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Non-Differentiable Functions Defined in Terms of Classical Representations of Real Numbers

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The present paper is devoted to the functions from a certain subclass of non-differentiable functions. The arguments and values of the considered functions are represented by the s-adic representation or the nega-s-adic representation of real numbers. The technique of modeling these functions is the simplest as compared with the well-known techniques of modeling non-differentiable functions. In other words, the values of these functions are obtained from the s-adic or nega-s-adic representation of the argument by a certain change of digits or combinations of digits.

Integral, fractal and other properties of the functions are described.

Key words: nowhere differentiable function, s-adic representation, nega-s-adic representation, non-monotonic function, Hausdorff–Besicovitch dimension.

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1. Introduction

A nowhere differentiable function is a function whose derivative equals infinity or does not exist at each point from the domain of definition.

The idea of the existence of continuous non-differentiable functions appeared in the nineteenth century. In 1854, Dirichlet speaking at lectures at Berlin University said on the existence of a continuous function without derivative. In 1830, the first example of a continuous non-differentiable function was modeled by Bolzano in "Doctrine on Function" but the paper was published one hundred years later [1, 2]. In 1861, Rieman gave the following example of a non-differentiable function without proof [37]:

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}. \tag{1.1}$$

It was also studied by Hardy [3], Gerver [4], and Du Bois-Reymond. The function has a finite derivative that equals $\frac{1}{2}$ at the points of the form $\xi\pi$, where ξ is a rational number with an odd numerator and an odd denominator. Function (1.1) does not have other points of differentiability.

In 1875, Du Bois-Reymond published the following example of the function [5]:

$$f(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$ and $b > 1$ is an odd integer number such that $ab > 1 + \frac{3}{2}\pi$. The last-mentioned function was modeled by Weierstrass in 1871. This function has the derivative that equals $(+\infty)$ or $(-\infty)$ on an uncountable everywhere dense set. The following example of non-differentiable function was modeled nearly simultaneously and independently by Darboux in the paper [6]:

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin((n+1)!x)}{n!}.$$

In the sequel, other examples of the functions were constructed and classes of non-differentiable functions were founded. The major contribution to these studies was made by the following scientists: Dini [9, p. 148–158], Darboux [7], Orlicz [8], Hankel [10, p. 61–65].

In 1929, the problem on the massiveness of the set of non-differentiable functions in the space of continuous functions was formulated by Steinhaus. In 1931, this problem was solved independently and by different ways by Banach [11] and Mazurkiewicz [13]. So the following statement is true.

Theorem 1.1 (Banach–Mazurkiewicz). *The set of non-differentiable functions in the space $C[0,1]$ of functions, that are continuous on $[0,1]$, with the uniform metric is a set of the second category.*

There also exist functions that do not have a finite or infinite one-sided derivative at any point. In 1922, an example of such function was modeled by Besicovitch in [12]. The set of continuous on $[0,1]$ functions whose right-sided derivative equals a finite number or equals $+\infty$ on an uncountable set is a set of the second Baire category in the space of all continuous functions. Hence the set of functions, that do not have a finite or infinite one-sided derivative at any point, is a set of the first category in the space of continuous on a segment functions. The last-mentioned statement was proved by Saks in 1932 (see [14]).

Now researchers are trying to find simpler examples of non-differentiable functions. Interest in such functions is explained by their connection with fractals, modeling of real objects, processes, and phenomena (in physics, economics, technology, etc.).

The present paper is devoted to the simplest examples of non-differentiable functions defined in terms of the s-adic or nega-s-adic representations.

In addition, we consider some examples of nowhere differentiable functions defined by other ways.

2. Certain examples of non-differentiable functions

Example 2.1. Consider the functions

$$f\left(\Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right)=\Delta^2_{\varphi_1(x)\varphi_2(x)\ldots\varphi_n(x)\ldots} \text{ and } g\left(\Delta^s_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right)=\Delta^2_{\varphi_1(x)\varphi_2(x)\ldots\varphi_n(x)\ldots},$$

where $s>2$ is a fixed positive integer number,

$$\Delta^s_{\alpha_1\alpha_2\ldots\alpha_n\ldots}=\sum_{n=1}^{\infty}\frac{\alpha_n}{s^n}, \qquad \alpha_n\in\{0,1,\ldots,s-1\},$$

$$\varphi_1(x)=\begin{cases}0 & \text{if } \alpha_1(x)=0\\ 1 & \text{if } \alpha_1(x)\neq 0\end{cases}, \quad \varphi_j(x)=\begin{cases}\varphi_{j-1}(x) & \text{for } \alpha_j(x)=\alpha_{j-1}(x)\\ 1-\varphi_{j-1}(x) & \text{for } \alpha_j(x)\neq\alpha_{j-1}(x)\end{cases}.$$

In 1952, the function g was introduced by Bush in [15], and the function f was modeled by Wunderlich in [38]. The functions f and g are non-differentiable.

In [17], Salem modeled the function

$$s(x)=s\left(\Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right)=\beta_{\alpha_1}+\sum_{n=2}^{\infty}\left(\beta_{\alpha_n}\prod_{i=1}^{n-1}q_{\alpha_i}\right)=y=\Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots},$$

where $q_0>0$, $q_1>0$, and $q_0+q_1=1$. That is, $\beta_{\alpha_n}=0$ whenever $\alpha_n=0$, $\beta_{\alpha_n}=q_0$ whenever $\alpha_n=1$, and $q_{\alpha_n}\in\{q_0,q_1\}$. This function is a singular function. However, generalizations of the Salem function can be non-differentiable functions or do not have the derivative on a certain set.

