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Lagrange Stability of Semilinear Differential-Algebraic Equations and Application to Nonlinear Electrical Circuits

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A semilinear differential-algebraic equation (DAE) is studied focusing on the Lagrange stability (instability). The conditions for the existence and uniqueness of global solutions (a solution exists on an infinite interval) of the Cauchy problem, as well as the conditions of the boundedness of the global solutions, are obtained. Furthermore, the obtained conditions of the Lagrange stability of the semilinear DAE guarantee that every its solution is global and bounded and, in contrast to the theorems on the Lyapunov stability, allow us to prove the existence and uniqueness of global solutions regardless of the presence and the number of equilibrium points. We also obtain the conditions for the existence and uniqueness of solutions with a finite escape time (a solution exists on a finite interval and is unbounded, i.e., is Lagrange unstable) for the Cauchy problem. The constraints of the type of global Lipschitz condition are not used which allows to apply efficiently the work results for solving practical problems. The mathematical model of a radio engineering filter with nonlinear elements is studied as an application. The numerical analysis of the model verifies theoretical studies.

Key words: differential-algebraic equation, Lagrange stability, instability, regular pencil, bounded global solution, finite escape time, nonlinear electrical circuit.

Mathematical Subject Classification 2010: 34A09, 34D23, 65L07.

1. Introduction

Differential-algebraic equations (DAEs), which are also called descriptor, algebraic-differential and degenerate differential equations, have a wide range of practical applications. Certain classes of mathematical models in radioelectronics, control theory, economics, robotics technology, mechanics and chemical kinetics are described by semilinear DAEs. Semilinear DAEs comprise in particular semiexplicit DAEs and in turn can be attributed to quasilinear DAEs. The Lagrange stability of a DAE guarantees that every its solution is global and bounded. The presence of a global solution of the equation guarantees a sufficiently long action term of the corresponding real system. The properties of boundedness and stability of solutions of the equations describing mathematical models are used in the design and synthesis of the corresponding real systems and processes. The application of the DAE theory to the study of electrical circuits can be found in various monographs and papers, [4, 5, 9, 11, 13–17] are among them.

In the present paper, the semilinear differential-algebraic equation (DAE)

$$\frac{d}{dt}[Ax] + Bx = f(t,x) \tag{1.1}$$

with a nonlinear function $f\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ and the linear operators A, B : $\mathbb{R}^n \to \mathbb{R}^n$ is considered. The operator A is degenerate (noninvertible), the operator B may also be degenerate. Note that the solutions of a semilinear DAE of the form $A\frac{d}{dt}x + Bx = f(t,x)$ must be smoother than the solutions of a semilinear DAE of the form (1.1). The availability of a noninvertible operator (matrix) at the derivative in the DAE means the presence of algebraic connections, which influence the trajectories of solutions and impose restrictions on the initial data. For the DAE (1.1) with the initial condition

$$x(t_0) = x_0, \tag{1.2}$$

the initial value x_0 must be chosen so that the initial point (t_0, x_0) belongs to the manifold $L_0 = \{(t,x) \in [0,\infty)\times\mathbb{R}^n \mid Q_2[Bx - f(t,x)] = 0\}$ (which is also defined in (3.1), where Q_2 is a spectral projector considered in Section 2). The initial value x_0 satisfying the consistency condition $(t_0, x_0) \in L_0$ is called a consistent initial value. A solution $x(t)$ of the Cauchy problem (1.1), (1.2) (see Definition 2.1) is called *global* if it exists on the whole interval $[t_0,\infty)$.

The influence of the linear part $\frac{d}{dt}[Ax] + Bx$ of the DAE (1.1) is determined by the properties of the pencil $\lambda A + B$ (λ is a complex parameter). It is assumed that $\lambda A + B$ is a regular pencil of index 1, i.e., there exists the resolvent of the pencil $(\lambda A + B)^{-1}$ and it is bounded for sufficiently large $|\lambda|$ (see Section 2). This property of the pencil allows one to use the spectral projectors P_1, P_2, Q_1, Q_2, which can be calculated by contour integration and reduce the DAE to the equivalent system of a purely differential equation and a purely algebraic equation (see Section 2). This is one of the reasons why we use the requirement of index 1 for the characteristic pencil $\lambda A + B$ of the linear part of the DAE and not for the DAE, as, for example, in [11, 12, 22]. Another reason is as follows. The requirement that the DAE have index 1 does not give us the necessary result and it is too restrictive for our research (this will be discussed in Section 2). It is also worth noting that semilinear DAEs of the form (1.1) arise in many practical problems, examples of which can be found in the books and papers by R. Riaza, A.G. Rutkas, A. Favini, L.A. Vlasenko, A.D. Myshkis, S.L. Campbell, L.R. Petzold, K.E. Brenan, E. Hairer, G. Wanner, J. Huang, J.F. Zhang, R.E. Showalter and other authors. However, in present literature these equations are often written in the form $\frac{d}{dt}[Ax] = g(t,x)$ or in the form of semiexplicit DAE.

The objective of the paper is to find the conditions of the Lagrange stability and instability of the semilinear DAE (see Definitions 2.4–2.6). A mathematical model of a radio engineering filter with nonlinear elements is considered as an application. It should be noticed that if the operator A in the semilinear DAE is invertible, then the results obtained in the paper remain valid (in this case, the semilinear DAE is equivalent to an ordinary differential equation).

In Section 3, the theorem on the Lagrange stability, which gives sufficient conditions for the existence and uniqueness of global solutions of the Cauchy problem for the semilinear DAE, as well as conditions of the boundedness of global solutions, is proved. Furthermore, the theorem gives conditions of the Lagrange stability of the semilinear DAE, which ensure that each solution of the DAE starting at the time moment $t_0 \in [0,\infty)$ exists on the whole infinite interval $[t_0,\infty)$ (is global) and is bounded. In Section 4, the theorem on the Lagrange instability, which gives sufficient conditions for the existence and uniqueness of solutions with a finite escape time for the Cauchy problem, is proved. It is important that the proved theorems do not contain restrictions of the type of global Lipschitz condition, including the condition of contractivity, which enable using them for solving more general classes of applied problems. Theorems on the unique global solvability of semilinear DAEs that comprise conditions equivalent to global Lipschitz conditions are known (cf. [23]). Also, the proved theorems do not contain the requirement that the DAE have index 1 globally (this requirement is found, for example, in [11, Theorem 6.7]). For comparison, the theorems from [11, 12, 22] are considered in Sections 1 and 2.

The Lagrange stability of the ordinary differential equation (ODE) $\frac{d}{dt}x = f(t,x)$ ($t \geq 0$, x is an n-dimensional vector) was studied in [10, Chapter 4] using the method obtained by extending the direct (second) method of Lyapunov. The results of [10, Chapter 4] concerning the Lagrange stability are extended to semilinear DAEs in the present paper. The existence and uniqueness theorem of a global solution of the Cauchy problem for the semilinear DAE with a singular pencil $\lambda A + B$ was proved in the author's paper [7]. The results on the Lagrange stability of the semilinear DAE with the regular pencil, obtained by the author in [6], have been improved and have been applied for a detailed study of evolutionary properties of the mathematical model for a radio engineering filter in the present paper.

The stability of linear DAEs and descriptor control systems described by linear DAEs was studied by many authors (see, for example, [5, 11, 16, 21] and references therein).

In [12], R. März studied the Lyapunov stability of an equilibrium point of the autonomous "quasilinear" DAE

$$A\frac{d}{dt}x + g(x) = 0, \tag{1.3}$$

where $A \in L(\mathbb{R}^n)$ is singular (noninvertible) and $g\colon D \to \mathbb{R}^n$, $D \subseteq \mathbb{R}^n$ open. The theorem [12, Theorem 2.1] *allows to prove the existence and uniqueness of global solutions only in some* (sufficiently small) *neighborhood of an equilibrium point* x^* of (1.3), i.e., $g(x^*) = 0$, $x^* \in D$. If there are the two equilibrium points $x_1^* \in D$ and $x_2^* \in D$, $x_1^* \neq x_2^*$, then *the theorem* [12, Theorem 2.1] *can* *not guarantee the existence of a unique global solution in D.* Namely, if the conditions of the theorem are fulfilled for the equilibria x_1^* and x_2^*, then for some initial time moment t_0 there exists the unique global solution $x = \varphi(t)$ of (1.3) (with the initial condition [12, (2.8)]) in some neighborhood of x_1^* and the unique global solution $x = \psi(t)$ of (1.3) in some neighborhood of x_2^*, but this does not guarantee the existence of a unique global solution in D. *Theorem 3.1 allows to prove the existence and uniqueness of global solutions for all possible initial points* (as noted in Remark 3.2), that is, *regardless of the presence of an equilibrium point, in the presence of several equilibrium points or the infinite number of equilibrium points, and for a more general equation than* (1.3).

A theorem similar to [12, Theorem 2.1] was proved by C. Tischendorf [22] for the autonomous nonlinear DAE $f(x'(t), x(t)) = 0$. The theorem [22, Theorem 3.3] gives conditions for the asymptotic stability (in Lyapunov's sense) of a stationary solution x^*, i.e., $f(0, x^*) = 0$. The definition of asymptotic stability from [22, Section 3] is equivalent to the fulfillment of conditions (i)–(iii) from [12, Theorem 2.1], and if we take $f(x'(t), x(t)) = Ax'(t) + g(x(t))$, then [22, Theorem 3.3] and [12, Theorem 2.1] are analogous.

For a global solution of a nonautonomous nonlinear DAE, the conditions for the asymptotic stability (in Lyapunov's sense) which can also be considered only locally (in a sufficiently small neighborhood of this solution) are given in the theorem [11, Theorem 6.16]. Under the conditions of the theorem, it is assumed that the regular index-1 DAE has the global solution, and a DAE linearized along this solution is strongly contractive [11, Definition 6.5].

It is important to note that the theorem on the Lagrange stability (Theorem 3.1) gives conditions for the existence and uniqueness of global solutions (as well as conditions of the boundedness) independently of the presence and the number of equilibrium points. In contrast to Lyapunov stability, Lagrange stability can be considered as the stability of the entire system, not just of its equilibria. From this, in particular, it follows that a globally stable dynamic system can be not only monostable (as in the case of the global stability in Lyapunov's sense), but also multistable (cf. [24, Section I]). In [24], A. Wu and Z. Zeng studied the Lagrange stability of neural networks, which are described by ODEs with delay. It is known that neural networks are also described by DAEs (including semilinear DAEs), therefore the research of the present paper is useful for the analysis and synthesis of the neural networks. Lagrange stability is also used for the analysis of ecological stability. The theorem on the Lagrange instability (Theorem 4.1) can be used, in particular, for the analysis of nonlinear control systems. For example, the study of the Lagrange instability allows to find such a property as a blow up of the solution for a nonlinear control system on a finite time interval.

It is also important to note that even for an ordinary differential equation containing a nonlinear part, the Lyapunov stability of a nontrivial solution does not imply that the solution is bounded, i.e., Lagrange stable. Since the DAE considered in the paper contains the nonlinear part, the Lyapunov stability of its solution does not imply the Lagrange stability. Also, in the general case, the

Lyapunov instability does not imply the Lagrange instability, but the converse assertion is true. Therefore, the proved Lagrange instability theorem can also be regarded as the Lyapunov instability theorem.

Thus, the results obtained on the Lagrange stability of semilinear DAEs are important for the development of the DAE theory and for applied problems. The Lagrange stability of various types of ODEs and its applications are considered in many works, e.g., [1,3,10,24]. However, in [11,12,22] and other cited works, the Lagrange stability of DAEs was not studied.

In Sections 5 and 6, the mathematical model of a radio engineering filter with nonlinear elements is studied with the help of the theorems proved in the previous sections. The restrictions on the initial data and parameters for the electrical circuit of the filter, which ensure the existence, uniqueness and boundedness of global solutions, and the existence and uniqueness of solutions with a finite escape time for the dynamics equation of the electrical circuit are obtained. Certain functions and quantities (including nonlinear functions that are not global Lipschitz) defining the circuit parameters and satisfying the obtained restrictions are given. The numerical analysis of the mathematical model is carried out.

The paper has the following structure. The main theoretical results are given in Sections 3, 4. Namely, the theorems on the Lagrange stability and instability of the DAE are proved. In Sections 5, 6, the mathematical model of the nonlinear radio engineering filter is studied with the help of the obtained theorems. The conclusions and explanations of the obtained results from a physical point of view are given in Subsections 5.1, 6.1, and the numerical analysis of the mathematical model is carried out in Subsections 5.2, 6.2. In Section 2, we give a problem setting, preliminary information and definitions. Section 7 contains general conclusions.

The following notation will be used in the paper: E_X is the identity operator in the space X; $A|_X$ is the restriction of the operator A to X; $L(X,Y)$ is the space of continuous linear operators from X to Y, $L(X,X) = L(X)$; the notation $\int_c^{+\infty} f(t)\,dt < +\infty$ ($\int_c^{+\infty} f(t)\,dt = \infty$) means that the integral converges (does not converge); x^T is the transpose of x. Sometimes the function f is denoted by the same symbol $f(x)$ as its value at the point x in order to explicitly indicate that the function depends on the variable x, but from the context it will be clear what exactly is meant.

