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Inverse Scattering on the Half Line for the Matrix Schrödinger Equation

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Dedicated to Professor V.A. Marchenko
for his 95th birthday

The matrix Schrödinger equation is considered on the half line with the general selfadjoint boundary condition at the origin described by two boundary matrices satisfying certain appropriate conditions. It is assumed that the matrix potential is integrable, is selfadjoint, and has a finite first moment. The corresponding scattering data set is constructed, and such scattering data sets are characterized by providing a set of necessary and sufficient conditions assuring the existence and uniqueness of the one-to-one correspondence between the scattering data set and the input data set containing the potential and boundary matrices. The work presented here provides a generalization of the classic result by Agranovich and Marchenko from the Dirichlet boundary condition to the general selfadjoint boundary condition.

Key words: matrix Schrödinger equation, selfadjoint boundary condition, Marchenko method, matrix Marchenko method, Jost matrix, scattering matrix, inverse scattering, characterization.

Mathematical Subject Classification 2010: 34L25, 34L40, 81U05.

1. Introduction

Our aim in this paper is to describe the direct and inverse scattering problems for the half-line matrix Schrödinger operator with a selfadjoint boundary condition. In the direct problem we are given an input data set **D** consisting of an $n \times n$ matrix-valued potential $V(x)$ and a selfadjoint boundary condition at $x = 0$, and our goal is to determine the corresponding scattering data set **S** consisting of the scattering matrix $S(k)$ and the bound-state data. In the inverse problem, we are given a scattering data set **S**, and our goal is to determine the corresponding input data set **D**. We would like to have a one-to-one correspondence between an input data set **D** and a scattering data set **S** so that both the direct and inverse problems are well posed. Thus, some restrictions are needed on **D** and **S** for a one-to-one correspondence.

Since the scattering and inverse scattering problems in the scalar case, i.e. when $n = 1$, are well understood, it is desirable that the analysis in the matrix

case reduces to the scalar case when $n = 1$. However, as elaborated in Section 8, the current formulation of the scattering and inverse scattering problems in the scalar case presents a problem. As a consequence, it becomes impossible to have a one-to-one correspondence between an input data set **D** and a scattering data set **S**, unless the Dirichlet and non-Dirichlet boundary conditions are analyzed separately and they are not mixed with each other. Although not ideal, this could perhaps be done in the scalar case because a given boundary condition in the scalar case is either Dirichlet or non-Dirichlet. On the other hand, in the matrix case with $n \geq 2$, a given boundary condition may partly be Dirichlet and partly non-Dirichlet, and this may be as a result of constraints in a physical problem. It turns out that the proper way to deal with the issue is to modify the definition of the scattering matrix in such a way that it is defined the same way regardless of the boundary condition, i.e. one should avoid defining the scattering matrix in one way in the Dirichlet case and in another way in the non-Dirichlet case.

There are four aspects related to the direct and inverse problems. These are the existence, uniqueness, construction, and characterization. In the existence aspect in the direct problem, given **D** in a specified class we determine whether a corresponding **S** exists in some specific class. The uniqueness aspect is concerned with whether there exists a unique **S** corresponding to a given **D**, or two or more distinct sets **S** may correspond to the same **D**. The construction deals with the recovery of **S** from **D**. In the inverse problem the existence problem deals with the existence of some **D** corresponding to a given **S** belonging to a particular class. The uniqueness deals with the question whether **D** corresponding to a given **S** is unique, and the construction consists of the recovery of **D** from **S**. After the existence and uniqueness aspects in the direct and inverse problems are settled, one then turns the attention to the characterization problem, which consists of the identification of the class to which **D** belongs and the identification of the class to which **S** belongs so that there is a one-to-one correspondence between **D** and **S** in the respective classes. One also needs to ensure that the scattering data set **S** uniquely constructed from a given **D** in the direct problem in turn uniquely constructs the same **D** in the inverse problem.

A viable characterization in the literature for the matrix Schrödinger operator on the half line can be found in the seminal work by Agranovich and Marchenko [1]. However, the analysis in [1] is restricted to the Dirichlet boundary condition, and hence our study can be viewed as a generalization of the characterization in [1]. A characterization for the case of the general selfadjoint boundary condition is recently provided [9, 10] by the authors of this paper, and in the current paper we present a summary of some of the results in [9, 10]. For brevity, we do not include any proofs because such proofs are already available in [9, 10].

We present the existence, uniqueness, reconstruction, and characterization issues related to the relevant direct and inverse problems under the assumption that **D** belongs to the *Faddeev class* and that **S** belongs to the *Marchenko class*. The Faddeev class consists of input data sets **D** as in (2.1), where the potential $V(x)$ and the boundary matrices A and B are as specified in Definition 2.1.

The Marchenko class consists of scattering data sets **S** as in (3.12), where the scattering matrix $S(k)$ and the bound-state data $\{\kappa_j, M_j\}_{j=1}^N$ are as specified in Definition 4.1.

Let us mention the relevant references [14–16], where the direct and inverse problems for (2.2) are formally studied with the general selfadjoint boundary condition, not as in (2.5)–(2.7) but in a form equivalent to (2.5)–(2.7). However, the study in [14–16] lacks the large-k analysis beyond the leading term and also lacks the small-k analysis of the scattering data, which are both essential for the analysis of the relevant inverse problem. Thus, our study can also be considered as a complement to the work by Harmer [14–16]. In our paper, which is essentially a brief summary of [9, 10], we rely on results from previous work [1, 4, 5, 8, 22–24], in particular [1, 4, 8, 22].