In October 2014, generalizations of the Salem function such that their arguments are represented in terms of positive [16] or alternating [36] Cantor series or the nega-$\tilde{Q}$-representation [29–31] were considered by Serbenyuk in [25–28,32,35].

Consider these generalizations of the Salem function.

Example 2.2 ([28]). Let (d_n) be a fixed sequence of positive integers, $d_n>1$, and (A_n) be a sequence of the sets $A_n=\{0,1,\ldots,d_n-1\}$.

Let $x\in[0,1]$ be an arbitrary number represented by a positive Cantor series

$$x=\Delta^D_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}=\sum_{n=1}^{\infty}\frac{\varepsilon_n}{d_1d_2\ldots d_n}, \quad \text{where } \varepsilon_n\in A_n.$$

Let $P=\|p_{i,n}\|$ be a fixed matrix such that $p_{i,n}\geq 0$, $n=1,2,\ldots$, and $i=\overline{0,d_n-1}$, $\sum_{i=0}^{d_n-1}p_{i,n}=1$ for an arbitrary $n\in\mathbb{N}$, and $\prod_{n=1}^{\infty}p_{i_n,n}=0$ for any sequence (i_n).

Suppose that elements of the matrix $P=\|p_{i,n,}\|$ can be negative numbers as well, but

$$\beta_{0,n}=0, \beta_{i,n}>0 \quad \text{for} \quad i\neq 0, \quad \text{and} \quad \max_i|p_{i,n}|<1.$$

Here

$$\beta_{\varepsilon_k,k}=\begin{cases}0 & \text{if } \varepsilon_k=0\\ \sum_{i=0}^{\varepsilon_k-1}p_{i,k} & \text{if } \varepsilon_k\neq 0\end{cases}.$$

Then the following statement is true.

Theorem 2.3. *Given the matrix P such that for all $n \in \mathbb{N}$ the following are true: $p_{\varepsilon_n,n}p_{\varepsilon_n-1,n} < 0$, moreover $d_n p_{d_n-1,n} \ge 1$ or $d_n p_{d_n-1,n} \le 1$; and the conditions*

$$\lim_{n\to\infty}\prod_{k=1}^{n} d_k p_{0,k} \ne 0, \lim_{n\to\infty}\prod_{k=1}^{n} d_k p_{d_k-1,k} \ne 0$$

hold simultaneously. Then the function

$$F(x) = \beta_{\varepsilon_1(x),1} + \sum_{k=2}^{\infty}\left(\beta_{\varepsilon_k(x),k}\prod_{n=1}^{k-1} p_{\varepsilon_n(x),n}\right)$$

is non-differentiable on $[0,1]$.

Example 2.4 ([35]). Let $P = \|p_{i,n}\|$ be a given matrix such that $n = 1, 2, \dots$ and $i = \overline{0, d_n - 1}$. For this matrix the following system of properties holds:

1°. $\forall n \in \mathbb{N} \quad p_{i,n} \in (-1,1)$;

2°. $\forall n \in \mathbb{N} \quad \sum_{i=0}^{d_n-1} p_{i,n} = 1$;

3°. $\forall (i_n), i_n \in A_{d_n} \quad \prod_{n=1}^{\infty} |p_{i_n,n}| = 0$;

4°. $\forall i_n \in A_{d_n} \setminus \{0\} \quad 1 > \beta_{i_n,n} = \sum_{i=0}^{i_n-1} p_{i,n} > \beta_{0,n} = 0$.

Let us consider the function

$$\tilde{F}(x) = \beta_{\varepsilon_1(x),1} + \sum_{n=2}^{\infty}\left(\tilde{\beta}_{\varepsilon_n(x),n}\prod_{j=1}^{n-1} \tilde{p}_{\varepsilon_j(x),j}\right),$$

where

$$\tilde{\beta}_{\varepsilon_n(x),n} = \begin{cases} \beta_{\varepsilon_n(x),n} & \text{if } n \text{ is odd} \\ \beta_{d_n-1-\varepsilon_n(x),n} & \text{if } n \text{ is even} \end{cases},$$

$$\tilde{p}_{\varepsilon_n(x),n} = \begin{cases} p_{\varepsilon_n(x),n} & \text{if } n \text{ is odd} \\ p_{d_n-1-\varepsilon_n(x),n} & \text{if } n \text{ is even} \end{cases},$$

$$\beta_{\varepsilon_n(x),n} = \begin{cases} 0 & \text{if } \varepsilon_n = 0 \\ \sum_{i=0}^{\varepsilon_n-1} p_{i,n} & \text{if } \varepsilon_n \ne 0 \end{cases}.$$

Here x is represented by an alternating Cantor series, i.e.,

$$x = \Delta^{-(d_n)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} = \sum_{n=1}^{\infty}\frac{1+\varepsilon_n}{d_1 d_2 \ldots d_n}(-1)^{n+1},$$

where (d_n) is a fixed sequence of positive integers, $d_n > 1$, and (A_{d_n}) is a sequence of the sets $A_{d_n} = \{0, 1, \dots, d_n - 1\}$, and $\varepsilon_n \in A_{d_n}$.