2. Problem setting and preliminaries

Consider the Cauchy problem (1.1), (1.2) for the semilinear DAE, where $t, t_0 \geq 0$, $x, x_0 \in \mathbb{R}^n$, $f\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, A, $B\colon \mathbb{R}^n \to \mathbb{R}^n$ are linear operators to which $n \times n$ matrices A, B correspond. The operator A is degenerate (noninvertible), the operator B may also be degenerate. The matrix pencil as well as the corresponding operator pencil $\lambda A + B$ is *regular*, i.e., $\det(\lambda A + B) \not\equiv 0$.

Definition 2.1. A function $x(t)$ is called a *solution* of the Cauchy problem (1.1), (1.2) on some interval $[t_0, t_1)$, $t_1 \leq \infty$, if $x \in C([t_0,t_1),\mathbb{R}^n)$,

$Ax \in C^1([t_0,t_1),\mathbb{R}^n)$, x satisfies equation (1.1) on $[t_0,t_1)$ and the initial condition (1.2).

It is assumed that $\lambda A+B$ is a regular *pencil of index 1*, that is, there exist constants C_1, $C_2>0$ such that

$$\left\|(\lambda A+B)^{-1}\right\| \le C_1, \quad |\lambda| \ge C_2. \tag{2.1}$$

For the pencil $\lambda A+B$ satisfying (2.1) there exist the two pairs of mutually complementary projectors P_1, P_2 and Q_1, Q_2 (i.e., $P_iP_j=\delta_{ij}P_i$, $P_1+P_2=E_{\mathbb{R}^n}$, and $Q_iQ_j=\delta_{ij}Q_i$, $Q_1+Q_2=E_{\mathbb{R}^n}$, $i,j=1,2$, δ_{ij} is the Kronecker delta) first introduced by A.G. Rutkas [17, Lemma 3.2]. The projectors can be constructively determined by the formulas similar to [19, (5), (6)] (where $X=Y=\mathbb{R}^n$) or [17, (3.4)] (for the real operators A, B the projectors are real). These projectors decompose the space $\mathbb{R}^n$ into direct sums of subspaces

$$\mathbb{R}^n = X_1\dot{+}X_2, \quad \mathbb{R}^n = Y_1\dot{+}Y_2, \quad X_j = P_j\mathbb{R}^n, \quad Y_j = Q_j\mathbb{R}^n, \quad j=1,2, \tag{2.2}$$

such that the operators A, B map X_j into Y_j, and the induced operators $A_j = A|_{X_j}: X_j \to Y_j$, $B_j = B|_{X_j}: X_j \to Y_j$, $j=1,2$, ($X_2 = Ker\,A$, $Y_1 = A\mathbb{R}^n = A_1X_1$) are such that $A_2=0$, the inverse operators $A_1^{-1} \in L(Y_1,X_1)$, $B_2^{-1} \in L(Y_2,X_2)$ exist (cf. [17, Lemma 3.2], [19, Sections 2,6]), and

$$AP_j = Q_jA, \quad BP_j = Q_jB, \quad j=1,2. \tag{2.3}$$

With respect to the decomposition (2.2) any vector $x\in\mathbb{R}^n$ can be uniquely represented as the sum

$$x = x_{p_1}+x_{p_2}, \quad x_{p_1}=P_1x\in X_1,\ x_{p_2}=P_2x\in X_2. \tag{2.4}$$

This representation will be used further. We will also use the auxiliary operator $G\in L(\mathbb{R}^n)$, (cf. [19, Sections 2, 6])

$$G = AP_1+BP_2 = A+BP_2, \quad GX_j = Y_j, \quad j=1,2,$$

which has the inverse operator $G^{-1}\in L(\mathbb{R}^n)$ with the properties $G^{-1}AP_1=P_1$, $G^{-1}BP_2=P_2$, $AG^{-1}Q_1=Q_1$, $BG^{-1}Q_2=Q_2$.

By using the projectors P_1, P_2, Q_1, Q_2, the DAE can be reduced to the equivalent system of a purely differential equation and a purely algebraic equation. Applying Q_1, Q_2 to (1.1) and taking into account (2.3), we obtain the equivalent system

$$\begin{cases} \dfrac{d}{dt}(AP_1x)+BP_1x = Q_1f(t,x), \\ Q_2f(t,x)-BP_2x=0. \end{cases}$$

Further, using G^{-1}, we obtain the system, which is equivalent to the DAE (1.1):

$$\begin{cases} \dfrac{d}{dt}(P_1x) = G^{-1}\big[-BP_1x+Q_1f(t,P_1x+P_2x)\big], \\ G^{-1}Q_2f(t,P_1x+P_2x)-P_2x=0. \end{cases} \tag{2.5}$$

Remark 2.1. We consider various notions of an index of the pencil, an index of the DAE, a relationship between them and their relationship with the mentioned notion of the pencil of index 1. In [23, Section 6.2], the maximum length of the chain of an eigenvector and adjoint vectors of the matrix pencil $A + \mu B$ at the point $\mu = 0$ is called the index of the matrix pencil $\lambda A + B$. Following [23, Sections 6.2, 2.3.1], the regular pencil $\lambda A + B$ with the property (2.1) is called a regular pencil of index 1. Taking into account the properties of the projectors P_j, Q_j and the induced operators A_j, B_j, $j = 1, 2$, if the condition (2.1) holds, then the index of the pencil (or the index of nilpotency of the matrix pencil) (A, B) is 1 in the sense as defined in the works of C.W. Gear, L.R. Petzold, for example, [8, p. 717–718] (it is easy to verify using [8, Theorem 2.2]). In [11, Definition 1.4], the index of nilpotency of the matrix pencil (A, B) [8] is called the Kronecker index of the regular matrix pair $\{A, B\}$ which forms the matrix pencil $\lambda A + B$. Also, by the index of the pencil one can determine the index of the corresponding system of differential-algebraic equations [8, p. 718]. In particular, the index of the pencil (A, B) (the Kronecker index of the regular matrix pair $\{A, B\}$) coincides with the index of the linear DAE (the Kronecker index of the regular DAE [11, Definition 1.4]) $A\frac{d}{dt}x + Bx = g(t)$. This is analogous to the fact that the pencil $\lambda A + B$ corresponds to the linear part $\frac{d}{dt}[Ax] + Bx$ of the DAE (1.1) and the influence of the linear part is determined by the properties of the corresponding pencil. For comparison with the notion of the "tractability index" from the works of R. März, C. Tischendorf and R. Lamour [11, 12, 22], note that the linear DAE $\frac{d}{dt}[Ax] + Bx = q(t)$ with the regular pencil $\lambda A + B$ of index 1 (i.e., (2.1) is fulfilled) is regular with tractability index 1 [11, p. 65, 91, Definition 2.25].

But if we consider a semilinear DAE $A\frac{d}{dt}x + Bx = f(t, x)$ (in this case the solution $x(t)$ must be smoother than the solution of the DAE (1.1)), then for it to have index 1 for all $t \geq 0$, $x \in D \subseteq \mathbb{R}^n$ (to be exact, $(t, x) \in L_0$, where L_0 is defined in (3.1)), it is necessary that the pencil $\big(A, B - \frac{\partial}{\partial x} f(t, x)\big)$ have index 1 for all $t \geq 0$, $x \in D$. This condition is too restrictive for our research and it does not allow us to prove the existence of a unique global solution since the uniqueness of the solution can be proved only locally (in [2, Ch. 9], the same is shown for a "semi-explicit index-1" DAE).

One of the conditions for proving the existence of a unique global solution of the Cauchy problem (1.1), (1.2) for any consistent initial value x_0 is the condition of the basis invertibility of an operator function (Definition 2.3) which will be discussed below. To begin, we introduce the definitions.

Definition 2.2. A system of one-dimensional projectors $\{\Theta_k\}_{k=1}^{s}$, $\Theta_k \colon Z \to Z$ such that $\Theta_i \Theta_j = \delta_{ij}\Theta_i$ (δ_{ij} is the Kronecker delta) and $E_Z = \sum_{k=1}^{s} \Theta_k$ is called an *additive resolution of the identity* in an s-dimensional linear space Z.

The additive resolution of the identity generates a direct decomposition of Z into the sum of s one-dimensional subspaces: $Z = Z_1 \dot{+} Z_2 \dot{+} \cdots \dot{+} Z_s$, $Z_k = \Theta_k Z$.

Definition 2.3. Let W, Z be s-dimensional linear spaces, $D \subset W$. An operator function (a mapping) $\Phi\colon D \to L(W, Z)$ is called *basis invertible* on the convex hull $\operatorname{conv}\{\hat{w}, \hat{\hat{w}}\}$ of vectors $\hat{w}, \hat{\hat{w}} \in D$ if for any set of vectors $\{w^k\}_{k=1}^s$, $w^k \in \operatorname{conv}\{\hat{w}, \hat{\hat{w}}\}$, and some additive resolution of the identity $\{\Theta_k\}_{k=1}^s$ in the space Z the operator

$$\Lambda = \sum_{k=1}^{s} \Theta_k \Phi(w^k) \in L(W, Z)$$

has the inverse operator $\Lambda^{-1} \in L(Z, W)$.

Let us represent the operator $\Phi(w) \in L(W, Z)$ as a matrix relative to some bases in the s-dimensional spaces W, Z:

$$\Phi(w) = \begin{pmatrix} \Phi_{11}(w) & \cdots & \Phi_{1s}(w) \\ \vdots & & \vdots \\ \Phi_{s1}(w) & \cdots & \Phi_{ss}(w) \end{pmatrix}.$$

Definition 2.3 can be stated as follows: the matrix function Φ is *basis invertible* on the convex hull $\operatorname{conv}\{\hat{w}, \hat{\hat{w}}\}$ of the vectors $\hat{w}, \hat{\hat{w}} \in D$ if for any set of vectors $\{w^k\}_{k=1}^s \subset \operatorname{conv}\{\hat{w}, \hat{\hat{w}}\}$, the matrix

$$\Lambda = \begin{pmatrix} \Phi_{11}(w^1) & \cdots & \Phi_{1s}(w^1) \\ \vdots & & \vdots \\ \Phi_{s1}(w^s) & \cdots & \Phi_{ss}(w^s) \end{pmatrix}$$

has the inverse Λ^{-1}.

Note that the property of basis invertibility does not depend on the choice of a basis or an additive resolution of the identity in Z. This statement follows directly from Definitions 2.2, 2.3.

Obviously, if the operator function Φ is basis invertible on $\operatorname{conv}\{\hat{w}, \hat{\hat{w}}\}$, then it is invertible at any point $w^* \in \operatorname{conv}\{\hat{w}, \hat{\hat{w}}\}$ ($w^* = \alpha\hat{\hat{w}} + (1-\alpha)\hat{w}$, $\alpha \in [0, 1]$), i.e., for each point $w^* \in \operatorname{conv}\{\hat{w}, \hat{\hat{w}}\}$, its image $\Phi(w^*)$ under the mapping Φ is an invertible continuous linear operator from W to Z. The converse is not true unless the spaces W, Z are one-dimensional. We give an example.

Example 2.1. Let $W = Z = \mathbb{R}^2$, $D = \operatorname{conv}\{\hat{w}, \hat{\hat{w}}\}$, $\hat{w} = (1, -1)^T$, $\hat{\hat{w}} = (1, 1)^T$, $w = (a, b)^T \in D$,

$$\Phi(w) = \begin{pmatrix} ab & 1 \\ -1 & ab \end{pmatrix}.$$

For the set of vectors $\{w^1, w^2\} \subset \operatorname{conv}\{\hat{w}, \hat{\hat{w}}\}$, $w^1 = (a_1, b_1)^T$, $w^2 = (a_2, b_2)^T$, the operator Λ has the form

$$\Lambda = \begin{pmatrix} a_1 b_1 & 1 \\ -1 & a_2 b_2 \end{pmatrix}.$$

Since $\det \Phi(w) = a^2 b^2 + 1 \neq 0$ for any $w \in D$, then $\Phi(w)$ is invertible on D. However, the operator Λ is not invertible for $\{w^1, w^2\} = \{\hat{w}, \hat{\hat{w}}\}$ and hence the operator function Φ is not basis invertible on D. If we take $\hat{w} = (1, 0)^T$, then Φ is basis invertible on D.

Now we will explain why this definition is needed. As shown above, the DAE (1.1) is equivalent to the system of a purely differential equation and a purely algebraic equation. The algebraic equation defines one of the components of a DAE solution as an implicitly given function. With the help of the implicit function theorem this component can be defined as a (unique) explicitly given function, but only locally, i.e., in some sufficiently small neighborhood. But we need a unique globally defined explicit function for further application of the results on Lagrange stability to the differential equation, which will be obtained by substitution of the found component (function). For this purpose, the condition of *the basis invertibility of an operator function* (Definition 2.3), which was first introduced in [18], is used. Note that this condition does not impose restrictions of a type of a global Lipschitz condition, including the condition of contractivity, and does not require the global boundedness of the norm for an inverse function on the whole domain of definition (see Remark 3.1).

