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Renormalized Solutions for Nonlinear Parabolic Systems in the Lebesgue–Sobolev Spaces with Variable Exponents

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The existence result of renormalized solutions for a class of nonlinear parabolic systems with variable exponents of the type

$$\partial_t e^{\lambda u_i(x,t)} - \operatorname{div}(|u_i(x,t)|^{p(x)-2}u_i(x,t)) \\ + \operatorname{div}(c(x,t)|u_i(x,t)|^{\gamma(x)-2}u_i(x,t)) = f_i(x,u_1,u_2) - \operatorname{div}(F_i),$$

for $i = 1, 2$, is given. The nonlinearity structure changes from one point to other in the domain Ω. The source term is less regular (bounded Radon measure) and no coercivity is in the nondivergent lower order term $\operatorname{div}(c(x,t)|u(x,t)|^{\gamma(x)-2}u(x,t))$. The main contribution of our work is the proof of the existence of renormalized solutions without the coercivity condition on nonlinearities which allows us to use the Gagliardo–Nirenberg theorem in the proof.

Key words: parabolic problems, Lebesgue–Sobolev space, variable exponent, renormalized solutions.

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1. Introduction

One of the driving forces for the rapid development of the theory of variable exponent function spaces was the model of electrorheological fluids introduced by Rajagopal and Rusička [25]. The model leads naturally to a functional setting involving function spaces with variable exponents. Electrorheological fluids change their mechanical properties dramatically when an external electric field is applied. In the mathematical community these materials have been intensively studied in the recent years. In the case of an isothermal, homogeneous, incompressible electrorheological fluid, the governing equations read

$$\partial_t v + - \operatorname{div} S + [\nabla v]v + \nabla\pi = g + [\nabla E]P \quad \text{in } \Omega, \\ \operatorname{div} v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

where v is the velocity, $[\nabla v]v$ is the convective term, π denotes the pressure, S denotes the extra stress tensor, g is the external body force, E is the electric field, and P is the electric polarization. The extra stress tensor is given by

$$S = \alpha_2 1((1+|D|^2)^{\frac{p-1}{2}} - 1)E \otimes E + (\alpha_2 1 + \alpha_2 1|E|^2)(1+|D|^2)^{\frac{p-2}{2}}),$$

$p = p(|E|^2)$ is a Hölder continuous function with $1 < p^- < p^+ < N$; this requirement also ensures that the operator induced by $-\operatorname{div} S(D,E)$ is coercive and satisfies appropriate growth conditions. For the mathematical treatment, we have additionally to assume that the operator induced by $-\operatorname{div} S(D,E)$ is strictly monotone.

The first systematic study of spaces with variable exponents was carried out by Nakano in [23], later in [22] Museilak and in [19] Kovacik investigated the modular spaces which are more general frameworks.

In the real line, the Lebesgue space with variable exponents was developed by Tsenov, Sharapudinov, and Zhikov [27, 28]. The reader can find numerous references in the overview paper by Antontsev [3] and in the monograph on evolution PDEs by Antontsev and Shmarev [7]. In this paper, we consider a problem with potential application in electrorheological fluids (smart fluids), the flow through the porous media [7].

Let Ω be a bounded-connected domain of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$, the generic cylinder of an arbitrary finite hight $T < \infty$, and consider the following strongly nonlinear parabolic system:

$$\begin{cases} \dfrac{\partial b_1(x,u_1)}{\partial t} - \operatorname{div}(a(x,t,u_1,\nabla u_1)) + \operatorname{div}(\phi_1(x,t,u_1)), & \\ \qquad = f_1(x,u_1,u_2) - \operatorname{div}(F_1) & \text{in } Q_T, \\ \dfrac{\partial b_2(x,u_2)}{\partial t} - \operatorname{div}(a(x,t,u_2,\nabla u_2)) + \operatorname{div}(\phi_2(x,t,u_2)), & \\ \qquad = f_2(x,u_1,u_2) - \operatorname{div}(F_2) & \text{in } Q_T, \\ u_1(x,t) = u_2(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ b_1(x,u_1(x,0)) = b_1(x,u_1^0(x)) & \text{in } \Omega, \\ b_2(x,u_2(x,0)) = b_2(x,u_2^0(x)) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Let $p : \overline{\Omega} \to [1,+\infty)$ be a continuous real-valued function and let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p^- < p^+ < N$. The operator $-\operatorname{div}\big(a(x,t,u,\nabla u)\big)$ is a Leray–Lions operator defined from some generalized Sobolev space V into its dual space V^* (the two functional spaces will be developed bellow, see (2.3), $\phi_i(x,t,u_i)$ are the Carathéodory functions (see assumptions (3.6)–(3.8)), and $b_i : \Omega \times \mathbb{R} \to \mathbb{R}$ are the Carathéodory functions such that for every $x \in \Omega$, $b_i(x,\cdot)$ is a strictly increasing C^1-function, the function $u_{0,i}$ is in $L^1(\Omega)$ such that $b_i(\cdot,u_{0,i})$ in $L^1(\Omega)$. The functions $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the Carathéodory functions (see assumptions (**H4**) below), and $F_i \in \big(L^{p'(\cdot)}(Q_T)\big)^N$.

Under our assumptions, problem (1.1) does not admit, in general, a solution in the sense of distribution since we cannot expect to have the field $\phi_i(x,t,u_i)$

in $\left(L^1_{\mathrm{loc}}(Q_T)\right)^N$. For this reason, we consider the framework of renormalized solutions (see definition 3.1). The notion of renormalized solutions was introduced by R.-J. DiPerna and P.-L. Lions [17] to study the Boltzmann equation. It was then adapted to the study of some nonlinear elliptic or parabolic problems and evolution problems in fluid mechanics [10, 14].

It should be mentioned that the existence and uniqueness of the renormalized solution for parabolic equations in the form

$$u_t = -\operatorname{div}(a(x,t,u,\nabla u) + \operatorname{div}(\phi(u)) = f$$

have been studied by many authors under various conditions on the data in the classical Sobolev spaces (see, e.g., [1, 2, 11, 15]), and by J. Bennouna [9] in the setting of Orlicz spaces.

In the framework of Sobolev spaces with variable exponents, S.N. Antontsev et al. in [4–6] studied the existence and blow up properties of energy weak solutions for parabolic equations with nonstandard growth conditions of the type

$$u_t = (a(x,t,u,\nabla u)_{x_i} + b(x,t,u)_{x_i} + f. \tag{1.2}$$

In [8, 26], P. Wittbold studied equations (1.2) with the $p(x)$-Laplacian operator. In this paper, we extend these results to nonlinear parabolic equations with the terms $b(x,u(x,t))_t$ and a lower order term of type $(c(x,t)|u(x,t)|^{\gamma(x)})_{x_i}$, where $\gamma(x)$ is suitably given (in terms of $p(\cdot)$ and the dimension N), and we overcome the lack of coercivity by using the approach of renormalized solutions.

The paper is organized as follows. In Section 2, we recall some basic notations and properties of Sobolev spaces with variable exponents. In Section 3, we give basic assumptions and introduce the definition of a renormalized solution. In Section 4, we prove the main result of this paper, Theorem 4.1, on the existence of a renormalized solution. In Appendix A, some technical results are given.

2. Functional spaces

We recall some definitions and basic properties of the generalized Lebesgue–Sobolev spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. We refer to Fan and Zhao [18] for further properties of Lebesgue–Sobolev spaces with variable exponents.

Let $p : \overline{\Omega} \to [1, +\infty)$ be a continuous real-valued function and let $p^- = \min_{x\in\overline{\Omega}} p(x)$ and, $p^+ = \max_{x\in\overline{\Omega}} p(x)$ with $1 < p(\cdot) < N$. We define the Lebesgue space with variable exponent

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R};\, u \,\text{is measurable with} \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

We define the norm, the so-called Luxemburg norm, on this space by the formula

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \left\{ \mu > 0; \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

The following inequality will be used later:

$$\min\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\} \leq \int_\Omega |u(x)|^{p(x)}\,dx \\ \leq \max\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(cdot)}(\Omega)}\right\}. \tag{2.1}$$

If $p^- > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive, and the dual space of $L^{p(\cdot)}(\Omega)$ can be identified with $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder type inequality

$$\int_\Omega |uv|\,dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \tag{2.2}$$

holds true.

Extending a variable exponent $p : \overline{\Omega} \to [1,\infty)$ to $\overline{Q_T} = [0,T] \times \overline{\Omega}$ by setting $p(t,x) := p(x)$ for all $(t,x) \in \overline{Q_T}$, we can also consider the generalized Lebesgue space

$$L^{p(\cdot)}(Q_T) = \left\{u : Q_T \to \mathbb{R}; u \,\text{is measurable with} \quad \int_{Q_T} |u(t,x)|^{p(x)}\,d(t,x) < \infty\right\}$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q_T)} = \inf\left\{\mu > 0; \int_{Q_T} \left|\frac{u(t,x)}{\mu}\right|^{p(x)} \quad d(t,x) \leq 1\right\}$$

which has the same properties as $L^{p(\cdot)}(\Omega)$. We also define the variable Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}.$$

In $W^{1,p(\cdot)}(\Omega)$ we may consider one of the following equivalent norms:

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

or

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \inf\left\{\mu > 0; \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right)\,dx \leq 1\right\}.$$

Then we define $W_0^{1,p(\cdot)}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}$. Assuming $1 < p^- \leq p^+ < \infty$, the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces. The space $\left(W_0^{1,p(\cdot)}(\Omega)\right)^*$ is denoted as the dual space of $W_0^{1,p(\cdot)}(\Omega)$.

For priori estimates, it is necessary to introduce more restrictions on the variable exponents supposing them to be log-Hölder continuous. This concept was introduced for the first time by V.V. Zhikov in [29] (see Theorem 2.2, Paragraph 2 for sufficient conditions for regularity; see also [16] for more details). Moreover, the concept is used to obtain several regularity results for Sobolev spaces with variable exponents, in particular that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$.

Remark 2.1 ([16,29]). The variable exponent $p : \overline{\Omega} \to [1, \infty)$ is said to satisfy the log-continuity condition if

$$\forall x_1, x_2 \in \overline{\Omega}, \quad |x_1 - x_2| < 1, \quad |p(x_1) - p(x_2)| < w(|x_1 - x_2|),$$

where $w : (0, \infty) \to \mathbb{R}$ is a nondecreasing function with

$$\limsup_{\alpha \to 0^+} w(\alpha) \ln(1/\alpha) < +\infty.$$

Lemma 2.1. *Let a variable exponent $p(\cdot)$ satisfy the log-continuity such that $1 \leq p^- \leq p^+ < N$,*

$$\forall u \in W_0^{1,p(\cdot)}(\Omega), \quad \|u\|_{L^{p^*(\cdot)}(\Omega)} \leq C \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

with $C = C(N, C_{log}(p), p^+)$ and $\frac{1}{p^*(\cdot)} = \frac{1}{p(\cdot)} - \frac{1}{N}$ for $p(x) < N$ a.e. in Ω, $p^*(\cdot) = \infty$ otherwise.