Theorem 2.5. *Let $p_{\varepsilon_n,n}p_{\varepsilon_n-1,n} < 0$ for all $n \in \mathbb{N}$, $\varepsilon_n \in A_{d_n} \setminus \{0\}$ and conditions*

$$\lim_{n\to\infty} \prod_{k=1}^{n} d_k p_{0,k} \neq 0, \lim_{n\to\infty} \prod_{k=1}^{n} d_k p_{d_k-1,k} \neq 0$$

hold simultaneously. Then the function $\tilde{F}$ is non-differentiable on $[0,1]$.

Example 2.6 ([32]). Let $\tilde{Q} = \|q_{i,n}\|$ be a fixed matrix, where $i = \overline{0, m_n}$, $m_n \in N^0_\infty = \mathbb{N} \cup \{0, \infty\}$, $n = 1, 2, \dots$, and the following system of properties is true for elements $q_{i,n}$ of the last-mentioned matrix:

1°. $q_{i,n} > 0$;

2°. $\forall n \in \mathbb{N} \quad \sum_{i=0}^{m_n} q_{i,n} = 1$;

3°. $\forall (i_n), i_n \in \mathbb{N} \cup \{0\} \quad \prod_{n=1}^{\infty} q_{i_n,n} = 0$.

The expansion of $x \in [0, 1)$,

$$x = \sum_{i=0}^{i_1-1} q_{i,1} + \sum_{n=2}^{\infty} \left[(-1)^{n-1} \tilde{\delta}_{i_n,n} \prod_{j=1}^{n-1} \tilde{q}_{i_j,j} \right] + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{2n-1} \tilde{q}_{i_j,j} \right), \tag{2.1}$$

is called *the nega-$\tilde{Q}$-expansion of* x. By $x = \Delta^{-\tilde{Q}}_{i_1 i_1 \dots i_n \dots}$ denote the nega-$\tilde{Q}$-expansion of x. The last-mentioned notation is called *the nega-$\tilde{Q}$-representation of* x. Here

$$\tilde{\delta}_{i_n,n} = \begin{cases} 1 & \text{if } n \text{ is even and } i_n = m_n \\ \sum_{i=m_n-i_n}^{m_n} q_{i,n} & \text{if } n \text{ is even and } i_n \neq m_n \\ 0 & \text{if } n \text{ is odd and } i_n = 0 \\ \sum_{i=0}^{i_n-1} q_{i,n} & \text{if } n \text{ is odd and } i_n \neq 0 \end{cases},$$

and the first sum in expression (2.1) is equal to 0 if $i_1 = 0$.

Suppose that $m_n < \infty$ for all positive integers n.

Numbers from some countable subset of $[0, 1]$ have two different nega-$\tilde{Q}$-representations, i.e.,

$$\Delta^{-\tilde{Q}}_{i_1 i_2 \dots i_{n-1} i_n m_{n+1} 0 m_{n+3} 0 m_{n+5} \dots} = \Delta^{-\tilde{Q}}_{i_1 i_2 \dots i_{n-1} [i_n - 1] 0 m_{n+2} 0 m_{n+4} \dots}, \quad i_n \neq 0.$$

These numbers are called *nega-$\tilde{Q}$-rationals*, and the rest of the numbers from $[0, 1]$ are called *nega-$\tilde{Q}$-irrationals*.

Suppose we have matrixes of the same dimension $\tilde{Q} = \|q_{i,n}\|$ (the properties of the last-mentioned matrix were considered earlier) and $P = \|p_{i,n}\|$, where $i = \overline{0, m_n}$, $m_n \in \mathbb{N} \cup \{0\}$, $n = 1, 2, \dots$, and for elements $p_{i,n}$ of P the following system of conditions is true:

1°. $p_{i,n} \in (-1, 1)$;

2°. $\forall n \in \mathbb{N} \quad \sum_{i=0}^{m_n} p_{i,n} = 1$;

3°. $\forall (i_n), i_n \in \mathbb{N} \cup \{0\} \quad \prod_{n=1}^{\infty} |p_{i_n,n}| = 0$;

4°. $\forall i_n \in \mathbb{N} \quad 0 = \beta_{0,n} < \beta_{i_n,n} = \sum_{i=0}^{i_n-1} p_{i,n} < 1.$

Theorem 2.7. *If the following properties of the matrix P hold:*

- *for all $n \in \mathbb{N}$, $i_n \in N^1_{m_n} = \{1, 2, \dots, m_n\}$,*

$$p_{i_n,n} p_{i_n-1,n} < 0;$$

- *the conditions*

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{p_{0,k}}{q_{0,k}} \neq 0, \lim_{n\to\infty} \prod_{k=1}^{n} \frac{p_{m_k,k}}{q_{m_k,k}} \neq 0$$

hold simultaneously, then the function

$$F(x) = \beta_{i_1(x),1} + \sum_{k=2}^{\infty} \left[\tilde{\beta}_{i_k(x),k} \prod_{j=1}^{k-1} \tilde{p}_{i_j(x),j} \right]$$

does not have a finite or infinite derivative at any nega-$\tilde{Q}$-rational point from the segment $[0, 1]$.