Our paper is organized as follows. In Section 2 we introduce the matrix Schrödinger equation on the half line, describe the general selfadjoint boundary condition in terms of two constant matrices A and B. We then describe the Faddeev class of input data sets **D** consisting of the matrix potential $V(x)$ and the boundary matrices A and B. In Section 3 we describe the solution to the direct problem, which uses an input data set **D** in the Faddeev class. We outline the construction of various quantities such as the Jost solution, the physical solution, the regular solution, the Jost matrix, the scattering matrix, and the bound-state data. In Section 4 we introduce the Marchenko class of scattering data sets. We present the solution to the inverse problem by starting with a scattering data set **S** in the Marchenko class, and we describe the construction of the potential and the boundary matrices. In Section 5 we provide a characterization of the scattering data by showing that there is a one-to-one correspondence between the Faddeev class of input data sets **D** and the Marchenko class of scattering data sets **S**. In Section 6 we provide an equivalent description of the Marchenko class, and we provide an alternate characterization of the scattering data with the help of Levinson's theorem. In Section 7 we provide yet another description of the Marchenko class based on an approach utilizing the so-called generalized Fourier map. Finally, in Section 8 we contrast our definition of the Jost matrix and the scattering matrix with those definitions in the previous literature. We indicate the similarities and differences occurring when the boundary condition used is Dirichlet or non-Dirichlet. We elaborate on the resulting nonuniqueness issue if the scattering matrix is defined differently when the Dirichlet boundary condition is used, as commonly done in the previous literature.

2. The matrix Schrödinger equation

In this section we introduce the matrix Schrödinger equation (2.2), the matrix potential $V(x)$, and the boundary matrices A and B used to describe the general selfadjoint boundary condition. We also indicate that the boundary matrices A and B can be uniquely specified modulo a postmultiplication by an invertible matrix. Our input data set **D** is defined as

$$\mathbf{D} := \{V, A, B\}. \tag{2.1}$$

Consider the matrix Schrödinger equation on the half line

$$-\psi'' + V(x)\,\psi = k^2\,\psi, \qquad x \in \mathbf{R}^+, \tag{2.2}$$

where $\mathbf{R}^+ := (0, +\infty)$, the prime denotes the derivative with respect to the spatial coordinate x, k^2 is the complex-valued spectral parameter, the potential $V(x)$ is an $n \times n$ selfadjoint matrix-valued function of x and belongs to class $L_1^1(\mathbf{R}^+)$, and n is any positive integer. We assume that the value of n is fixed and is known. The selfadjointness of $V(x)$ is expressed as

$$V(x) = V(x)^\dagger, \qquad x \in \mathbf{R}^+, \tag{2.3}$$

where the dagger denotes the matrix adjoint (complex conjugate and matrix transpose). We equivalently say Hermitian to describe a selfadjoint matrix. We remark that, unless we are in the scalar case, i.e. unless $n = 1$, the potential is not necessarily real valued. The condition $V \in L_1^1(\mathbf{R}^+)$ means that each entry of the matrix $V(x)$ is Lebesgue measurable on $\mathbf{R}^+$ and

$$\int_0^\infty dx\,(1+x)\,|V(x)| < +\infty, \tag{2.4}$$

where $|V(x)|$ denotes the matrix operator norm. Clearly, a matrix-valued function belongs to $L_1^1(\mathbf{R}^+)$ if and only if each entry of that matrix belongs to $L_1^1(\mathbf{R}^+)$.

The wavefunction $\psi(k, x)$ appearing in (2.2) may be either an $n \times n$ matrix-valued function or it may be a column vector with n components. We use $\mathbf{C}$ for the complex plane, $\mathbf{R}$ for the real line $(-\infty, +\infty)$, $\mathbf{R}^-$ for the left-half line $(-\infty, 0)$, $\mathbf{C}^+$ for the open upper-half complex plane, $\overline{\mathbf{C}^+}$ for $\mathbf{C}^+ \cup \mathbf{R}$, $\mathbf{C}^-$ for the open lower-half complex plane, and $\overline{\mathbf{C}^-}$ for $\mathbf{C}^- \cup \mathbf{R}$.

We are interested in studying (2.2) with an $n \times n$ selfadjoint matrix potential $V(x)$ in $L_1^1(\mathbf{R}^+)$ under the general selfadjoint boundary condition at $x = 0$. There are various equivalent formulations [4,8,14–18] of the general selfadjoint boundary condition at $x = 0$, and we find it convenient to state it [4, 8] in terms of two constant $n \times n$ matrices A and B as

$$-B^\dagger\,\psi(0) + A^\dagger\,\psi'(0) = 0, \tag{2.5}$$

where A and B satisfy

$$-B^\dagger\,A + A^\dagger\,B = 0, \tag{2.6}$$

$$A^\dagger\,A + B^\dagger\,B > 0. \tag{2.7}$$

The condition in (2.7) means that the $n \times n$ matrix $(A^\dagger A + B^\dagger B)$ is positive, which is also called positive definite. One can easily verify that (2.5)–(2.7) remain invariant if the boundary matrices A and B are replaced with AT and BT, respectively, where T is an arbitrary $n \times n$ invertible matrix. We express this fact by saying that the selfadjoint boundary condition (2.5) is uniquely determined by the matrix pair (A, B) modulo an invertible matrix T, and we equivalently state that (2.5) is equivalent to the knowledge of (A, B) modulo T. We remark

that the positivity condition (2.7) is equivalent to having the rank of the $2n \times n$ matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ equal to n.

In our analysis of the direct problem related to (2.2) and (2.5), we assume that our input data set **D** belongs to the Faddeev class defined below.

