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On Robust Feedback for Systems with Multidimensional Control

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The paper deals with local robust feedback synthesis problem for systems with multidimensional control and unknown bounded perturbations. Using V.I. Korobov's controllability function method, a bounded control which steers an arbitrary initial point to the origin at some finite time is constructed; an estimate from above for the time of motion is given. The range of a segment where the perturbations can vary is found. As an example, the problem of stopping the oscillations of the system of two coupled pendulums is considered.

Key words: controllability function method, systems with multidimensional control, robust feedback synthesis, finite-time stabilization, unknown bounded perturbations, uncertain systems.

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1. Introduction and Problem Statement

The paper deals with the synthesis problem, i.e., the problem of constructing a control which depends on the phase coordinates and steers an arbitrary initial point from some neighborhood of the origin to the origin at some finite time. Besides, the control should satisfy some preassigned constrains. In [1], the methods for solving the feedback synthesis problem for a linear system are given. Further we consider the synthesis problem for the linear system with continuous *bounded unknown perturbations*. In the present paper, we find a constraint for unknown perturbations such that the control which solves the synthesis problem for the system without perturbation also solves the synthesis problem for the perturbed system.

For the first time, the concept of the feedback synthesis was introduced and studied in [2] published in Russian. In this and other papers of the author, this concept was translated from Russian as a "positional synthesis". Later the concept was introduced and studied in [3, 4] where it was called a "feedback synthesis". Now the term "feedback synthesis" is generally used for the concept of the synthesis introduced in [2]. The controllability function method was introduced in [2]. In this method the angle between the direction of motion and the direction of decrease of the controllability function is not less than the corresponding angle in the dynamic programming method, and not more than in a method of Lyapunov function [1, p. 10]. The main advance of the controllability function method is the finiteness of the motion time. Among other authors developing this approach we would like to mention [5]. Herein the concept of finite time stability is to find the trajectories within specifical domains of the state space during a given finite time interval. A bit later, the problem of steering an arbitrary initial point from some neighborhood of the origin to the origin (or in general case an equilibrium point) in a finite time was called the "finite-time stabilization" (see, e.g., [6, 7]). In contrast to this problem, the controllability function method allows us to solve the problem of steering an arbitrary initial point to a generally non-equilibrium point in a finite time. The paper [8] is devoted to the problem of constructing a constrained control, which transfers a control system from any point to a given non-equilibrium point in a finite time in global sense.

Let us consider the system

$$\dot{x} = (A_0 + K + R(t,x))x + B_0 u, \tag{1}$$

where $t \geq 0,\ \ x \in Q \subset \mathbb{R}^n$, Q is a neighborhood of the origin; $u \in \mathbb{R}^r$ is a control satisfying the constraint $\|u\| \leq 1$; A_0 is an $(n \times n)$ matrix of the form $A_0 = \operatorname{diag}(A_{01}, \ldots, A_{0r})$, where A_{0i} are $(n_i \times n_i)$ matrices of the form $A_{0i} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$, $i = 1, \ldots, r;\ n_1 \geq n_2 \geq \ldots \geq n_r \geq 1,\ n_1 + \ldots + n_r = n;$ B_0 is an $(n \times r)$ matrix whose elements $(B_0)_{s_i i}$ are equal to 1, $s_i = n_1 + \ldots + n_i$, $i = 1, \ldots, r$ and others are equal to zero; the elements of the matrix K which are in row s_i (in other words, a row which contains a control) are equal to $k_{s_i j}$, and the other elements are equal to zero, $R(t,x) = \operatorname{diag}(R_1(t,x), \ldots, R_r(t,x)) + \hat{R}(t,x)$,

$R_i(t,x)$

$$= \begin{pmatrix} r_{(s_{i-1}+1)1} & r_{(s_{i-1}+1)2} & 0 & 0 & \dots & 0 & 0 \\ r_{(s_{i-1}+2)1} & r_{(s_{i-1}+2)2} & r_{(s_{i-1}+2)3} & 0 & \dots & 0 & 0 \\ & & & \dots & & & \\ r_{(s_i-2)1} & r_{(s_i-2)2} & r_{(s_i-2)3} & r_{(s_i-2)4} & \dots & r_{(s_i-2)(s_i-1)} & 0 \\ r_{(s_i-1)1} & r_{(s_i-1)2} & r_{(s_i-1)3} & r_{(s_i-1)4} & \dots & r_{(s_i-1)(s_i-1)} & r_{(s_i-1)s_i} \\ r_{s_i 1} & r_{s_i 2} & r_{s_i 3} & r_{s_i 4} & \dots & r_{s_i(s_i-1)} & r_{s_i s_i} \end{pmatrix}, \tag{2}$$

the elements of the matrix $\hat{R}(t,x)$ which are in row s_i (in other words, a row which contains a control) are equal to $r_{s_i j}$, and the other elements are equal to zero, $r_{mj} = r_{mj}(t,x)$. We assume that the functions $r_{mj}(t,x)$ are *unknown*, and we call these systems *robust systems*, see, for example, [9, p. 173]. We assume that the functions $r_{mj}(t,x)$ satisfy the constraints

$$\max_{1\le j\le m+1\le n_i,\ i=1,\dots,r} |r_{mj}(t,x)| \le \Delta. \tag{3}$$

The goal is to find Δ and to build *a bounded control* which steers an arbitrary initial point $x_0 \in Q$ to the origin *in a finite time* for any perturbation matrix $R(t,x)$ satisfying condition (3).

A classical example of this problem is a control over the motion of a car on the surface with an unknown bounded friction. The motion of this system is described by the equation

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = r_{22}(t,x_1,x_2)x_2 + u. \end{cases}$$

The term $r_{22}(t,x_1,x_2)x_2$ is a sliding frictional force and $r_{22}(t,x_1,x_2)$ is a coefficient of the nonlinear viscous friction which is an unknown function and satisfies the constraint $|r_{22}(t,x_1,x_2)| \le \Delta$. The constraint under consideration on $r_{22}(t,x_1,x_2)$ does not except a "negative" friction.

The general approach to the admissible control synthesis problem for an arbitrary nonlinear autonomous control system was given by V.I. Korobov in [2]. In this paper, an estimate for the time of motion (settling-time function) from an arbitrary initial point to the origin was given. Recently, the problem of finite-time stabilization has been formulated in several different ways [1], [9–15]. The paper [16] describes a method for solving the feedback synthesis problem for systems with multidimensional control and without perturbations (i. e., $R(t,x)\equiv 0$). Moreover, in this case, the controllability function is the time of motion. In [10], we solved the robust synthesis problem for a case with one perturbation and a scalar control. In [11], the case, where $R(t,x) = p(t,x)R,\ K \equiv 0$, and the control is scalar, was considered.

In [12], an adaptive fuzzy finite-time control scheme was proposed for a class of nonlinear systems with unknown nonlinearities. The scheme can guarantee that the states of the closed-loop system converge to a small neighborhood of the origin in a finite time. The book [9, p. 201] deals with the problem of robust stabilization for the systems with constant affine perturbations. In [14], the Lyapunov function method was suggested to study the finite-time stabilization of the system $\dot{x}(t) = A_0x + B_0u(t) + d(t, x(t))$, where $u(t)$ is a scalar function and $d(t, x)$ is measurable and uniformly bounded in the variable t function. In [13, 14], the finite-time stabilization conditions were formulated in the form of linear matrix inequalities. In [15], the problem of finite-time stabilization for the second order system of general form (or double integrator) with a scalar control was considered.

