + \mathcal{O}\left(\Theta^2\right), \tag{6}$$

$$\hat{\lambda} = \lambda + \lambda^1(\lambda, A_\mu) + \mathcal{O}\left(\Theta^2\right), \tag{7}$$

$$\hat{A}_\xi = A_\xi + A^1_\xi(A_\xi) + \mathcal{O}\left(\Theta^2\right), \tag{8}$$

$$\hat{F}_{\mu\xi} = F_{\mu\xi}(A_\xi) + F^1_{\mu\xi}(A_\xi) + \mathcal{O}\left(\Theta^2\right), \tag{9}$$

where

$$\psi^1 = -\frac{i}{2}\Theta^{\alpha\beta}(\{A_\alpha, \partial_\beta\psi\} + \frac{1}{2}\{[\psi, A_\alpha], A_\beta\}), \tag{10}$$

$$\lambda^1 = \Theta^{\alpha\beta}\partial_\alpha\lambda A_\beta, \tag{11}$$

$$A^1_\xi = \frac{1}{2}\Theta^{\alpha\beta}A_\alpha\left(\partial_\xi A_\beta - 2\partial_\beta A_\xi\right), \tag{12}$$

$$F^1_{\mu\xi} = -\Theta^{\alpha\beta}\left(A_\alpha\partial_\beta F_{\mu\xi} + F_{\mu\alpha}F_{\beta\xi}\right), \tag{13}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{14}$$

To begin, we consider an action for a free boson and fermion fields in the presence of an electrodynamic gauge field in a noncommutative space-time. We propose the following action [28]:

$$S = \int d^4x \left(\mathcal{L}_{\mathrm{MB}} + \mathcal{L}_{\mathrm{MF}} - \frac{1}{4}\hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} \right), \tag{15}$$

where $\mathcal{L}_{\mathrm{MB}}$ and $\mathcal{L}_{\mathrm{MF}}$ are the boson and fermion matter densities, respectively, in the non-commutative space-time and are given by

$$\mathcal{L}_{\mathrm{MB}} = \eta^{\mu\nu} \left(\hat{D}_\mu \hat{\varphi} \right)^\dagger * \hat{D}_\nu \hat{\varphi} + m^2 \hat{\varphi}^\dagger * \hat{\varphi}, \tag{16}$$

and

$$\mathcal{L}_{\mathrm{MF}} = \overline{\hat{\psi}} * \left(i\gamma^\nu \hat{D}_\nu - m \right) * \hat{\psi}, \tag{17}$$

where the gauge covariant derivative is defined as $\hat{D}_\mu = \partial_\mu + ie\hat{A}_\mu$.

From the action variational principle the generalized equations of Lagrange up to $\mathcal{O}\left(\Theta^2\right)$ are [29]:

$$\frac{\partial \mathcal{L}}{\partial \hat{\Phi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \hat{\Phi} \right)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \partial_\nu \hat{\Phi} \right)} + \mathcal{O}\left(\Theta^2\right) = 0, \tag{18}$$

where

$$\mathcal{L} = \mathcal{L}_{\mathrm{MB}} + \mathcal{L}_{\mathrm{MF}} - \frac{1}{4}\hat{F}_{\mu\nu} * \hat{F}^{\mu\nu}. \tag{19}$$

3. Noncommutative Time-Space KG Equation

Using the modified field equation (18) , with the generic boson field $\hat{\varphi}$, one can find in a free non-commutative space-time and in the presence of the external potential $\hat{A}_\mu$ the following modified Klein–Gordon equation:

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu - m_e^2 \right) \hat{\varphi} + \left(ie\eta^{\mu\nu}\partial_\mu \hat{A}_\nu - e^2 \eta^{\mu\nu} \hat{A}_\mu * \hat{A}_\nu + 2ie\eta^{\mu\nu}\hat{A}_\mu \partial_\nu \right) \hat{\varphi} = 0, \tag{20}$$

where the deformed external potential $\hat{A}_\mu \left({}^{-e/r}_{\;\;0} \right)$ in free noncommutative space-time is [30]:

$$\hat{a}_0 = -\frac{e}{r} - \frac{e^3}{r^4}\Theta^{0k} x_k + \mathcal{O}\left(\Theta^2\right), \tag{21}$$

$$\hat{a}_i = \frac{e^3}{4\,r^4}\Theta^{ik} x_k + \mathcal{O}\left(\Theta^2\right), \tag{22}$$

for a non-commutative time-space, $\Theta^{0k} \neq 0$ and $\Theta^{ki} = 0$, where $i, k = 1, 2, 3$. In this case, we can check that

$$\eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_0^2 + \Delta, \tag{23}$$

and

$$2ie\eta^{\mu\nu}\hat{A}_\mu\partial_\nu = i\frac{2e^2}{r}\partial_0 + 2i\frac{e^4}{r^4}\Theta^{0j}x_j\partial_0, \tag{24}$$

and

$$-e^2\eta^{\mu\nu}\hat{A}_\mu * \hat{A}_\nu = \frac{e^4}{r^2} + 2\frac{e^6}{r^5}\Theta^{0j}x_j, \tag{25}$$

then the Klein-Gordon equation (20) up to $\mathcal{O}\left(\Theta^2\right)$ takes the form

$$\left[-\partial_0^2 + \Delta - m_e^2 + \frac{e^4}{r^2} + i\frac{2e^2}{r}\partial_0 + 2i\frac{e^4}{r^4}\Theta^{0j}x_j\partial_0 + 2\frac{e^6}{r^5}\Theta^{0j}x_j\right]\hat{\varphi} = 0. \tag{26}$$

The solution to equation (26) in spherical polar coordinates (r, θ, ϕ) takes the separable form

$$\hat{\varphi}(r, \theta, \phi, t) = \frac{1}{r}\hat{R}(r)\hat{Y}\left(\theta, \phi\right)\exp(-iEt). \tag{27}$$

Then (26) reduces to the radial equation

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1) - e^4}{r^2} + \frac{2Ee^2}{r} + \right. \\ \left. + E^2 - m_e^2 + 2E\frac{e^4}{r^4}\Theta^{0j}x_j + 2\frac{e^6}{r^5}\Theta^{0j}x_j\right]\hat{R}(r) = 0. \tag{28}$$

In (28), the Coulomb potential in noncommutative space-time appears within the perturbation terms [31]:

$$H^{\Theta}_{\text{pert}} = 2E\frac{e^4}{r^4}\Theta^{0j}x_j + 2\frac{e^6}{r^5}\Theta^{0j}x_j, \tag{29}$$

where the first term is the electric dipole–dipole interaction created by the non-commutativity, the second term is the electric dipole–quadruple interaction. These interactions show us that the effect of space-time noncommutativity on the interaction of the electron and the proton is equivalent to an extension of two nuclei interactions at a considerable distance. This idea effectively confirms the presence of gravity at this level. To investigate the modification of the energy levels by equation (29), we use the first-order perturbation theory. The spectrum of H_0 and the corresponding wave functions are well-known and given by

$$R_{nl}(r) = \sqrt{\frac{a}{n+\nu+1}}\left(\frac{n!}{\Gamma\left(n+2\nu+2\right)}\right)^{1/2} x^{\nu+1}e^{-x/2}L_n^{2\nu+1}(x)\,, \tag{30}$$

where the relativistic energy levels are given by

$$E = E_{n,l} = \frac{m_e\left(n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\alpha^2}\right)}{\left[\left(n+\frac{1}{2}\right)^2+\left(l+\frac{1}{2}\right)^2+2\left(n+\frac{1}{2}\right)\sqrt{\left(l+\frac{1}{2}\right)^2-\alpha^2}\right]^{\frac{1}{2}}}, \tag{31}$$

and $L_n^{2\nu+1}$ are the associated Laguerre polynomials [32], with the following notations:

$$\nu = -\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\alpha^2}, \qquad \alpha = e^2, \qquad a = \sqrt{m_e^2-E^2}. \tag{32}$$