Definition 2.1. The input data set **D** given in (2.1) is said to belong to the Faddeev class if the potential $V(x)$ satisfies (2.3) and (2.4) and the boundary matrices A and B appearing in (2.5) satisfy (2.6) and (2.7). In other words, **D** belongs to the Faddeev class if the $n \times n$ matrix-valued potential $V(x)$ appearing in (2.2) is Hermitian and belongs to class $L_1^1(\mathbf{R}^+)$ and the constant $n \times n$ matrices A and B appearing in (2.5) satisfy (2.6) and (2.7).

It is possible to formulate the general selfadjoint boundary condition by using a unique $n \times n$ constant matrix instead of using the pair of matrices A and B appearing in (2.5)–(2.7). For example, in [15] a unitary $n \times n$ matrix U is used to describe the selfadjoint boundary condition as

$$\frac{i}{2}\left(U^\dagger - I\right)\psi(0) + \frac{1}{2}\left(U^\dagger + I\right)\psi'(0) = 0,$$

where I is the $n \times n$ identity matrix. Without loss of any generality, one could also use [4] a diagonal representation of the selfadjoint boundary condition by choosing the matrices A and B as

$$A = \operatorname{diag}\{-\sin\theta_1, \ldots, -\sin\theta_n\}, \quad B = \operatorname{diag}\{\cos\theta_1, \ldots, \cos\theta_n\}, \tag{2.8}$$

where the θ_j are some real constants in the interval $(0, \pi]$. In fact, through the representation (2.8), one can directly identify [5] the three integers n_D, n_N, and n_M, where n_D is the number of θ_j-values equal to π, n_N is the number of θ_j-values equal to $\pi/2$, and n_M is the number of θ_j-values in the union $(0, \pi/2) \cup (\pi/2, \pi)$. One can informally call n_D the number of Dirichlet boundary conditions, n_N the number of Neumann boundary conditions, and n_M the number of mixed boundary conditions.

We find it more convenient to write the general selfadjoint boundary condition in terms of the two constant $n \times n$ matrices A and B, with the understanding that A and B are unique up to a postmultiplication by an invertible matrix T. For example, the so-called Kirchhoff boundary condition is easier to recognize if expressed in terms of A and B, rather than written in terms of a single unique $n \times n$ constant matrix.

3. The solution to the direct problem

In this section we summarize the solution to the direct scattering problem associated with (2.2) and (2.5) when the related input data set **D** given in (2.1) belongs to the Faddeev class. In other words, we start with an $n \times n$ Hermitian potential $V(x)$ belonging to $L_1^1(\mathbf{R}^+)$ and a pair of constant boundary matrices A and B satisfying (2.6) and (2.7), and we construct the relevant quantities

leading to the scattering data set $\mathbf{S}$. The unique construction of the scattering data set $\mathbf{S}$ also enables us to determine the basic properties of $\mathbf{S}$. The steps of the construction are given below:

(a) When our input data set $\mathbf{D}$ belongs to the Faddeev class, regardless of the boundary matrices A and B, the matrix Schrödinger equation (2.2) has an $n \times n$ matrix-valued solution, usually called the Jost solution and denoted by $f(k,x)$, satisfying the asymptotic condition

$$f(k,x) = e^{ikx}\,[I + o(1)], \qquad x \to +\infty. \tag{3.1}$$

The solution $f(k,x)$ is uniquely determined by the potential $V(x)$. For each fixed $x \in [0,+\infty)$, the Jost solution $f(k,x)$ has an extension from $k \in \mathbf{R}$ to $k \in \overline{\mathbf{C}^+}$, and such an extension is continuous in $k \in \overline{\mathbf{C}^+}$ and analytic in $k \in \mathbf{C}^+$ and has the asymptotic behavior

$$e^{-ikx} f(k,x) = I + o(1), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}.$$

(b) In terms of the boundary matrices A and B in $\mathbf{D}$ and the Jost solution $f(k,x)$ obtained as in (a), we construct the Jost matrix $J(k)$ as

$$J(k) := f(-k^*,0)^\dagger\, B - f'(-k^*,0)^\dagger\, A, \qquad k \in \mathbf{R}, \tag{3.2}$$

where the asterisk denotes complex conjugation. We remark that $J(k)$ is an $n \times n$ matrix-valued function of k. The redundant appearance of k^* instead of k in (3.2) when $k \in \mathbf{R}$ is useful in extending the Jost matrix analytically from $k \in \mathbf{R}$ to $k \in \overline{\mathbf{C}^+}$. We recall that the boundary matrices A and B can be postmultiplied by any invertible matrix T without affecting (2.5)–(2.7) and hence the definition given in (3.2) yields the Jost matrix $J(k)$, which is unique up to a postmultiplication by T.

(c) In terms of the Jost matrix $J(k)$, obtained from $\mathbf{D}$ as indicated in (3.2), we construct the scattering matrix $S(k)$ as

$$S(k) := -J(-k)\,J(k)^{-1}, \qquad k \in \mathbf{R}. \tag{3.3}$$

We remark that $S(k)$ is an $n \times n$ matrix-valued function of k. Even though the Jost matrix in (3.2) is uniquely determined up to a postmultiplication by an invertible matrix T, from (3.3) we see that the scattering matrix $S(k)$ is uniquely determined irrespective of T.