First, we describe the conditions which the perturbations $r_{mj}(t, x)$ must satisfy.

Definition 1.1. *By $\mathcal{R}$, we denote a set of matrices $R(t, x)$ whose elements are the functions $r_{mj}(t, x) : [0, +\infty) \times Q \to \mathbb{R}$, satisfying the following conditions:*

1) *$r_{mj}(t, x)$ are continuous in the variables t and x;*

2) $\max\limits_{1\le j\le m+1\le n_i,\, i=1,\ldots,r} |r_{mj}(t, x)| \le \Delta$ *for all $(t, x) \in [0, +\infty) \times Q$;*

3) *in any domain $\overline{K}_1(\rho_2) = \{(t, x) : \ 0 \le t < +\infty, \ \|x\| \le \rho_2\}$, the vector function $R(t, x)x$ satisfies the Lipschitz condition*

$$|R(t, x'')x'' - R(t, x')x'| \le \ell_1(\rho_2)\|x'' - x'\|.$$

If $R(t, x) \equiv 0$, then (1) is a canonical system: $\dot{x} = (A_0 + K)x + B_0u$. For the first time this concept was introduced in [2]. This system was also called “integrator system” (for the second order system see, for example, [7]). Solving the synthesis problem for an arbitrary linear system with a multidimensional control can be reduced to solving the synthesis problem for the canonical system [1, p. 105]. The canonical system is completely controllable. The control $u(x)$, which solves the synthesis problem for the canonical system, is given in [1, Theorem 2.3]; [16] .

Definition 1.2. *The problem of finding a range of perturbations Δ such that the trajectory $x(t)$ of the closed-loop system with the control $u(x)$*

$$\dot{x} = (A_0 + K + R(t, x))x + B_0u(x), \tag{4}$$

starting at an arbitrary initial point $x(0) = x_0 \in Q$, ends at the origin at some finite time $T(x_0, \mathcal{R})$, i. e., $\lim\limits_{t\to T(x_0,\mathcal{R})} x(t) = 0$, is said to be the local robust feedback synthesis. If $Q = \mathbb{R}^n$, this problem is called the global robust feedback synthesis.

Obviously, if $r_{11}(t,x) \equiv 0$ and $r_{12}(t,x) \equiv -1$, then the first coordinate x_1 in (1) is uncontrollable, and thus the problem is not solvable for any value of Δ.

The paper is organized as follows. In Section 2, some basic concepts of the controllability function method are given. Section 3 represents the main results. In Example 3.1 we consider the problem of stopping of oscillations of the system of two coupled pendulums.

2. Background: the Controllability Function Method

In this Section we introduce some basic concepts and some results of the controllability function method [1, 2]. Let us consider a nonlinear system of the form

$$\dot{x} = f(x,u), \tag{5}$$

where $x \in Q \subset \mathbb{R}^n$, and $u \in \Omega \subset \mathbb{R}^r$, Ω is such that $0 \in \ int\, \Omega,\ f(0,0) = 0$.

Definition 2.3. *The problem of constructing a control of the form $u = u(x)$, $x \in Q$ is said to be the local feedback synthesis if:*
1) $u(x) \in \Omega$;
2) *the trajectory $x(t)$ of the closed-loop system $\dot{x} = f(x, u(x))$, starting at an arbitrary initial point $x_0 \in Q$, ends at the origin at some finite time $T(x_0)$. If $Q = \mathbb{R}^n$, the problem is called the global feedback synthesis.*

The sufficient conditions for solving the problem of feedback synthesis for system (5) are given in [1, Theorem 1.1].

Let us describe one of possible approaches to solving the feedback synthesis problem for the canonical system [1, Theorem 2.3]; [16]:

$$\dot{x} = (A_0 + K)x + B_0 u, \tag{6}$$

where $x \in \mathbb{R}^n$, $\ u \in \mathbb{R}^r$ is a control which satisfies the constraint $\|u\| \leq 1$. It should be noted that system (1) coincides with completely controllable system (6) when $R(t,x) \equiv 0$. Let us set

$$F^{-1} = \int_0^1 (1-t) e^{-A_0 t} B_0 B_0^* e^{-A_0^* t} dt. \tag{7}$$

Let $D(\Theta)$ be a diagonal matrix of the form

$$D(\Theta) = \operatorname{diag}(D_1(\Theta), \ldots, D_r(\Theta)), \quad \text{where} \quad D_i(\Theta) = \operatorname{diag}\left(\Theta^{-\frac{2n_i - 2j + 1}{2}}\right)_{j=1}^{n_i}. \tag{8}$$

Theorem 1. [1, Theorem 2.3]; [16]. *Let the controllability function* $\Theta = \Theta(x)$, $x \neq 0$, *be a unique positive solution of the equation*

$$2a_0\Theta = (D(\Theta)FD(\Theta)x, x), \tag{9}$$

where the constant a_0 *satisfies the inequality*

$$0 < a_0 \leq \frac{1}{\|F^{-1}\| \cdot (\|B_0^*F\| + 2\max\{c^{n_1}, c\}\|B_0^*K\|)^2}, \tag{10}$$

besides the domain of solvability synthesis problem is of the form $Q = \{x : \Theta(x) \leq c\}$, *where* Q *is an ellipsoid. At* $x = 0$, *we put* $\Theta(0) = 0$.

Then in the domain Q *the control*

$$u(x) = -\left(\frac{1}{2}\, B_0^*D(\Theta(x))FD(\Theta(x)) + B_0^*K\right)x \tag{11}$$

solves the local feedback synthesis problem for system (6) *and satisfies the constraint* $\|u(x)\| \leq 1$. *Moreover, in this case, the equation* $\dot{\Theta}(x) = -1$ *holds, i. e., the controllability function* $\Theta(x)$ *is equal to the time of motion from any initial point* $x \in Q$ *to the origin.*

In the case where $K \equiv 0$, *the synthesis is global.*

3. The Solution of the Robust Feedback Synthesis Problem

Let us consider system (1). Equation (4) with control (11) takes the form

$$(A_0 + K + R(t,x))x + B_0u(x)$$
$$= (A_0 + K + R(t,x))x - \left(\frac{1}{2}\, B_0B_0^*D(\Theta(x))FD(\Theta(x)) + B_0B_0^*K\right)x.$$

Due to the fact that $B_0B_0^*K = K$, the last equation takes the form

$$(A_0 + K + R(t,x))x + B_0u(x) = (A_0 + R(t,x))x - \frac{1}{2}\, B_0B_0^*D(\Theta(x))FD(\Theta(x)x.$$