Here

$$\tilde{p}_{i_n,n} = \begin{cases} p_{i_n,n} & \text{if } n \text{ is odd} \\ p_{m_n-i_n,n} & \text{if } n \text{ is even} \end{cases},$$

$$\tilde{\beta}_{i_n,n} = \begin{cases} \beta_{i_n,n} & \text{if } n \text{ is odd} \\ \beta_{m_n-i_n,n} & \text{if } n \text{ is even} \end{cases},$$

$$\beta_{i_n,n} = \begin{cases} \sum_{i=0}^{i_n-1} p_{i,n} > 0 & \text{if } i_n \neq 0 \\ 0 & \text{if } i_n = 0 \end{cases}.$$

The last-mentioned examples of non-differentiable functions are difficult. However, there exist elementary examples of these functions.

3. The simplest example of non-differentiable function and its analogues

In 2012, the main results of this subsection were represented by the author of the present paper in [18–20, 34]

We will not consider numbers whose ternary representation has the period (2) (without the number 1). Let us consider a certain function f defined on $[0, 1]$ in the following way:

$$x = \Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \xrightarrow{f} \Delta^3_{\varphi(\alpha_1)\varphi(\alpha_2)\ldots\varphi(\alpha_n)\ldots} = f(x) = y,$$

where $\varphi(i) = \frac{-3i^2+7i}{2}$, $i \in N^0_2 = \{0, 1, 2\}$, and $\Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ is the ternary representation of $x \in [0, 1]$. That is, the values of this function are obtained from the ternary representation of the argument by the following change of digits: 0 by 0, 1 by 2, and 2 by 1. This function preserves the ternary digit 0.

In this subsection, differential, integral, fractal, and other properties of the function f are described; equivalent representations of this function by additionally defined auxiliary functions are considered.

We begin with the definitions of some auxiliary functions.

Let i, j, k be pairwise distinct digits of the ternary numeral system. First, let us introduce a function $\varphi_{ij}(\alpha)$ defined on the alphabet of the ternary numeral system by the following:

	i	j	k
$\varphi_{ij}(\alpha)$	0	0	1

That is, f_{ij} is a function given on $[0,1]$ in the form

$$x = \Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \stackrel{f_{ij}}{\rightarrow} \Delta^3_{\varphi_{ij}(\alpha_1)\varphi_{ij}(\alpha_2)\ldots\varphi_{ij}(\alpha_n)\ldots} = f_{ij}(x) = y.$$

Remark 3.1. From the definition of f_{ij} it follows that $f_{01} = f_{10}$, $f_{02} = f_{20}$, and $f_{12} = f_{21}$. Since it is true, we will use only the notations f_{01}, f_{02}, f_{12}.

Lemma 3.2. *The function f can be represented by:*

1. $f(x) = 2x - 3f_{01}(x)$, *where* $\Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \stackrel{f_{01}}{\rightarrow} \Delta^3_{\varphi_{01}(\alpha_1)\varphi_{01}(\alpha_2)\ldots\varphi_{01}(\alpha_n)\ldots}$, $\varphi_{01}(i) = \frac{i^2-i}{2}$, $i \in N^0_2$;

2. $f(x) = \frac{3}{2} - x - 3f_{12}(x)$, *where* $\Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \stackrel{f_{12}}{\rightarrow} \Delta^3_{\varphi_{12}(\alpha_1)\varphi_{12}(\alpha_2)\ldots\varphi_{12}(\alpha_n)\ldots}$, $\varphi_{12}(i) = \frac{i^2-3i+2}{2}$, $i \in N^0_2$.

3. $f(x) = \frac{x}{2} + \frac{3}{2}f_{02}(x)$, *where* $\Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \stackrel{f_{02}}{\rightarrow} \Delta^3_{\varphi_{02}(\alpha_1)\varphi_{02}(\alpha_2)\ldots\varphi_{02}(\alpha_n)\ldots}$, $\varphi_{02}(i) = -i^2 + 2i$, $i \in N^0_2$.

Lemma 3.3. *The functions $f, f_{01}, f_{02}, f_{12}$ have the properties:*

1. $[0,1] \stackrel{f}{\rightarrow} \left([0,1] \setminus \{\Delta^3_{\alpha_1\alpha_2\ldots\alpha_n 111\ldots}\}\right) \cup \left\{\dfrac{1}{2}\right\}$;

2. *the point $x_0 = 0$ is the unique invariant point of the function f;*

3. *the function f is not bijective on a certain countable subset of $[0,1]$;*

4. *the following relationships hold for all $x \in [0,1]$:*

$$\begin{aligned} f(x) - f(1-x) &= f_{01}(x) - f_{12}(x), \\ f(x) + f(1-x) &= \frac{1}{2} + 3f_{02}(x), \\ f_{01}(x) + f_{02}(x) + f_{12}(x) &= \frac{1}{2}, \\ 2f_{01}(x) + f_{02}(x) &= x, \\ f_{01}(x) - f_{12}(x) &= x - \frac{1}{2}; \end{aligned}$$

5. *the function f is not monotonic on the domain of definition; in particular, the function f is a decreasing function on the set*

$$\{x : x_1 < x_2 \Rightarrow (x_1 = \Delta^3_{c_1\ldots c_{n_0}1\alpha_{n_0+2}\alpha_{n_0+3}\ldots} \wedge x_2 = \Delta^3_{c_1\ldots c_{n_0}2\beta_{n_0+2}\beta_{n_0+3}\ldots})\},$$

where $n_0 \in \mathbb{Z}_0 = \mathbb{N} \cup \{0\}$, $c_1, c_2, \ldots, c_{n_0}$ is an ordered set of the ternary digits, $\alpha_{n_0+i} \in N_2^0$, $\beta_{n_0+i} \in N_2^0$, $i \in \mathbb{N}$; and the function f is an increasing function on the set

$$\{x : x_1 < x_2 \Rightarrow (x_1 = \Delta^3_{c_1\ldots c_{n_0}0\alpha_{n_0+2}\alpha_{n_0+3}\ldots} \wedge x_2 = \Delta^3_{c_1\ldots c_{n_0}r\beta_{n_0+2}\beta_{n_0+3}\ldots})\},$$

where $r \in \{1, 2\}$.