(d) In terms of the Jost solution $f(k,x)$ obtained in (a) and the scattering matrix $S(k)$ obtained in (c), we construct the so-called physical solution to (2.2). The physical solution, denoted by $\Psi(k,x)$, is constructed as

$$\Psi(k,x) := f(-k,x) + f(k,x)\,S(k), \qquad k \in \mathbf{R}. \tag{3.4}$$

We remark that $\Psi(k,x)$ is an $n \times n$ matrix-valued function of k and x. The physical solution, as the name implies, has the physical interpretation of a

scattering solution; namely, the initial $n \times n$ matrix-valued plane wave $e^{-ikx}I$ sent from $x = +\infty$ onto the potential yields the $n \times n$ matrix-valued scattered wave $S(k)\, e^{ikx}$ at $x = +\infty$ with the amplitude $S(k)$. This interpretation is seen by using (3.1) in (3.4), i.e. for each fixed $k \in \mathbf{R}$, we get

$$\Psi(k,x) = e^{-ikx} + S(k)\, e^{ikx} + o(1), \qquad x \to +\infty.$$

We also remark that each column of the physical solution satisfies the boundary condition (2.5), and hence the physical solution itself satisfies (2.5) and we have

$$-B^{\dagger}\, \Psi(k,0) + A^{\dagger}\, \Psi'(k,0) = 0. \tag{3.5}$$

Even though the boundary matrices A and B appearing in (2.5)–(2.7) can be modified by a postmultiplication by an invertible matrix T, the definition given in (3.4) uniquely determines the physical solution irrespective of T. In the definition of the physical solution given in (3.4), one could multiply the right-hand side of (3.4) by a scalar function of k without affecting the physical interpretation of a physical solution. Nevertheless, we prefer to use (3.4) to define the physical solution in a unique manner.

(e) Instead of constructing the physical solution via (3.4), one can alternatively construct it in an equivalent way as follows. When our input data set $\mathbf{D}$ belongs to the Faddeev class, there exists [4] an $n \times n$ matrix-valued solution to (2.1), called the regular solution and denoted by $\varphi(k,x)$, satisfying the initial conditions

$$\varphi(k,0) = A, \quad \varphi'(k,0) = B.$$

The solution $\varphi(k,x)$ is uniquely determined by the input data set $\mathbf{D}$ given in (2.1). We remark that $\varphi(k,x)$ depends on the choice of A and B. The solution $\varphi(k,x)$ is known as the regular solution because it is entire in k for each fixed $x \in [0,+\infty)$. In terms of the regular solution $\varphi(k,x)$ and the Jost matrix $J(k)$ appearing in (3.2) we can introduce the physical solution as

$$\Psi(k,x) = -2ik\, \varphi(k,x)\, J(k)^{-1}. \tag{3.6}$$

One can show that the expressions given in (3.4) and (3.6) are equivalent, and this can be shown by using the relationship given in (3.5) of [4], i.e.

$$\varphi(k,x) = \frac{1}{2ik}\, f(k,x)\, J(-k) - \frac{1}{2ik}\, f(-k,x)\, J(k), \tag{3.7}$$

where we recall that $f(k,x)$ is the Jost solution appearing in (3.1).

(f) When the input data set $\mathbf{D}$ belongs to the Faddeev class, the Jost matrix $J(k)$ constructed as in (3.2) has an analytic extension from $k \in \mathbf{R}$ to $k \in \mathbf{C}^+$ and its determinant $\det[J(k)]$ is nonzero in $\mathbf{C}^+$ except perhaps at a finite number of k-values on the positive imaginary axis. Let us use N to denote the number of distinct zeros of $\det[J(k)]$ in $\mathbf{C}^+$ without counting multiplicities of those zeros, by realizing that the integer N may be zero for some input data sets $\mathbf{D}$.

Let us use N distinct positive numbers κ_j so that the zeros of $\det[J(k)]$ occur at $k = i\kappa_j$ and use m_j to denote the multiplicity of the zero of $\det[J(k)]$ at $k = i\kappa_j$. Thus, the nonnegative integer N, the set of distinct positive values $\{\kappa_j\}_{j=1}^N$, and the set of positive integers $\{m_j\}_{j=1}^N$ are all uniquely determined by the input data set $\mathbf{D}$. Each m_j satisfies $1 \leq m_j \leq n$. It is appropriate to call N the number of bound states without counting the multiplicities. The nonnegative integer $\mathcal{N}$ defined as

$$\mathcal{N} := \sum_{j=1}^{N} m_j, \tag{3.8}$$

can be referred to as the number of bound states including the multiplicities.

(g) Having determined the sets $\{\kappa_j\}_{j=1}^N$ and $\{m_j\}_{j=1}^N$, let us use $\mathrm{Ker}[J(i\kappa_j)^\dagger]$ to denote the kernel of the $n \times n$ constant matrix $J(i\kappa_j)^\dagger$. Next, we construct the orthogonal projection matrices P_j onto $\mathrm{Ker}[J(i\kappa_j)^\dagger]$ for $j = 1, \ldots, N$. The $n \times n$ matrices P_j are Hermitian and idempotent, i.e.

$$P_j^\dagger = P_j, \quad P_j^2 = P_j, \qquad j = 1, \ldots, N.$$

Furthermore, the rank of P_j is equal to m_j. We then construct the constant $n \times n$ matrices A_j, B_j, and M_j defined as

$$A_j := \int_0^\infty dx\, f(i\kappa_j, x)^\dagger\, f(i\kappa_j, x), \qquad j = 1, \ldots, N,$$

$$B_j := (I - P_j) + P_j\, A_j\, P_j, \qquad j = 1, \ldots, N, \tag{3.9}$$

$$M_j := B_j^{-1/2}\, P_j, \qquad j = 1, \ldots, N, \tag{3.10}$$

where $f(k, x)$ is the Jost solution constructed in (a). We remark that when $\mathbf{D}$ belongs to the Faddeev class, the matrices B_j given in (3.9) are Hermitian and positive definite and hence the matrices $B_j^{-1/2}$ are well defined as positive definite matrices. Since each projection matrix P_j has rank m_j, it follows from (3.10) that each matrix M_j is Hermitian, nonnegative, and has rank m_j. The matrices M_j are usually called the bound-state normalization matrices.