We introduce the functional space

$$V = \left\{ v \in L^{p^-}(0, T; W_0^{1,p(\cdot)}(\Omega));\ |\nabla v| \in L^{p(\cdot)}(Q_T) \right\}, \tag{2.3}$$

which, endowed with the norm

$$\|v\|_V := \|\nabla v\|_{L^{p(\cdot)}(Q_T)},$$

or, the equivalent norm

$$\|v\|_V := \|v\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|\nabla v\|_{L^{p(\cdot)}(Q_T)},$$

is a separable Banach space. We state some further properties of V in the following lemma.

Lemma 2.2 (see [8])**.** *Let V be defined as in (2.3) and its dual space be denoted by V^*. Then*

(i) *we have the following continuous dense embedding:*

$$L^{p^+}\left(0, T; W_0^{1,p(\cdot)}(\Omega)\right) \hookrightarrow V \hookrightarrow L^{p^-}\left(0, T; W_0^{1,p(\cdot)}(\Omega)\right). \tag{2.4}$$

In particular, since $D(Q_T)$ is dense in $L^{p^+}\left(0, T; W_0^{1,p(\cdot)}(\Omega)\right)$, it is also dense in V, and for the corresponding dual space, we have

$$L^{(p^-)'}\left(0, T; \left(W_0^{1,p(\cdot)}(\Omega)\right)^*\right) \hookrightarrow V^* \hookrightarrow L^{(p^+)'}\left(0, T; \left(W_0^{1,p(\cdot)}(\Omega)\right)^*\right); \tag{2.5}$$

(ii) *one can represent the elements of V^* as follows: if $\widetilde{F} \in V^*$, then there exists $F = (f_1, \ldots, f_N) \in \left(L^{p(\cdot)}(Q_T)\right)^N$ such that $\widetilde{F} = \operatorname{div}_x F$ and, for any $v \in V$,*

$$\langle \widetilde{F}, v \rangle_{V^*,V} = \langle \operatorname{div}_x F, v \rangle_{V^*,V} = \int_0^T \int_\Omega F \nabla v \, dx dt,$$

moreover, we have, $\|\widetilde{F}\|_{V^*} = \max\left\{ \|f_i\|_{L^{p(\cdot)}(Q_T)},\ i = 1, \ldots, n \right\}$.

Remark 2.2. Notice that $V \cap L^\infty(Q_T)$, endowed with the norm

$$\|v\|_{V\cap L^\infty(Q_T)} := \max\left\{\|v\|_V, \|v\|_{L^\infty(Q_T)}\right\}, \quad v \in V \cap L^\infty(Q_T),$$

is a Banach space. In fact, it is the dual space of the Banach space $V^* + L^1(Q_T)$, endowed with the norm

$$\|v\|_{V^*+L^1(Q_T)} := \inf\left\{\|v_1\|_{V^*} + \|v_2\|_{L^1(Q_T)};\ v = v_1 + v_2, v_1 \in V^*, v_2 \in L^1(Q_T)\right\}.$$

Lemma 2.3. *The following holds:*

$$W := \left\{u \in V;\ u_t \in V^* + L^1(Q_T)\right\} \hookrightarrow C\left([0,T]; L^1(\Omega)\right), \tag{2.6}$$

and

$$W \cap L^\infty(Q_T) \hookrightarrow C\left([0,T]; L^2(\Omega)\right). \tag{2.7}$$

Proof. The proof of this lemma follows the same lines as the proof of the corresponding result for the case of a constant exponent p, Theorem 1.1 from [8]. □

3. Assumptions on the data and definition of renormalized solution

Throughout this paper, we will assume that the following assumptions hold true:

Assumption (H1):

$$b_i : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function} \tag{3.1}$$

such that for every $x \in \Omega$, $b_i(x,\cdot)$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function with $b_i(x,0) = 0$; for any $k > 0$, there exists a constant $\lambda_k^i > 0$ and the functions $A_k^i \in L^\infty(\Omega)$ and $B_k^i \in L^{p(\cdot)}(\Omega)$ such that for almost every x in Ω,

$$\lambda_k^i \le \frac{\partial b_i(x,s)}{\partial s} \le A_k^i(x) \quad \text{and} \quad \left|\nabla_x\left(\frac{\partial b_i(x,s)}{\partial s}\right)\right| \le B_k^i(x), \quad |s| \le k. \tag{3.2}$$

Assumption (H2): Let $a:\ Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that for any $k > 0$, there exists $h_k \in L^{p'(\cdot)}(Q_T)$ with

$$|a(x,t,s,\xi)| \le \nu[h_k(x,t) + |\xi|^{p(x)-1}], \qquad |s| \le k \text{ with } \nu > 0, \tag{3.3}$$

$$a(x,t,s,\xi)\xi \ge \alpha|\xi|^{p(x)} \qquad \text{with } \alpha > 0, \tag{3.4}$$

$$(a(x,t,s,\xi) - a(x,t,s,\eta))(\xi - \eta) > 0 \qquad \text{with } \xi \ne \eta. \tag{3.5}$$

Assumption (H3): For $i = 1, 2$, let $\phi_i:\ Q_T \times \mathbb{R} \to \mathbb{R}^N$ be a Carathéodory function such that

$$|\phi_i(x,t,s)| \le c_i(x,t)|s|^{\gamma(x)}, \tag{3.6}$$

$$c_i(x,t) \in \left(L^{\tau(x)}(Q_T)\right)^N, \quad \tau(\cdot) = \frac{N+p(\cdot)}{p(\cdot)-1}, \tag{3.7}$$

$$\gamma(\cdot) = \frac{N+2}{N+p(\cdot)}(p(\cdot)-1), \quad \gamma^- = \min_{x\in\overline{\Omega}} \gamma(x), \quad \gamma^+ = \max_{x\in\overline{\Omega}} \gamma(x) \tag{3.8}$$

for almost every $(x,t) \in Q_T$, for every $s \in \mathbb{R}$, and every $\xi, \eta \in \mathbb{R}^N$.

Assumption (H4): For $i = 1, 2$, $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f_1(x, 0, s) = f_2(x, s, 0) = 0$ a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$. And for almost every $x \in \Omega$, for every $s_1, s_2 \in \mathbb{R}$,

$$\operatorname{sign}(s_i) f_i(x, s_1, s_2) \geq 0. \tag{3.9}$$

The growth assumptions on f_i are as follows: for each $k > 0$, there exists $\sigma_k > 0$ and a function $H_{1,k}$ in $L^1(\Omega)$ such that

$$|f_1(x, s_1, s_2)| \leq H_{1,k}(x) + \sigma_k |b_2(x, s_2)| \quad \text{a.e. in } \Omega,\ |s_1| \leq k,\ s_2 \in \mathbb{R}; \tag{3.10}$$

for each $k > 0$, there exists $\mu_k > 0$ and a function $H_{2,k}$ in $L^1(\Omega)$ such that

$$|f_2(x, s_1, s_2)| \leq H_{2,k}(x) + \mu_k |b_1(x, s_1)| \quad \text{a.e. in } \Omega,\ |s_2| \leq k,\ s_1 \in \mathbb{R}, \tag{3.11}$$

$$F_i \in L^{p'(\cdot)}(Q_T) \quad \text{for } i = 1, 2, \tag{3.12}$$

$$u_{0,i} \in L^1(\Omega) \quad \text{such that } b_i(x, u_{0,i}) \in L^1(\Omega). \tag{3.13}$$

The definition of the renormalized solution for problem (1.1) can be stated as follows.

Definition 3.1. A couple of measurable functions (u_1, u_2) defined on Q_T is called a *renormalized* solution of (1.1) if for $i = 1, 2$, the function u_i satisfies

$$b_i(x, u_i) \in L^\infty\left(0, T; L^1(\Omega)\right), \tag{3.14}$$

$$T_k(u_i) \in L^{p^-}\left((0,T); W_0^{1,p(\cdot)}(\Omega)\right), \quad k > 0, \tag{3.15}$$

$$\nabla T_k(u_i) \in \left(L^{p(\cdot)}(Q_T)\right)^N, \quad k > 0, \tag{3.16}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{|u_i| \leq n\}} a(x, t, u_i, \nabla u_i) \nabla u_i \, dx\, dt = 0, \tag{3.17}$$

and if for every function S in $W^{2,\infty}(\mathbb{R})$, which is piecewise C^1 and such that S' has a compact support, the following holds:

$$\begin{aligned}
\frac{\partial B_{i,S}(x,u_i)}{\partial t} &- \operatorname{div}\Big(a(x,t,u_i,\nabla u_i) S'(u_i)\Big) + S''(u_i) a(x,t,u_i,\nabla u_i)\nabla u_i \\
&+ \operatorname{div}\Big(\phi_i(x,t,u_i) S'(u_i)\Big) - S''(u_i)\phi_i(x,t,u_i)\nabla u_i \\
&\qquad = f_i(x,u_1,u_2) S'(u_i) - \operatorname{div}(S'(u_i) F_i) \\
&\qquad\quad + S''(u_i) F_i \nabla u_i \qquad \text{in } D'(Q_T),
\end{aligned} \tag{3.18}$$

$$B_{i,S}(x, u_i)|_{t=0} = B_{i,S}(x, u_{i,0}) \quad \text{in } \Omega, \tag{3.19}$$

where $B_{i,S}(x,z) = \displaystyle\int_0^z \frac{\partial b_i(x,s)}{\partial s} S'(s)\, ds.$

Remark 3.1. Equation (3.18) is formally obtained through multiplication of (1.1) by $S'(u)$. However, as $a(x,t,u_i,\nabla u_i)$ and $\phi_i(x,t,u_i)$ do not in general make sense in $D'(Q_T)$, all the terms in (3.18) have a meaning in $D'(Q_T)$ (see, e.g., [14]).

We have

$$\frac{\partial B_{i,S}(x,u_i)}{\partial t} \quad \text{belongs to} \quad L^1(Q_T)+V^*. \tag{3.20}$$

The properties of S, assumptions (3.2) and (3.15) imply that if K is such that $\operatorname{supp} S^{'} \subset [-K,K]$,

$$\left|\nabla B_{i,S}(x,u_i)\right| \le \|A_K^i\|_{L^\infty(\Omega)}|DT_K(u_i)|\|S'\|_{L^\infty(\mathbb{R})} + K\|S'\|_{L^\infty(\mathbb{R})}B_K^i(x), \tag{3.21}$$

and

$$B_{i,S}(x,u_i) \quad \text{belongs to} \quad V \cap L^\infty(Q_T), \tag{3.22}$$

then (3.20) and (3.22) imply that $B_{i,S}(x,u_i)$ belongs to $C^0([0,T];L^1(\Omega))$ (for the proof of this trace result see [24]). Hence the initial condition (3.19) makes sense.

4. Main result: existence of renormalized solution

Our main results are collected in the following theorem.