Put $y(\Theta, x) = D(\Theta)x$. Then equation (9) has the form

$$2a_0\Theta = (Fy(\Theta,x), y(\Theta,x)). \tag{12}$$

Let us set

$$H = \operatorname{diag}(H_1, \dots, H_r), \quad \text{where} \quad H_i = \operatorname{diag}\left(-\frac{2n_i - 2j + 1}{2}\right)_{j=1}^{n_i}$$

and

$$F^1 = F - FH - HF = ((2n - i - j + 2)f_{ij})_{i,j=1}^n. \tag{13}$$

If the matrix F is positive defined, then equation (12) has a unique positive solution $\Theta = \Theta(y)$ [1, p. 108]. Since the controllability function is the time of motion, then the matrix F^1 is positive defined [1, p. 106]. Let the constant a_0 satisfy inequality (10). Consider the closed-loop system (4) with the control given by (11). Let us denote the trajectory of this system by $x(t)$ and find the derivative of the controllability function with respect to system (4): $\dot{\Theta} = \frac{d}{dt}\Theta(x(t))$. From equation (12) it follows that

$$2a_0\dot{\Theta} = (F\dot{y}(\Theta, x), y(\Theta, x)) + (Fy(\Theta, x), \dot{y}(\Theta, x)). \tag{14}$$

Let us find $\dot{y}(\Theta, x)$. We obtain that $\dfrac{d}{d\Theta}D(\Theta) = \dfrac{1}{\Theta}HD(\Theta)$. Therefore,

$$\dot{y}(\Theta, x) = \dot{D}(\Theta)x + D(\Theta)\dot{x} = \frac{\dot{\Theta}}{\Theta}Hy(\Theta, x) + D(\Theta)A_0D^{-1}(\Theta)y(\Theta, x) +$$

$$+ D(\Theta)R(t, x)D^{-1}(\Theta)y(\Theta, x) - \frac{1}{2}D(\Theta)B_0B_0^*D(\Theta)Fy(\Theta, x).$$

Let us set

$$S(\Theta, t, x) = \Theta(FD(\Theta)R(t, x)D^{-1}(\Theta) + D^{-1}(\Theta)R^*(t, x)D(\Theta)F). \tag{15}$$

In [1, p. 109], it was proved that

$$D(\Theta)A_0D^{-1}(\Theta) = \Theta^{-1}A_0, \quad D(\Theta)b_0 = \Theta^{-1/2}b_0, \quad FA_0 + A_0^*F - FB_0B_0^*F = -F^1.$$

From (14), we can see that

$$\dot{\Theta}(2a_0 - \frac{1}{\Theta}((FH + HF)y(\Theta, x), y(\Theta, x))) = \frac{1}{\Theta}((-F^1 + S(\Theta, t, x))y(\Theta, x), y(\Theta, x)).$$

Taking into account equation (12), we obtain that the derivative of the controllability function with respect to system (4) is of the form

$$\dot{\Theta} = -1 + \frac{(S(\Theta, t, x)y(\Theta, x), y(\Theta, x))}{(F^1y(\Theta, x), y(\Theta, x))}. \tag{16}$$

Let us introduce the following notation:

- M^* is a transpose matrix to the matrix M;
- $\sigma(M)$ is the spectrum of the matrix M;
- $\lambda_{min}(M) = \min\{\lambda : \lambda \in \sigma(M)\}$;
- $\lambda_{max}(M) = \max\{\lambda : \lambda \in \sigma(M)\}$;
- $\rho(M) = \max\{|\lambda|, \lambda \in \sigma(M)\}$ is a spectral radius of the matrix M;

• $|M| = (|m_{ij}|)_{i,j=1}^n$ is the absolute value of the matrix M, i. e., the matrix which consists of absolute values of the elements of the matrix M;
• $\widetilde{G} = |(F^1)^{-1}| \cdot (F\widetilde{R} + \widetilde{R}^* F)$, where the matrix $\widetilde{R}$ coincides with the matrix $R(t,x)$ at $r_{mj}(t,x) = 1$.

Denote $y = y(\Theta, x)$. Let us find the exact estimate for $\dot{\Theta}$. To this end, we find the largest and the smallest values of the ratio $(S(\Theta,t,x)y,y)/(F^1 y,y)$ at $y \neq 0$. Let us consider the problem

$$(S(\Theta,t,x)y,y) \to extr, \quad y \in \{y : \ (F^1 y, y) = c\}.$$

We solve this problem using the method of Lagrange multipliers. The Lagrange function takes the form

$$\mathcal{L}(y,\lambda) = (S(\Theta,t,x)y,y) - \lambda[(F^1 y,y) - c].$$

From the necessary condition of the extremum, we obtain that $S(\Theta,t,x)y - \lambda F^1 y = 0$. So at the extremum point the following condition holds: $(S(\Theta,t,x)y,y) = \lambda(F^1 y,y)$, moreover, $\lambda \in \sigma((F^1)^{-1} S(\Theta,t,x))$. Therefore,

$$\lambda_{min}((F^1)^{-1} S(\Theta,t,x)) \le \frac{(S(\Theta,t,x)y,y)}{(F^1 y,y)} \le \lambda_{max}((F^1)^{-1} S(\Theta,t,x)).$$

Thus, from (16) we obtain that

$$\dot{\Theta} \le -1 + \lambda_{max}((F^1)^{-1} S(\Theta,t,x)). \tag{17}$$

3.1. Perturbations of the superdiagonal elements

Suppose that the $(n_i \times n_i)$ matrices $R_i(t,x)$ have nonzero elements only at the main superdiagonal and $\hat{R}(t,x) \equiv 0$. Then system (1) has the form

$$\begin{cases} \dot{x}_{s_{i-1}+j} = (1 + r_{(s_{i-1}+j)(s_{i-1}+j+1)}(t,x))x_{s_{i-1}+j+1}, \ j = 1,\dots,n_i - 1, \\ \dot{x}_{s_i} = \sum\limits_{j=1}^{n} k_{s_i j} x_j + u_i, \ i = 1,\dots,r. \end{cases} \tag{18}$$

Similarly to [1, p. 109], one can show that $D(\Theta)R(t,x)D^{-1}(\Theta) = \Theta^{-1}R(t,x)$ (due to the fact that in the case under consideration the matrix $R(t,x)$ has the same structure as A_0). We obtain that

$$S(\Theta,t,x) = S_0(t,x) = FR(t,x) + R^*(t,x)F. \tag{19}$$

It should be noted that the matrix $S_0(t,x)$ does not depend on Θ. This observation is crucial for our method of solving the robust feedback synthesis problem. Indeed, the explicit form of $S_0(t,x)$ is $S_0(t,x) = \operatorname{diag}(S_1(t,x),\dots,S_r(t,x))$, where

$S_i(t,x)$

$$\begin{pmatrix} 0 & f_{11}r_{12} & \dots & f_{1(n_i-1)}r_{(n_i-1)n_i} \\ f_{11}r_{12} & 2f_{12}r_{12} & \dots & f_{1n_i}r_{12}+f_{2(n_i-1)}r_{(n_i-1)n_i} \\ f_{12}r_{23} & f_{13}r_{12}+f_{22}r_{23} & \dots & f_{2n_i}r_{23}+f_{3(n_i-1)}r_{(n_i-1)n_i} \\ & \dots & & \\ f_{1(n_i-1)}r_{(n_i-1)n_i} & f_{1n_i}r_{12}+f_{2(n_i-1)}r_{(n_i-1)n_i} & \dots & 2f_{(n_i-1)n_i}r_{(n_i-1)n_i} \end{pmatrix},$$

and $r_{mj}=r_{mj}(t,x)$.