In the theorem [11, Theorem 6.7], the conditions of global solvability are given for the nonlinear DAE $f\big((D(t)x)', x, t\big) = 0$ [11, (4.1)]. It is assumed that the DAE [11, (4.1)] is globally regular of index 1, i.e., for all $Dx \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $t \in [0,\infty)$ [11, Theorem 6.7]. This condition means that the pencil $\lambda \frac{\partial}{\partial y} f(y,x,t)D(t) + \frac{\partial}{\partial x} f(y,x,t)$ is regular with Kronecker index 1 for all $y \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $t \in [0,\infty)$ [11, p. 318-320]. Therefore, there must exist the constants C_1, $C_2 > 0$ independent of t, x, y and such that $\left\|\big(\lambda \frac{\partial}{\partial y} f(y,x,t)D(t) + \frac{\partial}{\partial x} f(y,x,t)\big)^{-1}\right\| \le C_1$ for all $y \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $t \in [0,\infty)$, $|\lambda| \ge C_2$, i.e., the norm is globally bounded. Also, the theorem contains the requirement of the contractivity of the regular index-1 DAE (see [11, Definition 6.1, 6.5]) which is an additional condition. Taking into account Remark 3.1, in the case of a semilinear DAE *these conditions are more restrictive than those of global solvability from Theorem 3.1*.

Concerning the theorems [12, Theorem 2.1], [22, Theorem 3.3], note that they are obtained for the autonomous DAE. If we consider the nonautonomous DAEs, namely, $Ax' + g(t,x) = 0$ or $f(x',x,t) = 0$, where $f(x',x,t) = Ax' + g(t,x)$, then, as said above, the requirement that the pencil $\lambda A + \frac{\partial}{\partial x} g(t,x^*)$ have index 1 means that there exist the constants $C_1, C_2 > 0$ independent of t and such that $\left\|\big(\lambda A + \frac{\partial}{\partial x} g(t,x^*)\big)^{-1}\right\| \le C_1$, $|\lambda| \ge C_2$ for all $t \in [0,\infty)$, i.e., the norm is globally bounded in t. Hence, this requirement is more restrictive than the requirement that the operator function Φ is basis invertible and the pencil $\lambda A + B$ has index 1.

Also note that in [12, Theorem 2.1] and [22, Theorem 3.3], the nonlinear function is required to be twice continuously differentiable, while in Theorem 3.1 f is required to be continuous and have the continuous $\frac{\partial}{\partial x} f(t,x)$.

Definition 2.4. A solution $x(t)$ of the Cauchy problem (1.1), (1.2) has a *finite escape time* if it exists on some finite interval $[t_0, T)$ and is unbounded, i.e., there exists $T < \infty$ $(T > t_0)$ such that $\lim_{t \to T-0} \|x(t)\| = +\infty$.

If the solution has a finite escape time, it is called *Lagrange unstable*.

Definition 2.5. A solution $x(t)$ of the Cauchy problem (1.1), (1.2) is called *Lagrange stable* if it is global and bounded, i.e., the solution $x(t)$ exists on $[t_0,\infty)$ and $\sup_{t\in[t_0,\infty)}\|x(t)\|<+\infty$.

Definition 2.6. *Equation* (1.1) *is Lagrange stable* if every solution of the Cauchy problem (1.1), (1.2) is Lagrange stable.

Equation (1.1) *is Lagrange unstable* if every solution of the Cauchy problem (1.1), (1.2) is Lagrange unstable.

3. Lagrange stability of the semilinear DAE

The theorem on the Lagrange stability of the DAE (1.1), which gives sufficient conditions for the existence and uniqueness of global solutions of the Cauchy problem (1.1), (1.2), where the initial points satisfy the consistency condition $(t_0,x_0)\in L_0$ (the manifold L_0 is defined in (3.1)), and gives conditions of the boundedness of the global solutions, is given below.

Theorem 3.1. *Let $f\in C([0,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$ have the continuous partial derivative $\frac{\partial}{\partial x}f(t,x)$ on $[0,\infty)\times\mathbb{R}^n$, $\lambda A+B$ be a regular pencil of index 1 and*

$$\forall t\ge 0\,\forall x_{p_1}\in X_1\,\exists x_{p_2}\in X_2\quad (t,x_{p_1}+x_{p_2})\in L_0=\{(t,x)\in[0,\infty)\times\mathbb{R}^n\mid Q_2[Bx-f(t,x)]=0\},\tag{3.1}$$

where X_1, X_2 from (2.2). Let for any $\hat{x}_{p_2}$, $\hat{\hat{x}}_{p_2}\in X_2$ such that $(t_*,x^*_{p_1}+\hat{x}_{p_2})$, $(t_*,x^*_{p_1}+\hat{\hat{x}}_{p_2})\in L_0$ the operator function

$$\Phi\colon X_2\to L(X_2,Y_2),\quad \Phi(x_{p_2})=\left[\frac{\partial}{\partial x}\big(Q_2f(t_*,x^*_{p_1}+x_{p_2})\big)-B\right]P_2,\tag{3.2}$$

be basis invertible on the convex hull $\operatorname{conv}\{\hat{x}_{p_2},\hat{\hat{x}}_{p_2}\}$. Suppose that for some self-adjoint positive operator $H\in L(X_1)$ and some number $R>0$ there exist functions $k\in C([0,\infty),\mathbb{R})$, $U\in C((0,\infty),(0,\infty))$ such that

$$\int_c^{+\infty}\frac{dv}{U(v)}=+\infty\quad(c>0),$$

$$\big(HP_1x,G^{-1}[-BP_1x+Q_1f(t,x)]\big)\le k(t)\,U\big(\tfrac{1}{2}(HP_1x,P_1x)\big),\quad (t,x)\in L_0,\ \|P_1x\|\ge R.\tag{3.3}$$

Then for each initial point $(t_0,x_0)\in L_0$, there exists a unique solution $x(t)$ of the Cauchy problem (1.1), (1.2) on $[t_0,\infty)$.

If, additionally,

$$\int_{t_0}^{+\infty}k(t)\,dt<+\infty,$$

there exists $\tilde{x}_{p_2}\in X_2$ such that for any $\tilde{\tilde{x}}_{p_2}\in X_2$ such that $(t_*,x^*_{p_1}+\tilde{\tilde{x}}_{p_2})\in L_0$ the operator function (3.2) is basis invertible on $\operatorname{conv}\{\tilde{x}_{p_2},\tilde{\tilde{x}}_{p_2}\}\setminus\{\tilde{x}_{p_2}\}$, and

$$\sup_{t\in[0,\infty),\,\|x_{p_1}\|\le M}\|Q_2f(t,x_{p_1}+\tilde{x}_{p_2})\|<+\infty,\qquad M>0\text{ is a number},\tag{3.4}$$

then for the initial points $(t_0,x_0)\in L_0$ the equation (1.1) is Lagrange stable.

Remark 3.1. Now we explain the restriction which is imposed on Φ (3.2) (for the existence and uniqueness of global solutions). In the case when the space X_2 is one-dimensional (then the basis invertibility is equivalent to the invertibility), it is required that the continuous linear operator $\Lambda = \Phi(x^*_{p_2})$, $x^*_{p_2} \in \operatorname{conv}\{\hat{x}_{p_2}, \hat{\hat{x}}_{p_2}\}$, have a continuous linear inverse operator for any fixed $\hat{x}_{p_2}$, $\hat{\hat{x}}_{p_2}$, t^*, $x^*_{p_1}$ such that $(t^*, x^*_{p_1} + \hat{x}_{p_2}), (t^*, x^*_{p_1} + \hat{\hat{x}}_{p_2}) \in L_0$. In the case when the dimension of X_2 is greater than 1, the operator $\Lambda \in L(X_2, Y_2)$, which is constructed from the operator function Φ (as shown in Definition 2.3) for fixed $\hat{x}_{p_2}$, $\hat{\hat{x}}_{p_2}$, t^*, $x^*_{p_1}$ such that $(t^*, x^*_{p_1} + \hat{x}_{p_2}), (t^*, x^*_{p_1} + \hat{\hat{x}}_{p_2}) \in L_0$, is required to be invertible. At the same time, the global boundedness of the norm of the mapping $[\Phi]^{-1}$ on X_2 and the global boundedness of the norm of the function $\big[\frac{\partial}{\partial x}\big(Q_2 f(t, x_{p_1} + x_{p_2})\big) P_2 - BP_2\big]^{-1}$ on $[0,\infty)\times\mathbb{R}^n$ are not required (i.e., the norm of the function is not required to be bounded by a constant for all t, x_{p_1}, x_{p_2}). For comparison, the condition of index 1 for the DAE was discussed above.

Proof. The DAE (1.1) is equivalent to system (2.5) (as shown in Section 2). Denote $\dim X_1 = a$, $\dim X_2 = d$ $(d = n - a)$. Any vector $x \in \mathbb{R}^n$ can be represented as $x = \begin{pmatrix} z \\ u \end{pmatrix} \in \mathbb{R}^a \times \mathbb{R}^d$, where $z \in \mathbb{R}^a$, $u \in \mathbb{R}^d$ are column vectors. We introduce the operators (the method of the construction of the operators is given in [18, Section 2]) $P_a \colon \mathbb{R}^a \to X_1$, $P_d \colon \mathbb{R}^d \to X_2$, which have the inverse operators $P_a^{-1} \colon X_1 \to \mathbb{R}^a$, $P_d^{-1} \colon X_2 \to \mathbb{R}^d$. Then $z = P_a^{-1} P_1 x$, $u = P_d^{-1} P_2 x$, $x = P_a z + P_d u$ (recall that (2.4)), and $P_a^{-1} P_1 P_a = E_{\mathbb{R}^a}$, $P_d^{-1} P_2 P_d = E_{\mathbb{R}^d}$. Multiplying the equations of system (2.5) by P_a^{-1}, P_d^{-1}, and replacing $P_1 x$ and $P_2 x$ by $P_a z$ and $P_d u$, respectively, we get the equivalent system

$$\frac{d}{dt} z = P_a^{-1} G^{-1} \big[- BP_a z + Q_1 \tilde{f}(t, z, u) \big], \tag{3.5}$$

$$P_d^{-1} G^{-1} Q_2 \tilde{f}(t, z, u) - u = 0, \tag{3.6}$$

where $\tilde{f}(t, z, u) = f(t, P_a z + P_d u)$.

Thus, the semilinear DAE (1.1) is equivalent to system (3.5), (3.6).

Further we are going to prove the theorem in two steps.

I (The existence and uniqueness). We prove the first part of the theorem, that is, the existence and uniqueness of global solutions.

Consider the mapping

$$F(t, z, u) = P_d^{-1} G^{-1} Q_2 \tilde{f}(t, z, u) - u. \tag{3.7}$$

It is continuous on $[0, \infty) \times \mathbb{R}^a \times \mathbb{R}^d$ and has continuous partial derivatives

$$\frac{\partial}{\partial z} F(t, z, u) = P_d^{-1} G^{-1} \frac{\partial}{\partial x} (Q_2 f(t, x)) P_a,$$

$$\frac{\partial}{\partial u} F(t, z, u) = P_d^{-1} \left[G^{-1} \frac{\partial}{\partial x} (Q_2 f(t, x)) - P_2 \right] P_d = P_d^{-1} G^{-1} \Phi(P_d u) P_d,$$

where Φ is the operator function (3.2), $\Phi(P_d u) = \Phi(x_{p_2})$, $x_{p_2} = P_d u \in X_2$.