(h) When the input data set $\mathbf{D}$ belongs to the Faddeev class, at each $k = i\kappa_j$ with $j = 1, \ldots, N$, the Schrödinger equation (2.2) has m_j linearly independent column-vector solutions, where each of those column-vector solutions is square integrable in $x \in \mathbf{R}^+$. It is possible to rearrange those m_j linearly independent column-vector solutions to form an $n \times n$ matrix $\Psi_j(x)$, in such a way that $\Psi_j(x)$ can be uniquely constructed as

$$\Psi_j(x) := f(i\kappa_j, x)\, M_j, \qquad j = 1, \ldots, N, \tag{3.11}$$

where M_j is the $n \times n$ normalization matrix defined in (3.10). We can refer to $\Psi_j(x)$ as the normalized bound-state matrix solution to (2.1) at $k = i\kappa_j$.

We remark that each $\Psi_j(x)$ satisfies the boundary condition (2.5) and has rank equal to m_j.

Having constructed all the relevant quantities starting with the input data set $\mathbf{D}$, we now define the scattering data set $\mathbf{S}$ as

$$\mathbf{S} := \{S, \{\kappa_j, M_j\}_{j=1}^N\}, \tag{3.12}$$

where S denotes the scattering matrix $S(k)$ for $k \in \mathbf{R}$ constructed as in (3.3), the N distinct positive constants κ_j are as described in (f), and the N Hermitian, nonnegative, rank-m_j matrices M_j are as in (3.10).

4. The solution to the inverse problem

In this section, given the scattering data set $\mathbf{S}$ in (3.12), our goal is to construct the input data set $\mathbf{D}$ given in (2.1), with the understanding that the potential $V(x)$ is uniquely constructed and that the boundary matrices A and B are uniquely constructed up to a postmultiplication by an invertible matrix. The construction is given when $\mathbf{S}$ belongs to the so-called Marchenko class. We first present the construction and provide the definition of the Marchenko class at the end of the construction procedure. Later in the section we show that the Marchenko class can also be described in various equivalent ways.

We summarize the steps in the construction of $\mathbf{D}$ from $\mathbf{S}$ as follows:

(a) From the large-k asymptotics of the scattering matrix $S(k)$, we determine the constant $n \times n$ matrix S_∞ via

$$S_\infty := \lim_{k \to \pm\infty} S(k), \tag{4.1}$$

and the constant $n \times n$ matrix G_1 via

$$S(k) = S_\infty + \frac{G_1}{ik} + o\left(\frac{1}{k}\right), \qquad k \to \pm\infty. \tag{4.2}$$

(b) Using $S(k)$ and S_∞, we uniquely construct the $n \times n$ matrix $F_s(y)$ via

$$F_s(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, [S(k) - S_\infty]\, e^{iky}, \qquad y \in \mathbf{R}. \tag{4.3}$$

(c) Using $F_s(y)$ constructed as in (4.3) and the bound-state data $\{\kappa_j, M_j\}_{j=1}^N$ appearing in $\mathbf{S}$, we construct the $n \times n$ matrix $F(y)$ via

$$F(y) := F_s(y) + \sum_{j=1}^{N} M_j^2\, e^{-\kappa_j y}, \qquad y \in \mathbf{R}^+. \tag{4.4}$$

Note that we have $F_s(y)$ for $y \in \mathbf{R}$, but we need $F(y)$ only for $y \in \mathbf{R}^+$.

(d) We use the matrix $F(y)$ given in (4.4) as input to the Marchenko integral equation

$$K(x,y) + F(x+y) + \int_x^\infty dz\, K(x,z)\, F(z+y) = 0, \qquad 0 \le x < y, \tag{4.5}$$

and uniquely solve (4.5) and obtain $K(x,y)$ for $0 \le x < y < +\infty$. We remark that $K(x,y)$ is continuous in the region $0 \le x < y < +\infty$. We note that $K(0,0)$, which is used to denote $K(0,0^+)$, is well defined as a constant $n \times n$ matrix.

(e) Having obtained $K(x,y)$ for $0 \le x < y < +\infty$ uniquely from $\mathbf{S}$ as described in (d), we construct the potential $V(x)$ via

$$V(x) = -2\,\frac{dK(x,x)}{dx}, \qquad x \in \mathbf{R}^+. \tag{4.6}$$

By $K(x,x)$ we mean $K(x,x^+)$. We remark that, in general, $V(x)$ constructed as in (4.6) may exists only a.e. and it may not be continuous in x.

(f) Having constructed the potential $V(x)$ from the scattering data set $\mathbf{S}$, we turn our attention to the construction of the boundary matrices A and B appearing in (2.1). We recall that we need to construct A and B uniquely, where the uniqueness is understood in the sense of being up to a postmultiplication by an arbitrary invertible $n \times n$ matrix T. Such a construction is carried out as follows. We use the already-constructed $n \times n$ constant matrices S_∞, G_1, and $K(0,0)$ as input in the linear, homogeneous algebraic system

$$\begin{cases} (I - S_\infty)\, A = 0, \\ (I + S_\infty)\, B = [G_1 - S_\infty\, K(0,0) - K(0,0)\, S_\infty]\, A, \end{cases} \tag{4.7}$$

and determine A and B as the general solution to (4.7). Such a general solution is equivalent to finding A and B satisfying (4.7) in such a way that the rank of the $2n \times n$ matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is equal to n.