Let us consider the fractal properties of all level sets of the functions f_{01}, f_{02}, f_{12}.

The set

$$f^{-1}(y_0) = \{x : g(x) = y_0\},$$

where y_0 is a fixed element of the range of values $E(g)$ of the function g, is called *a level set of g*.

Theorem 3.4. *The following statements are true:*

- *if there exists at least one digit 2 in the ternary representation of y_0, then $f_{ij}^{-1}(y_0) = \varnothing$;*
- *if $y_0 = 0$ or y_0 is a ternary-rational number from the set $C[3, \{0, 1\}] = \{y : y = \Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \alpha_n \in \{0, 1\}\}$, then*

$$\alpha_0(f_{ij}^{-1}(y_0)) = \log_3 2;$$

- *if y_0 is a ternary-irrational number from the set $C[3, \{0, 1\}]$, then*

$$0 \leq \alpha_0(f_{ij}^{-1}(y_0)) \leq \log_3 2,$$

where $\alpha_0(f_{ij}^{-1}(y_0))$ is the Hausdorff–Besicovitch dimension of $f_{ij}^{-1}(y_0)$.

Let us describe the main properties of the function f.

Theorem 3.5. *The function f is continuous at ternary-irrational points, and ternary-rational points are points of discontinuity of the function. Furthermore, a ternary-rational point $x_0 = \Delta^3_{\alpha_1\alpha_2\ldots\alpha_n000\ldots}$ is a point of discontinuity $\frac{1}{2\cdot 3^{n-1}}$ whenever $\alpha_n = 1$, and is a point of discontinuity $\left(-\frac{1}{2\cdot 3^{n-1}}\right)$ whenever $\alpha_n = 2$.*

Theorem 3.6. *The function f is non-differentiable.*

Let us consider one fractal property of the graph of f. Suppose that

$$X = [0,1] \times [0,1] = \left\{(x, y) : x = \sum_{m=1}^{\infty} \frac{\alpha_m}{3^m},\ \alpha_m \in N_2^0,\ y = \sum_{m=1}^{\infty} \frac{\beta_m}{3^m},\ \beta_m \in N_2^0\right\}.$$

Then the set

$$\sqcap_{(\alpha_1\beta_1)(\alpha_2\beta_2)\ldots(\alpha_m\beta_m)} = \Delta^3_{\alpha_1\alpha_2\ldots\alpha_m} \times \Delta^3_{\beta_1\beta_2\ldots\beta_m}$$

is a square with a side length of 3^{-m}. This square is called *a square of rank m with base* $(\alpha_1\beta_1)(\alpha_2\beta_2)\ldots(\alpha_m\beta_m)$.

If $E \subset X$, then the number

$$\alpha^K(E) = \inf\{\alpha : \widehat{H}_\alpha(E) = 0\} = \sup\{\alpha : \widehat{H}_\alpha(E) = \infty\},$$

where

$$\widehat{H}_\alpha(E) = \lim_{\varepsilon \to 0} \left[\inf_{d \le \varepsilon} K(E,d) d^\alpha \right],$$

and $K(E,d)$ is the minimum number of squares of the diameter d required to cover the set E, is called *the fractal cell entropy dimension of the set E.* It is easy to see that $\alpha^K(E) \ge \alpha_0(E)$.

The notion of the fractal cell entropy dimension is used for the calculation of the Hausdorff–Besicovitch dimension of the graph of f, because, in the case of the function f, we obtain that $\alpha^K(E) = \alpha_0(E)$ (it follows from the self-similarity of the graph of f).

Theorem 3.7. *The Hausdorff–Besicovitch dimension of the graph of f is equal to 1.*

The integral properties of f are described in the theorem below.

Theorem 3.8. *The Lebesgue integral of the function f is equal to $\frac{1}{2}$.*

There exist several analogues of the function f which have the same properties and are defined by analogy. Let us consider these functions.

One can define $3! = 6$ functions determined on $[0,1]$ in terms of the ternary numeral system in the following way:

$$\Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \xrightarrow{f_m} \Delta^3_{\varphi_m(\alpha_1)\varphi_m(\alpha_2)\ldots\varphi_m(\alpha_n)\ldots},$$

where the function $\varphi_m(\alpha_n)$ is determined on an alphabet of the ternary numeral system, and $f_m(x)$ is defined by using the table for each $m = \overline{1,6}$.

	0	1	2
$\varphi_1(\alpha_n)$	0	1	2
$\varphi_2(\alpha_n)$	0	2	1
$\varphi_3(\alpha_n)$	1	0	2
$\varphi_4(\alpha_n)$	1	2	0
$\varphi_5(\alpha_n)$	2	0	1
$\varphi_6(\alpha_n)$	2	1	0

Thus one can model a class of functions whose values are obtained from the ternary representation of the argument by a certain change of ternary digits.