3.1. Noncommutative corrections of the relativistic energy

Now to obtain the modification to the energy levels as a result of the terms (29) due to the noncommutativity of space-time, we use the perturbation theory. For simplicity, first of all, we choose the coordinate system (t, r, θ, φ) so that $\Theta^{0j} = -\Theta^{j0} = \Theta\delta^{01}$, such that $\Theta^{0j}x_j = \Theta r$ and assume that the other components are all zero and also the fact that in first-order perturbation theory the expectation values of $1/r^3$ and $1/r^4$ are as follows:

$$\begin{aligned}
\langle nlm \mid r^{-3} \mid nlm'\rangle &= \int_0^\infty R_{nl}^2(r) r^{-3} dr \delta_{mm'} \\
&= \frac{4a^3 n!}{(n+\nu+1)\Gamma(n+2\nu+2)} \int_0^\infty x^{2\nu-1} e^{-x}\left[L_n^{2\nu+1}(x)\right]^2 dx\delta_{mm'} \\
&= \frac{4a^3 n!}{(n+\nu+1)\Gamma(n+2\nu+2)}\left[\frac{\Gamma(n+2\nu+2)}{\Gamma(n+1)\Gamma(2\nu+2)}\right]^2 \\
&\quad\times \int_0^\infty x^{2\nu-1}e^{-x}\left[F(-n;2\nu+2;x)\right]^2 dx\delta_{mm'} \\
&= \frac{2a^3}{\nu(2\nu+1)(n+\nu+1)}\left\{1+\frac{n}{(\nu+1)}\right\}\delta_{mm'} = f(3), \qquad (33)\\
\langle nlm \mid r^{-4} \mid nlm'\rangle &= \frac{4a^4}{(2\nu-1)\nu(2\nu+1)(n+\nu+1)}\left[1+\frac{3n}{(\nu+1)}\right. \\
&\quad\left.+\frac{3n(n-1)}{(\nu+1)(2\nu+3)}\right]\delta_{mm'} = f(4), \qquad (34)
\end{aligned}$$

Now, the correction to the energy to first order in Θ is

$$E^{\Theta(1)} = \langle \psi^0_{nlm} \left| H^{\Theta(1)}_{\text{pert}} \right| \psi^0_{nlm} \rangle. \tag{35}$$

where $H^{\Theta(1)}_{\text{pert}}$ is the noncommutative correction to the first order in Θ of the perturbation Hamiltonian, which is given in the following relation:

$$H^{\Theta(1)}_{\text{pert}} = 2E \frac{e^4}{r^3}\Theta + 2\frac{e^6}{r^4}\Theta. \tag{36}$$

To calculate $E^{\Theta(1)}$, we use the radial function in Eq. (30) to obtain

$$E^{\Theta(1)} = 2\Theta\alpha^2 \left(E^0_{n,l} f\left(3\right) + \alpha f\left(4\right) \right).$$

Finally, the energy correction of the hydrogen atom in the framework of the non-commutative KG equation is

$$\begin{aligned} \Delta E^{\text{NC}} &= \frac{E^{\Theta(1)}}{2E} \\ &= \Theta\alpha^2 \left(f\left(3\right) + \frac{\alpha}{E^0_{n,l}} f\left(4\right) \right). \end{aligned} \tag{37}$$

This result is important because it reflects the existence of Lamb shift, which is induced by the noncommutativity of the space. Obviously, when $\Theta = 0$, then $\Delta E^{\text{NC}} = 0$, which is exactly the result of the space-space commuting case, where the energy-levels are not shifted.

We showed that the energy-level shift for $1S$ is

$$\Delta E^{\text{NC}}_{1S} = \Theta\alpha^2 \left(f_{1S}\left(3\right) + \frac{\alpha}{E^0_{1,0}} f_{1S}\left(4\right) \right). \tag{38}$$

In our analysis, we simply identify spin up if the noncommutativity parameter takes the eigenvalue $+\Theta$ and spin down if the noncommutativity parameter takes the eigenvalue $-\Theta$. Also we can say that the Lamb shift is actually induced by the space-time noncommutativity which plays the role of a magnetic field and spin in the same moment (Zemann effect). This represents Lamb shift corrections for $l = 0$. The result is very important: as a possible means of introducing electron spin we replace $l \to \pm\left(j + \frac{1}{2}\right)$ and $n \to n - j - 1 - \frac{1}{2}$, where j is the quantum number associated to the total angular momentum. Then the $l = 0$ state has the same total quantum number $j = \frac{1}{2}$. In this case, the noncommutative value of the energy levels indicates the splitting of the $1s$ states.