Theorem 2. *Let γ be an arbitrary number which satisfies inequality $0<\gamma<1$. Let*

$$\Delta=\frac{(1-\gamma)}{\rho(\widetilde{G})}. \tag{20}$$

Let the controllability function $\Theta=\Theta(x)$, $x\neq 0$, be a unique positive solution of equation (9) *where the constant a_0 satisfies inequality* (10).

Then, in the domain Q defined by $Q=\{x:\ \Theta(x)\le c\}$, the control given by (11) *solves the local robust feedback synthesis problem for system* (18). *Moreover, the trajectory $x(t)$ of the closed-loop system* (4), *starting at an arbitrary initial point $x(0)=x_0\in Q$, ends at the origin at some finite time $T(x_0,\mathcal{R})$, where the time of motion $T(x_0,\mathcal{R})$ is bounded as follows:*

$$T(x_0,\mathcal{R})\le\frac{\Theta(x_0)}{\gamma}. \tag{21}$$

In the case where $K\equiv 0$, the robust feedback synthesis problem is global.

P r o o f. Since $B_0=\operatorname{diag}(B_{01},\dots,B_{0r})$, then the matrices A_0 and B_0 have a block structure. So the matrix F^{-1} given by (7) is of the form

$$F^{-1}=\operatorname{diag}(F_1^{-1},\dots,F_r^{-1}),$$

where (see [1, p. 98])

$$F_i^{-1}=\int_0^1(1-t)e^{-A_{0i}t}B_{0i}B_{0i}^*e^{-A_{0i}^*t}dt$$
$$\left(\frac{(-1)^{m+j}}{(n_i-m)!(n_i-j)!(2n_i-m-j+1)(2n_i-m-j+2)}\right)_{m,j=1}^{n_i}. \tag{22}$$

Let us fix the value of i and consider the matrix F_i which is inverse to the matrix F_i^{-1}. Let us prove that the elements of the matrix F_i are positive. To this end, we analyze the matrix

$$\widetilde{M}=\left(\frac{1}{(2n_i-m-j+1)(2n_i-m-j+2)}\right)_{m,j=1}^{n_i}.$$

Put $d_m = (-1)^m(n_i - m)$. The elements of the matrix F_i^{-1} can be calculated from the elements of the matrix $\widetilde{M}$ by multiplying every element of the row m by d_m and every element of the column j by d_j. It is known that if every element of the row m of the matrix is multiplied by $\varepsilon \neq 0$, then every element of the column m in the inverse matrix will be divided by ε. A similar assertion is true for the columns. To determine the elements of the matrix F_i, we should divide every element of the column m of the matrix $\widetilde{M}^{-1}$ by d_m, and every element of the row j of the matrix $\widetilde{M}^{-1}$ by d_j. Therefore, the element with the number mj will be divided by $d_m d_j$, sign $d_m d_j = (-1)^{m+j}$.

Let us prove that all the minors of the matrix $\widetilde{M}$ are positive. It is known that all the minors of the $n_i \times n_i$ matrix $\widetilde{M}$ are positive if its s order minors composed of the consecutive s rows and the consecutive s columns are positive [17, Theorem 3.3]. This theorem was first proved in [18]. So, in the matrix $\widetilde{M}$ we consider only submatrices composed of the consecutive s rows $\bar{r}+1, \bar{r}+2, \dots, \bar{r}+s$ and the consecutive s columns $\bar{c}+1, \bar{c}+2, \dots, \bar{c}+s$. In addition, any of these submatrices is the Schur product of the Cauchy matrices. A Cauchy matrix is a matrix of the form $\left(\dfrac{1}{x_m + y_j}\right)_{m,j=1}^{n}$ [19, Theorem 1.2.12.1]. Each consecutive submatrix of the matrices $\left(\dfrac{1}{2n_i - m - j + 1}\right)_{m,j=1}^{n_i}$ and $\left(\dfrac{1}{2n_i - m - j + 2}\right)_{m,j=1}^{n_i}$ is a Cauchy matrix (put for the first matrix $x_m = n_i - m,\ y_j = n_i - j + 1$). The determinant of the Cauchy matrix can be found by the formula [19, Theorem 1.2.12.1],

$$\frac{\prod\limits_{m>j} (x_m - x_j)(y_m - y_j)}{\prod\limits_{m,j} (x_m + y_j)}. \tag{23}$$

Each consecutive submatrix of the matrices $\left(\dfrac{1}{2n_i - m - j + 1}\right)_{m,j=1}^{n_i}$ and $\left(\dfrac{1}{2n_i - m - j + 2}\right)_{m,j=1}^{n_i}$ is a positive definite matrix by the Silvester criteria and formula (23). The Schur product of the matrices $\left(\dfrac{1}{2n_i - m - j + 1}\right)_{m,j=1}^{n_i}$ and $\left(\dfrac{1}{2n_i - m - j + 2}\right)_{m,j=1}^{n_i}$ is the matrix of the form

$$\left(\frac{1}{(2n_i - m - j + 1)(2n_i - m - j + 2)}\right)_{m,j=1}^{n_i}$$

and it is equal to $\widetilde{M}$. The Schur product of positive definite matrices is a positive definite matrix [19, Theorem 6.4.2.1]. Hence, in the matrix $\widetilde{M}$ all the submatrices composed of the consecutive s rows $\bar{r}+1, \bar{r}+2, \ldots, \bar{r}+s$ and the consecutive s columns $\bar{c}+1, \bar{c}+2, \ldots, \bar{c}+s$ are positive. Therefore, all minors of the matrix $\widetilde{M}$ are positive. Then the minors of order $n_i - 1$ and n_i, in particular, are also positive. Hence, the elements of the matrix inverse to the matrix $\widetilde{M}$ have the sign $(-1)^{m+j}$. This implies that all the elements of the matrix F_i inverse to the matrix F_i^{-1} are positive.

It is known that $\lambda_{max}((F^1)^{-1} S_0(t,x)) \leq \rho((F^1)^{-1} S_0(t,x))$. We claim that $\rho((F^1)^{-1} S_0(t,x)) \leq \rho|(F^1)^{-1} S_0(t,x)|$. To prove this inequality we need the following theorem.

Theorem 3. [20, Theorem 8.1.18] *Let M and N be some matrices. Then*
1. $|M \cdot N| \leq |M| \cdot |N|$;
2. *If* $|M| \leq N$, *then* $\rho(M) \leq \rho(|M|) \leq \rho(N)$.

Therefore, $\rho((F^1)^{-1} S_0(t,x)) \leq \rho|(F^1)^{-1} S_0(t,x)| \leq \Delta\rho(\widetilde{G})$. Here we use the fact that the elements of the matrix F are positive. Let us substitute the last inequality into inequality (17). We obtain that

$$\dot{\Theta} \leq -1 + \Delta\rho(\widetilde{G}). \tag{24}$$

If we assume that $-1 + \Delta\rho(\widetilde{G}) \leq -\gamma$, then $\dot{\Theta} \leq -\gamma$. Similarly to [1, Theorem 1.2], the estimate for the time of motion (21) follows from the last inequality.