Theorem 4.1. *For $i=1,2$, let $b_i(x,u_{0,i}) \in L^1(\Omega)$. Assume that* (**H1**)–(**H4**) *hold true, then there exists at least one renormalized solution (u_1,u_2) of problem* (1.1) *(in the sense of Definition 3.1).*

Proof. The above theorem is to be proved in five steps.

Step 1: A regularized problem. For $i=1,2$ for each $\epsilon>0$, let us introduce the following regularization of the data:

$$b_{i,\epsilon}(x,r) = b_i(x,T_{1/\epsilon}(r)) + \epsilon r \qquad \text{a.e. in } \Omega, \qquad r\in\mathbb{R}, \tag{4.1}$$

$$a_\epsilon(x,t,s,\xi) = a(x,t,T_{1/\epsilon}(s),\xi) \qquad \text{a.e. in } Q_T, \qquad s\in\mathbb{R},\ \xi\in\mathbb{R}^N, \tag{4.2}$$

$$\phi_{i,\epsilon}(x,t,r) = \phi_i(x,t,T_{1/\epsilon}(r)) \qquad \text{a.e. in } Q_T, \qquad r\in\mathbb{R}, \tag{4.3}$$

$$\begin{aligned} f_{1,\epsilon}(x,s_1,s_2) &= f_1(x,T_{1/\epsilon}(s_1),s_2) && \text{a.e. in } \Omega, && s_1,s_2\in\mathbb{R},\\ f_{2,\epsilon}(x,s_1,s_2) &= f_2(x,s_1,T_{1/\epsilon}(s_2)) && \text{a.e. in } \Omega, && s_1,s_2\in\mathbb{R}. \end{aligned} \tag{4.4}$$

Let $u_{i,0\epsilon}\in D(\Omega)$ such that

$$b_{i,\epsilon}(x,u_{i,0\epsilon}) \to b_i(x,u_{i,0}) \text{ strongly in } L^1(\Omega). \tag{4.5}$$

In view of (4.1), for $i=1,2$, $b_{i,\epsilon}$ is a Carathéodory function which satisfies (3.2). There exists $\lambda_i+\epsilon>0$ and functions $A_\epsilon^i\in L^\infty(\Omega)$ and $B_\epsilon^i\in L^{p(\cdot)}(\Omega)$ such that

$$\lambda_i+\epsilon \le \frac{\partial b_{i,\epsilon}(x,s)}{\partial s} \text{ and } |b_{i,\epsilon}(x,s)| \le \max_{|s|\le 1/\epsilon}|b_i(x,s)| \quad \text{a.e. in } \Omega,\ s\in\mathbb{R}. \tag{4.6}$$

Let us now consider the regularized problem

$$\begin{cases} \dfrac{\partial b_{i,\epsilon}(x,u_{i,\epsilon})}{\partial t} - \operatorname{div}(a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})) & \\ \qquad + \operatorname{div}(\phi_{i,\epsilon}(x,t,u_{i,\epsilon})) & \\ \qquad = f_{i,\epsilon}(x,u_1,u_2) - \operatorname{div}(F_i) & \text{in } Q_T, \\ u_{i,\epsilon}(x,t) = 0 & \text{on } \partial\Omega\times(0,T), \\ b_{i,\epsilon}(x,u_{i,\epsilon})|_{t=0} = b_{i,\epsilon}(x,u_{i,0\epsilon}) & \text{in } \Omega. \end{cases} \tag{4.7}$$

In view of (3.10)–(3.11), there exist $H_{1,\epsilon}\in L^1(\Omega)$, $H_{2,\epsilon}\in L^1(\Omega)$, $\sigma_\epsilon>0$, and $\mu_\epsilon>0$ such that

$$\begin{aligned} &|f_{1,\epsilon}(x,s_1,s_2)| \le H_{1,\epsilon}(x) + \sigma_\epsilon \max_{|s|\le 1/\epsilon} |b_i(x,s)| && \text{a.e. in } \Omega,\ s_1,s_2\in\mathbb{R}, \\ &|f_{2,\epsilon}(x,s_1,s_2)| \le H_{2,\epsilon}(x) + \mu_\epsilon \max_{|s|\le 1/\epsilon} |b_i(x,s)| && \text{a.e. in } \Omega,\ s_1,s_2\in\mathbb{R}. \end{aligned} \tag{4.8}$$

As a consequence, it is easy to prove the existence of a weak solution $u_\epsilon\in V$ of (4.7) (see [21]).

Step 2: A priori estimates for the solutions and their gradients. Let $t_1\in(0,T)$ and t be fixed in $(0,t_1)$. Using in (4.7) $T_k(u_{i,\epsilon})\chi_{(0,t)}$ as a test function, we integrate in the interval $(0,t)$. By the conditions (4.3) and (3.6), we have

$$\begin{aligned} \int_\Omega B^\epsilon_{i,k}(x,u_{i,\epsilon}(t))\,dx &+ \int_{Q_t} a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\nabla T_k(u_{i,\epsilon})\,dx\,ds \\ &\le \int_{Q_t} c_i(x,t)|u_{i,\epsilon}|^{\gamma(x)}|\nabla T_k(u_{i,\epsilon})|\,dx\,ds \\ &\quad + \int_{Q_t} f_{i,\epsilon}(x,u_{1,\epsilon},u_{2,\epsilon})T_k(u_{i,\epsilon})\,dx\,ds \\ &\quad + \int_\Omega B^\epsilon_{i,k}(x,u_{i,0\epsilon})dx + \int_{Q_t} F_i\nabla T_k(u_{i,\epsilon})\,dx\,ds, \end{aligned} \tag{4.9}$$

where $B^\epsilon_{i,k}(x,r)=\int_0^r T_k(s)\frac{\partial b_{i,\epsilon}(x,s)}{\partial s}\,ds$. By (4.6),

$$\int_\Omega B^\epsilon_{i,k}(x,u_{i,\epsilon}(t))\,dx \ge \frac{\lambda_i+\epsilon}{2}\int_\Omega |T_k(u_{i,\epsilon})|^2\,dx \ge \frac{\lambda_i}{2}\int_\Omega |T_k(u_{i,\epsilon})|^2\,dx. \tag{4.10}$$

Under the definition of $B^\epsilon_{i,k}$, the inequality

$$0\le \int_\Omega B^\epsilon_{i,k}(x,u_{i,0\epsilon})\,dx \le k\int_\Omega |b_{i,\epsilon}(x,u_{i,0\epsilon})|\,dx, \quad k>0, \tag{4.11}$$

holds. According to (4.8)–(4.11), and (3.4), we obtain

$$\begin{aligned} \frac{\lambda_i}{2}\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_{i,\epsilon})|^2\,dx &+ \alpha\int_{Q_t} |\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds \\ &\le \int_{Q_t} c_i(x,t)|u_{i,\epsilon}|^{\gamma(x)}|\nabla T_k(u_{i,\epsilon})|\,dx\,ds \end{aligned}$$

$$+ k(\|b_{i,\epsilon}(x,u_{i,0\epsilon})\|_{L^1(\Omega)} + \|f_{i,\epsilon}\|_{L^1(Q_T)}) + \int_{Q_t} F_i \nabla T_k(u_{i,\epsilon})\,dx\,ds. \tag{4.12}$$

If we take the supremum for $t \in (0,t_1)$ and define $M_i = (\|f_{i,\epsilon}\|_{L^1(Q_T)} + \|b_{i,\epsilon}(x,u_{i,0\epsilon})\|_{L^1(\Omega)})$, we can deduce

$$\begin{aligned}\frac{\lambda_i}{2}\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_{i,\epsilon})|^2\,dx + \alpha\int_{Q_{t_1}} |\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds &\le M_i k + \int_{Q_{t_1}} c_i(x,t)|u_{i,\epsilon}|^{\gamma(x)}|\nabla T_k(u_{i,\epsilon})|\,dx\,ds \\ &\quad + \int_{Q_{t_1}} F_i\nabla T_k(u_{i,\epsilon})\,dx\,ds.\end{aligned} \tag{4.13}$$

Now we estimate $\int_{Q_{t_1}} c_i(x,t)|u_{i,\epsilon}|^{\gamma(x)}|\nabla T_k(u_\epsilon)|\,dx\,ds$. Using the generalized Hölder inequality, we have

$$\begin{aligned}&\int_{Q_{t_1}} c_i(x,t)|u_{i,\epsilon}|^{\gamma(x)}|\nabla T_k(u_{i,\epsilon})|\,dx\,dt \\ &\quad\le C\|c_i(x,t)\|_{L^{\tau(\cdot)}(Q_{t_1})}\||T_k u_{i,\epsilon}|^{\gamma(x)}\|_{L^{\omega(\cdot)}(Q_{t_1})}\|\nabla T_k(u_{i,\epsilon})\|_{L^{p(\cdot)}(Q_{t_1})},\end{aligned} \tag{4.14}$$

where $\omega(\cdot) > \frac{p(\cdot)(N+p(\cdot))}{N(p(\cdot)-1)}$, $\max_{x\in\overline{\Omega}}\omega(x) = \omega^+$, and $\min_{x\in\overline{\Omega}}\omega(x) = \omega^-$.

By applying Gagliardo–Niremberg generalized inequalities (see Appendix, Corollary A.1), one has

$$\begin{aligned}&\int_{Q_{t_1}} |T_k(u_{i,\epsilon})|^{\frac{p(x)(N+2)}{N}}\,dx\,ds \\ &\le C\max\left\{\left(\int_{Q_{t_1}}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\right)^{\frac{p^+}{p^-}}, \left(\int_{Q_{t_1}}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\right)^{\frac{p^-}{p^+}}\right\} \\ &\times\max\left\{\left(\sup_{t\in(0,t_1)}\int_\Omega|T_k(u_{i,\epsilon})|^2dx\right)^{\frac{p^+}{N}}, \left(\sup_{t\in(0,t_1)}\int_\Omega|T_k(u_{i,\epsilon})|^2dx\right)^{\frac{p^-}{N}}\right\}.\end{aligned} \tag{4.15}$$

Since

$$\min\left(\frac{1}{C^{p^+}}\|T_k u_{i,\epsilon}\|^{p^+}_{L^{p^*(\cdot)}(Q_{t_1})}, \frac{1}{C^{p^-}}\|T_k u_{i,\epsilon}\|^{p^-}_{L^{p^*(\cdot)}(Q_{t_1})}\right) \le \int_{Q_{t_1}}|\nabla T_K u_{i,\epsilon}|^{p(x)}\,dx\,ds$$

and

$$\|T_k(u_{i,\epsilon})\|^2_{L^2(\Omega)} \le \sup_{t\in(0,t_1)}\int_\Omega |T_k(u_{i,\epsilon})|^2dx,$$

we get

$$\beta_1\left(\sup_{t\in(0,t_1)}\int_\Omega|T_k(u_{i,\epsilon})|^2dx\right) \ge 1 \quad\text{and}\quad \beta_2\int_{Q_{t_1}}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds \ge 1, \tag{4.16}$$

where

$$\beta_1 = \frac{1}{\|T_k u_{i,\epsilon}\|^2_{L^2(\Omega)}},$$

$$\beta_2 = \left(\min\left(\frac{1}{C^{p^+}}\|T_k u_{i,\epsilon}\|^{p^+}_{L^{p^*(\cdot)}(Q_{t_1})}, \frac{1}{C^{p^-}}\|T_k u_{i,\epsilon}\|^{p^-}_{L^{p^*(\cdot)}(Q_{t_1})}\right)\right)^{-1}.$$