Let us prove that for any $\hat{u}$, $\hat{\hat{u}} \in \mathbb{R}^d$ such that $(t_*, z_*, \hat{u})$, $(t_*, z_*, \hat{\hat{u}}) \in \tilde{L}_0$, where

$$\tilde{L}_0 = \left\{(t,z,u) \in [0,\infty) \times \mathbb{R}^a \times \mathbb{R}^d \mid P_d^{-1}G^{-1}Q_2\tilde{f}(t,z,u) - u = 0\right\}, \tag{3.8}$$

the operator function $\Psi\colon \mathbb{R}^d \to L(\mathbb{R}^d)$, $\Psi(u) = \frac{\partial}{\partial u}F(t_*, z_*, u)$, is basis invertible on $\mathrm{conv}\{\hat{u}, \hat{\hat{u}}\}$. Since (3.2) is basis invertible on $\mathrm{conv}\{\hat{x}_{p_2}, \hat{\hat{x}}_{p_2}\}$ for any $\hat{x}_{p_2}, \hat{\hat{x}}_{p_2} \in X_2$ such that $(t_*, x^*_{p_1} + \hat{x}_{p_2})$, $(t_*, x^*_{p_1} + \hat{\hat{x}}_{p_2}) \in L_0$, there exists an additive resolution of the identity $\{\Theta_k\}_{k=1}^d$ in Y_2 such that the operator $\Lambda_1 = \sum_{k=1}^d \Theta_k \Phi(x^k_{p_2}) \in L(X_2, Y_2)$ is invertible for any set of vectors $\{x^k_{p_2}\}_{k=1}^d \subset \mathrm{conv}\{\hat{x}_{p_2}, \hat{\hat{x}}_{p_2}\}$. With the help of the invertible operator $N = P_d^{-1}G^{-1}\colon Y_2 \to \mathbb{R}^d$ we introduce the system of one-dimensional projectors $\hat{\Theta}_k = N\Theta_k N^{-1}$, which form the additive resolution of the identity $\{\hat{\Theta}_k\}_{k=1}^d$ in $\mathbb{R}^d$. Chose any $\hat{u}$, $\hat{\hat{u}} \in \mathbb{R}^d$ such that $(t_*, z_*, \hat{u})$, $(t_*, z_*, \hat{\hat{u}}) \in \tilde{L}_0$ and any $u^k \in \mathrm{conv}\{\hat{u}, \hat{\hat{u}}\}$, $k = \overline{1,d}$. Taking into account that $(t,z,u) \in \tilde{L}_0 \Leftrightarrow (t, x_{p_1} + x_{p_2}) \in L_0$ and for $\hat{x}_{p_2} = P_d\hat{u}$, $\hat{\hat{x}}_{p_2} = P_d\hat{\hat{u}}$, $x^k_{p_2} = P_d u^k$, $x^*_{p_1} = P_a z_*$ the operator Λ_1 is invertible, the operator

$$\Lambda_2 = \sum_{k=1}^d \hat{\Theta}_k \frac{\partial}{\partial u} F(t_*, z_*, u^k) = \sum_{k=1}^d \hat{\Theta}_k P_d^{-1} G^{-1} \Phi(P_d u^k) P_d = N \Lambda_1 P_d$$

acting in $\mathbb{R}^d$ is also invertible. Hence, Ψ is basis invertible on $\mathrm{conv}\{\hat{u}, \hat{\hat{u}}\}$.

Let (t_*, z_*) be an arbitrary (fixed) point of $[0,\infty) \times \mathbb{R}^a$. Due to the condition (3.1), choose $u_* \in \mathbb{R}^d$ such that $(t_*, z_*, u_*) \in \tilde{L}_0$. From the basis invertibility of Ψ, it follows that there exists a continuous linear inverse operator $\left[\frac{\partial}{\partial u}F(t_*, z_*, u_*)\right]^{-1}$. By the implicit function theorems [20], there exist neighborhoods $U_\delta(t_*, z_*) = U_{\delta_1}(t_*) \times U_{\delta_2}(z_*)$ (if $t_* = 0$, then $U_{\delta_1}(t_*) = [0, \delta_1)$), $U_\varepsilon(u_*)$ and a unique function $u = u(t,z) \in C(U_\delta(t_*, z_*), U_\varepsilon(u_*))$, which is continuously differentiable in z such that $F(t, z, u(t,z)) = 0$, $(t,z) \in U_\delta(t_*, z_*)$, and $u(t_*, z_*) = u_*$. We define a global function $u = \eta(t,z)\colon [0,\infty) \times \mathbb{R}^a \to \mathbb{R}^d$ at the point (t_*, z_*) as $\eta(t_*, z_*) = u(t_*, z_*)$.

Let us prove that

$$\forall (t,z) \in [0,\infty) \times \mathbb{R}^a \; \exists!\, u \in \mathbb{R}^d \; (t,z,u) \in \tilde{L}_0. \tag{3.9}$$

Consider arbitrary (fixed) points $(t_*, z_*, \hat{u}), (t_*, z_*, \hat{\hat{u}}) \in \tilde{L}_0$. Clearly, $F(t_*, z_*, \hat{u}) = 0$, $F(t_*, z_*, \hat{\hat{u}}) = 0$. The projections $F_k(t,z,u) = \hat{\Theta}_k F(t,z,u)$, $k = \overline{1,d}$, are the functions with values in the one-dimensional spaces $R_k = \hat{\Theta}_k \mathbb{R}^d$ isomorphic to $\mathbb{R}$. According to the formula of finite increments [20], $F_k(t_*, z_*, \hat{\hat{u}}) - F_k(t_*, z_*, \hat{u}) = \frac{\partial}{\partial u} F_k(t_*, z_*, u^k)(\hat{\hat{u}} - \hat{u}) = 0$, $u^k \in \mathrm{conv}\{\hat{u}, \hat{\hat{u}}\}$, $k = \overline{1,d}$. Hence, $\hat{\Theta}_k \frac{\partial}{\partial u} F(t_*, z_*, u^k)(\hat{\hat{u}} - \hat{u}) = 0$, $k = \overline{1,d}$, from which, by summing these expressions over k, we obtain that $\Lambda_2(\hat{\hat{u}} - \hat{u}) = 0$, where the operator $\Lambda_2 = \sum_{k=1}^d \hat{\Theta}_k \frac{\partial}{\partial u} F(t_*, z_*, u^k) = \sum_{k=1}^d \hat{\Theta}_k \Psi(u^k)$ is invertible by virtue of the basis invertibility of Ψ (see above). Consequently, $\hat{\hat{u}} = \hat{u}$.

Thus, (3.9) is proved. It is also proved that in some neighborhood of each point $(t_*, z_*) \in [0,\infty) \times \mathbb{R}^a$ there exists a unique solution $u = \nu(t,z)$ of (3.6), which is continuous in (t,z) and continuously differentiable in z. So, the function

$u=\eta(t,z)$ coincides with $\nu(t,z)$ in this neighborhood and it is a solution of (3.6) with the corresponding smoothness properties. Let us show that the function $u=\eta(t,z)$ is unique on the whole domain of definition. Indeed, if there exists a function $u=\mu(t,z)$ having the same properties as $u=\eta(t,z)$ at some point $(t_*,z_*)\in[0,\infty)\times\mathbb{R}^a$, then by (3.9), $\eta(t_*,z_*)=\mu(t_*,z_*)=u_*$. Therefore, $\eta(t,z)=\mu(t,z)$ on $[0,\infty)\times\mathbb{R}^a$.

Substituting the function $u=\eta(t,z)$ into (3.5) and denoting $g(t,z)=Q_1\tilde{f}(t,z,\eta(t,z))$, we get

$$\frac{d}{dt}z=P_a^{-1}G^{-1}[-BP_az+g(t,z)]. \tag{3.10}$$

By the properties of η, $Q_1\tilde{f}$, the function $g(t,z)$ is continuous in (t,z) and continuously differentiable in z on $[0,\infty)\times\mathbb{R}^a$. Hence, for each initial point (t_0,z_0) such that $(t_0,z_0,\eta(t_0,z_0))\in\tilde{L}_0$ there exists a unique solution $z(t)$ of the Cauchy problem for equation (3.10) on some interval $[t_0,\varepsilon)$ with the initial condition $z(t_0)=z_0$. Note that if $(t_0,x_0)\in L_0$ and $x_0=P_az_0+P_d\eta(t_0,z_0)$, then $(t_0,z_0,\eta(t_0,z_0))\in\tilde{L}_0$.

Introduce the function

$$V(P_1x)=\frac{1}{2}(HP_1x,P_1x)=\frac{1}{2}(HP_az,P_az)=\frac{1}{2}(P_a^*HP_az,z)=\frac{1}{2}(\hat{H}z,z)=\hat{V}(z),$$

where $\hat{H}=P_a^*HP_a$ and H is an operator from (3.3). Then $\operatorname{grad}\hat{V}(z)=\hat{H}z$, where $\operatorname{grad}\hat{V}$ is the gradient of the function $\hat{V}$. Since $\big(HP_az,G^{-1}[-BP_az+Q_1f(t,P_az+P_d\eta(t,z))]\big)=\big(\hat{H}z,P_a^{-1}G^{-1}[-BP_az+g(t,z)]\big)$, then, by (3.3), there exists $\hat{R}>0$ such that

$$(\hat{H}z,P_a^{-1}G^{-1}[-BP_az+g(t,z)])\le k(t)\,U(\hat{V}),\quad t\ge0,\ \|z\|\ge\hat{R}, \tag{3.11}$$

where $k\in C([0,\infty),\mathbb{R})$ and $U\in C((0,\infty),(0,\infty))$ such that $\int_c^{+\infty}\frac{dv}{U(v)}=+\infty$.

Taking into account (3.11), for all $t\ge0$ and all z such that $\|z\|\ge\hat{R}$, the derivative $\dot{\hat{V}}\big|_{(3.10)}$ of the function $\hat{V}$ along the trajectories of (3.10) (see the definition in [10, Chapter 2]) satisfies the estimate

$$\dot{\hat{V}}\Big|_{(3.10)}=(\hat{H}z,P_a^{-1}G^{-1}[-BP_az+g(t,z)])\le k(t)\,U(\hat{V}).$$

It follows from the properties of the functions k, U that the inequality $\dot{v}\le k(t)U(v)$, $t\ge0$, has no positive solution $v(t)$ with finite escape time (see [10, Chapter 4]). Then, by [10, Ch. 4, Theorem XIII], every solution $z(t)$ of (3.10) is defined in the future (i.e., the solution is defined on $[t_0,\infty)$). Thus, the function $x(t)=P_az(t)+P_d\eta(t,z(t))$ is a solution of the Cauchy problem (1.1), (1.2) on $[t_0,\infty)$.

Let us verify the *uniqueness* of the global solution. It follows from what has been proved that the global solution $x(t)$ is unique on some interval $[t_0,\varepsilon)$. Assume that the solution is not unique on $[t_0,\infty)$. Then there exists $t_*\ge\varepsilon$ and two different global solutions $x(t)$, $\tilde{x}(t)$ with the common value $x_* = x(t_*) = \tilde{x}(t_*)$. Let us take the point (t_*, x_*) as the initial point. Then there must be a unique solution of (1.1) on some interval $[t_*, \varepsilon_1)$ with the initial value $x(t_*) = x_*$, which contradicts the assumption.

II (Boundedness). We prove the second part of the theorem, that is, the Lagrange stability of the DAE. Suppose that the additional conditions of the theorem are satisfied.

Since $\int_{t_0}^{+\infty} k(t)\,dt < +\infty$, the inequality $\dot{v} \le k(t)U(v)$, $t \ge 0$, has no unbounded positive solution for $t \ge 0$ [10, Ch. 4]. Then by [10, Chapter 4, Theorem XV], equation (3.10) is Lagrange stable. Hence, $\sup_{t\in[t_0,\infty)} \|z(t)\| < +\infty$, i.e.,

$$\exists M_* \in (0,\infty)\ \forall\, t \in [t_0,\infty)\ \|z(t)\| \le M_*. \tag{3.12}$$

Taking into account the properties of Φ (3.2) and the connection between Φ and the operator function $\Psi\colon \mathbb{R}^d \to L(\mathbb{R}^d)$ introduced in part I of the proof, we get that there exists a point $\tilde{u} \in \mathbb{R}^d$ ($\tilde{u} = P_d^{-1}\tilde{x}_{p_2}$) such that for any $\tilde{\tilde{u}} \in \mathbb{R}^d$, satisfying $(t_*, z_*, \tilde{\tilde{u}}) \in \tilde{L}_0$, the operator function Ψ is basis invertible on $\operatorname{conv}\{\tilde{u}, \tilde{\tilde{u}}\} \setminus \{\tilde{u}\}$. Let $(t_*, z_*, \tilde{\tilde{u}}) \in \tilde{L}_0$ be an arbitrary (fixed) point and $\tilde{u}$ be a point with the property imposed above. Then using the formula of finite increments for $F_k(t_*, z_*, \tilde{\tilde{u}})$ and $F_k(t_*, z_*, \tilde{u})$, where $F_k(t, z, u) = \hat{\Theta}_k F(t, z, u)$, F is the mapping (3.7) and $\{\hat{\Theta}_k\}_{k=1}^d$ is an additive resolution of the identity in $\mathbb{R}^d$, and summing the obtained equalities over k, we get that $F(t_*, z_*, \tilde{\tilde{u}}) - F(t_*, z_*, \tilde{u}) = \Lambda_2(\tilde{\tilde{u}} - \tilde{u})$, where $\Lambda_2 = \sum_{k=1}^d \hat{\Theta}_k \Psi(u^k)$, $\Psi(u^k) = \frac{\partial}{\partial u} F(t_*, z_*, u^k)$, $u^k \in \operatorname{conv}\{\tilde{u}, \tilde{\tilde{u}}\} \setminus \{\tilde{u}\}$, i.e., $u^k = \alpha\tilde{\tilde{u}} + (1-\alpha)\tilde{u}$, $\alpha \in (0,1]$, $k = \overline{1,d}$. It follows from the basis invertibility of Ψ on $\operatorname{conv}\{\tilde{u}, \tilde{\tilde{u}}\} \setminus \{\tilde{u}\}$ that there exists the inverse operator $\Lambda_2^{-1} \in L(\mathbb{R}^d)$. The mentioned above and the fact that $F(t_*, z_*, \tilde{\tilde{u}}) = 0$, give us $\tilde{\tilde{u}} = \tilde{u} - \Lambda_2^{-1}[P_d^{-1}G^{-1}Q_2\tilde{f}(t_*, z_*, \tilde{u}) - \tilde{u}]$, which is fulfilled for an arbitrary point $(t_*, z_*, \tilde{\tilde{u}}) \in \tilde{L}_0$. Consequently, for each $t_* \in [t_0,\infty)$, the equality $\eta(t_*, z(t_*)) = \tilde{u} - \Lambda_2^{-1}[P_d^{-1}G^{-1}Q_2\tilde{f}(t_*, z(t_*), \tilde{u}) - \tilde{u}]$, where $z(t)$ and $\eta(t, z(t))$ are components of the global solution $x(t) = P_a z(t) + P_d\eta(t, z(t))$ of the Cauchy problem (1.1), (1.2), holds. Denote $\tilde{M} = \|\tilde{u}\|$. Taking into account that Λ_2^{-1} is a bounded linear operator (since $\Lambda_2^{-1} \in L(\mathbb{R}^d)$), there exists a constant $N > 0$ such that $\|\eta(t_*, z(t_*))\| \le (1+N)\tilde{M} + N\|P_d^{-1}G^{-1}\|\, \|Q_2\tilde{f}(t_*, z(t_*), \tilde{u})\|$ for each $t_* \in [t_0,\infty)$. Then it follows from (3.12), (3.4) that there exists a constant $C > 0$ such that $\|\eta(t_*, z(t_*))\| \le C$ for each $t_* \in [t_0,\infty)$.