It is easy to see that the function $f_1(x)$ is the function $y = x$ and the function $f_6(x)$ is the function $y = 1 - x$, i.e.,

$$y = f_1(x) = f_1\left(\Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots} = x,$$
$$y = f_6(x) = f_6\left(\Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \Delta^3_{[2-\alpha_1][2-\alpha_2]\ldots[2-\alpha_n]\ldots} = 1 - x.$$

We will describe some application of the function of the last-mentioned form in the next subsection.

Lemma 3.9. *Any function f_m can be represented by the functions f_{ij} in the form*

$$f_m = a^{(ij)}_m x + b^{(ij)}_m + c^{(ij)}_m f_{ij}(x), \text{ where } a^{(ij)}_m, b^{(ij)}_m, c^{(ij)}_m \in \mathbb{Q}.$$

One can formulate the following corollary.

Theorem 3.10. *The function f_m such that $f_m(x) \neq x$ and $f_m(x) \neq 1-x$ is:*

- *continuous almost everywhere;*
- *non-differentiable on $[0,1]$;*
- *a function whose Hausdorff–Besicovitch dimension of the graph is equal to 1;*
- *a function whose Lebesgue integral is equal to $\frac{1}{2}$.*

Generalizations of the results described in this subsection will be considered in the following subsection.

4. Generalizations of the simplest example of non-differentiable function

In 2013, the investigations of the last subsection were generalized by the author in several papers [21, 22, 33]. Consider these results.

We begin with the definitions.

Let $s > 1$ be a fixed positive integer number, and let the set $A = \{0, 1, \ldots, s-1\}$ be an alphabet of the s-adic or nega-s-adic numeral system. The notation $x = \Delta^{\pm s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ means that x is represented by the s-adic or nega-s-adic representation, i.e.,

$$x = \sum_{n=1}^{\infty} \frac{\alpha_n}{s^n} = \Delta^s_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$$

or

$$x = \sum_{n=1}^{\infty} \frac{(-1)^n \alpha_n}{s^n} = \Delta^{-s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \alpha_n \in A.$$

Let Λ_s be a class of functions of the type

$$f : x = \Delta^{\pm s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \to \Delta^{\pm s}_{\beta_1\beta_2\ldots\beta_n\ldots} = f(x) = y,$$

where $\left(\beta_{km+1}, \beta_{km+2}, \ldots, \beta_{(m+1)k}\right) = \theta\left(\alpha_{km+1}, \alpha_{km+2}, \ldots, \alpha_{(m+1)k}\right)$, the number k is a fixed positive integer for a specific function f, $m = 0, 1, 2, \ldots$, and $\theta(\gamma_1, \gamma_2, \ldots, \gamma_k)$ is some function of the k variables (it is the bijective correspondence) such that the set

$$A^k = \underbrace{A \times A \times \ldots \times A}_{k}$$

is its domain of definition and range of values.

Each combination $(\gamma_1, \gamma_2, \ldots, \gamma_k)$ of k s-adic or nega-s-adic digits (according to the number representation of the argument of a function f) is assigned to the single combination $\theta(\gamma_1, \gamma_2, \ldots, \gamma_k)$ of the k s-adic or nega-s-adic digits (according to the number representation of the value of a function f). The combination $\theta(\gamma_1, \gamma_2, \ldots, \gamma_k)$ is assigned to the unique combination $(\gamma_1^{'}, \gamma_2^{'}, \ldots, \gamma_k^{'})$ that may not match with $(\gamma_1, \gamma_2, \ldots, \gamma_k)$. The θ is a bijective function on A^k.

It is clear that any function $f \in \Lambda_s$ is one of the functions:

$$f^s_k, \qquad f_+, \qquad f^{-1}_+, \qquad f_+ \circ f^s_k, \qquad f^s_k \circ f^{-1}_+, \qquad f_+ \circ f^s_k \circ f^{-1}_+,$$

where

$$f^s_k\left(\Delta^s_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \Delta^s_{\beta_1\beta_2\ldots\beta_n\ldots},$$
$$\left(\beta_{km+1}, \beta_{km+2}, \ldots, \beta_{(m+1)k}\right) = \theta\left(\alpha_{km+1}, \alpha_{km+2}, \ldots, \alpha_{(m+1)k}\right)$$

for $m = 0, 1, 2, \ldots$, and some fixed positive integer number k, i.e.,

$$\left(\beta_1, \beta_2, \ldots, \beta_k\right) = \theta\left(\alpha_1, \alpha_2, \ldots, \alpha_k\right),$$
$$\left(\beta_{k+1}, \beta_{k+2}, \ldots, \beta_{2k}\right) = \theta\left(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_{2k}\right),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\left(\beta_{km+1}, \beta_{km+2}, \ldots, \beta_{(m+1)k}\right) = \theta\left(\alpha_{km+1}, \alpha_{km+2}, \ldots, \alpha_{(m+1)k}\right),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and

$$f_+\left(\Delta^s_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \Delta^{-s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots},$$
$$f^{-1}_+\left(\Delta^{-s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \Delta^s_{\alpha_1\alpha_2\ldots\alpha_n\ldots}.$$

Let us consider several examples.