To complete the proof of the theorem, the boundedness of the control has to be established. Since $B_0^* D(\Theta) = \Theta^{-\frac{1}{2}} B_0^*$, the control given by (11) can be written in the form

$$u(x) = -\left(\frac{\Theta^{-\frac{1}{2}}}{2} B_0^* F + B_0^* K D^{-1}(\Theta(x)) \right) y(\Theta, x).$$

Since $\|y(\Theta,x)\|^2 \leq 2a_0 \Theta(x) \|F^{-1}\|$ and

$$\|D^{-1}(\Theta(x))\| = \begin{cases} \Theta^{\frac{1}{2}} & \text{if } 0 < \Theta < 1, \\ \Theta^{\frac{2n_1-1}{2}} & \text{if } \Theta \geq 1, \end{cases}$$

at $\Theta(x) \leq c$ we get

$$\|u(x)\| \leq \left(\frac{1}{2} \|B_0^* F\| + \max\{c^{n_1}, c\} \|B_0^* K\| \right) \sqrt{2a_0 \|F^{-1}\|}.$$

Let the constant a_0 satisfy inequality (10). Then from the last inequality we obtain that $\|u(x)\| \leq 1$ for all $x \in Q$. Due to [1, Theorem 2.3], the control $u(x)$ of the form (11) solves the local feedback synthesis problem for system (18). The proof of the theorem is completed. ∎

3.2. The general case

Let the matrix $R(t,x)$ has the form given in (2). Then the elements of $S(\Theta,t,x)$ defined by (15) are polynomials in Θ with a degree not exceeding n_1. As in the case with the perturbations of the superdiagonal elements, from inequality (17) it follows that

$$\dot{\Theta} \leq -1 + \rho((F^1)^{-1} S(\Theta,t,x)) \leq -1 + \Delta \max\{c^{n_1}, c\}\rho(\widetilde{G})$$

at $\Theta(x) \leq c$. If we assume that

$$-1 + \Delta \max\{c^{n_1}, c\}\rho(\widetilde{G}) \leq -\gamma, \tag{25}$$

then $\dot{\Theta} \leq -\gamma$. Thus, the following theorem is valid.

Theorem 4. *Let the controllability function $\Theta = \Theta(x)$, $x \neq 0$, be a unique positive solution of equation* (9)*, where the constant a_0 satisfies inequality* (10)*. Let the solvability domain be defined by $Q = \{x : \ \Theta(x) \leq c\}$, where Q is an ellipsoid. Let γ be an arbitrary number which satisfies the inequality $0 < \gamma < 1$. Let*

$$\Delta = \frac{(1-\gamma)}{\max\{c^{n_1}, c\}\rho(\widetilde{G})}. \tag{26}$$

Then in the domain Q, the control given by (11) *solves the local robust feedback synthesis problem for system* (1)*. Moreover, the trajectory $x(t)$ of the closed-loop system* (4)*, starting at an arbitrary initial point $x(0) = x_0 \in Q$, ends at the origin at some finite time $T(x_0,\mathcal{R})$, where the time of motion $T(x_0,\mathcal{R})$ satisfies inequality* (21)*.*

R e m a r k 3.1. If we solve inequality (25) with respect to c and consider r to be arbitrary, then we obtain the following solvability domain of the synthesis problem: $Q = \{x : \ \Theta(x) \leq c\}$.

R e m a r k 3.2. The value of Δ is monotonically decreasing in γ. In addition, the inequality for the time of motion $T(x_0,\mathcal{R})$ given by (21) is also monotonically decreasing in γ. The value $\Delta \to \max$ at $\gamma \to 0$. Moreover, $T(x_0,\mathcal{R}) \to +\infty$ at $\Delta \to 0$.

R e m a r k 3.3. Let $R(t,x) \in \mathcal{R}$. To find the trajectory starting at an initial point $x_0 \in Q$, we solve equation (9) at $x = x_0$ and find its unique positive solution $\Theta(x_0) = \Theta_0$. Put $\theta(t) = \Theta(x(t))$. The trajectory satisfies the system

$$\begin{cases} \dot{x} = (A_0 + R(t,x))x - \frac{1}{2}\, B_0^* D(\theta(x)) F D(\theta(x))x, \\ \dot{\theta} = \dfrac{(-F^1 + S(\Theta,t,x))D(\theta)x, D(\theta)x)}{(F^1 D(\theta)x, D(\theta)x)}, \\ x(0) = x_0, \ \theta(0) = \Theta_0. \end{cases} \tag{27}$$

It should be noted that in order to determine Θ_0 it is enough to solve equation (9) only once.

E x a m p l e 3.1. **The stopping of oscillations of the system of two coupled pendulums.**

Let us consider a mechanical system which consists of two pendulums coupled by a spring. The pendulums oscillate in the same plane. We denote by l_1 and l_2 the lengths of pendulums and by m_1 and m_2, theirs masses. The lengths from the suspension points of two pendulums to the spring attachment points are considered to be equal to each other, and we denote them by h. The spring stiffness is equal to k. The oscillations of the system without a control were considered in many books (see, for example, [21, Sect. 6.1]; [22, Sect. 132]).

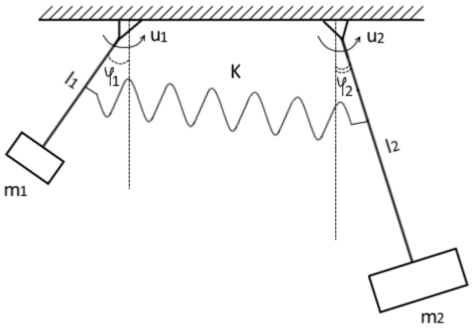

Fig. 1. The system which consists of two coupled pendulums.

Let us consider the controllable motion of this system. The pairs of forces u_1 and u_2 act as shown in Fig. 1. The linearized equations of the motion of these pendulums are of the form

$$\begin{cases} \ddot{\varphi}_1 = -\dfrac{m_1 g l_1 + kh^2}{m_1 l_1^2}\,\varphi_1 + \dfrac{kh^2}{m_1 l_1^2}\,\varphi_2 + u_1, \\ \ddot{\varphi}_2 = \dfrac{kh^2}{m_2 l_2^2}\,\varphi_1 - \dfrac{m_2 g l_2 + kh^2}{m_2 l_2^2}\,\varphi_2 + u_2. \end{cases} \tag{28}$$

The pairs of forces u_1 and u_2 satisfy the inequality $\|(u_1, u_2)^*\| = \sqrt{u_1^2 + u_2^2} \le 1$. We assume that a positive value of u_i corresponds to the case there the moments of the force act in a clockwise direction. The force acts tangentially to the trajectory of motion.

The first case. Suppose that the values of m_1, m_2, l_1, l_2 and h are known. Suppose that the spring stiffness k is unknown. Let us set

$$\frac{kh^2}{m_1 l_1^2} = r_{21},\ \frac{kh^2}{m_2 l_2^2} = r_{41},\ \frac{g}{l_1} = k_{21},\ \frac{g}{l_2} = k_{43}.$$

By changing the variables

$$x_1 = \varphi_1, \quad x_2 = \dot{\varphi}_1, \quad x_3 = \varphi_2, \quad x_4 = \dot{\varphi}_2,$$

system (28) is reduced to the following form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -(r_{21} + k_{21})x_1 + r_{21}x_3 + u_1, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = r_{41}x_1 - (r_{41} + k_{43})x_3 + u_2. \end{cases} \tag{29}$$

The coefficients r_{21} and r_{41} are unknown constants.