Since the estimate $\|x(t)\| = \|P_a z(t) + P_d\eta(t, z(t))\| \le \|P_a\|M_* + \|P_d\|C$ is fulfilled for all $t \in [t_0,\infty)$, the solution $x(t)$ of the Cauchy problem (1.1), (1.2) is Lagrange stable, which holds for each initial point $(t_0, x_0) \in L_0$. Hence, for the initial points $(t_0, x_0) \in L_0$, equation (1.1) is Lagrange stable. The theorem is proven. □

Remark 3.2. *The consistency condition* $(t_0, x_0) \in L_0$ for the initial point (t_0, x_0) is one of the necessary conditions for the existence of a solution of the Cauchy problem (1.1), (1.2).

Remark 3.3. If Φ (3.2) is basis invertible on $\operatorname{conv}\{\hat{x}_{p_2}, \hat{\hat{x}}_{p_2}\}$ for any $\hat{x}_{p_2}, \hat{\hat{x}}_{p_2} \in X_2$, $t_* \in [0,\infty)$, $x_{p_1}^* \in X_1$, then obviously it is basis invertible on $\operatorname{conv}\{\hat{x}_{p_2}, \hat{\hat{x}}_{p_2}\}$

for any $\hat{x}_{p_2}, \hat{\hat{x}}_{p_2}$ such that $(t_*, x^*_{p_1}+\hat{x}_{p_2}), (t_*, x^*_{p_1}+\hat{\hat{x}}_{p_2}) \in L_0$ and on $\operatorname{conv}\{\tilde{x}_{p_2}, \tilde{\tilde{x}}_{p_2}\} \setminus \{\tilde{x}_{p_2}\}$ for any $\tilde{x}_{p_2}$ and any $\tilde{\tilde{x}}_{p_2}$ such that $(t_*, x^*_{p_1} + \tilde{\tilde{x}}_{p_2}) \in L_0$. The verification of the basis invertibility of Φ on $\operatorname{conv}\{\hat{x}_{p_2}, \hat{\hat{x}}_{p_2}\}$ for any $\hat{x}_{p_2}, \hat{\hat{x}}_{p_2}$, t_*, $x^*_{p_1}$ may be more convenient for applications.

4. Lagrange instability of the semilinear DAE

Below is the theorem on the Lagrange instability of the DAE (1.1), which gives sufficient conditions for the existence and uniqueness of solutions with a finite escape time for the Cauchy problem (1.1), (1.2), where the initial points (t_0, x_0) satisfy the consistency condition $(t_0, x_0) \in L_0$ and the corresponding components $P_1 x_0$ belong to a certain region Ω.

Theorem 4.1. *Let $f \in C([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$ have a continuous partial derivative $\frac{\partial}{\partial x} f(t,x)$ on $[0,\infty) \times \mathbb{R}^n$, $\lambda A + B$ be a regular pencil of index 1 and (3.1) be fulfilled. Let for any $\hat{x}_{p_2}, \hat{\hat{x}}_{p_2} \in X_2$ such that $(t_*, x^*_{p_1} + \hat{x}_{p_2}), (t_*, x^*_{p_1} + \hat{\hat{x}}_{p_2}) \in L_0$ the operator function (3.2) be basis invertible on $\operatorname{conv}\{\hat{x}_{p_2}, \hat{\hat{x}}_{p_2}\}$. Further, let there exist a region $\Omega \subset X_1$ such that $P_1 x = 0 \notin \Omega$ and the component $P_1 x(t)$ of each existing solution $x(t)$ with the initial point $(t_0, x_0) \in L_0$, where $P_1 x_0 \in \Omega$, remains all the time in Ω. Suppose for some self-adjoint positive operator $H \in L(X_1)$ there exist the functions $k \in C([0,\infty), \mathbb{R})$, $U \in C((0,\infty), (0,\infty))$ such that*

$$\int_c^{+\infty} \frac{dv}{U(v)} < +\infty \quad (c > 0), \qquad \int_{t_0}^{+\infty} k(t)\, dt = \infty,$$

$$(HP_1 x, G^{-1}[-BP_1 x + Q_1 f(t,x)]) \geq k(t) U\big(\tfrac{1}{2}(HP_1 x, P_1 x)\big), \quad (t,x) \in L_0,\ P_1 x \in \Omega. \tag{4.1}$$

Then for each initial point $(t_0, x_0) \in L_0$, where $P_1 x_0 \in \Omega$, there exists a unique solution of the Cauchy problem (1.1), (1.2) and this solution has a finite escape time.

Proof. The beginning of the proof of Theorem 4.1 coincides with the proof of Theorem 3.1 up to the following statement. For each initial point (t_0, z_0) such that $(t_0, z_0, \eta(t_0, z_0)) \in \tilde{L}_0$, there exists a unique solution $z(t)$ of the Cauchy problem for equation (3.10) on some interval $[t_0, \varepsilon)$ with the initial condition $z(t_0) = z_0$. Hence, for each initial point $(t_0, x_0) \in L_0$, where $x_0 = P_a z_0 + P_d \eta(t_0, z_0)$, there exists a unique solution $x(t) = P_a z(t) + P_d \eta(t, z(t))$ of the Cauchy problem (1.1), (1.2) on $[t_0, \varepsilon)$.

Further, the proof takes the form.

By the condition of Theorem 4.1, there exists a region $\Omega \subset X_1$ such that $P_1 x = 0 \notin \Omega$ and the component $P_1 x(t)$ of each solution $x(t)$ with the initial point $(t_0, x_0) \in L_0$, where $P_1 x_0 \in \Omega$, remains all the time in Ω. Taking into account that $P_1 x = P_a z$, each solution $z(t)$ of equation (3.10) starting in the region $\hat{\Omega} = \{z \in \mathbb{R}^a \mid P_a z \in \Omega\} = P_a^{-1}\Omega$ remains all the time in it, and $z = 0 \notin \hat{\Omega}$. Introduce the function $\hat{V}(z) = \frac{1}{2}(\hat{H} z, z)$, where $\hat{H} = P_a^* H P_a$ and H is an operator from

(4.1). Clearly, the function $\hat{V}(z)$ is positive for all $z \in \hat{\Omega}$. It follows from (4.1) that

$$(\hat{H}z, P_a^{-1}G^{-1}[-BP_a z + g(t,z)]) \geq k(t)U(\hat{V}), \quad t \geq 0,\ z \in \hat{\Omega}, \tag{4.2}$$

where $k \in C([0,\infty),\mathbb{R})$, $U \in C((0,\infty),(0,\infty))$ such that $\int_c^{+\infty} \frac{dv}{U(v)} < +\infty$, $\int_{t_0}^{+\infty} k(t)\,dt = \infty$.

By (4.2), for all $t \geq 0$ and all $z \in \hat{\Omega}$, the derivative of $\hat{V}$ along the trajectories of (3.10) satisfies the estimate

$$\dot{\hat{V}}\Big|_{(3.10)} = (\hat{H}z, P_a^{-1}G^{-1}[-BP_a z + g(t,z)]) \geq k(t)U(\hat{V}).$$

It follows from the properties of the functions k, U that the inequality $\dot{v} \geq k(t)U(v)$, $t \geq 0$, has no positive solution defined in the future (see [10, Chapter 4]). By [10, Chapter 4, Theorem XIV], each solution $z(t)$ of (3.10) satisfying the condition $z(t_0) = z_0$, where $z_0 \in \hat{\Omega}$ and $(t_0, z_0, \eta(t_0, z_0)) \in \tilde{L}_0$, has a finite escape time, i.e., it exists on some finite interval $[t_0, T)$ and $\lim_{t\to T-0} \|z(t)\| = +\infty$. Then each function $x(t) = P_a z(t) + P_d \eta(t, z(t))$ with the corresponding initial values (t_0, x_0), where $x_0 = P_a z_0 + P_d \eta(t_0, z_0)$, is a solution of the Cauchy problem (1.1), (1.2) with a finite escape time, i.e., the solution $x(t)$ is defined on the corresponding finite interval $[t_0, T)$ and $\lim_{t\to T-0} \|x(t)\| = +\infty$.

Let us verify the *uniqueness* of the solution $x(t)$, $t \in [t_0, T)$. It was proved that the solution $x(t)$ is unique on some interval $[t_0, \varepsilon)$. Assume that the solution is not unique on $[t_0, T)$. Then there exists $t_* \in [\varepsilon, T)$ and two different solutions $x(t)$, $\tilde{x}(t)$ with the common value $x_* = x(t_*) = \tilde{x}(t_*)$ such that $(t_*, x_*) \in L_0$ and $P_1 x_* \in \Omega$. Let us take the point (t_*, x_*) as the initial point. Then there must be a unique solution of (1.1) on some interval $[t_*, \varepsilon_1) \subset [t_0, T)$ with the initial value $x(t_*) = x_*$, which contradicts the assumption. The theorem is proven. □

5. Lagrange stability of the mathematical model of a radio engineering filter

Let us consider the electrical circuit of a radio engineering filter given in Fig. 5.1. A voltage source e, nonlinear resistances φ, φ_0, ψ, a nonlinear conductance h, a linear resistance r, a linear conductance g, an inductance L and a capacitance C are given.

The currents and voltages in the circuit satisfy the Kirchhoff equations, as well as the constraint equations which describe operation modes of the electric circuit elements:

$$I_L = I + I_\psi, \quad U_\psi = U_\varphi + U_r + U_C, \quad e = U_{\varphi_0} + U_L + U_\psi,$$
$$U_L = \frac{d(LI_L)}{dt}, \quad I = \frac{d(CU_C)}{dt} + gU_C + h(U_C),$$
$$U_r = rI, \quad U_\varphi = \varphi(I), \quad U_{\varphi_0} = \varphi_0(I_L),\ U_\psi = \psi(I_\psi).$$

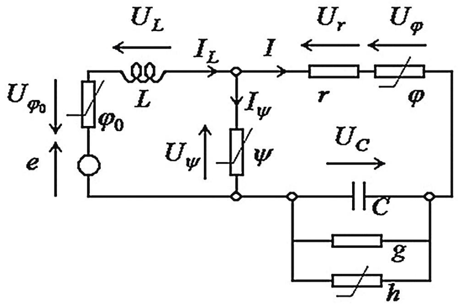


Fig. 5.1: The electric circuit diagram of the radio engineering filter.

From these equations we obtain the system with the variables $x_1 = I_L$, $x_2 = U_C$, $x_3 = I$:

$$L\frac{d}{dt}x_1 + x_2 + r\,x_3 = e(t) - \varphi_0(x_1) - \varphi(x_3), \tag{5.1}$$

$$C\frac{d}{dt}x_2 + gx_2 - x_3 = -h(x_2), \tag{5.2}$$

$$x_2 + rx_3 = \psi(x_1 - x_3) - \varphi(x_3). \tag{5.3}$$

The system describes a transient process in the electrical circuit (i.e., the process of transition from one operation mode of the electric circuit to another).

It is assumed that the linear parameters L, C, r, g are positive and real, $\varphi_0 \in C^1(\mathbb{R})$, $\varphi \in C^1(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and $e \in C([0,\infty),\mathbb{R})$.

The vector form of system (5.1)–(5.3) is the semilinear DAE

$$\frac{d}{dt}[Ax] + Bx = f(t,x), \tag{5.4}$$

where $x = (x_1, x_2, x_3)^T = (I_L, U_C, I)^T \in \mathbb{R}^3$,

$$f(t,x) = \begin{pmatrix} e(t) - \varphi_0(x_1) - \varphi(x_3) \\ -h(x_2) \\ \psi(x_1 - x_3) - \varphi(x_3) \end{pmatrix}, \quad A = \begin{pmatrix} L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & r \\ 0 & g & -1 \\ 0 & 1 & r \end{pmatrix}.$$

It is easy to verify that $\lambda A + B$ is a regular pencil of index 1.