The function f considered in the last subsection is a function of the f^3_1 type. In fact,

$$x = \Delta^3_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \xrightarrow{f} \Delta^3_{\varphi(\alpha_1)\varphi(\alpha_2)\ldots\varphi(\alpha_n)\ldots} = f(x) = y,$$

where $\varphi\left(\alpha_n\right)$ is a function defined in terms of the s-adic numeral system in the following way:

α_n	0	1	2
$\varphi(\alpha_n)$	0	2	1

.

Now we give the example of the function f^2_2. The function

$$f^2_2 : \Delta^2_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \to \Delta^2_{\beta_1\beta_2\ldots\beta_n\ldots},$$

where $(\beta_{2m+1}, \beta_{2(m+1)}) = \theta(\alpha_{2m+1}, \alpha_{2(m+1)})$, $m = 0, 1, 2, 3, \ldots$, and

$\alpha_{2m+1}\alpha_{2(m+1)}$	00	01	10	11
$\beta_{2m+1}\beta_{2(m+1)}$	10	11	00	01

,

is an example of the f^2_2-type function.

It is obvious that the set of f^2_1 functions consists only of the functions $y = x$ and $y = 1 - x$ in the binary numeral system. But the set of f^2_2 functions has the order, which is equal to 4!, and includes the functions $y = x$ and $y = 1 - x$ as well.

Remark 4.1. The class Λ_s of functions includes the following linear functions:

$$\begin{aligned}
&y = x,\\
&y = f(x) = f\left(\Delta^s_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \Delta^s_{[s-1-\alpha_1][s-1-\alpha_2]\ldots[s-1-\alpha_n]\ldots} = 1 - x,\\
&y = f(x) = f\left(\Delta^{-s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \Delta^{-s}_{[s-1-\alpha_1][s-1-\alpha_2]\ldots[s-1-\alpha_n]\ldots} = -\frac{s-1}{s+1} - x.
\end{aligned}$$

These functions are called Λ_s*-linear functions.*

Remark 4.2. The last-mentioned two functions in the last remark are interesting for applications in certain investigations. For example, in the case of a positive Cantor series, the function may have the form

$$\begin{aligned}
f(x) = f\left(\Delta^D_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}\right) &= f\left(\sum_{n=1}^{\infty} \frac{\varepsilon_n}{d_1d_2\ldots d_n}\right)\\
&= \Delta^D_{[d_1-1-\varepsilon_1][d_2-1-\varepsilon_2]\ldots[d_n-1-\varepsilon_n]\ldots} = \sum_{n=1}^{\infty} \frac{d_n - 1 - \varepsilon_n}{d_1d_2\ldots d_n}.
\end{aligned}$$

It is easy to see that this function is a transformation preserving the Hausdorff–Besicovitch dimension.

Consider the following representations by the alternating Cantor series:

$$\Delta^{-D}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} = \sum_{n=1}^{\infty} \frac{(-1)^n\varepsilon_n}{d_1d_2\ldots d_n},$$

$$\Delta^{-(d_n)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} = \sum_{n=1}^{\infty} \frac{1+\varepsilon_n}{d_1d_2\ldots d_n}(-1)^{n+1}.$$

In 2013, the study of the relations between positive and alternating Cantor series, as well as other investigations of alternating Cantor series, were presented in [23, 24]. These results were later published in [36].

Consider the following results that follow from the relations between positive and alternating Cantor series.

Lemma 4.3. *The following functions are identity transformations:*

$$x = \Delta^D_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} \xrightarrow{f} \Delta^{-(d_n)}_{\varepsilon_1[d_2-1-\varepsilon_2]\ldots\varepsilon_{2n-1}[d_{2n}-1-\varepsilon_{2n}]\ldots} = f(x) = y,$$
$$x = \Delta^{-(d_n)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} \xrightarrow{g} \Delta^D_{\varepsilon_1[d_2-1-\varepsilon_2]\ldots\varepsilon_{2n-1}[d_{2n}-1-\varepsilon_{2n}]\ldots} = g(x) = y.$$

Therefore the functions below are the DP-functions (the functions preserving the fractal Hausdorff–Besicovitch dimension):

$$x = \Delta^D_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} \xrightarrow{f} \Delta^{-(d_n)}_{[d_1-1-\varepsilon_1]\varepsilon_2\ldots[d_{2n-1}-1-\varepsilon_{2n-1}]\varepsilon_{2n}\ldots} = f(x) = y,$$
$$x = \Delta^{-(d_n)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} \xrightarrow{g} \Delta^D_{[d_1-1-\varepsilon_1]\varepsilon_2\ldots[d_{2n-1}-1-\varepsilon_{2n-1}]\varepsilon_{2n}\ldots} = g(x) = y.$$

A new method for the construction of the metric, probabilistic and dimensional theories for the families of representations of real numbers via studies of special mappings (G-isomorphisms of representations), under which the symbols of a given representation are mapped onto the same symbols of the other representation from the same family, when these mappings preserve the Lebesgue measure and the Hausdorff–Besicovitch dimension, follows from Remark 4.2 and investigations of the functions f_+, f_+^{-1}.

Let us describe the main properties of the functions $f \in \Lambda_s$.

Lemma 4.4. *For any function f from Λ_s, except for Λ_s-linear functions, the values of the function f for different representations of s-adic rational numbers from $[0,1]$ (nega-s-adic rational numbers from $[-\frac{s}{s+1}, \frac{1}{s+1}]$, respectively) are different.*

Remark 4.5. From the unique representation for each s-adic irrational number from $[0,1]$, it follows that the function f^s_k is well-defined at s-adic irrational points.