System (29) can be written in the matrix form:

$$\dot{x} = (A_0 + K + R)x + B_0 u, \tag{30}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{31}$$

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -k_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -k_{43} & 0 \end{pmatrix}, \ R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -r_{21} & 0 & r_{21} & 0 \\ 0 & 0 & 0 & 0 \\ r_{41} & 0 & -r_{41} & 0 \end{pmatrix},$$

and $n_1 = 2\ , n_2 = 2\ , s_1 = 2\ , s_2 = n = 4$.

Let us consider the robust feedback synthesis problem for system (30). Since for any fixed stiffness k the following equation holds: $rg(B_0, (A_0+K+R)B_0) = 4$, then this system is completely controllable.

The matrices F and $D(\Theta)$ given by (7) and (8), respectively, are of the form

$$F = \begin{pmatrix} 36 & 12 & 0 & 0 \\ 12 & 6 & 0 & 0 \\ 0 & 0 & 36 & 12 \\ 0 & 0 & 12 & 6 \end{pmatrix}, \quad D(\Theta) = \begin{pmatrix} \Theta^{-\frac{3}{2}} & 0 & 0 & 0 \\ 0 & \Theta^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & \Theta^{-\frac{3}{2}} & 0 \\ 0 & 0 & 0 & \Theta^{-\frac{1}{2}} \end{pmatrix}. \tag{32}$$

Let $x = (x_1, x_2, x_3, x_4)$. Define the controllability function $\Theta = \Theta(x)$ at $x \neq 0$ as a unique positive solution of equation (9),

$$2a_0\Theta^4 = 36x_1^2 + 24\Theta x_1 x_2 + 6\Theta^2 x_2^2 + 36x_3^2 + 24\Theta x_3 x_4 + 6\Theta^2 x_4^2. \tag{33}$$

At $x = 0$, we put $\Theta(0) = 0$. We consider the solution of the problem of robust feedback synthesis in the domain $Q = \{x : \Theta(x) \le c\}$, where Q is an ellipsoid. The constant $c > 0$ is defined below. The constant a_0 satisfies inequality (10) which takes the form

$$0 < a_0 \le \frac{3.58}{(13.42 + 2\max\{c^2, c\}\max\{k_{21}, k_{43}\})^2}. \tag{34}$$

For the solvability domain to contain the ellipsoid of the largest size, we will choose a_0 to be the largest value which satisfies (34).

The control given by (11) which solves the problem of robust feedback synthesis is of the form

$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} -\dfrac{6x_1}{\Theta^2(x)} - \dfrac{3x_2}{\Theta(x)} + k_{21}x_1 \\ -\dfrac{6x_3}{\Theta^2(x)} - \dfrac{3x_4}{\Theta(x)} + k_{43}x_3 \end{pmatrix},$$

where $\Theta = \Theta(x)$ is a unique positive solution of equation (33). For any value of k, this control steers an arbitrary initial point x_0 to the origin at some finite time $T(x_0, k) \le \Theta(x_0)/\gamma$, where γ is an arbitrary number which satisfies the inequality $0 < \gamma < 1$.

The matrix $S = S(\Theta, t, x)$ given by (15) has the form

$$S(\Theta) = \begin{pmatrix} -24r_{21}\Theta^2 & -6r_{21}\Theta^2 & 12(r_{21}+r_{41})\Theta^2 & 6r_{41}\Theta^2 \\ -6r_{21}\Theta^2 & 0 & 6r_{21}\Theta^2 & 0 \\ 12(r_{21}+r_{41})\Theta^2 & 6r_{21}\Theta^2 & -24r_{41}\Theta^2 & -6r_{41}\Theta^2 \\ 6r_{41}\Theta^2 & 0 & -6r_{41}\Theta^2 & 0 \end{pmatrix},$$

where $\Theta = \Theta(x)$ is a unique positive solution of equation (33).

Let us find an estimate for the solvability domain. To this end, we find c from inequality (25), which takes the form

$$-1 + \Delta\max\{c^2, c\}\rho(\widetilde{G}) \le -\gamma, \tag{35}$$

where $\widetilde{G} = \begin{pmatrix} \frac{7}{6} & \frac{1}{6} & \frac{7}{6} & \frac{1}{6} \\ 4 & \frac{1}{2} & 4 & \frac{1}{2} \\ \frac{7}{6} & \frac{1}{6} & \frac{7}{6} & \frac{1}{6} \\ 4 & \frac{1}{2} & 4 & \frac{1}{2} \end{pmatrix}$, $\rho(\widetilde{G}) \approx 8.4$, $\Delta = k\max\left\{\dfrac{h^2}{m_1 l_1^2}; \dfrac{h^2}{m_2 l_2^2}\right\}$.

From (35), it follows that

$$\max\{c^2, c\} \le \frac{0.12(1-\gamma)}{\Delta} = \frac{0.12(1-\gamma)}{k\max\left\{\dfrac{h^2}{m_1 l_1^2}; \dfrac{h^2}{m_2 l_2^2}\right\}}.$$

Taking into account inequality (17), let us find a more precise estimate for c. At $x \in Q$, from (17) it follows that

$$\dot{\Theta} \leq -1 + \lambda_{max}((F^1)^{-1} S(\Theta)) = -1 + \frac{\left(r_{21} + r_{41} + 2\sqrt{2(r_{21}^2 + r_{41}^2)}\right)\Theta^2}{6}$$

$$\leq -1 + \frac{\left(r_{21} + r_{41} + 2\sqrt{2(r_{21}^2 + r_{41}^2)}\right)c^2}{6}.$$

Let $c > 0$ be such that the following inequality holds:

$$-1 + \frac{\left(r_{21} + r_{41} + 2\sqrt{2(r_{21}^2 + r_{41}^2)}\right)c^2}{6} \leq -\gamma. \tag{36}$$

Then $\dot{\Theta} \leq -\gamma$. From (36), it follows that $c \leq \sqrt{\frac{6(1-\gamma)}{(r_{21}+r_{41}+2\sqrt{2(r_{21}^2+r_{41}^2)})}}$. For the solvability domain to contain the ellipsoid of the largest size, we will choose c to be the largest value which satisfies (36). So, we obtain the following solvability domain:

$$Q = \left\{ x : \ \Theta(x) \leq \sqrt{\frac{6(1-\gamma)}{k\left(\frac{h^2}{m_1 l_1^2} + \frac{h^2}{m_2 l_2^2} + 2\sqrt{\frac{2h^4}{m_1^2 l_1^4} + \frac{2h^4}{m_2^2 l_2^4}}\right)}} \right\}. \tag{37}$$

Let us consider the values of the parameters

$$m_1 = 1, \ m_2 = 2, \ l_1 = 60, \ l_2 = 30, \ h = 7.5, \ \gamma = 0.001.$$

Then $\dfrac{h}{l_1} = \dfrac{1}{8}, \ \dfrac{h}{l_2} = \dfrac{1}{4}, \ k_{21} = \dfrac{g}{l_1} \approx 0.16, \ k_{43} = \dfrac{g}{l_2} \approx 0.32, \ r_{21} = \dfrac{k}{64}, \ r_{41} = \dfrac{k}{32}.$

Let the stiffness k satisfy the constraint $k \leq 4$, but the value of k be unknown. Then the set of points (37) from which we can steer to the origin is an ellipsoid of the form $Q = \{x : \ \Theta(x) \leq 3.2\}$. Notice that from (37) it follows that the stiffness k decreases as the values of axes of ellipsoid Q increase. At $c = 3.2$, inequality (34) on a_0 takes the form: $a_0 \leq 0.0088\ldots$ Put $a_0 = 0.0088$.