After doing some calculations and by using (4.15) and (4.16), we obtain

$$\begin{aligned}
&\int_{Q_{t_1}} |T_k(u_{i,\epsilon})|^{\frac{p(x)(N+2)}{N}}\,dx\,ds \\
&\quad\le \frac{C}{\beta_1\beta_2}\max\left\{\beta_1^{1-\frac{p^+}{N}}\left(\beta_1 \sup_{t\in(0,t_1)}\int_\Omega |T_k(u_{i,\epsilon})|^2 dx\right)^{\frac{p^+}{N}}, \right. \\
&\qquad\qquad \left. \beta_1^{1-\frac{p^-}{N}}\left(\beta_1 \sup_{t\in(0,t_1)}\int_\Omega |T_k(u_{i,\epsilon})|^2 dx\right)^{\frac{p^-}{N}}\right\} \\
&\qquad \times \max\left\{\beta_2^{1-\frac{p^+}{p^-}}\left(\beta_2\int_{Q_{t_1}} |\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\right)^{\frac{p^+}{p^-}}, \right. \\
&\qquad\qquad \left. \beta_2^{1-\frac{p^-}{p^+}}\left(\beta_2\int_{Q_{t_1}} |\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\right)^{\frac{p^-}{p^+}}\right\} \\
&\quad\le C_1\left(\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_{i,\epsilon})|^2 dx\right)^{\frac{p^+}{N}}\left(\int_{Q_{t_1}} |\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\right)^{\frac{p^+}{p^-}}. \qquad (4.17)
\end{aligned}$$

In the same way, we arrive to the following inequality:

$$\|\nabla T_k(u_{i,\epsilon})\|_{L^{p(\cdot)}(Q_{t_1})} \le C_2\left(\int_{Q_{t_1}} |\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\right)^{\frac{1}{p^-}}. \qquad (4.18)$$

Combining (4.17) and (4.18), we can conclude that

$$\begin{aligned}
&\||T_k u_{i,\epsilon}|^{\gamma(x)}\|_{L^{\omega(\cdot)}(Q_{t_1})}\|\nabla T_k(u_{i,\epsilon})\|_{L^{p(\cdot)}(Q_{t_1})} \\
&\qquad\qquad \le C_3\left(\sup_{t\in(0,t_1)}\int_\Omega |T_k(u_{i,\epsilon})|^2 dx\right)^{\frac{\lambda p^+}{N}} \\
&\qquad\qquad\quad \times\left(\int_{Q_{t_1}} |\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\right)^{\frac{\lambda p^+}{p^-}+\frac{1}{p^-}}, \qquad (4.19)
\end{aligned}$$

where

$$\lambda = \begin{cases} \frac{1}{\omega^+} & \text{if } \||T_k u_{i,\epsilon}|^{\gamma(x)}\|_{L^{\omega(\cdot)}(Q_{t_1})} \ge 1, \\ \frac{1}{\omega^-} & \text{if } \||T_k u_{i,\epsilon}|^{\gamma(x)}\|_{L^{\omega(\cdot)}(Q_{t_1})} \le 1. \end{cases}$$

Since $\gamma(x) = \frac{(N+2)(p(x)-1)}{N+p(x)}$ for all $x \in \overline{\Omega}$ and $\frac{\gamma^-}{\gamma^+}p^- < p(x)$, we have $\gamma(x) < \frac{(N+2)(p(x)-1)}{N+\frac{\gamma^-}{\gamma^+}p^-}$ for all $x \in \overline{\Omega}$. Moreover, if $\frac{p(\cdot)(N+p(\cdot))}{N(p(\cdot)-1)} = \frac{p(x)(N+2)}{N\gamma(x)}$, then we can find that $\frac{p(x)(N+2)}{N\gamma^+} < \omega(x)$ for all $x \in \overline{\Omega}$.

Then the continuity of $\gamma(\cdot)$ and $p(\cdot)$ on $\overline{\Omega}$ implies that for all $x \in \Omega$, there exist some constants $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\max_{x\in \overline{B(x,\delta_1)\cap\Omega}} \frac{p(x)(N+2)}{N\gamma^+} < \min_{x\in \overline{B(x,\delta_1)\cap\Omega}} \omega(x), \tag{4.20}$$

$$\max_{x\in \overline{B(x,\delta_2)\cap\Omega}} \gamma(x) < \min_{x\in \overline{B(x,\delta_2)\cap\Omega}} \frac{(N+2)(p(x)-1)}{N+\frac{\gamma^-}{\gamma^+}p^-}. \tag{4.21}$$

By taking $\delta = \min(\delta_1, \delta_2)$, we can see that inequalities (4.20) and (4.21) hold on $\overline{B(x,\delta)\cap\Omega}$ for all $x \in \Omega$. So, recalling that $\overline{\Omega}$ is compact, we can cover it with a finite number of balls $(B_j)_{j=1,\dots,k}$. By p_j^+, $p_j'^+$, γ_j^+, ω_j^+ and λ_j^+ we denote the local maximum of p, p', γ, ω, and λ on $\overline{B_j\cap\Omega}$ and by p_j^-, $p_j'^-$, γ_j^-, ω_j^- and λ_j^- we denote the local minimum of p, p', γ, ω and λ on $\overline{B_j\cap\Omega}$). Hence $\frac{p_j^+(N+2)}{N\gamma_j^+} < \omega_j^- < \omega_j^+$, which implies

$$\frac{\lambda_j p_j^+}{p_j^-} < \frac{N\gamma_j^+}{p_j^-(N+2)} \quad \text{and} \quad \frac{\lambda_j p^+}{N} < \frac{\gamma_j^+}{N+2}. \tag{4.22}$$

From (4.16), (4.19), and (4.22), it easy to check that instead of global estimate we can find

$$\begin{aligned}
&\||T_k u_{i,\epsilon}|^{\gamma(x)}\|_{L^{\omega(\cdot)}(Q_{t_1}^j)}\|\nabla T_k(u_{i,\epsilon})\|_{L^{p(\cdot)}(Q_{t_1}^j)} \\
&\quad\le C_4\left(\sup_{t\in(0,t_1)}\int_{B_j\cap\Omega}|T_k(u_{i,\epsilon})|^2dx\right)^{\frac{\gamma_j^+}{N+2}}\left(\int_{Q_{t_1}^j}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\right)^{\frac{N\gamma_j^+}{p_j^-(N+2)}+\frac{1}{p_j^-}} \\
&\quad\le C_4\left(k^2\operatorname{mes}(\Omega)\right)^{\frac{\gamma_j^+\gamma_j^-}{\gamma_j^-(N+2)}}\left(\left(\frac{1}{k^2\operatorname{mes}(\Omega)}\sup_{t\in(0,t_1)}\int_{B_j\cap\Omega}|T_k(u_{i,\epsilon})|^2dx\right)^{\frac{\gamma_j^+}{\gamma_j^-}}\right)^{e\frac{\gamma_j^-}{N+2}} \\
&\qquad\times\left(\int_{Q_{t_1}^j}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\right)^{\frac{N\gamma_j^+}{p_j^-(N+2)}+\frac{1}{p_j^-}}.
\end{aligned}$$

Therefore, by $\int_\Omega |T_k(u_{i,\epsilon})|^2dx \le k^2\operatorname{mes}(\Omega)$, we get

$$\frac{1}{k^2\operatorname{mes}(\Omega)}\sup_{t\in(0,t_1)}\int_{B_j\cap\Omega}|T_k(u_{i,\epsilon})|^2dx \le 1,$$

and as $\frac{\gamma_j^+}{\gamma_j^-} > 1$, we can claim that

$$\||T_k u_{i,\epsilon}|^{\gamma(x)}\|_{L^{\omega(\cdot)}(Q^j_{t_1})}\|\nabla T_k(u_{i,\epsilon})\|_{L^{p(\cdot)}(Q^j_{t_1})}$$
$$\leq C_5 \left(\sup_{t\in(0,t_1)} \int_{B_j\cap\Omega} |T_k(u_{i,\epsilon})|^2 \, dx \right)^{\frac{\gamma_j^-}{N+2}}$$
$$\times \left(\int_{Q^j_{t_1}} |\nabla T_k(u_{i,\epsilon})|^{p(x)} \, dx \, ds \right)^{\frac{N\gamma_j^+}{p_j^-(N+2)} + \frac{1}{p_j^-}}. \quad (4.23)$$

Finally, using (4.14), we get the estimate

$$\int_{Q^j_{t_1}} c_i(x,t)|u_{i,\epsilon}|^{\gamma(x)}|\nabla T_k(u_{i,\epsilon})| \, dx \, ds$$
$$\leq C\|c_i(x,t)\|_{L^{\tau(\cdot)}(Q^j_{t_1})} \left(\sup_{t\in(0,t_1)} \int_{B_j\cap\Omega} |T_k(u_{i,\epsilon})|^2 dx \right)^{\frac{\gamma_j^-}{N+2}}$$
$$\times \left(\int_{Q^j_{t_1}} |\nabla T_k(u_{i,\epsilon})|^{p(x)} \, dx \, ds \right)^{\frac{N\gamma_j^+}{p_j^-(N+2)} + \frac{1}{p_j^-}}. \quad (4.24)$$

According to (4.24) and using Young inequalities, we obtain

$$\int_{Q^j_{t_1}} c_i(x,t)|u_{i,\epsilon}|^{\gamma(x)}|\nabla T_k(u_{i,\epsilon})| \, dx \, ds$$
$$\leq C\|c_i(x,t)\|_{L^{\tau(\cdot)}(Q^j_{t_1})}(\frac{\gamma_j^-}{N+2}) \sup_{t\in(0,t_1)} \int_{B_j\cap\Omega} |T_k u_{i,\epsilon}|^2 dx + C\|c_i(x,t)\|_{L^{\tau(\cdot)}(Q^j_{t_1})}$$
$$\times (\frac{N+2-\gamma_j^-}{N+2} + \epsilon_1) \left(\int_{Q^j_{t_1}} |\nabla T_k u_{i,\epsilon}|^{p(x)} \, dx \, ds \right)^{\frac{N+2}{N+2-\gamma_j^-}\left(\frac{1}{p_j^-} + \frac{N\gamma_j^+}{p_j^-(N+2)}\right)}. \quad (4.25)$$