Even though the steps outlined above complete the construction of the input data set $\mathbf{D}$ from the scattering data set $\mathbf{S}$, we can construct various auxiliary quantities relevant to the corresponding direct and inverse scattering problems as described below.

(g) Having constructed the solution $K(x,y)$ to the Marchenko integral equation (4.5), we obtain the Jost solution $f(k,x)$ via

$$f(k,x) = e^{ikx} I + \int_x^\infty dy\, K(x,y)\, e^{iky}. \tag{4.8}$$

(h) Having the Jost solution $f(k,x)$ and the scattering matrix $S(k)$ at hand, we construct the physical solution $\Psi(k,x)$ as in (3.4).

(i) Having the Jost solution $f(k,x)$ and the boundary matrices A and B at hand, we construct Jost matrix $J(k)$ as in (3.2). Note that the constructed A and B are unique up to a postmultiplication by an arbitrary invertible matrix T, and hence the constructed Jost matrix $J(k)$ is also unique up to a postmultiplication by T.

(j) Having the Jost solution $f(k,x)$ and the Jost matrix $J(k)$ at hand, we construct the regular solution $\varphi(k,x)$ as in (3.7). Since the constructed A and B as well as the constructed $J(k)$ are each unique up to a postmultiplication by an arbitrary invertible matrix T, the constructed regular solution $\varphi(k,x)$ is also unique up to a postmultiplication by T. For each particular choice of the pair (A,B), we have a particular choice of the regular solution.

(k) Having the Jost solution $f(k,x)$ and the bound-state data $\{\kappa_j, M_j\}_{j=1}^N$ appearing in $\mathbf{S}$, we construct the normalized bound-state matrix solutions $\Psi_j(x)$ as in (3.11).

Next we define the Marchenko class of scattering data sets $\mathbf{S}$. The importance of the Marchenko class is that there exists [9, 10] a one-to-one correspondence between the Faddeev class of input data sets $\mathbf{D}$ and the Marchenko class of scattering data sets $\mathbf{S}$.

Definition 4.1. Consider a scattering data set $\mathbf{S}$ as in (3.12), which consists of an $n\times n$ scattering matrix $S(k)$ for $k\in\mathbf{R}$, a set of N distinct positive constants κ_j, and a set of N constant $n\times n$ Hermitian and nonnegative matrices M_j with respective positive ranks m_j, where N is a nonnegative integer. In case $N=0$, it is understood that $\mathbf{S}$ consists only of $S(k)$ for $k\in\mathbf{R}$. We say that $\mathbf{S}$ belongs to the Marchenko class if $\mathbf{S}$ satisfies the following four conditions, listed below as **(1)**, **(2)**, **(3$_a$)**, **(4$_a$)**:

(1) The scattering matrix $S(k)$ satisfies

$$S(-k)=S(k)^\dagger=S(k)^{-1},\qquad k\in\mathbf{R}, \tag{4.9}$$

and there exist constant $n\times n$ matrices S_∞ and G_1 in such a way that (4.2) holds. Furthermore, the $n\times n$ matrix quantity $F_s(y)$ defined in (4.3) is bounded in $y\in\mathbf{R}$ and integrable in $y\in\mathbf{R}^+$.

(2) For the matrix $F_s(y)$ defined in (4.3), the derivative $F_s'(y)$ exists a.e. for $y\in\mathbf{R}^+$ and it satisfies

$$\int_0^\infty dy\,(1+y)\,|F_s'(y)|<+\infty, \tag{4.10}$$

where we recall that the norm in the integrand of (4.10) is the operator norm of a matrix.

(3$_a$) The physical solution $\Psi(k,x)$ satisfies the boundary condition (2.5), i.e. it satisfies (3.5). We clarify this property as follows: The scattering matrix

appearing in **S** yields a particular $n \times n$ matrix-valued solution $\Psi(k,x)$ to (2.2) known as the physical solution given in (3.4) and also yields a pair of matrices A and B (modulo an invertible matrix) satisfying (2.6) and (2.7). Our statement **(3$_a$)** is equivalent to saying that (2.5) is satisfied if we use in (2.5) the quantities $\Psi(k,x)$, A, and B constructed from **S**.

(4$_a$) The Marchenko equation (4.5) at $x = 0$ given by

$$K(0,y) + F(y) + \int_0^\infty dz\, K(0,z)\, F(z+y) = 0, \qquad y \in \mathbf{R}^+,$$

has a unique solution $K(0,y)$ in $L^1(\mathbf{R}^+)$. Here, $F(y)$ is the $n \times n$ matrix related to $F_s(y)$ as in (4.4).

Let us mention a slight drawback in the description of the Marchenko class given in Definition 4.1. The property **(3$_a$)** cannot be checked from the scattering data set **S** directly because it requires the construction of the corresponding boundary matrices A and B as well as the physical solution $\Psi(k,x)$. It is already known [9,10] that one can replace **(3$_a$)** by an equivalent pair of conditions, listed as **(III$_a$)** and **(V$_c$)** as indicated in the next theorem.