The projection matrices P_i, Q_i and the matrix G^{-1} have the form

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -r^{-1} & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r^{-1} & 1 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & r^{-1} \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -r^{-1} \\ 0 & 0 & 1 \end{pmatrix},$$

$$G^{-1} = \begin{pmatrix} L^{-1} & 0 & -L^{-1} \\ 0 & C^{-1} & (Cr)^{-1} \\ 0 & -(Cr)^{-1} & (Cr-1)C^{-1}r^{-2} \end{pmatrix}.$$

The projections of the vector x have the form

$$x_{p_1} = P_1 x = (x_1, x_2, -r^{-1}x_2)^T = (a, -rb, b)^T,$$
$$x_{p_2} = P_2 x = (0, 0, r^{-1}x_2 + x_3)^T = (0, 0, u)^T,$$

where $a = x_1$, $b = -r^{-1}x_2$, $u = r^{-1}x_2 + x_3 \in \mathbb{R}$.

The equation $Q_2[Bx - f(t,x)] = 0$, determining the manifold L_0 from (3.1), is equivalent to equation (5.3). Taking into account the new notation, the condition (3.1) holds if for any $a, b \in \mathbb{R}$ there exists $u \in \mathbb{R}$ such that

$$ru = \psi(a - b - u) - \varphi(b + u). \tag{5.5}$$

Consider the operator function $\tilde{\Phi}\colon X_2 \to L(\mathbb{R}^3, Y_2)$,

$$\tilde{\Phi}(x_{p_2}) = \left[\frac{\partial}{\partial x}\big(Q_2 f(t_*, x_{p_1}^* + x_{p_2})\big) - B\right] P_2$$
$$= \big(\psi'(a_* - b_* - u) + \varphi'(b_* + u) + r\big)\begin{pmatrix} 0 & -r^{-1} & -1 \\ 0 & r^{-2} & r^{-1} \\ 0 & -r^{-1} & -1 \end{pmatrix},$$

where $\psi'(a - b - u) = \frac{d\psi(y)}{dy}\Big|_{y=a-b-u}$, $\varphi'(b + u) = \frac{d\varphi(y)}{dy}\Big|_{y=b+u}$, $t_* \in [0, \infty)$, $a_*, b_* \in \mathbb{R}$, $x_{p_1}^* = (a_*, -rb_*, b_*)^T$. Since the spaces X_2, Y_2 are one-dimensional, the invertibility of the operator function $\Phi = \tilde{\Phi}\Big|_{X_2} : X_2 \to L(X_2, Y_2)$ (i.e., the operator $\Phi(x_{p_2}) \in L(X_2, Y_2)$ is the restriction of the operator $\tilde{\Phi}(x_{p_2}) \in L(\mathbb{R}^3, Y_2)$ to X_2) is equivalent to the basis invertibility of Φ. Let for any (fixed) $\hat{u}, \hat{\hat{u}}, a_*, b_* \in \mathbb{R}$ satisfying (5.5), the condition $\psi'(a_* - b_* - u_*) + \varphi'(b_* + u_*) \neq -r$ be fulfilled for any $u_* \in \operatorname{conv}\{\hat{u}, \hat{\hat{u}}\}$. Then the operator $\Lambda = \tilde{\Lambda}\Big|_{X_2} \in L(X_2, Y_2)$, where $\tilde{\Lambda} = \tilde{\Phi}(x_{p_2}^*)$, $x_{p_2}^* = (0, 0, u_*)^T$, is invertible since from $\tilde{\Lambda} x_{p_2} = 0$, $x_{p_2} \in X_2$, it follows that $x_{p_2} = 0$. Hence, for any $\hat{u}, \hat{\hat{u}}, a_*, b_* \in \mathbb{R}$ satisfying (5.5), the operator function Φ (3.2) is basis invertible on the convex hull $\operatorname{conv}\{\hat{x}_{p_2}, \hat{\hat{x}}_{p_2}\}$, where $\hat{x}_{p_2} = (0, 0, \hat{u})^T$, $\hat{\hat{x}}_{p_2} = (0, 0, \hat{\hat{u}})^T$.

Choose

$$H = \begin{pmatrix} 2L & 0 & 0 \\ 0 & Cr & 0 \\ 0 & 0 & Cr^3 \end{pmatrix}.$$

Then

$$\big(HP_1 x, G^{-1}[-BP_1 x + Q_1 f(t,x)]\big)$$
$$= 2\big[-(gr+1)x_2^2 - x_1\varphi_0(x_1) + (x_2 - x_1)\psi(x_1 - x_3) - rx_2 h(x_2) - x_2\varphi(x_3) + x_1 e(t)\big].$$

Since $\varphi, \psi \in C^1(\mathbb{R})$, there exists a constant C such that for any fixed $\tilde{x}_{p_2} = (0, 0, \tilde{u})^T$, where $\tilde{u} \in \mathbb{R}$, and for all $t \in [0, \infty)$, $\|x_{p_1}\| \le M$, where M is a number, the estimate

$$\|Q_2 f(t, x_{p_1} + \tilde{x}_{p_2})\| \le \sqrt{2 + r^{-2}} \max_{\|x_{p_1}\| \le M} |\psi(a - b - \tilde{u}) - \varphi(b + \tilde{u})| \le C$$

is fulfilled. Hence, the condition (3.4) is satisfied for any fixed $\tilde{x}_{p_2} = (0,0,\tilde{u})^T$ (i.e., any fixed $\tilde{u} \in \mathbb{R}$).

5.1. Conclusions. By Theorem 3.1 for each initial point $(t_0, x^0) \in [0,\infty)\times \mathbb{R}^3$ ($x^0 = (x_1^0, x_2^0, x_3^0)^T$) satisfying the consistency condition (the equation (5.3))

$$x_2^0 + r x_3^0 = \psi(x_1^0 - x_3^0) - \varphi(x_3^0), \tag{5.6}$$

there exists a unique solution $x(t)$ of the Cauchy problem for the DAE (5.4) with the initial condition

$$x(t_0) = x^0 \tag{5.7}$$

on the whole interval $[t_0, \infty)$ if:

1) for any $a, b \in \mathbb{R}$ there exists $u \in \mathbb{R}$ such that (5.5) is fulfilled;
2) for any $\hat{u}, \hat{\hat{u}}, a_*, b_* \in \mathbb{R}$ satisfying (5.5), the condition $\psi'(a_* - b_* - u_*) + \varphi'(b_* + u_*) \neq -r$ is fulfilled for any $u_* \in \operatorname{conv}\{\hat{u}, \hat{\hat{u}}\}$;
3) for some number $R > 0$, there exist the functions $k \in C([0,\infty),\mathbb{R})$, $U \in C((0,\infty),(0,\infty))$ such that $\int_c^{+\infty} \frac{dv}{U(v)} = +\infty$ and

$$-(gr+1)x_2^2 - x_1\varphi_0(x_1) + (x_2 - x_1)\psi(x_1 - x_3) - r x_2 h(x_2) - x_2\varphi(x_3) + \\ + x_1 e(t) \le k(t)\, U\big(L x_1^2 + C r x_2^2\big)$$

for any $t \ge 0$, $x \in \mathbb{R}^3$ such that (5.3), $\|P_1 x\| = \sqrt{x_1^2 + (1 + r^{-2})x_2^2} \ge R$.

If, additionally, $\int_{t_0}^{+\infty} k(t)\, dt < +\infty$ and

4) there exists $\tilde{u} \in \mathbb{R}$ such that for any $\tilde{\tilde{u}}, a_*, b_* \in \mathbb{R}$ satisfying (5.5), the condition $\psi'(a_* - b_* - u_*) + \varphi'(b_* + u_*) \neq -r$ is fulfilled for any $u_* \in \operatorname{conv}\{\tilde{u}, \tilde{\tilde{u}}\} \setminus \{\tilde{u}\}$ (i.e., $u_* = \alpha\tilde{\tilde{u}} + (1-\alpha)\tilde{u}$, $\alpha \in (0,1]$),

then for the initial points (t_0, x^0) equation (5.4) is Lagrange stable.

In terms of physics it means that if the input voltage $e(t) \in C([0,\infty),\mathbb{R})$, the nonlinear resistances $\varphi, \varphi_0, \psi \in C^1(\mathbb{R})$ and the nonlinear conductance $h \in C^1(\mathbb{R})$ satisfy the aforementioned conditions 1)–3), then for any initial time moment $t_0 \ge 0$ and any initial values $I_L(t_0)$, $U_C(t_0)$, $I(t_0)$ satisfying $U_C(t_0) + rI(t_0) = \psi(I_L(t_0) - I(t_0)) - \varphi(I(t_0))$, there exist the currents $I_L(t)$, $I(t)$ and voltage $U_C(t)$ in the circuit (Fig. 5.1) for all $t \ge t_0$, which are uniquely determined by the initial values. The functions $I_L(t)$, $U_C(t)$ are continuously differentiable and the function $I(t)$ is continuous on $[t_0, \infty)$. The currents and voltage are bounded for all $t \ge t_0$ (Lagrange stability) if, additionally, $\int_{t_0}^{+\infty} k(t)\, dt < +\infty$, and condition 4) is satisfied. The remaining currents and voltages in the circuit are uniquely expressed in terms of $I_L(t)$, $I(t)$, $U_C(t)$.

Let us consider *the particular cases*:

$$\varphi_0(y) = \alpha_1 y^{2k-1}, \quad \varphi(y) = \alpha_2 y^{2l-1}, \quad \psi(y) = \alpha_3 y^{2j-1}, \quad h(y) = \alpha_4 y^{2s-1}, \tag{5.8}$$

$$\varphi_0(y) = \alpha_1 y^{2k-1}, \quad \varphi(y) = \alpha_2 \sin y, \quad \psi(y) = \alpha_3 \sin y, \quad h(y) = \alpha_4 \sin y, \tag{5.9}$$

where $k, l, j, s \in \mathbb{N}$, $\alpha_i > 0$, $i = \overline{1,4}$, $y \in \mathbb{R}$. Note that the functions of the types (5.8), (5.9) for nonlinear resistances and conductances are encountered in real radio engineering devices.

For the functions of the form (5.8) and each initial point (t_0, x^0) satisfying (5.6), there exists a unique solution of the Cauchy problem (5.4), (5.7) on $[t_0, \infty)$ if $j \le k$, $j \le s$ and α_3 is sufficiently small. For the functions of the form (5.9) and each initial point (t_0, x^0) satisfying (5.6), there exists a unique solution of the Cauchy problem (5.4), (5.7) on $[t_0, \infty)$ if $\alpha_2 + \alpha_3 < r$. If, additionally, $\sup_{t\in[0,\infty)} |e(t)| < +\infty$ or $\int_{t_0}^{+\infty} |e(t)|\, dt < +\infty$, then for the initial points (t_0, x^0) the DAE (5.4) is Lagrange stable (in both cases), i.e., every solution of the DAE is bounded. In particular, these requirements are fulfilled for voltages of the form

$$e(t) = \beta(t+\alpha)^{-n},\ e(t) = \beta e^{-\alpha t},\ e(t) = \beta e^{-\frac{(t-\alpha)^2}{\sigma^2}},\ e(t) = \beta \sin(\omega t + \theta), \quad (5.10)$$

where $\alpha > 0$, β, σ, $\omega \in \mathbb{R}$, $n \in \mathbb{N}$, $\theta \in [0, 2\pi]$. For voltage having the form

$$e(t) = \beta(t+\alpha)^n,\ \alpha,\ \beta \in \mathbb{R},\ n \in \mathbb{N}, \quad (5.11)$$

global solutions exist, but they are not bounded on the whole interval $[t_0, \infty)$.

5.2. Numerical analysis. We find approximate solutions of the DAE (5.4) (system (5.1)–(5.3)) with the initial condition (5.7) using the numerical method given in [6].

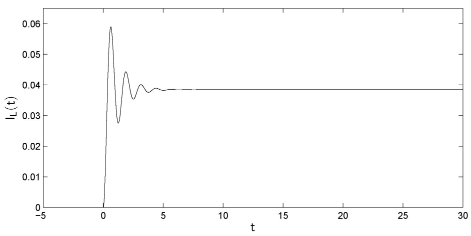
Choose the parameters $L = 500$, $C = 0.5$, $r = 2$, $g = 0.2$ and the input voltage $e(t) = 100\, e^{-t} \sin(5\,t)$. For the nonlinear resistances and conductance of the form (5.8) with $k = l = j = s = 2$, $\alpha_i = 1$, $i = \overline{1,4}$, the numerical solution with the initial values $t_0 = 0$, $x^0 = (0,0,0)^T$ is obtained. The components of the obtained solution are shown in Fig. 5.2.