In view of (4.21), we deduce

$$\frac{N+2+\gamma_j^+ N}{p_j^-(N+2) - \gamma_j^- p_j^-} \leq 1. \quad (4.26)$$

Since

$$\left(\frac{1}{p_j^-} + \frac{N\gamma_j^+}{P_j^-(N+2)} \right) \frac{N+2}{N+2-\gamma^-} = \frac{N+2+\gamma_j^+ N}{p_j^-(N+2) - \gamma_j^- p_j^-}$$

and $\beta_2 \int_{Q^j_{t_1}} |\nabla T_k u_{i,\epsilon}|^{p(x)} \, dx \, ds \geq 1$, using (4.25), we obtain that

$$
\begin{aligned}
&\int_{Q^j_{t_1}} c_i(x,t)|u_{i,\epsilon}|^{\gamma(x)}|\nabla T_k(u_{i,\epsilon})|\,dx\,ds\\
&\qquad\qquad \le C\frac{\gamma_j^-}{N+2}\|c_i(x,t)\|_{L^{\tau(\cdot)}(Q^j_{t_1})}\sup_{t\in(0,t_1)}\int_{B_j\cap\Omega}|T_k u_{i,\epsilon}|^2dx\\
&\qquad\qquad + C\frac{N+2-\gamma_j^-}{N+2}\|c_i(x,t)\|_{L^{\tau(\cdot)}(Q^j_{t_1})}\int_{Q^j_{t_1}}|\nabla T_k u_{i,\epsilon}|^{p(x)}\,dx\,ds. \qquad (4.27)
\end{aligned}
$$

Combining (4.13) and (4.27) and using Young inequality, we have

$$
\begin{aligned}
&\frac{\lambda_i}{2}\int_{B_j\cap\Omega}|T_k(u_{i,\epsilon})||^2\,dx+\alpha\int_{Q^j_{t_1}}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\\
&\qquad \le M_ik+C\frac{\gamma_j^-}{N+2}\|c_i(x,t)\|_{L^{\tau(\cdot)}(Q^j_{t_1})}\sup_{t\in(0,t_1)}\int_{B_j\cap\Omega}|T_k(u_{i,\epsilon})|^2\,dx\\
&\qquad + C\frac{N+2-\gamma_j^-}{N+2}\|c_i(x,t)\|_{L^{\tau(\cdot)}(Q^j_{\tau_1})}\int_{Q^j_{t_1}}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\\
&\qquad\qquad + \frac{1}{p_j'^-}\|F_i\|^{\beta}_{L^{p'(\cdot)}(Q_T)}+\frac{1}{p_j^-}\int_{Q^j_{t_1}}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds,
\end{aligned}
$$

where

$$
\beta=\begin{cases} p_j'^+ & \text{if } \|F_i\|_{L^{p'(\cdot)}(Q_T)}\ge 1,\\ p_j'^- & \text{if } \|F_i\|_{L^{p'(\cdot)}(Q_T)}\le 1,\end{cases}
$$

which is equivalent to

$$
\begin{aligned}
&\left(\frac{\lambda_i}{2}-C\frac{\gamma_j^-}{N+2}||c_i(x,t)||_{L^{\tau(\cdot)}(Q^j_{t_1})}\right)\sup_{t\in(0,t_1)}\int_{B_j\cap\Omega}|T_k(u_{i,\epsilon})|^2\,dx\\
&\quad+\left(\alpha-C\frac{N+2-\gamma_j^-}{N+2}||c_i(x,t)||_{L^{\tau(\cdot)}(Q^j_{t_1})}-\frac{1}{p_j^-}\right)\int_{Q^j_{t_1}}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,ds\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le M'_{i,j}k,
\end{aligned}
$$

where $M'_{i,j}=M_i+\frac{1}{kp_j'^-}\|F_i\|^{\beta}_{L^{p'(\cdot)}(Q_T)}$.

If we choose t_1 such that

$$
\left(\frac{\lambda_i}{2}-C\frac{\gamma_j^-}{N+2}||c_i(x,t)||_{L^{\tau(\cdot)}(Q^j_{t_1})}\right)>0, \qquad (4.28)
$$

$$
\left(\alpha-C\frac{N+2-\gamma_j^-}{N+2}||c_i(x,t)||_{L^{\tau(\cdot)}(Q^j_{t_1})}-\frac{1}{p_j^-}\right)>0, \qquad (4.29)
$$

then, denoting by $C_{i,j}$ the minimum between (4.28) and (4.29), we obtain

$$\sup_{t\in(0,t_1)}\int_{B_j\cap\Omega}|T_k(u_{i,\epsilon})|^2\,dx+\int_{Q^j_{t_1}}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,dt\le C_{i,j}M'_{i,j}k$$
$$\text{for all}\, j=1,\ldots,k.$$

Hence we obtain the desired result

$$\sup_{t\in(0,t_1)}\int_{\Omega}|T_k(u_{i,\epsilon})|^2\,dx+\int_{Q_{t_1}}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,dt\le C_iM'_ik. \tag{4.30}$$

Then, by (4.30), we conclude that $T_k(u_{i,\epsilon})$ is bounded in V independently of ϵ and for any $k\ge 0$. Thus, there exists a subsequence still denoted by $u_{i,\epsilon}$ such that

$$T_k(u_{i,\epsilon})\rightharpoonup m_{i,k}\quad\text{in } L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega)). \tag{4.31}$$

We turn now to proving the almost every convergence of $u_{i,\epsilon}$ and $b_{i,\epsilon}(u_{i,\epsilon})$. Consider a nondecreasing function $g_k\in C^2(\mathbb{R})$ such that $g_k(s)=s$ for $|s|\le\frac{k}{2}$ and $g_k(s)=k$ for $|s|\ge k$. Multiplying the approximate equation by $g'_k(u_{i,\epsilon})$, we get

$$\begin{aligned}\frac{\partial B_g^{i,\epsilon}(x,u_{i,\epsilon})}{\partial t}&-\operatorname{div}\Big(a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})g'_k(u_{i,\epsilon})\Big)\\&+a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})g''_k(u_{i,\epsilon})\nabla u_{i,\epsilon}+\operatorname{div}\Big(\phi_{i,\epsilon}(x,t,u_{i,\epsilon})g'_k(u_{i,\epsilon})\Big)\\&-g''_k(u_{i,\epsilon})\nabla u_{i,\epsilon}\phi_{i,\epsilon}(x,t,u_{i,\epsilon})\nabla u_{i,\epsilon}\\=f_{i,\epsilon}g'_k(u_{i,\epsilon})&-\operatorname{div}(F_ig'_k(u_{i,\epsilon}))+F_ig''_k(u_{i,\epsilon})\nabla u_{i,\epsilon}\quad\text{in } D'(Q_T),\end{aligned} \tag{4.32}$$

where $B_g^{i,\epsilon}(x,z)=\int_0^z\frac{\partial b_{i,\epsilon}(x,s)}{\partial s}g'_k(s)ds$.

In view of (3.3) and (4.2) and taking into account that $T_k(u_{i,\epsilon})$ is bounded in V, we deduce that $g_k(u_{i,\epsilon})$ is bounded in V and $\frac{\partial B_g^{i,\epsilon}(x,u_{i,\epsilon})}{\partial t}$ is bounded in $L^1(Q_T)+V^*$. Indeed, since $\operatorname{supp}(g'_k)$ and $\operatorname{supp}(g''_k)$ are both included into $[-k,k]$, by (4.3), it follows that for all $0<\epsilon<\frac{1}{k}$, we have

$$\begin{aligned}&\left|\int_{Q_T}\phi_{i,\epsilon}(x,t,u_{i,\epsilon})^{p'(x)}g'_k(u_{i,\epsilon})^{p'(x)}\,dx\,dt\right|\\&\qquad\le\int_{Q_T}c_i(x,t)^{p'(x)}|T_{\frac{1}{\epsilon}}(u_{i,\epsilon})|^{p'(x)\gamma(x)}|g'_k(u_{i,\epsilon})|^{p'(x)}\,dx\,dt\\&\qquad=\int_{\{|u_{i,\epsilon}|\le k\}}c_i(x,t)^{p'(x)}|T_k(u_{i,\epsilon})|^{p'(x)\gamma(x)}|g'_k(u_{i,\epsilon})|^{p'(x)}\,dx\,dt.\end{aligned}$$

Furthermore, by the Hölder and the Gagliard–Niremberg inequalities, it results

$$\begin{aligned}&\int_{\{|u_{i,\epsilon}|\le k\}}c_i(x,t)^{p'(x)}|T_k(u_{i,\epsilon})|^{p'(x)\gamma(x)}|g'_k(u_{i,\epsilon})|^{p'(x)}\,dx\,dt\\&\le\|g'_k\|_{L^\infty(\mathbb{R})}\|c_i(x,t)\|^{\eta_1}_{L^{\tau(\cdot)}(Q_T)}\\&\quad\times\left[\sup_{t\in(0,T)}\left(\int_\Omega|T_k(u_{i,\epsilon})|^2\,dx\right)^{\frac{p^-}{N}}\left(\int_{Q_T}|\nabla T_k(u_{i,\epsilon})|^{p(x)}\,dx\,dt\right)^{\frac{p^+}{p^-}}\right]^{\eta_2}\le c_k,\end{aligned}$$

where

$$\eta_1 = \begin{cases} \frac{\tau^+}{\mu^-} & \text{if } \|c_i(x,t)\|_{L^{\tau(\cdot)}(Q_T)} \geq 1, \\ \frac{\tau^-}{\mu^+} & \text{if } \|c_i(x,t)\|_{L^{\tau(\cdot)}(Q_T)} \leq 1, \end{cases}$$

$$\eta_2 = \begin{cases} \frac{1}{\nu^-} & \text{if } \||T_k(u_{i,\epsilon})|^{\gamma(x)p'(x)}\|_{L^{\nu(\cdot)}(Q_T)} \geq 1, \\ \frac{1}{\nu^+} & \text{if } \||T_k(u_{i,\epsilon})|^{\gamma(x)p'(x)}\|_{L^{\nu(\cdot)}(Q_T)} \leq 1, \end{cases}$$

$$\mu(\cdot) = \frac{\tau(\cdot)}{p'(\cdot)}, \quad \text{and} \quad \nu(\cdot) = \frac{\tau(\cdot)}{\tau(\cdot) - p'(\cdot)},$$

and c_k is a constant independent of ϵ which varies from line to line. In the same way, by (4.3), we have

$$\left| \int_{Q_T} \phi_{i,\epsilon}(x,t,u_{i,\epsilon})^{p'(x)} (g_k''(u_{i,\epsilon}) \nabla u_{i,\epsilon})^{p'(x)} \, dx \, dt \right| \\ \leq \int_{Q_T} (g_k''(u_{i,\epsilon})^{p'(x)} |c_i(x,t)|^{p'(x)} |T_{\frac{1}{\epsilon}}(u_{i,\epsilon})|^{p'(x)} |\nabla u_{i,\epsilon}|^{p'(x)} \, dx \, dt. \quad (4.33)$$

Furthermore, by the Hölder and the Gagliardo–Niremberg inequalities, for $0 < \epsilon < \frac{1}{k}$, we obtain

$$\begin{aligned} &\int_{Q_T} (g_k''(u_{i,\epsilon})^{p'(x)} |c_i(x,t)|^{p'(x)} |T_{\frac{1}{\epsilon}}(u_{i,\epsilon})|^{\gamma(x)p'(x)} |\nabla u_{i,\epsilon}|^{p'(x)} \, dx \, dt \\ &\quad = \int_{Q_T} (g_k''(u_{i,\epsilon})^{p'(x)} |c_i(x,t)|^{p'(x)} |T_k(u_{i,\epsilon})|^{\gamma(x)p'(x)} |\nabla T_k(u_{i,\epsilon})|^{p'(x)} \, dx \, dt \\ &\quad \leq \|g_k''\|_{L^\infty(\mathbb{R})} \sup_{|r| \leq k} |b'(r)| \int_{Q_T} |c_i(x,t)|^{p'(x)} |T_k(u_{i,\epsilon})|^{\gamma(x)p'(x)} |\nabla T_k(u_{i,\epsilon})|^{p'(x)} \, dx \, dt \\ &\quad \leq c_k. \end{aligned}$$

By (4.32), we may conclude that

$$\frac{\partial g_k(b_{i,\epsilon}(x,u_{i,\epsilon}))}{\partial t} \quad \text{is bounded in } L^1(Q_T) + V^*. \quad (4.34)$$

Arguing again as in [12], estimates (4.31) and (4.34) imply that there exists a subsequence, still indexed by $u_{i,\epsilon}$,

$$u_{i,\epsilon} \to u_i \text{ a.e. in } Q_T, \quad (4.35)$$

where u_i is a measurable function defined on Q_T.