To reach that any function $f \in \Lambda_s$ such that $f(x) \neq x$ and $f(x) \neq 1-x$ is well-defined on the set of s-adic rational numbers from $[0,1]$, we will not consider the s-adic representation with period $(s-1)$.

Analogously, we will not consider the nega-s-adic representation with period $(0[s-1])$.

Lemma 4.6. *The set of functions f^s_k with the defined operation "composition of functions" is a finite group of order $(s^k)!$.*

Lemma 4.7. *The function $f \in \Lambda_s$ such that $f(x) \neq x$, $f(x) \neq -\frac{s-1}{s+1} - x$, and $f(x) \neq 1-x$ has the following properties:*

1) *f reflects $[0,1]$ or $[-\frac{s}{s+1}, \frac{1}{s+1}]$ (according to the number representation of the argument of a function f) into one of the segments $[0,1]$ or $[-\frac{s}{s+1}, \frac{1}{s+1}]$ without enumerable subset of points (according to the number representation of the value of a function f);*

2) *the function f is not monotonic on the domain of definition;*

3) *the function f is not a bijective mapping on the domain of definition.*

Lemma 4.8. *The following properties of the set of invariant points of the function f_k^s are true:*

- *the set of invariant points of f_k^s is a continuum set, and its Hausdorff–Besicovitch dimension is equal to $\frac{1}{k}\log_s j$, when there exists a set $\{\sigma_1, \sigma_2, \dots, \sigma_j\}$ $(j \geq 2)$ of k-digit combinations $\sigma_1, \dots, \sigma_j$ of s-adic digits such that*

$$\theta(a_1^{(i)}, a_2^{(i)}, \dots, a_k^{(i)}) = (a_1^{(i)}, a_2^{(i)}, \dots, a_k^{(i)}),$$

  where $\sigma_i = (a_1^{(i)} a_2^{(i)} \dots a_k^{(i)})$, $i = \overline{1, j}$;

- *the set of invariant points of f_k^s is a finite set, when there exists a unique k-digit combination σ of s-adic digits such that*

$$\theta(a_1, a_2, \dots, a_k) = (a_1, a_2, \dots, a_k), \ \ \sigma = (a_1 a_2 \dots a_k);$$

- *the set of invariant points of f_k^s is an empty set, when there does not exist any k-digit combination σ of s-adic digits such that*

$$\theta(a_1, a_2, \dots, a_k) = (a_1, a_2, \dots, a_k), \ \ \sigma = (a_1 a_2 \dots a_k).$$

In addition, the functions f_+ and f_+^{-1} have the following properties.

Lemma 4.9. *For each $x \in [0, 1]$, the function f_+ satisfies the equation*

$$f(x) + f(1-x) = -\frac{s-1}{s+1}.$$

Lemma 4.10. *For each $y \in [-\frac{s}{s+1}, \frac{1}{s+1}]$, the function f_+^{-1} satisfies the equation*

$$f^{-1}(y) + f^{-1}\left(-\frac{s-1}{s+1} - y\right) = 1.$$

Lemma 4.11. *The set of invariant points of the function f_+, as well as f_+^{-1}, is a self-similar fractal, and its Hausdorff–Besicovitch dimension is equal to $\frac{1}{2}$.*

The following theorems are the main theorems about the properties of the functions $f \in \Lambda_s$.

Theorem 4.12. *A function $f \in \Lambda_s$ such that $f(x) \neq x$, $f(x) \neq -\frac{s-1}{s+1} - x$, and $f(x) \neq 1 - x$ is:*

- *continuous at s-adic irrational or nega-s-adic irrational points, and s-adic rational or nega-s-adic rational points are points of discontinuity of this function (according to the number representation of the argument of the function f);*
- *a non-differentiable function.*

Theorem 4.13. *Let $f \in \Lambda_s$. Then the following are true:*

- *the Hausdorff–Besicovitch dimension of the graph of any function from the class Λ_s is equal to 1;*

- $\displaystyle\int_{D(f)} f(x)\,dx = \frac{1}{2}$, *where* $D(f)$ *is the domain of* f.

So, in the present paper, we considered historical moments of the development of the theory of non-differentiable functions, difficult and simplest examples of such functions. Integral, fractal, and other properties of the simplest example of a nowhere differentiable function and its analogues and generalizations are described. Equivalent representations of the considered simplest example by additionally defined auxiliary functions were reviewed.

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Недиференційовні функції, визначені в термінах класичних представлень дійсних чисел

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Цю роботу присвячено деякому підкласу недиференційовних функцій. Аргументи і значення функцій, що розглядаються, подано через s-ве або нега-s-ве зображення дійсних чисел. Техніка моделювання таких функцій є простішою в порівнянні з добре відомими техніками моделювання недиференційовних функцій. Іншими словами, значення цих функцій отримано з s-го або нега-s-го зображення аргументу за допомоги певної заміни цифр чи комбінацій цифр.

Описано інтегральні, фрактальні та інші властивості розглянутих функцій.

Ключові слова: ніде недиференційовні функції, s-адичні представлення, нега-s-адичні представлення, немонотонні функції, розмірність Гаусдорфа–Безіковича.