The components of the solution for the electrical circuit with the linear parameters $L = 50$, $C = 1$, $r = 0.001$, $g = 1$, the nonlinear parameters (5.8), where $k = l = j = s = 2$, $\alpha_i = 1$, $i = \overline{1,3}$, $\alpha_4 = 0.01$, and the input voltage $e(t) = 2\sin t$, and for the initial values $t_0 = 0$, $x^0 = (0,0,0)^T$, are shown in Fig. 5.3.

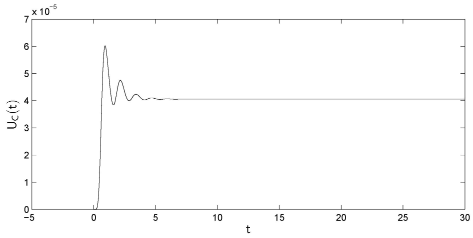
For the linear parameters $L = 300$, $C = 0.5$, $r = 2.6$, $g = 0.2$, the nonlinear resistances and conductance (5.9), where $k = 2$, $\alpha_1 = 0.5$, $\alpha_2 = 1.5$, $\alpha_3 = 1$, $\alpha_4 = 3$, and the voltage $e(t) = 200\sin(0.5\,t) - 0.2$, the solution components with the initial values $t_0 = 0$, $x^0 = (\pi/6, 0.5, 0)^T$ are shown in Fig. 5.4.

For the linear parameters $L = 1$, $C = 5$, $r = 1.51$, $g = 5$, the nonlinear parameters (5.9), where $k = 2$, $\alpha_i = 1$, $i = 1,2,4$, $\alpha_3 = 0.5$, the voltage $e(t) = (t+30)^{-2}$ and the initial values $t_0 = 0$, $x^0 = (0,0,0)^T$ the solution components are shown in Fig. 5.5.

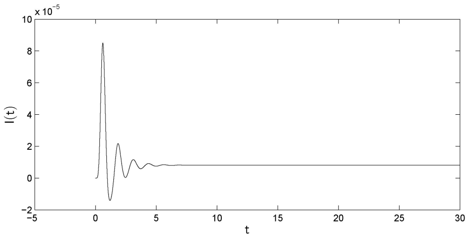
The components of the solution for the electrical circuit with the linear parameters $L = 1000$, $C = 0.5$, $r = 2$, $g = 0.3$, the nonlinear parameters (5.8) with $k = l = j = s = 2$, $\alpha_i = 1$, $i = \overline{1,4}$, the input voltage $e(t) = -t^2$, and for the initial values $t_0 = 0$, $x^0 = (0,0,0)^T$ are shown in Fig. 5.6.



(a) The current $I_L(t)$

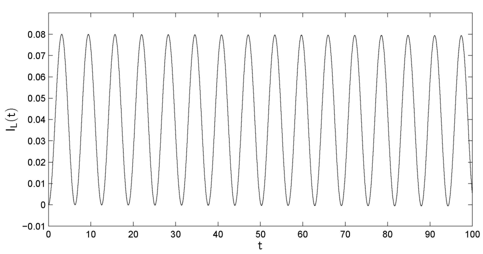


(b) The voltage $U_C(t)$

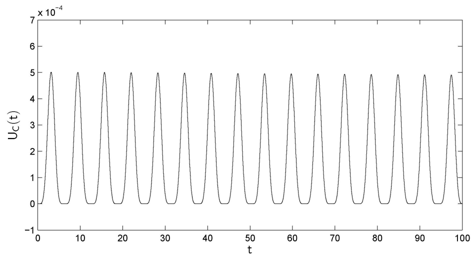


(c) The current $I(t)$

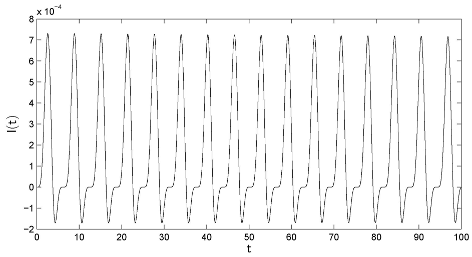
Fig. 5.2: (a)–(c): The components of the numerical solution.



(a) The current $I_L(t)$



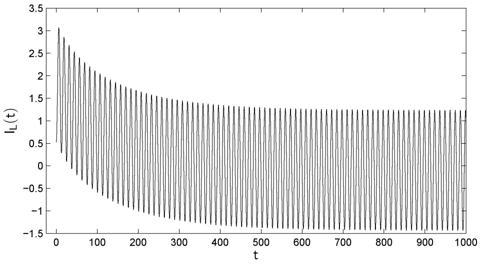
(b) The voltage $U_C(t)$



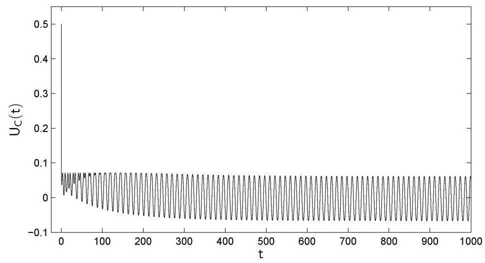
(c) The current $I(t)$

Fig. 5.3: (a)–(c): The components of the numerical solution.

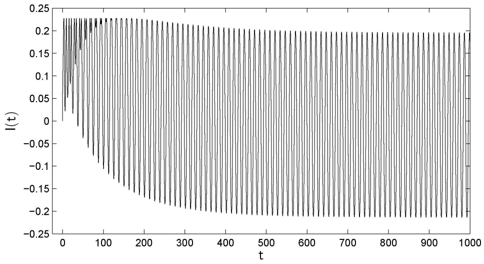
For the linear parameters $L = 100$, $C = 5$, $r = 3$, $g = 4$, the nonlinear parameters (5.9), where $k = 2$, $\alpha_1 = 1$, $\alpha_2 = 0.9$, $\alpha_3 = 2$, $\alpha_4 = 5$, the voltage $e(t) = (t - 50)^3$ and the initial values $t_0 = 0$, $x^0 = (0, 0, 0)^T$ the solution components are shown in Fig. 5.7.



(a) The current $I_L(t)$

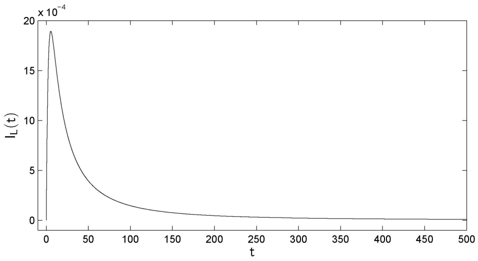


(b) The voltage $U_C(t)$

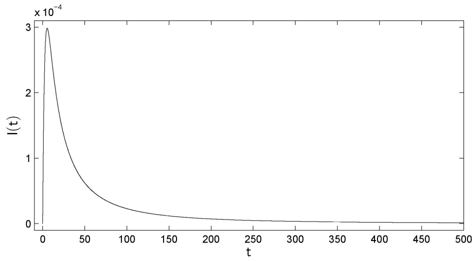


(c) The current $I(t)$

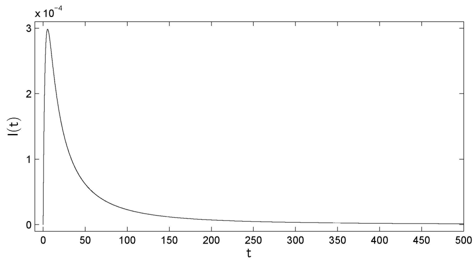
Fig. 5.4: (a)–(c): The components of the numerical solution.



(a) The current $I_L(t)$



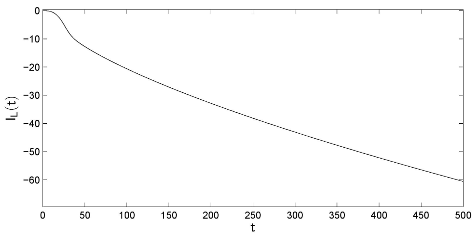
(b) The voltage $U_C(t)$



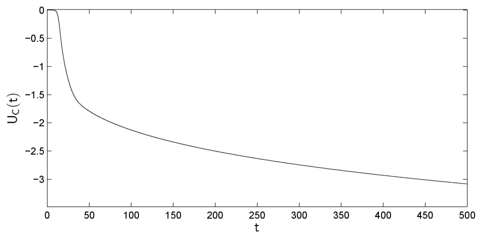
(c) The current $I(t)$

Fig. 5.5: (a)–(c): The components of the numerical solution.

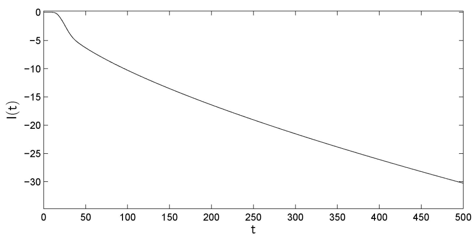
The numerical solutions shown in Figs. 5.2–5.5 are bounded on the corresponding time intervals. When we increase the time intervals by a factor of 5–10, the solutions are also bounded. The analysis of these numerical solutions indicates



(a) The current $I_L(t)$

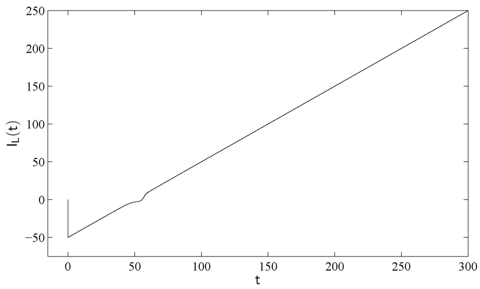


(b) The voltage $U_C(t)$

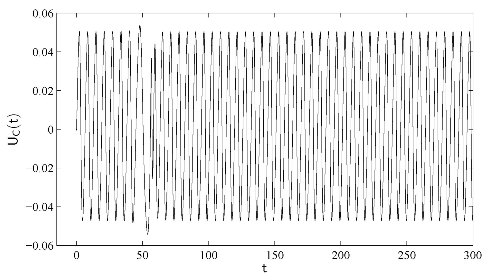


(c) The current $I(t)$

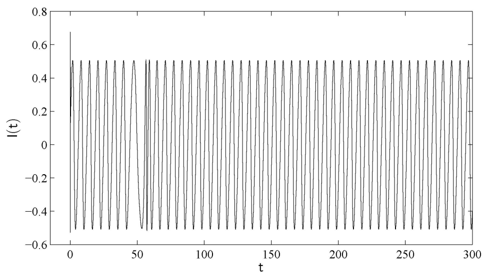
Fig. 5.6: (a)–(c): The components of the numerical solution.



(a) The current $I_L(t)$



(b) The voltage $U_C(t)$



(c) The current $I(t)$

Fig. 5.7: (a)–(c): The components of the numerical solution.

that there exist bounded global solutions of equation (5.4) (system (5.1)–(5.3)) with the input voltage of the form (5.10) and the nonlinear resistances and conductance of the form (5.8), (5.9). The analysis of the numerical solutions shown in Figs. 5.6, 5.7 indicates that there exist global solutions, increasing without bound with increasing time (as $t \to \infty$), for equation (5.4) (system (5.1)–(5.3)) with the input voltage of the form (5.11) and the nonlinear parameters of the form (5.8), (5.9). Similar results follow from the application of Theorem 3.1. Therefore, the conclusions obtained with the help of this theorem are verified by a numerical experiment.

6. Lagrange instability of the mathematical model of a radio engineering filter

Consider system (5.1)–(5.3) (the DAE (5.4)) with the nonlinear resistances and conductance

$$\varphi_0(x_1) = -x_1^2, \quad \varphi(x_3) = x_3^3, \quad \psi(x_1 - x_3) = (x_1 - x_3)^3, \quad h(x_2) = x_2^2. \quad (6.1)$$

It is assumed that there exists $M_e = \sup_{t\in[t_0,\infty)} |e(t)| < +\infty$.

The verification of the condition (3.1) and the condition for the operator function (3.2) is similar to that given in Section 5. Thus, it is easy to verify that these requirements are fulfilled.

Denote $z = (x_1, x_2)^T \in \mathbb{R}^2$. Choose

$$\Omega_{\mathbb{R}^2} = \Big\{ z = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 > m_1,$$
$$m_1 = \max\left\{1 + \sqrt{M_e},\ \sqrt[3]{g + \frac{1}{r}},\ \frac{3C}{L},\ \sqrt{\max\left\{\frac{L}{3rC} - \frac{r}{3}, 0\right\}}\right\},$$
$$x_2 < -rx_1 - x_1^3 - m_2,\ m_2 = \max\left\{g - \frac{2Cr}{L},\ 0\right\}\Big\}, \quad (6.2)$$
$$\Omega = \{x_{p_1} = P_1 x \in X_1 \mid z \in \Omega_{\mathbb{R}^2}\}.$$

Since $x_{p_1} = (x_1, x_2, -r^{-1}x_2)^T$, then $x_{p_1} \in \Omega \Leftrightarrow z \in \Omega_{\mathbb{R}^2}$. Obviously, $x_{p_1} = 0 \notin \Omega$.