Theorem 4.2. *Consider a scattering data set* **S** *as in* (3.12)*, which consists of an $n \times n$ scattering matrix $S(k)$ for $k \in \mathbf{R}$, a set of N distinct positive constants κ_j, and a set of N constant $n \times n$ Hermitian and nonnegative matrices M_j with respective positive ranks m_j, where N is a nonnegative integer. In case $N = 0$, it is understood that* **S** *consists only of $S(k)$ for $k \in \mathbf{R}$ and that $\mathcal{N}$ appearing in* (3.8) *is zero. The scattering data set* **S** *belongs to the Marchenko class if and only if* **S** *satisfies the five conditions, three of which are listed as* (1)*,* (2)*, and* **(4$_a$)** *in Definition 4.1, and the two additional conditions* **(III$_a$)** *and* **(V$_c$)** *are given as:*

(III$_a$) *For the matrix-valued function $F_s(y)$ given in* (4.3)*, the derivative $F_s'(y)$ for $y \in \mathbf{R}^-$ can be written as a sum of two matrix-valued functions, one of which is integrable and the other is square integrable in $y \in \mathbf{R}^-$. Furthermore, the only solution $X(y)$, which is a row vector with n square-integrable components in $y \in \mathbf{R}^-$, to the linear homogeneous integral equation*

$$-X(y) + \int_{-\infty}^0 dz\, X(z)\, F_s(z+y) = 0, \qquad y \in \mathbf{R}^-,$$

is the trivial solution $X(y) \equiv 0$.

(V$_c$) *The linear homogeneous integral equation*

$$X(y) + \int_0^\infty dz\, X(z)\, F_s(z+y) = 0, \qquad y \in \mathbf{R}^+, \tag{4.11}$$

has precisely $\mathcal{N}$ linearly independent row-vector solutions for some nonnegative integer $\mathcal{N}$, with n components which are integrable in $y \in \mathbf{R}^+$. Here $F_s(y)$ is the matrix defined in (4.3) *and $\mathcal{N}$ is the nonnegative integer readily constructed from* **S** *as in* (3.8)*. If $\mathcal{N} = 0$, it is understood that the only solution in $L^1(\mathbf{R}^+)$ to* (4.11) *is the trivial solution $X(y) \equiv 0$.*

We remark that Theorem 4.2 is a special case of Theorem 4.5, but we still prefer to state it as a separate result. This is because Theorem 4.2 is closely related to the characterization result stated by Agranovich and Marchenko in the Dirichlet case on pp. 4–5 of their manuscript [1].

The next theorem shows that in the description of the Marchenko class specified in Definition 4.1, we can replace the condition **(3$_a$)** with another equivalent condition.

Theorem 4.3. *Consider a scattering data set* **S** *as in* (3.12)*, which consists of an* $n\times n$ *scattering matrix* $S(k)$ *for* $k\in\mathbf{R}$*, a set of* N *distinct positive constants* κ_j*, and a set of* N *constant* $n\times n$ *Hermitian and nonnegative matrices* M_j *with respective positive ranks* m_j*, where* N *is a nonnegative integer. In case* $N=0$*, it is understood that* **S** *consists only of* $S(k)$ *for* $k\in\mathbf{R}$*. The scattering data set* **S** *belongs to the Marchenko class if and only if* **S** *satisfies the four conditions, three of which are listed as* (**1**)*,* (**2**)*, and* **(4$_a$)** *in Definition 4.1 and one additional condition* **(3$_b$)** *replacing* **(3$_a$)***, which is given by*

(3$_b$) *The Jost matrix* $J(k)$ *satisfies*

$$J(-k)+S(k)\,J(k)=0,\qquad k\in\mathbf{R}. \tag{4.12}$$

We clarify this property as follows: The scattering matrix $S(k)$ *given in* **S** *yields a Jost matrix* $J(k)$ *constructed as in* (3.2)*, unique up to a postmultiplication by an invertible matrix. Using the scattering matrix* $S(k)$ *given in* **S** *and the Jost matrix constructed from* $S(k)$*, we find that* (4.12) *is satisfied.*

Let us use $\hat{L}^1(\mathbf{C}^+)$ to denote the Banach space of all complex-valued functions $\xi(k)$ that are analytic in $k\in\mathbf{C}^+$ in such a way that there exists a corresponding function $\eta(x)$ belonging to $L^1(\mathbf{R}^+)$ satisfying

$$\xi(k)=\int_0^\infty dx\,\eta(x)\,e^{ikx}.$$

We note that if $\xi(k)$ belongs to $\hat{L}^1(\mathbf{C}^+)$, then $\xi(k)$ is continuous in $k\in\mathbf{R}$ and it satisfies $\xi(k)=o(1)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. If $\xi(k)$ is vector valued or matrix valued instead of being scalar valued, then it belongs to $\hat{L}^1(\mathbf{C}^+)$ if and only if each entry of $\xi(k)$ belongs to $\hat{L}^1(\mathbf{C}^+)$.

We remark that the result of Theorem 4.3 is included in the next theorem presented. However, we have stated Theorem 4.3 separately in order to emphasize the importance of (**3$_b$**) of Theorem 4.3 and its connection to **(3$_a$)** of Definition 4.1.