Let us prove that $b_i(x,u_i)$ belongs to $L^\infty((0,T);L^1(\Omega))$. Using (4.9), (4.27), and (4.30), we deduce that

$$\int_\Omega B_k^{i,\epsilon}(x,u_{i,\epsilon})dx \leq M_i' k C_i + C_1. \quad (4.36)$$

In view of (4.35) and passing to the limit-inf in (4.36), as ϵ tends to zero, we obtain that

$$\frac{1}{k} \int_\Omega B_{i,k}(x,u_i(\tau)) \, dx \leq C_2 \quad (4.37)$$

for almost any τ in $(0,T)$, with

$$B_{i,k}(x,r) = \int_0^r \frac{\partial b_i(x,s)}{\partial s} T_k(s)\, ds.$$

Due to the definition of $B_{i,k}(x,s)$ and the fact that $\frac{1}{k}B_k(x,u_i)$ converges pointwise to

$$\int_0^{u_i} \operatorname{sign}(s)\frac{\partial b_i(x,s)}{\partial s}\, ds = |b_i(x,u_i)|,$$

as k tends to $+\infty$, it is possible to show that $b_i(x,u_i) \in L^\infty(0,T;L^1(\Omega))$.

Step 3: The limit of the solution of the approximated problem.

Lemma 4.1. *For $i = 1,2$, the subsequence of $u_{i,\epsilon}$ defined in Step 1 satisfies*

$$\lim_{n\to+\infty} \limsup_{\epsilon\to 0} \frac{1}{n}\int_{|u_{i,\epsilon}|\le n\}} a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\nabla u_{i,\epsilon}\, dx\, dt = 0. \tag{4.38}$$

Proof. Using the admissible test function $\frac{1}{n}T_n(u_{i,\epsilon})$ from (4.7), and by (4.3), (4.24), (4.26) and using again the elliptic condition on a, the Young inequality and the boundedness of $T_n(u_{i,\epsilon)}$ in V, we can claim that for all $R < n$ for $\epsilon < \frac{1}{n}$:

$$\begin{aligned}
&\frac{1}{2n}\int_{\{|u_{i,\epsilon}|\le n\}} a(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\nabla T_n(u_{i,\epsilon})\, dx\, dt \\
&\qquad\qquad \le \frac{1}{n}\int_{\{|u_{i,\epsilon}|\le R\}} c_i(x,t)|T_R(u_{i,\epsilon})|^{\gamma(x)}|\nabla T_R(u_{i,\epsilon})|\, dx\, dt \\
&\qquad\qquad + \frac{1}{n}\int_\Omega B_{i,n}(x,u_{i,0\epsilon})dx + C_6\|c_i(x,t)\chi_{\{|u_{i,\epsilon}|>R\}}\|_{L^{\tau(\cdot)}(Q_T)}^{\frac{N+2}{\gamma^-}} \\
&\qquad\qquad + \frac{1}{n}\int_{Q_T} f_{i,\epsilon}T_n(u_{i,\epsilon})\, dx\, dt + \frac{1}{n}\left(\frac{2}{\omega}\right)^{\frac{p'^+}{p^-}} \|F_i\|_{Q_T}^\lambda.
\end{aligned} \tag{4.39}$$

Due to the fact that $u_{i,\epsilon}$ converges to u almost everywhere, for $|T_n(r)| \le r$, the Lebesgue dominated convergence theorem implies that $T_n(u_{i,\epsilon})$ converges to $T_n(u_i)$ in $L^\infty(Q_T)$ weakly-$*$.

By (4.37) and (3.2), we obtain

$$\int_\Omega |u_i(x,t)|dx \le \frac{3}{2}C_1 k|\Omega| + C_2$$

for almost any $t \in (0,T)$, which shows that $u_i \in L^\infty(0,T;L^1(\Omega))$.

As a consequence, we have that $T_n(u_i)/n$ tends to zero almost everywhere in Q_T.

In view of (3.9), (4.4), (4.5), (4.30), (4.31) and (4.35), using the Lebesgue convergence theorem and passing to limit in (4.39), as ϵ tends to zero (then n tends to $+\infty$ and then R tends to $+\infty$), it is obvious that $u_{i,\epsilon}$ satisfies Lemma 4.1.

□

Step 4: Here we are to prove that the weak limit h_k of $a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_{i,\epsilon}))$ can be identified with $a(x,t,T_k(u_i),\nabla T_k(u_i))$ for $i=1,2$. In order to show this result we recall the lemma below.

Lemma 4.2. *For $i=1,2$, the subsequence of $u_{i,\epsilon}$ satisfies*

$$\limsup_{\epsilon\to 0}\int_{Q_T}\int_0^t a(x,s,u_{i,\epsilon},\nabla T_k(u_{i,\epsilon}))\nabla T_k(u_{i,\epsilon})\,ds\,dx\,dt \le \int_{Q_T}\int_0^t h_k\nabla T_k(u_i)\,dx\,ds\,dt, \tag{4.40}$$

$$\lim_{\epsilon\to 0}\int_{Q_T}\int_0^t \Big(a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_{i,\epsilon}))-a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_i))\Big) \times\Big(\nabla T_k(u_{i,\epsilon})-\nabla T_k(u_i)\Big)=0, \tag{4.41}$$

$$h_k=a(x,t,T_k(u_i),\nabla T_k(u_i)) \text{ a.e. in } Q_T, \text{for any } k\ge 0, \text{ as } \epsilon \text{ tends to } 0, \tag{4.42}$$

$$a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_{i,\epsilon}))\nabla T_k(u_{i,\epsilon}) \rightharpoonup a(x,t,T_k(u_i),\nabla T_k(u_i))\nabla T_k(u_i) \quad \text{weakly in } L^1(Q_T). \tag{4.43}$$

Proof. For $i=1,2$, we introduce a time regularization of the $T_k(u_i)$ for $k>0$ in order to perform the monotonicity method. This specific time regularization was first introduced by R. Landes in [20]. By $(T_k(u_i))_\mu$, we denote the regularized function of $T_k(u_i)$ with $\mu>0$. Thus, by using the same argument as in [11], we can show the following lemma.

Lemma 4.3. *Let $k\ge 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})$-function such that $S(r)=r$ for $|r|\le k$, and $\operatorname{supp} S'$ is compact. Then,*

$$\liminf_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\left\langle \frac{\partial b_{i,\epsilon}(x,u_{i,\epsilon})}{\partial t}, S'(u_{i,\epsilon})(T_k(u_{i,\epsilon})-(T_k(u_i))_\mu)\right\rangle \ge 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^1(\Omega)+W^{-1,p'(\cdot)}(\Omega)$ and $L^\infty(\Omega)\cap V(\Omega)$.

Let S_n be a sequence of increasing $C^\infty(\mathbb{R})$-functions such that

$$S_n(r)=r \text{ for } |r|\le n,\ \operatorname{supp} S_n'\subset[-2n,2n] \text{ and } \|S_n''\|_{L^\infty(\mathbb{R})}\le\frac{3}{n} \text{ for any } n\ge 1.$$

For $i=1,2$, we use the sequence $(T_k(u_i))_\mu$ of approximation of $T_k(u_i)$ and consider the test function $S_n'(u_{i,\epsilon})(T_k(u_{i,\epsilon})-(T_k(u_i))_\mu)$ for $n>0$ and $\mu>0$. For fixed $k\ge 0$, we define $W_\mu^{i,\epsilon}=T_k(u_{i,\epsilon})-(T_k(u_i))_\mu$ and by integrating over $(0,t)$ and then over $(0,T)$, we get

$$\int_0^T\int_0^t\left\langle\frac{\partial b_{i,\epsilon}(x,u_{i,\epsilon})}{\partial t}, S_n'(u_{i,\epsilon})W_\mu^{i,\epsilon}\right\rangle ds\,dt + \int_0^T\int_0^t\int_\Omega a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})S_n'(u_{i,\epsilon})\nabla W_\mu^{i,\epsilon}\,dx\,ds\,dt$$

$$
\begin{aligned}
&+\int_0^T\int_0^t\int_\Omega a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})S_n''(u_{i,\epsilon})\nabla u_{i,\epsilon}W_\mu^{i,\epsilon}\,dx\,ds\,dt\\
&-\int_0^T\int_0^t\int_\Omega \phi_{i,\epsilon}(x,t,u_{i,\epsilon})S_n'(u_{i,\epsilon})\nabla W_\mu^{i,\epsilon}\,dx\,ds\,dt\\
&-\int_0^T\int_0^t\int_\Omega S_n''(u_{i,\epsilon})\phi_{i,\epsilon}(x,t,u_{i,\epsilon})\nabla u_{i,\epsilon}W_\mu^{i,\epsilon}\,dx\,ds\,dt\\
=&\int_0^T\int_0^t\int_\Omega f_{i,\epsilon}S_n'(u_{i,\epsilon})W_\mu^{i,\epsilon}\,dx\,ds\,dt.\\
&+\int_{Q_T}\int_0^t F_iS_n'(u_{i,\epsilon})\nabla W_\mu^{i,\epsilon}\,ds\,dt\,dx\\
&+\int_{Q_T}\int_0^t F_iS_n''(u_{i,\epsilon})\nabla u_{i,\epsilon}W_\mu^{i,\epsilon}\,ds\,dt\,dx.
\end{aligned}
\tag{4.44}
$$