3.2. Non-relativistic limit

The non-relativistic limit of noncommutative K-G Eq. (26) is written as [33, 34]:

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{2m_e e^2}{r} + 2m_e\epsilon + 2m_e\frac{e^4}{r^3}\Theta + 2\frac{e^6}{r^4}\Theta\right]\hat{R}(r) = 0. \tag{39}$$

In this non-relativistic limit the charged boson does not represent a single charged particle, but is a distribution of positive and negative charges which are different and extended in space linearly in $\sqrt{\Theta}$. The absence of a perturbation term of the form Θ/r^2 in the noncommutative Coulomb interaction shows that the distribution of positive and negative charges is spherically symmetric. This can be interpreted as the spherically symmetric distribution of charges of the quarks inside the proton.

Now to obtain the modification of energy levels as a result of the noncommutative terms in (39), we use the first-order perturbation theory. The spectrum of $H_0\,(\Theta = 0)$ and the corresponding wave functions are well-known and given by

$$\varepsilon_n = -\frac{m_e\alpha^2}{2\hbar^2 n^2}, \tag{40}$$

and

$$R_{nl}(r) = \frac{1}{n}\left(\frac{(n-l-1)!}{a\,(n+l)!}\right)^{1/2} x^{l+1}e^{-x/2}L_{n-l-1}^{2l+1}(x), \qquad x = \frac{2}{an}r, \tag{41}$$

where $a = \hbar^2/(m_e\alpha)$ is the Bohr radius of the hydrogen atom. The Coulomb potential in noncommutative space-time appears within the perturbation terms

$$H_{\text{pert}}^{\Theta} = 2\Theta\alpha^2\left(\frac{m_e}{r^3} + \frac{\alpha}{r^4}\right) + \mathcal{O}\left(\Theta^2\right), \tag{42}$$

where the expectation values of $1/r^3$ and $1/r^4$ are as follows:

$$\langle nlm \mid r^{-3} \mid nlm'\rangle_{l>0} = \frac{2}{a^3n^3l(l+1)(2l+1)}\delta_{mm'}, \tag{43}$$

$$\langle nlm \mid r^{-4} \mid nlm'\rangle_{l>0} = \left[\frac{4\left(3n^2 - l(l+1)\right)}{a^4n^5l(l+1)(2l-1)(2l+1)(2l+3)} + \frac{35\left(3n^2 - l(l+1)\right)}{3(l-1)(l+2)(2l-1)(2l+1)(2l+3)}\right]\delta_{mm'}. \tag{44}$$

Hence the modification to the energy levels is given by

$$\Delta E^{\text{NC}} = \Theta\alpha^2\left[f\,(3) + \frac{\alpha}{m_e}f\,(4)\right] + \mathcal{O}\left(\Theta^2\right). \tag{45}$$

We can also compute the correction to the Lamb shift of the $2P$ level where we have

$$\Delta E_{2P}^{\mathrm{NC}} = 0.243\,156\,\Theta\,(\mathrm{MeV})^3\,. \tag{46}$$

According to [35], the current theoretical result for the Lamb shift is 0.08 kHz. From the splitting (46), this then gives the following bound on Θ:

$$\Theta \leq (8.5\,\mathrm{TeV})^{-2}\,. \tag{47}$$

This corresponds to a lower bound for the energy scale of 8.5 TeV, which is in the range that was obtained in [36–39], namely 1–10 TeV.