Let the initial point be equal to $x(0) = (-0.3, 0.3, 0, 0)$, $x(0) \in Q$. The unique positive solution Θ_0 of equation (33) is $\Theta_0 \approx 3.2$. Let $x = x(t, k_0)$ be the trajectory of system (27), which is realized at some coefficient of stiffness k_0 which satisfies inequality $k_0 \leq 4$. Put $\theta(t) = \Theta(x(t, k_0))$. The trajectory $x = x(t, k_0)$

satisfies the system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \dfrac{k_0}{64}(-x_1 + x_3) - \dfrac{6\,x_1}{\theta^2} - \dfrac{3x_2}{\theta}, \\ \dot{x}_3 = x_4, \\ \dot{x}_2 = \dfrac{k_0}{32}(x_1 - x_3) - \dfrac{6\,x_1}{\theta^2} - \dfrac{3x_2}{\theta}, \\ \dot{\theta} = \phi, \\ x_1(0) = -0.3, \quad x_2(0) = 0.3, \quad x_2(0) = 0, \quad x_2(0) = 0, \quad \theta(0) = 3.2, \end{cases} \tag{38}$$

where

$$\begin{aligned} \phi = \; & -((12 + 0.03\,k_0\,\theta^2)\,x_1^2 + (6 + 0.02\,k_0\,\theta^2)\,x_1\,x_2\,\theta + x_2^2\,\theta^2 \\ & +(12 + 0.06\,k_0\,\theta^2)\,x_3^2 + (6 + 0.03\,k_0\,\theta^2)\,x_3\,x_4\,\theta + x_4^2\,\theta^2 \\ & -0.09\,k_0\,x_1\,x_3\,\theta^2 - 0.02\,k_0\,x_2\,x_3\,\theta^3 - 0.03\,k_0\,x_1\,x_4\,\theta^3) \\ & /(12\,x_1^2 + 6x_1\,x_2\,\theta + x_2^2\,\theta^2 + 12\,x_3^2 + 6x_3\,x_4\,\theta + x_4^2\,\theta^2). \end{aligned}$$

A two-dimensional projection of the domain Q on the plane $Ox_1x_2 = O\varphi_1\dot{\varphi}_1$ or equally $\bar{Q} = \{(x_1^0, x_2^0, 0, 0) : \ \Theta(x_1^0, x_2^0, 0, 0) \leq 3.2\}$ is given in Fig. 2. Let $(x_1^0(t), x_2^0(t), x_3^0(t), x_4^0(t), \theta(t))$ be the solution of system (38) at $k_0 = 4$. The curve $(x_1^0(t), x_2^0(t))$ is also given in Fig. 2 (the solid line). The curve $(\bar{x}_1^0(t), \bar{x}_2^0(t))$ (the dashed line), which corresponds to the case $k_0 = 0$, is also given in Fig. 2.

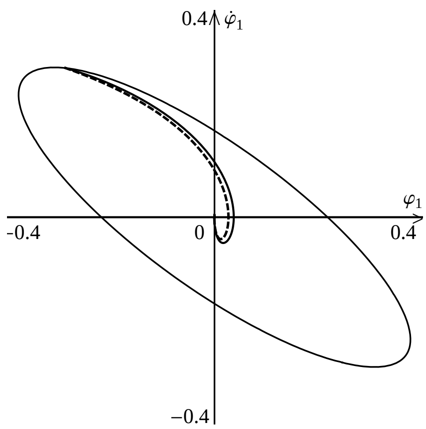

Fig. 2. The projection of the phase trajectory and the ellipsoid Q on the plane $O\varphi_1\dot{\varphi}_1$

.

All the other trajectories fill the domain between the trajectories corresponding to $k_0 = 0$ and $k_0 = 4$ if the stiffness k_0 satisfies the inequality $0 \leq k_0 \leq 4$ and the trajectories begin from $x(0)$. At $k_0 = 0$, the trajectory can be found from the system

$$\begin{cases} \dot{x}_1 = x_2, \quad \dot{x}_2 = -\dfrac{6\,x_1}{\theta^2} - \dfrac{3x_2}{\theta}, \quad \dot{x}_3 = x_4, \quad \dot{x}_2 = -\dfrac{6\,x_1}{\theta^2} - \dfrac{3x_2}{\theta}, \quad \dot{\theta} = -1, \\ x_1(0) = -0.3, \quad x_2(0) = 0.3, \quad x_2(0) = 0, \quad x_2(0) = 0, \quad \theta(0) = 3.2. \end{cases}$$

The graphs of the components of the control on the trajectory

$$u_1 = u_1(x_1^0(t), x_2^0(t), x_3^0(t), x_4^0(t)) = -\frac{6x_1^0(t)}{\theta^2(t)} - \frac{3x_2^0(t)}{\theta(t)} + 0.16x_1^0(t),$$

$$u_2 = u_2(x_1^0(t), x_2^0(t), x_3^0(t), x_4^0(t)) = -\frac{6x_3^0(t)}{\theta^2(t)} - \frac{3x_4^0(t)}{\theta(t)} + 0.32x_3^0(t)$$

are given in Fig. 3. The norm of the control $\|(u_1, u_2)^*\| = \sqrt{u_1^2 + u_2^2}$ is given in Fig. 4, and we can see that $\|(u_1, u_2)^*\| \le 1$. The controllability function $\theta(t)$ is close to the linear function $(y = 3.2 - t)$ and is given in Fig. 5. The derivative of the controllability function with respect to the system (38) is given in Fig. 6, and we can see that it is negative. The estimate for the time of motion (21) is of the form: $T \le 3206$. It is fulfilled for all $0 \le k_0 \le 4$, but at a particular value of k_0 the value of T is less than 3206. It can be shown numerically that the time of motion T from the point $x(0)$ for $k_0 = 4$ is $T \approx 3.43$, besides it can be shown numerically that for $0 \le k_0 \le 4$ the following inequality holds: $3.2 \le T \le 3.43$. All graphs are given on the trajectory for $k_0 = 4$. For other values of k_0, the graphs are similarly to those given in Fig. 2–6.

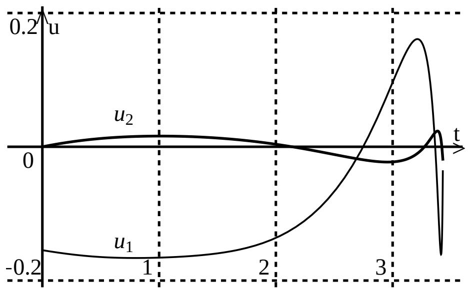

Fig. 3. The components of the control.

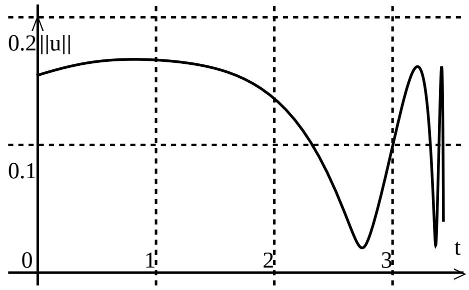

Fig. 4. The norm of the control.