The boundary of the region $\Omega_{\mathbb{R}^2}$ consists of the parts $x_1 = m_1$ and $x_2 + rx_1 + x_1^3 + m_2 = 0$. Since $x_1 \ge m_1$, $\frac{d}{dt}x_1 > 0$ and $x_2 + rx_1 + x_1^3 + m_2 \le 0$, $\frac{d}{dt}(x_2 + rx_1 + x_1^3 + m_2) < 0$ for all $t \ge 0$, $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ satisfying (5.3) (the condition $(t, x) \in L_0$), where $z = (x_1, x_2)^T \in \overline{\Omega}_{\mathbb{R}^2}$ ($\overline{\Omega}_{\mathbb{R}^2}$ is the closure of $\Omega_{\mathbb{R}^2}$), the component $z(t) = (x_1(t), x_2(t))^T$ of each existing solution, which starts at time $t_0 \ge 0$ in the region $\Omega_{\mathbb{R}^2}$, cannot leave this region. Consequently, the component $x_{p_1}(t) = P_1 x(t)$ of each existing solution $x(t)$ with the initial point $(t_0, x^0) \in [0, \infty) \times \mathbb{R}^3$ ($x^0 = (x_1^0, x_2^0, x_3^0)^T$) satisfying (5.6), where $P_1 x^0 \in \Omega$ ($(x_1^0, x_2^0)^T \in \Omega_{\mathbb{R}^2}$), remains all the time in Ω.

We choose $H = \begin{pmatrix} 2L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & Cr^2 \end{pmatrix}$. Then for any $x = (x_1, x_2, x_3)^T$ satisfying (5.3) and such that $(x_1, x_2)^T \in \Omega_{\mathbb{R}^2}$, the condition

$$\big(HP_1x, G^{-1}[-BP_1x + Q_1 f(t, x)]\big) = 2\big[e(t)x_1 - (g + r^{-1})x_2^2 + x_1^3 +$$

$$+\,(r^{-1}x_2 - x_1)(x_1 - x_3)^3 - x_2^3 - r^{-1}x_2x_3^3\big] > 2\left[-(g + r^{-1})x_2^2 + x_2^3\right] \ge \alpha\, v^{3/2},$$

where $v = \frac{1}{2}(HP_1x, P_1x) = Lx_1^2 + Cx_2^2$ and $\alpha > 0$ is a certain constant, is fulfilled. Hence, the condition (4.1), where $k(t) \equiv 1$, $U(v) = \alpha\, v^{3/2}$, is fulfilled.

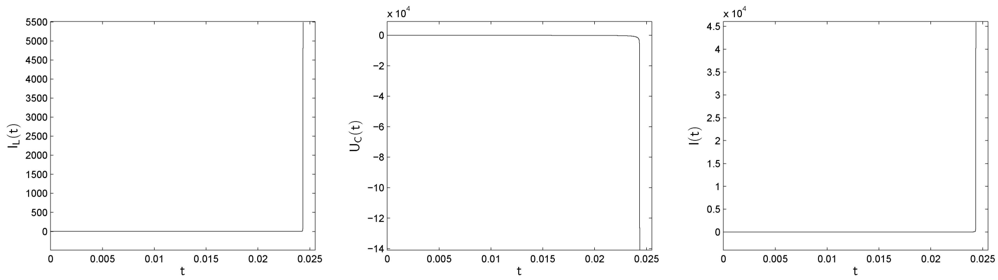
Thus, all the conditions of Theorem 4.1 are satisfied.

6.1. Conclusions. By Theorem 4.1, for each initial point $(t_0, x^0) \in [0,\infty) \times \mathbb{R}^3$ satisfying (5.6) and such that $(x_1^0, x_2^0)^T \in \Omega_{\mathbb{R}^2}$, where $\Omega_{\mathbb{R}^2}$ is the region (6.2), there exists a unique solution of the Cauchy problem for the DAE (5.4) with the initial condition (5.7), where the functions φ_0, φ, ψ, h have the form (6.1) and $\sup_{t\in[t_0,\infty)} |e(t)| < +\infty$, and this solution has a finite escape time (the solution exists on some finite interval and it is unbounded).

In terms of physics it means that if $\sup_{t\in[t_0,\infty)} |e(t)| < +\infty$ and the nonlinear resistances and conductance have the form (6.1), then for any initial time moment $t_0 \ge 0$ and any initial values $I_L(t_0)$, $U_C(t_0)$, $I(t_0)$ satisfying $U_C(t_0) + rI(t_0) = \psi(I_L(t_0) - I(t_0)) - \varphi(I(t_0))$ and such that $(I_L(t_0), U_C(t_0))^T \in \Omega_{\mathbb{R}^2}$, on some finite interval $t_0 \le t < T$ there exist the currents $I_L(t)$, $I(t)$ and the voltage $U_C(t)$ in the circuit in Fig. 5.1, which are uniquely determined by the initial values, and $\lim_{t\to T-0} \left\|(I_L(t), U_C(t), I(t))^T\right\| = +\infty$.

6.2. Numerical analysis. We find approximate solutions for the DAE (5.4) (system (5.1)–(5.3)) with the functions of nonlinear resistances and conductance (6.1) and the initial condition (5.7). The initial values t_0, $x^0 = (x_1^0, x_2^0, x_3^0)^T$ are chosen such that (5.6) is satisfied and $(x_1^0, x_2^0)^T \in \Omega_{\mathbb{R}^2}$, where $\Omega_{\mathbb{R}^2}$ is (6.2).

Choose the parameters $L = 10$, $C = 0.5$, $r = 2$, $g = 0.2$, the input voltage $e(t) = 2\sin t$ and the initial values $t_0 = 0$, $x^0 = (2.45, -20.625125, 2.5)^T$. The components of the obtained numerical solution are shown in Fig. 6.1.



(a) The current $I_L(t)$ (b) The voltage $U_C(t)$ (c) The current $I(t)$

Fig. 6.1: (a)–(c): The components of the numerical solution.

For the electrical circuit with the linear parameters $L = 5$, $C = 0.5$, $r = 2$, $g = 0.5$ and the input voltage $e(t) = 0$, the components of the numerical solution with the initial values $t_0 = 0$, $x^0 = (1.1, -4.129, 1.2)^T$ have the form similar to that shown in Fig. 6.1.

The analysis of the obtained numerical solutions shows that the corresponding exact solutions have a finite escape time and verifies the results obtained with the help of Theorem 4.1.

7. Conclusions

The theorems, enabling to prove the existence and boundedness of global solutions (Lagrange stability) of the semilinear DAE (1.1) or their non-existence (solutions have a finite escape time, i.e., they are Lagrange unstable), are obtained. Using these theorems, we have found the restrictions on the initial data and the parameters of the electrical circuit (Fig. 5.1) of the nonlinear radio engineering filter under which the mathematical model (the DAE (5.4)) of the circuit is Lagrange stable, and the conditions under which the mathematical model is Lagrange unstable. The functions and quantities defining the circuits parameters (resistances, conductivities and others) and satisfying the obtained conditions have been given. It has been checked that the mentioned conditions of the Lagrange stability are fulfilled for certain classes of nonlinear functions which do not satisfy the global Lipschitz condition. In particular, it has been proven that the presence of nonlinear resistances and conductivities of the form (5.8), (5.9) in electric circuits admits the Lagrange stability of the corresponding mathematical models. Notice that nonlinear resistances and conductivities of this type are often encountered in real radio engineering systems.

The results of the study of the mathematical model have shown that the obtained theorems can be effectively applied in practice. The analysis of the numerical solutions of the mathematical model verifies the results of theoretical studies.

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References

[1] R. Andrzejewski and J. Awrejcewicz, *Nonlinear Dynamics of a Wheeled Vehicle*, Advances in Mechanics and Mathematics **10**, Springer, New York, NY, 2005.

[2] U.M. Ascher and L.R. Petzold, *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations*, SIAM, Philadelphia, PA, 1998.

[3] A. Bacciotti and L. Rosier, *Liapunov and Lagrange Stability: Inverse Theorems for Discontinuous Systems*, Mathematics of Control, Signals and Systems **11** (1998), 101–128.

[4] K.E. Brenan, S.L. Campbell, and L.R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, SIAM, Philadelphia, PA, 1996.

[5] L. Dai, *Singular Control Systems. Lecture Notes in Control and Information Sciences*, **118**, Springer-Verlag, Berlin, Heidelberg, 1989.

[6] M.S. Filipkovska, *Lagrange stability and numerical method for solving semilinear descriptor equations,* Visn. Kharkiv. Nats. Univ. Mat. Model. Inform. Tekh. Avt. Syst. Upr. **26**, No. 1156 (2015), 152–167 (Russian).

[7] M. Filipkovskaya, *Global solvability of singular semilinear differential equations and applications to nonlinear radio engineering,* Challenges of Modern Technology **6** (2015), 3–13.

[8] C.W. Gear and L.R. Petzold, *ODE methods for the solution of differential/algebraic systems,* SIAM J. Numer. Anal. **21** (1984), 716–728.

[9] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations. Analysis and Numerical Solution,* EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2006.

[10] J. La Salle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications,* Mathematics in Science and Engineering, **4**, Academic Press, New York-London, 1961.

[11] R. Lamour, R. März and C. Tischendorf, *Differential-Algebraic Equations: A Projector Based Analysis,* Differential-Algebraic Equations Forum, Springer, Heidelberg, 2013.

[12] R. März, *Practical Lyapunov stability criteria for differential algebraic equations,* Numerical Analysis and Mathematical Modelling, Banach Center Publ., **29**, Polish Acad. Sci. Inst. Math., Warsaw, 1994, 245–266.

[13] R.E. O'Malley and L.V. Kalachev, *Regularization of nonlinear differential-algebraic equations,* SIAM J. Math. Anal. **25** (1994), 615–629.

[14] P.J. Rabier and W.C. Rheinboldt, *Discontinuous solutions of semilinear differential-algebraic equations. II. P-consistency,* Nonlinear Anal. **27** (1996), 1257–1280.

[15] T. Reis and T. Stykel, *Lyapunov balancing for passivity-preserving model reduction of RC circuits,* SIAM J. Appl. Dyn. Syst. **10** (2011), 1–34.

[16] R. Riaza, *Differential-Algebraic Systems. Analytical Aspects and Circuit Applications,* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[17] A.G. Rutkas, *Cauchy problem for the equation $Ax'(t)+Bx(t) = f(t)$,* Differ. Uravn. **11** (1975), 1996–2010 (Russian).

[18] A.G. Rutkas and M.S. Filipkovska, *Extension of solutions of one class of differential-algebraic equations,* Zh. Obchysl. Prykl. Mat. (2013), no. 2, 135–145 (Russian).

[19] A.G. Rutkas and L.A. Vlasenko, *Existence, uniqueness and continuous dependence for implicit semilinear functional differential equations,* Nonlinear Anal. **55** (2003), 125–139.

[20] L. Schwartz, *Analyse Mathématique, I,* Hermann, Paris, 1967 (French).

[21] A.A. Shcheglova and V.F. Chistyakov, *Stability of linear differential-algebraic systems,* Differ. Uravn. **40** (2004), 47–57; Engl. transl.: Differ. Equ. **40** (2004), 50–62.

[22] C. Tischendorf, *On the stability of solutions of autonomous index-1 tractable and quasilinear index-2 tractable DAEs,* Circuits Systems Signal Process. **13** (1994), 139–154.

[23] L.A. Vlasenko, *Evolution models with implicit and degenerate differential equations,* Sistemnye Tekhnologii, Dniepropetrovsk, 2006 (Russian).

[24] A. Wu and Z. Zeng, *Lagrange stability of memristive neural networks with discrete and distributed delays*, IEEE Trans. Neural Netw. Learn. Syst. **25** (2014), 690–703.

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Стійкість за Лагранжем напівлінійних диференціально-алгебраїчних рівнянь та застосування до нелінійних електричних кіл

Maria S. Filipkovska

Проводиться дослідження напівлінійного диференціально-алгебраїчного рівняння (ДАР) з акцентом на стійкість (нестійкість) за Лагранжем. Отримано умови існування та єдиності глобальних розв'язків (розв'язок існує на нескінченному інтервалі) задачі Коші, а також умови обмеженості глобальних розв'язків. Більш того, отримані умови стійкості за Лагранжем напівлінійного ДАР гарантують, що кожний його розв'язок є глобальним і обмеженим, та, на відміну від теорем про стійкість за Ляпуновим, дозволяють довести існування та єдиність глобальних розв'язків незалежно від наявності та кількості точок рівноваги. Також отримано умови існування та єдиності розв'язків зі скінченним часом визначення (розв'язок існує на скінченному інтервалі та є необмеженим, тобто нестійким за Лагранжем) для задачі Коші. Не використовуються обмеження типу глобальної умови Ліпшиця, що дозволяє ефективно використовувати результати роботи у практичних застосуваннях. В якості застосування досліджено математичну модель радіотехнічного фільтру з нелінійними елементами. Чисельний аналіз моделі підтверджує результати теоретичних досліджень.

Ключові слова: диференціально-алгебраїчне рівняння, стійкість за Лагранжем, нестійкість, регулярний жмуток, обмежений глобальний розв'язок, скінченний час визначення, нелінійне електричне коло.