Now we pass to the limit in (4.44) as $\epsilon\to 0$, $\mu\to+\infty$, $n\to+\infty$ for k real number fixed. In order to perform this task, we prove below the following results for any fixed $k\geq 0$:

$$
\liminf_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\left\langle\frac{\partial b_{i,\epsilon}(x,u_{i,\epsilon})}{\partial t},S_n'(u_{i,\epsilon})W_\mu^{i,\epsilon}\right\rangle ds\,dt\geq 0 \text{ for any } n\geq k, \tag{4.45}
$$

$$
\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega \phi_{i,\epsilon}(x,t,u_{i,\epsilon})S_n'(u_{i,\epsilon})\nabla W_\mu^{i,\epsilon}\,dx\,ds\,dt=0 \quad \text{for any } n\geq 1, \tag{4.46}
$$

$$
\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega \phi_{i,\epsilon}(x,t,u_{i,\epsilon})\nabla u_{i,\epsilon}W_\mu^{i,\epsilon}\,dx\,ds\,dt=0 \text{ for any } n\geq 1, \tag{4.47}
$$

$$
\begin{aligned}
\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega &a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\\
&\times S_n''(u_{i,\epsilon})\nabla u_{i,\epsilon}W_\mu^{i,\epsilon}\,dx\,ds\,dt=0,
\end{aligned}
\tag{4.48}
$$

$$
\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega f_{i,\epsilon}S_n'(u_{i,\epsilon})W_\mu^{i,\epsilon}\,dx\,ds\,dt=0, \tag{4.49}
$$

$$
\lim_{\mu\to+\infty}\int_{Q_T}\int_0^t F_iS_n'(u_{i,\epsilon})\nabla W_\mu^{i,\epsilon}\,ds\,dt\,dx=0, \tag{4.50}
$$

$$
\lim_{\mu\to+\infty}\int_{Q_T}\int_0^t F_iS_n''(u_{i,\epsilon})\nabla u_{i,\epsilon}W_\mu^{i,\epsilon}\,ds\,dt\,dx=0. \tag{4.51}
$$

Proof of (4.45)*:* The function S_n belongs to $C^\infty(\mathbb{R})$ and it is increasing. We have $n\geq k$, $S_n(r)=r$ for $|r|\leq k$ because $\operatorname{supp} S_n'$ is compact. In view of the definition of $W_\mu^{i,\epsilon}$ we apply Lemma 4.3 with $S=S_n$ for fixed $n\geq k$. As a consequence, (4.45) holds true. □

Proof of (4.46)*:* Let us recall the main properties of $W_\mu^{i,\epsilon}$. For fixed $\mu>0$, $W_\mu^{i,\epsilon}$ converges to $T_k(u_i)-(T_k(u_i))_\mu$ weakly in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ as $\epsilon\to 0$. Taking into account that

$$
\|W_\mu^{i,\epsilon}\|_{L^\infty(Q_T)}\leq 2k \quad \text{for any } \epsilon>0,\ \mu>0, \tag{4.52}
$$

we can deduce that

$$W_\mu^{i,\epsilon} \rightharpoonup T_k(u_i) - (T_k(u_i))_\mu \quad \text{a.e. in } Q_T \text{ and } L^\infty(Q_T) \text{ weakly-}* \text{ as } \epsilon \to 0. \tag{4.53}$$

For any fixed $n \geq 1$ and $0 < \epsilon < \frac{1}{2n}$,

$$\phi_{i,\epsilon}(x,t,u_\epsilon) S_n'(u_{i,\epsilon}) \nabla W_\mu^{i,\epsilon} = \phi_{i,\epsilon}(x,t,T_{2n}(u_{i,\epsilon})) S_n'(u_{i,\epsilon}) \nabla W_\mu^{i,\epsilon} \quad \text{a.e. in } Q_T,$$

since $\operatorname{supp} S' \subset [-2n, 2n]$. On the other hand,

$$\phi_{i,\epsilon}(x,t,T_{2n}(u_{i,\epsilon})) S_n'(u_{i,\epsilon}) \to \phi(x,t,T_{2n}(u)) S_n'(u) \quad \text{a.e. in } Q_T,$$
$$|\phi_{i,\epsilon}(x,t,T_{2n}(u_{i,\epsilon})) S_n'(u_{i,\epsilon})| \leq C c(x,t) (2n)^{\gamma^+} \quad \text{for } n \geq 1.$$

By (4.53) and the strong convergence of $T_k(u_{i,\epsilon})_\mu$ to $T_k(u_i)$ in $L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega))$, we obtain (4.46). □

Proof of (4.47): For any fixed $n \geq 1$ and $0 < \epsilon < \frac{1}{2n}$, we have

$$\phi_{i,\epsilon}(x,t,u_{i,\epsilon}) S_n''(u_{i,\epsilon}) \nabla u_{i,\epsilon} W_\mu^{i,\epsilon} = \phi_{i,\epsilon}(x,t,T_{2n}(u_{i,\epsilon})) S_n''(u_{i,\epsilon}) \nabla T_{2n}(u_{i,\epsilon}) W_\mu^\epsilon \quad \text{a.e. in } Q_T;$$

by (4.53) and (4.35), as in the previous step, it is possible to pass to the limit for $\epsilon \to 0$:

$$\phi_{i,\epsilon}(x,t,T_{n+1}(u_{i,\epsilon})) S_n''(u_{i,\epsilon}) W_\mu^{i,\epsilon} \to \phi_i(x,t,T_{2n}(u_i)) S_n''(u_i) W_\mu \quad \text{a.e. in } Q_T.$$

Since

$$|\phi_i(x,t,T_{n+1}(u_i)) S_n''(u_i) W_\mu| \leq 2Ck|c(x,t)|(2n)^{\gamma^+} \quad \text{a.e. in } Q_T$$

and $(T_k(u_i))_\mu$ converges to $T_k(u_i)$ in $L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, we obtain (4.47). □

Proof of (4.48): In view of the definition of S_n, we have $\operatorname{supp} S' \subset [-2n,-n] \cup [n,2n]$ for any $n \geq 1$. Thus,

$$\left| \int_0^T \int_0^t \int_\Omega a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon}) S_n''(u_{i,\epsilon}) W_\mu^{i,\epsilon} \, dx\, ds\, dt \right|$$
$$\leq T \|S_n''(u_{i,\epsilon})\|_{L^\infty(\mathbb{R})} \|W_\mu^{i,\epsilon}\|_{L^\infty(Q_T)} \int_{n \leq |u_{i,\epsilon}| \leq 2n} a(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon}) \nabla u_{i,\epsilon} \, dx\, ds\, dt$$

for any $n \geq 1$, any $0 < \epsilon < \frac{1}{2n}$ and any $\mu > 0$. Since $\|S_n''\|_{L^\infty(\mathbb{R})} \leq \frac{3}{n}$, by (4.38), it is possible to establish (4.48). □

Proof of (4.49): By (4.4), due to the pointwise convergence of $u_{i,\epsilon}$ and $W_\mu^{i,\epsilon}$, and their boundness, it is possible to pass to the limit for $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \int_0^T \int_0^t \int_\Omega f_{i,\epsilon} S_n'(u_i)(T_k(u_i) - (T_k(u_i))_\mu) \, dx\, ds\, dt$$
$$= \int_0^T \int_0^t \int_\Omega f_i S_n'(u_i) \left(T_k(u_i) - (T_k(u_i))_\mu\right) dx\, ds\, dt$$

for any $\mu > 0$ and any $n \geq 1$. Now, for fixed $n \geq 1$, it is possible to pass to the limit as μ tends to $+\infty$ in the above inequality. □

Proof of (4.50)*:* We have

$$
\begin{aligned}
&F_i S'_n(u_{i,\epsilon}) \to F_i.S'_n(u_i) && \text{a.e. in } Q_T,\\
&|F_i S'_n(u_{i,\epsilon})| \le 2n\|F_i\|_{L^{p'(\cdot)}(Q_T)} && \text{a.e. in } Q_T.
\end{aligned}
$$

We obtain (4.50), by (4.53) and the strong convergence of $T_k(u_{i,\epsilon})_\mu$ to $T_k(u_i)$ in $L^{p^-}\left(0,T,W_0^{1,p(\cdot)}(\Omega)\right)$. □

Proof of (4.51)*:* For any fixed $n \ge 1$ and $0 < \epsilon < \frac{1}{2n}$, we have

$$F_i S''(u_{i,\epsilon})\nabla u_{i,\epsilon} W_\mu^{i,\epsilon} = F_i S''(u_{i,\epsilon})\nabla T_{2n}(u_{i,\epsilon}) W_\mu^{\epsilon} \quad \text{a.e. in } Q_T.$$

As in the previous step, we can obtain (4.51) by passing to the limit for $\epsilon \to 0$, where (4.52), (4.53), and the strong convergence of $T_k(u_{i,\epsilon})_\mu$ in $L^{p^-}\left(0,T,W_0^{1,p(\cdot)}(\Omega)\right)$ are taken into account. Recalling (4.45)–(4.51), we can pass to the limit-sup in (4.44) as μ tends to $+\infty$, and to the limit as n tends to $+\infty$. Using the definition of $W_\mu^{i,\epsilon}$, we deduce that for any $k \ge 0$,

$$
\begin{aligned}
\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega & S'_n(u_{i,\epsilon}) a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\\
&\times \left(\nabla T_k(u_{i,\epsilon}) - \nabla(T_k(u_i))_\mu\right)\,dx\,ds\,dt \le 0.
\end{aligned}
$$

Since

$$S'_n(u_{i,\epsilon}) a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\nabla T_k(u_{i,\epsilon}) = a(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\nabla T_k(u_{i,\epsilon})$$

for $k \le \frac{1}{\epsilon}$ and $k \le n$, by using the properties of S'_n, the above inequality implies that for $k \le n$,

$$
\begin{aligned}
\limsup_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega & a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\nabla T_k(u_{i,\epsilon})\,dx\,ds\,dt\\
&\le \lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega S'_n(u_{i,\epsilon}) a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\\
&\qquad\times \nabla(T_k(u_{i,\epsilon}))_\mu\,dx\,ds\,dt. \qquad (4.54)
\end{aligned}
$$

On the other hand, for $\epsilon < \frac{1}{2n}$,

$$S'_n(u_{i,\epsilon}) a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon}) = S'_n(u_{i,\epsilon}) a(x,t,T_{2n}(u_{i,\epsilon}),\nabla T_{2n}(u_{i,\epsilon})) \quad \text{a.e. in } Q_T.$$

Furthermore, we have

$$a_\epsilon(x,t,T_k u_{i,\epsilon},\nabla T_k u_{i,\epsilon}) \rightharpoonup h_k \quad \text{weakly in } \left(L^{p'(\cdot)}(Q_T)\right)^N. \qquad (4.55)$$

It follows that for a fixed $n \ge 1$,

$$S'_n(u_{i,\epsilon}) a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon}) \to S'_n(u_{i,\epsilon}) h_{2n} \quad \text{weakly in } L^{p'(\cdot)}(Q_T)$$

as ϵ tends to zero. Finally, using the strong convergence of $(T_k(u_i))_\mu$ to $T_k(u_i)$ in $L^{p^-}\big(0,T;W_0^{1,p(\cdot)}(\Omega)\big)$ as μ tends to $+\infty$, we get

$$\begin{aligned}\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega S_n'(u_{i,\epsilon})a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\nabla(T_k(u_{i,\epsilon}))_\mu\,dx\,ds\,dt\\ =\int_0^T\int_0^t\int_\Omega S_n'(u_{i,\epsilon})h_{2n}\nabla T_k(u_i)\,dx\,ds\,dt \quad (4.56)\end{aligned}$$

as soon as $k\le n$. Now, for $k\le n$, we have

$$\begin{aligned}a(x,t,T_{2n}(u_{i,\epsilon}),\nabla T_{2n}(u_{i,\epsilon}))\chi_{\{|u_{i,\epsilon}|\le k\}}\\ =a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_{i,\epsilon}))\chi_{\{|u_{i,\epsilon}|\le k\}}\quad\text{a.e. in } Q_T \quad (4.57)\end{aligned}$$

which implies that, by (4.35), (4.55) and by passing to the limit when ϵ tends to 0,

$$h_{2n}\chi_{|u_i|\le k}=h_k\chi_{\{|u_i|\le k\}}\quad\text{a.e. in } Q_T-\{|u_i|=k\}\text{ for } k\le n. \qquad (4.58)$$

Finally, by (4.55) and (4.58), we have $h_{2n}\nabla T_k(u_i)=h_k\nabla T_k(u_i)$ a.e. in Q_T for $k\le n$. Thus the proof of (4.51) is complete. □

Now we are able to prove Lemma 4.2.