The second case. Let $l_1 = l_2 = l$. Let us consider that the values m_1, m_2 and k are known. We will also consider that the pendulum length l is unknown. Besides, the ratio $\dfrac{h}{l}$ is known. Let us set

$$\frac{kh^2}{m_1 l_1^2} = k_{21}, \; \frac{kh^2}{m_2 l_2^2} = k_{41}, \; \frac{g}{l} = r_{21}.$$

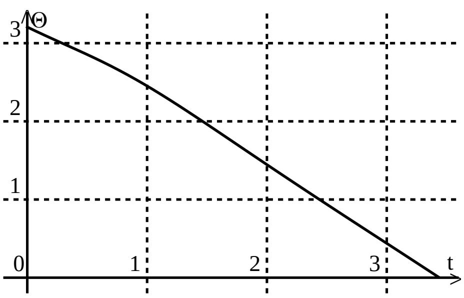

Fig. 5. The controllability function.

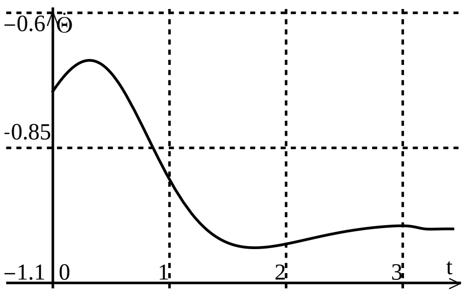

Fig. 6. The derivative of the controllability function w. r. t. the system.

By changing the variables

$$x_1 = \varphi_1, \quad x_2 = \dot{\varphi}_1, \quad x_3 = \varphi_2, \quad x_4 = \dot{\varphi}_2,$$

we reduce (28) to the system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -(r_{21} + k_{21})x_1 + k_{21}x_3 + u_1, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = k_{41}x_1 - (r_{21} + k_{41})x_3 + u_2. \end{cases}$$

The coefficient r_{21} is an unknown constant.

The system can be written in the matrix form (30), where the matrices A_0 and B_0 are given by (31) and the matrices K and R are of the form

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -k_{21} & 0 & k_{21} & 0 \\ 0 & 0 & 0 & 0 \\ k_{41} & 0 & -k_{41} & 0 \end{pmatrix}, \ R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -r_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r_{21} & 0 \end{pmatrix}.$$

The matrices F and $D(\Theta)$ are given by (32). Let us define the controllability function $\Theta = \Theta(x)$ at $x \neq 0$ as a unique positive solution of equation (33). At $x = 0$, we put $\Theta(0) = 0$. Similarly to the first case, we consider the solution of the problem of robust feedback synthesis in the domain $Q = \{x : \Theta(x) \leq c\}$, where Q is ellipsoid. The constant $c > 0$ is defined below. The constant a_0 satisfies inequality (10) that takes the form

$$0 < a_0 \leq \frac{3.58}{(13.42 + 2.83 \max\{c^2, c\}\sqrt{k_{21}^2 + k_{41}^2})^2}. \tag{39}$$

For the solvability domain to contain the ellipsoid of the largest size, we will choose a_0 to be the largest value which satisfies (39).

The control given by (11), which solves the robust feedback synthesis problem, is of the form

$$u(x)=\begin{pmatrix} u_1(x)\\ u_2(x)\end{pmatrix}=\begin{pmatrix} -\dfrac{6x_1}{\Theta^2(x)}-\dfrac{3x_2}{\Theta(x)}+k_{21}(x_1-x_3)\\ -\dfrac{6x_3}{\Theta^2(x)}-\dfrac{3x_4}{\Theta(x)}+k_{41}(-x_1+x_3)\end{pmatrix},$$

where $\Theta=\Theta(x)$ is a unique positive solution of equation (33). For any value of l, this control steers an arbitrary initial point x_0 to the origin at some finite time $T(x_0,l)\le\Theta(x_0)/\gamma$, where γ is an arbitrary number which satisfies the inequality $0<\gamma<1$.

The matrix $S=S(\Theta,t,x)$ given by (15) has the form

$$S=\begin{pmatrix} -\dfrac{24g\Theta^2}{l_1} & -\dfrac{6g\Theta^2}{l_1} & 0 & 0\\ -\dfrac{6g\Theta^2}{l_1} & 0 & 0 & 0\\ 0 & 0 & -\dfrac{24g\Theta^1}{l_1} & -\dfrac{6g\Theta^2}{l_1}\\ 0 & 0 & -\dfrac{6g\Theta^2}{l_1} & 0\end{pmatrix},$$

where $\Theta=\Theta(x)$ is a unique positive solution of equation (33).

Taking into account inequality (17), let us find an exact estimate for c. Since $\lambda_{max}((F^1)^{-1}S(\Theta))=\dfrac{g\Theta^2}{2l}$, then at $x\in Q$ from (17) it follows that

$$\dot\Theta\le-1+\lambda_{max}((F^1)^{-1}S(\Theta))=-1+\frac{g\Theta^2}{2l}\le-1+\frac{gc^2}{2l}.$$

Let $c>0$ be such that the following inequality holds:

$$-1+\frac{gc^2}{2l}\le-\gamma. \tag{40}$$

Then $\dot\Theta\le-\gamma$. From (40) it follows that $c\le\sqrt{0.2\,l(1-\gamma)}$. For the solvability domain to contain the ellipsoid of the largest size, we choose c to be the largest value which satisfies (40). So, we obtain the solvability domain

$$Q=\{x:\ \Theta(x)\le\sqrt{0.2\,l(1-\gamma)}\}. \tag{41}$$

Let

$$m_1=1,\ m_2=2,\ k=1,\ \frac{h}{l}=\frac{1}{4},\ \gamma=0.001.$$

Then $k_{21} = \dfrac{kh^2}{m_1 l^2} = \dfrac{1}{16}$, $k_{41} = \dfrac{kh^2}{m_2 l^2} = \dfrac{1}{32}$, $r_{21} = \dfrac{9.8}{l}$.

Let the length l satisfy the constraint $l \geq 30$, but the value of l be unknown. Then the set of points (41), from which we can steer to the origin, is an ellipsoid of the form $Q = \{x : \Theta(x) \leq 2.47\}$. Besides, from (41) it follows that the length l decreases as the values of axes of ellipsoid Q decrease. At $c = 2.47$, inequality (39) on a_0 takes the form: $a_0 \leq 0.016\ldots$ Put $a_0 = 0.016$.

Similarly to the first case, let the initial point be equal to $x(0) = (-0.3, 0.3, 0, 0)$, $x(0) \in Q$. The unique positive solution Θ_0 of equation (33) is $\Theta_0 \approx 2.44$. The estimate for the time of motion (21) is of the form: $T \leq 2438$. It is fulfilled for all $l \geq 30$, but at a particular value of l, the value of T is less than 2438. It can be shown numerically that the time of motion T from the point $x(0)$ at $l = 30$ is $T \approx 3$, besides it can be shown numerically that at $l \geq 30$ the following inequality holds: $2.44 \leq T \leq 3$. The further considerations are similar to those in the first case.

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