4. Noncommutative Time-Space Dirac Equation

Now, concerning the Dirac equation in the free non-commutative time-space and in the presence of the vector potential $\hat{A}_\mu$ and using the modified field Eq. (18), with the generic field $\hat{\psi}$, we can find the modified Dirac equation up to $\mathcal{O}\left(\Theta^2\right)$ as

$$\left(i\gamma^\mu\partial_\mu - m_e\right)\hat{\psi} - e\gamma^\mu A_\mu\hat{\psi} - e\gamma^\mu A_\mu^1\hat{\psi} + \frac{ie}{2}\Theta^{\alpha\beta}\gamma^\mu\partial_\alpha A_\mu\partial_\beta\hat{\psi} = 0. \tag{48}$$

For a noncommutative time-space ($\Theta^{ki} = 0$, where $i, k = 1, 2, 3$), in this case we can write:

$$i\gamma^\mu\partial_\mu - m_e = i\gamma^0\partial_0 + i\gamma^i\partial_i - m_e, \tag{49}$$

$$-e\gamma^\mu\hat{A}_\mu = \frac{e^2}{r}\gamma^0 + \frac{e^4}{r^4}\gamma^0\Theta^{0k}x_k, \tag{50}$$

$$\frac{ie}{2}\Theta^{\alpha\beta}\gamma^\mu\partial_\alpha A_\mu\partial_\beta = -i\frac{e^2}{2}\gamma^0\frac{\Theta^{0k}x_k}{r^3}\partial_0. \tag{51}$$

Then the noncommutative Dirac equation (48) up to $\mathcal{O}\left(\Theta^2\right)$ takes the following form:

$$\left[i\gamma^0\partial_0 + i\gamma^i\partial_i - m_e + \frac{e^2}{r}\gamma^0 + \frac{e^4}{r^4}\gamma^0\Theta^{0k}x_k - i\frac{e^2}{2}\gamma^0\frac{\Theta^{0k}x_k}{r^3}\partial_0\right]\hat{\psi} = 0. \tag{52}$$

We can write this equation as

$$\hat{H}\hat{\psi}\left(t, r, \theta, \varphi\right) = i\partial_0\hat{\psi}\left(t, r, \theta, \varphi\right). \tag{53}$$

Then replacing

$$\hat{\psi}\left(t, r, \theta, \varphi\right) \to \exp\left(-iEt\right)\hat{\psi}\left(r, \theta, \varphi\right) \tag{54}$$

gives the stationary noncommutative Dirac equation

$$\hat{H}\hat{\psi}(r,\theta,\varphi) = E\hat{\psi}(r,\theta,\varphi),$$

where E is the ordinary energy of the electron and $\hat{H}$ is the noncommutative Hamiltonian of the form

$$\hat{H} = \hat{H}_0 + \hat{H}^{\Theta}_{\text{pert}}, \tag{55}$$

where H_0 is the relativistic Hamiltonian for the hydrogen atom

$$\hat{H}_0 = \vec{\boldsymbol{\alpha}}.\left(-i\vec{\boldsymbol{\nabla}}\right) + \beta m_e - \frac{e^2}{r}, \tag{56}$$

and H^{Θ}_{pert} is the leading-order perturbation

$$\hat{H}^{\Theta}_{\text{pert}} = \left(\frac{E}{2} - \frac{e^2}{r}\right) e^2 \frac{\vec{\boldsymbol{\Theta}}_t \cdot \vec{\boldsymbol{r}}}{r^3}. \tag{57}$$

The leading long-distance part of H^{Θ}_{pert} behaves like that of a magnetic dipole potential where the noncommutativity plays the role of a magnetic moment. So the noncommutative Coulomb potential is the multipolar contribution and this means that the distribution is not spherically symmetric. In the above the matrices $\vec{\boldsymbol{\alpha}}$ and β are given by

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix},$$

where σ^i are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To investigate the modification of the energy levels by equation (57), we use the first-order perturbation theory, where, by restoring the constants c and $\hbar$, the spectrum of $\hat{H}_0$ and the corresponding wave functions are well-known and are given by (see [35, 40–45]):

$$\psi(r,\theta,\varphi) = \begin{pmatrix} \phi(r,\theta,\varphi) \\ \chi(r,\theta,\varphi) \end{pmatrix} = \begin{pmatrix} f(r)\,\Omega_{jlM}(\theta,\varphi) \\ g(r)\,\Omega_{jlM}(\theta,\varphi) \end{pmatrix}, \tag{58}$$