Proof of (4.41)*:* Using (3.4), we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega\big(a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_{i,\epsilon}))-a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_i))\big)\\ \times\big(\nabla T_k(u_{i,\epsilon})-\nabla T_k(u_i)\big)\ge 0. \quad (4.59)\end{aligned}$$

Further, by (3.3), (4.35), and the growth condition, we show

$$\begin{aligned}a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_i))\to a(x,t,T_k(u_i),\nabla T_k(u_i))\\ \text{strongly in }\Big(L^{p'(\cdot)}(Q_T)\Big)^N. \quad (4.60)\end{aligned}$$

Furthermore, we have

$$a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\rightharpoonup h_k\quad\text{weakly in }\Big(L^{p'(\cdot)}(Q_T)\Big)^N. \qquad (4.61)$$

Finally, the use of (4.35), (4.60), and (4.61) makes it possible to pass to the limit-sup as ϵ tends to 0 in (4.59), and we have (4.41). □

Proof of (4.42)*:* We observe that for for any $k>0$, any $0<\epsilon<\frac{1}{k}$ and any $\xi\in\mathbb{R}^N$:

$$a_\epsilon(x,t,T_k(u_{i,\epsilon}),\xi)=a(x,t,T_k(u_{i,\epsilon}),\xi)=a_{\frac{1}{k}}(x,t,T_k(u_{i,\epsilon}),\xi)\quad\text{a.e. in } Q_T.$$

Since

$$T_k(u_{i,\epsilon}) \rightharpoonup T_k(u_i) \quad \text{weakly in } L^{p^-}\left((0,T), W_0^{1,p(\cdot)}(\Omega)\right), \tag{4.62}$$

then by (4.41), we obtain

$$\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega a_{\frac{1}{k}}(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_{i,\epsilon}))\nabla T_k(u_{i,\epsilon})\,dx\,ds\,dt \\ = \int_0^T\int_0^t\int_\Omega h_k\nabla T_k(u_i)\,dx\,ds\,dt. \tag{4.63}$$

Since, for fixed $k>0$, the function $a_{\frac{1}{k}}(x,t,s,\xi)$ is continuous and bounded with respect to s, by applying the usual Minty's argument in view of (4.61)–(4.63), it follows that (4.42) holds true. In order to prove (4.43), by (3.4), (4.61) and proceeding as in [11, 12], it is easy to obtain (4.43). □

Thus Lemma 4.2 is proved. □

Passing to the limit: Using the same argument as in [1], we can prove that u_i for $i=1,2$ satisfies (3.17)–(3.19). The proof of Theorem 4.1 is complete. □

A. Appendix

Theorem A.1 (Gagliardo–Nirenberg generalized inequality)**.** *Let v be a function in $W_0^{1,q(\cdot)}(\Omega)\cap L^{\rho(\cdot)}(\Omega)$ with $q,\rho\in P^{\log}(\Omega)$, $1<q^-\le q(x)\le q^+\le N$, $1<\rho^-\le\rho(x)\le\rho^+\le N$.*

Then there exists a positive constant C, depending on N, $q(x)$ and $\rho(x)$, such that

$$\|v\|_{L^{\gamma(\cdot)}(\Omega)} \le C\|\nabla v\|^{\theta}_{(L^{q(\cdot)}(\Omega))^N}\|v\|^{1-\theta}_{L^{\rho(\cdot)}(\Omega)}$$

for every θ and $\gamma(\cdot)$ satisfying: $0\le\theta\le 1$, $1\le\gamma(\cdot)\le+\infty$, $\frac{1}{\gamma(\cdot)}=\theta(\frac{1}{q(\cdot)}-\frac{1}{N})+\frac{1-\theta}{\rho(\cdot)}$.

The proof follows the same lines as the proof for the case of constant exponent [13, p. 147].

Corollary A.1. *Let $v\in L^{q^-}\big((0,T),W_0^{1,q(\cdot)}(\Omega)\big)\cap L^\infty\big((0,T),L^2(\Omega)\big)$, with $q\in P^{\log}(\Omega)$ and $1<q^-\le q(x)\le q^+\le N$. Then $v\in L^{\sigma(\cdot)}(\Omega)$ with $\sigma(\cdot)=q(\cdot)\frac{N+2}{N}$ and*

$$\int_{Q_T}|v|^{\sigma(x)}\,dx\,dt \le C\max\left(\|v\|^{\frac{2q^+}{N}}_{L^\infty(0,T,L^2(\Omega))};\|v\|^{\frac{2q^-}{N}}_{L^\infty(0,T,L^2(\Omega))}\right) \\ \times\max\left(\left(\int_{Q_T}|\nabla v|^{q(x)}\,dxdt\right)^{\frac{q^+}{q^-}};\left(\int_{Q_T}|\nabla v|^{q(x)}\,dx\,dt\right)^{\frac{q^-}{q^+}}\right).$$

Proof. We set $\theta = \frac{N}{N+2}$ in Theorem A.1, then $\sigma(\cdot) = q(\cdot)(\frac{N+2}{N})$, and

$$\|v\|_{L^{\sigma(\cdot)}(\Omega)} \leq C\|v\|^{(1-\theta)}_{L^2(\Omega)}\|v\|^{\theta}_{L^{q(\cdot)}(\Omega)}.$$

We have

$$\int_\Omega |v|^{\sigma(x)}dx \leq \max\left(\|v\|^{\sigma^+}_{L^{\sigma(\cdot)}(\Omega)}; \|v\|^{\sigma^-}_{L^{\sigma(\cdot)}(\Omega)}\right).$$

By Gagliardo–Niremberg generalized inequalities (see Theorem 4.1), one has

$$\|v\|^{\sigma^+}_{L^{\sigma(\cdot)}(\Omega)} \leq C\|v\|^{(1-\theta)\sigma^+}_{L^2(\Omega)} \max\left(\left(\int_\Omega |\nabla v|^{q(x)}\,dx\right)^{\frac{\theta\sigma^+}{q^+}}; \left(\int_\Omega |\nabla v|^{q(x)}\,dx\right)^{\frac{\theta\sigma^+}{q^-}}\right),$$

$$\|v\|^{\sigma^-}_{L^{\sigma(\cdot)}(\Omega)} \leq C\|v\|^{(1-\theta)\sigma^-}_{L^2(\Omega)(\Omega)} \max\left(\left(\int_\Omega |\nabla v|^{q(x)}\,dx\right)^{\frac{\theta\sigma^-}{q^+}}; \left(\int_\Omega |\nabla v|^{q(x)}\,dx\right)^{\frac{\theta\sigma^-}{q^-}}\right).$$

Moreover, $\sigma(\cdot) = q(\cdot)(\frac{N+2}{N})$ implies $\frac{\theta\sigma^+}{q^+} = \frac{\theta\sigma^-}{q^-} = 1$, then

$$\|v\|^{\sigma^+}_{L^{\sigma(\cdot)}(\Omega)} \leq C\|v\|^{2\frac{q^+}{N}}_{L^2(\Omega)} \max\left(\int_\Omega |\nabla v|^{q(x)}\,dx; \left(\int_\Omega |\nabla v|^{q(x)}\,dx\right)^{\frac{q^+}{q^-}}\right),$$

$$\|v\|^{\sigma^-}_{L^{\sigma(\cdot)}(\Omega)} \leq C\|v\|^{2\frac{q^-}{N}}_{L^2(\Omega)} \max\left(\int_\Omega |\nabla v|^{q(x)}\,dx; \left(\int_\Omega |\nabla v|^{q(x)}\,dx\right)^{\frac{q^-}{q^+}}\right).$$

Finally,

$$\begin{aligned}\max&\left(\|v\|^{\sigma^+}_{L^{\sigma(\cdot)}(\Omega)}; \|v\|^{\sigma^-}_{L^{\sigma(\cdot)}(\Omega)}\right)\\ &\leq C\max\left(\|v\|^{\frac{2q^+}{N}}_{L^2(\Omega)}; \|v\|^{\frac{2q^-}{N}}_{L^2(\Omega)}\right)\\ &\quad\times\max\left(\left(\int_\Omega \|\nabla v|^{q(x)}dx\right)^{\frac{q^+}{q^-}}; \left(\int_\Omega |\nabla v|^{q(x)}dx\right)^{\frac{q^-}{q^+}}\right),\end{aligned}$$

and thus the proof of the corollary is complete. □

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Перенормовані розв'язки нелінійних параболічних систем у просторах Лебега–Соболєва з експонентою, що змінюється

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Наведено результат існування перенормованих розв'язків для класу нелінійних параболічних систем з експонентою, що змінюється, типу

$$\partial_t e^{\lambda u_i(x,t)} - \operatorname{div}(|u_i(x,t)|^{p(x)-2}u_i(x,t)) \\ + \operatorname{div}(c(x,t)|u_i(x,t)|^{\gamma(x)-2}u_i(x,t)) = f_i(x,u_1,u_2) - \operatorname{div}(F_i),$$

для $i = 1, 2$. Структура нелінійності змінюється від точки до точки в області Ω. Член джерела менш регулярний (обмежена міра Радона) і в недивергентному члені низшого порядку $\operatorname{div} v(c(x,t)|u(x,t)|^{\gamma(x)-2}u(x,t))$

відсутня коерцитивність. Основний внесок нашої роботи — це доведення існування перенормованих розв'язків без умов коерцитивності на нелінійності, що дозволяє нам скористатися для доведення теоремою Гальярдо–Ніренберга.

Ключові слова: параболічні задачі, простір Лебега–Соболєва, експонента, що змінюється, перенормовані розв'язки.