where the bi-spinors $\Omega_{jlM}(\theta,\varphi)$ are defined by

$$\Omega_{jlM}(\theta,\varphi) = \begin{pmatrix} \mp\sqrt{\frac{(j+1/2)\mp(M-1/2)}{2j+(1\pm1)}}Y_{j\pm1/2,M-1/2}(\theta,\varphi) \\ \sqrt{\frac{(j+1/2)\pm(M+1/2)}{2j+(1\pm1)}}Y_{j\pm1/2,M+1/2}(\theta,\varphi) \end{pmatrix}, \tag{59}$$

with the radial functions $f(r)$ and $g(r)$ given as

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = \left(a\frac{mc}{\hbar}\right)^2 \frac{1}{\nu}\sqrt{\frac{\hbar c\left(E\varkappa - m_e c^2\nu\right)n!}{\left(m_e c^2\right)^2 \alpha\left(\varkappa-\nu\right)\Gamma\left(n+2\nu\right)}} e^{-\frac{1}{2}x} x^{\nu-1} \times \\ \times \begin{pmatrix} f_1 x L_{n-1}^{2\nu+1}(x) + f_2 L_n^{2\nu-1}(x) \\ g_1 x L_{n-1}^{2\nu+1}(x) + g_2 L_n^{2\nu-1}(x) \end{pmatrix}, \tag{60}$$

where the ordinary relativistic energy levels are given by

$$E = E_{n,j} = \frac{m_e c^2 (n+\nu)}{\sqrt{\alpha^2 + (n+\nu)^2}}, \qquad n = 0, 1, 2\cdots \tag{61}$$

and $L_n^{\alpha}(x)$ are the associated Laguerre polynomials [32], with the following notations:

$$a = \frac{1}{m_e c^2}\sqrt{(m_e c^2)^2 - E^2}, \qquad x = \frac{2}{\hbar c}\sqrt{(m_e c^2)^2 - E^2}\, r,$$
$$\varkappa = \pm\left(j + \frac{1}{2}\right), \qquad \nu = \sqrt{\varkappa^2 - \alpha^2},$$
$$f_1 = \frac{a\alpha}{\frac{E}{m_e c^2}\varkappa - \nu}, \qquad f_2 = \varkappa - \nu,$$
$$g_1 = \frac{a(\varkappa - \nu)}{\frac{E}{m_e c^2}\varkappa - m_e\nu}, \qquad g_2 = \frac{e^2}{\hbar c} = \alpha.$$

In the above, m_e is the mass of the electron and α is the fine structure constant.

4.1. Noncommutative Corrections to the Dirac Energy

Now to obtain the modification to the energy levels as a result of the terms (57) due to the noncommutativity of time-space, we use the perturbation theory up to the first order. With respect to the selection rule $\Delta l = 0$ and choosing the coordinate system (t, r, θ, φ) so that $\Theta^{0k} = -\Theta^{k0} = \Theta\delta^{01}$, we have

$$\Delta E_{n,j}^{(\Theta)} = \Delta E_{n,j}^{(1)} + \Delta E_{n,j}^{(2)}, \tag{62}$$

where

$$\Delta E_{n,j}^{(1)} = \frac{E}{2\hbar c}e^2 \int_0^{4\pi} \Theta d\Omega \int_0^{\infty} dr[\psi_{njlM}^{\dagger}(r,\theta,\varphi)\,\psi_{nj'l'M'}(r,\theta,\varphi)] \\ = \frac{E}{2\hbar c}e^2\Theta_{MM'}\langle\frac{1}{r^2}\rangle, \tag{63}$$

and

$$\Delta E_{n,j}^{(2)} = -\frac{e^4}{\hbar c}\int\limits_0^{4\pi} \Theta d\Omega \int\limits_0^{\infty} drr^{-2}[\psi_{njlM}^{\dagger}(r,\theta,\varphi)\,\psi_{nj'l'M'}(r,\theta,\varphi)]$$
$$= -\frac{e^4}{\hbar c}\Theta_{MM'}\langle\frac{1}{r^3}\rangle, \tag{64}$$

where

$$\langle\frac{1}{r^2}\rangle = 2\hbar c\left(\frac{m_e c}{\hbar}a\right)^3\left[\frac{\varkappa\left(2E\varkappa - m_e c^2\right)}{\left(m_e c^2\right)^2\alpha\nu\left(4\nu^2-1\right)}\right], \tag{65}$$

$$\langle\frac{1}{r^3}\rangle = \left(\frac{mc}{\hbar}a\right)^3\left[\frac{3E\varkappa\left(E\varkappa - m_e c^2\right)}{\left(m_e c^2\right)^2\nu\left(4\nu^2-1\right)\left(\nu^2-1\right)} - \frac{\left(m_e c^2\right)^2\left(\nu^2-1\right)}{\left(m_e c^2\right)^2\nu\left(4\nu^2-1\right)\left(\nu^2-1\right)}\right]. \tag{66}$$

From Eq. (62) we obtain the modified energy levels in noncommutative space-time to the first order of Θ as:

$$\Delta E_{n,j}^{(\Theta)} = \left(\frac{\alpha}{\hbar c}\right)^2\frac{m_e c^2 a^3}{\nu\left(4\nu^2-1\right)}\times$$
$$\times\left[E\varkappa\left(\frac{\left(2E\varkappa - m_e c^2\right)}{\alpha^2} - \frac{3\left(E\varkappa - m_e c^2\right)}{\left(\nu^2-1\right)}\right) + \left(m_e c^2\right)^2\right]\Theta_{MM'}. \tag{67}$$

The selection rules for the transitions between the levels $\left(Nl_j^M \to Nl_j^{M'}\right)$ are $\Delta l = 1$ and $\Delta M = 0, \pm 1$, where $N = n + |\varkappa|$ describes the principal quantum number. The $2S_{1/2}$ and $2P_{1/2}$ levels correspond respectively to

$$\left(N = 1, j = 1/2, \varkappa = \pm 1, M = \pm 1/2\right). \tag{68}$$

From (67) and (68) we can write

$$\Delta E_{2S_{1/2}}^{(\Theta)} = 1.944\,64\times 10^{-8}\Theta\ (\mathrm{MeV})^3\,, \tag{69}$$

$$\Delta E_{2P_{1/2}}^{(\Theta)} = \pm 2.160\,75\times 10^{-9}\Theta\ (\mathrm{MeV})^3\,. \tag{70}$$

The non-commutative correction to the transition follows as

$$\Delta E_{2,1/2}^{(\Theta)}\left(2P_{1/2}\to 2S_{1/2}\right) = 2.160715\times 10^{-8}\Theta\ \ (\mathrm{MeV})^3\,.$$

Now again using the current theoretical result on the $2P$ Lamb shift from [39], which is about 0.08 kHz, and from the splitting (69), we get the bound

$$\Theta \lesssim (0.25\,\text{TeV})^{-2}\,. \tag{71}$$

Restoring the constants c and $\hbar$ in (71), we write the bound on the non-commutativity parameter as

$$\Theta \lesssim 1.7 \times 10^{-35}\text{m}^2. \tag{72}$$

It is interesting that the value of the upper bound on the time-space noncommutativity parameter as derived here is better than the results of [12, 21, 36, 46]. This value is only in the sense of an upper bound and not the value of the parameter itself for which the fundamental length $\sqrt{\Theta}$ is compatible with the value of the electroweak length scale (ℓ_ω). This effectively confirms the presence of gravity at this level.

5. Conclusions

In this work we started from quantum relativistic charged scalar and fermion particles in a canonical noncommutative space-time to find the action which is invariant under the infinitesimal gauge transformations. By using the Seiberg–Witten maps and the Moyal product, we derived the deformed KG and Dirac equations for noncommutative Coulomb potential up to first order in the noncommutativity parameter Θ. By solving the deformed KG and Dirac equations, we found the Θ-correction energy shift. This proves that the non-commutativity has an effect similar to that of the magnetic field. The corrections induced to the energy levels by this noncommutative effect and the Lamb shift were induced and compared with experimental results from high precision hydrogen spectroscopy to obtain a new bound for the noncommutativity parameter of around $(0.25\,\text{TeV})^{-2}$, for which the fundamental length $\sqrt{\Theta}$ is compatible with the value of the electroweak length scale (ℓ_ω). This effectively confirms the presence of gravity at this level.

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