Theorem 4.4. *Consider a scattering data set* **S** *as in* (3.12)*, which consists of an* $n\times n$ *scattering matrix* $S(k)$ *for* $k\in\mathbf{R}$*, a set of* N *distinct positive constants* κ_j*, and a set of* N *constant* $n\times n$ *Hermitian and nonnegative matrices* M_j *with respective positive ranks* m_j*, where* N *is a nonnegative integer. In case* $N=0$*, it is understood that* **S** *consists only of* $S(k)$ *for* $k\in\mathbf{R}$*. The scattering data set* **S** *belongs to the Marchenko class if and only if* **S** *satisfies the four conditions* (**1**)*,* (**2**)*,* (**3**)*, and* (**4**)*, where* (**3**) *can be either one of* **(3$_a$)** *and* (**3$_b$**)*; and* (**4**) *can be*

any one of $(\mathbf{4}_a)$, $(\mathbf{4}_b)$, $(\mathbf{4}_c)$, $(\mathbf{4}_d)$, $(\mathbf{4}_e)$. *Note that* $(\mathbf{1})$, $(\mathbf{2})$, $(\mathbf{3}_a)$, $(\mathbf{4}_a)$ *are listed in Definition 4.1;* $(\mathbf{3}_b)$ *is listed in Theorem 4.3; and the remaining conditions* $(\mathbf{4}_b)$, $(\mathbf{4}_c)$, $(\mathbf{4}_d)$, $(\mathbf{4}_e)$ *are listed below:*

$(\mathbf{4}_b)$ *The only solution in* $L^1(\mathbf{R}^+)$ *to the homogeneous Marchenko integral equation at* $x = 0$ *given by*

$$K(0,y) + \int_0^\infty dz\, K(0,z)\, F(z+y) = 0, \qquad y \in \mathbf{R}^+, \tag{4.13}$$

is the trivial solution $K(0,y) \equiv 0$. *Note that* (4.13) *is the homogeneous version at* $x = 0$ *of the Marchenko equation given by* (4.5). *We remark that* $F(y)$ *appearing in* (4.13) *is the quantity defined in* (4.4).

$(\mathbf{4}_c)$ *The only integrable solution* $X(y)$, *which is a row vector with* n *integrable components in* $y \in \mathbf{R}^+$, *to the linear homogeneous integral equation*

$$X(y) + \int_0^\infty dz\, X(z)\, F(z+y) = 0, \qquad y \in \mathbf{R}^+, \tag{4.14}$$

is the trivial solution $X(y) \equiv 0$. *Again, we recall that* $F(y)$ *is the quantity defined in* (4.4).

$(\mathbf{4}_d)$ *The only solution* $\hat{X}(k)$ *to the system*

$$\begin{cases} \hat{X}(i\kappa_j)\, M_j = 0\,, & j = 1, \dots, N, \\ \hat{X}(-k) + \hat{X}(k)\, S(k) = 0\,, & k \in \mathbf{R}, \end{cases} \tag{4.15}$$

where $\hat{X}(k)$ *is a row vector with* n *components belonging to the class* $\hat{L}^1(\mathbf{C}^+)$, *is the trivial solution* $\hat{X}(k) \equiv 0$.

$(\mathbf{4}_e)$ *The only solution* $h(k)$ *to the system*

$$\begin{cases} M_j\, h(i\kappa_j) = 0\,, & j = 1, \dots, N, \\ h(-k) + S(k)\, h(k) = 0\,, & k \in \mathbf{R}, \end{cases} \tag{4.16}$$

where $h(k)$ *is a column vector with* n *components belonging to the class* $\hat{L}^1(\mathbf{C}^+)$, *is the trivial solution* $h(k) \equiv 0$.

We use $\mathbf{H}^2(\mathbf{C}^\pm)$ to denote the Hardy space of all complex-valued functions $\xi(k)$ that are analytic in $k \in \mathbf{C}^\pm$ with a finite norm defined as

$$\|\xi\|_{\mathbf{H}^2(\mathbf{C}^\pm)} := \sup_{\rho>0} \left[\int_{-\infty}^{\infty} d\alpha\, |\xi(\alpha \pm i\rho)|^2 \right]^{1/2}.$$

Thus, $\xi(k)$ is square integrable along all lines in $\mathbf{C}^\pm$ that are parallel to the real axis. The value of $\xi(k)$ for $k \in \mathbf{R}$ is defined to be the non-tangential limit of $\xi(k \pm i\rho)$ as $\rho \to 0^+$. Such a non-tangential limit exists a.e. in $k \in \mathbf{R}$ and belongs

to $L^2(\mathbf{R})$. It is known that $\xi(k)$ belongs to $\mathbf{H}^2(\mathbf{C}^+)$ if and only if there exists a corresponding function $\eta(x)$ belonging to $L^2(\mathbf{R}^+)$ in such a way that

$$\xi(k) = \int_0^\infty dx\, \eta(x)\, e^{ikx}.$$

Similarly, $\xi(k)$ belongs to $\mathbf{H}^2(\mathbf{C}^-)$ if and only if there exists a corresponding function $\eta(x)$ belonging to $L^2(\mathbf{R}^-)$ in such a way that

$$\xi(k) = \int_{-\infty}^0 dx\, \eta(x)\, e^{ikx}.$$

If $\xi(k)$ is vector valued or matrix valued instead of being scalar valued, then it belongs to $\mathbf{H}^2(\mathbf{C}^\pm)$ if and only if each entry of $\xi(k)$ belongs to $\mathbf{H}^2(\mathbf{C}^\pm)$.

The next theorem shows that in the equivalent description of the Marchenko class specified in Theorem 4.2, we can replace the condition $(\mathbf{III}_a)$ with one of two other equivalent conditions and we can also replace the condition $(\mathbf{V}_c)$ with any one of various other equivalent conditions.

Theorem 4.5. *Consider a scattering data set* $\mathbf{S}$ *as in* (3.12)*, which consists of an* $n \times n$ *scattering matrix* $S(k)$ *for* $k \in \mathbf{R}$*, a set of* N *distinct positive constants* κ_j*, and a set of* N *constant* $n \times n$ *Hermitian and nonnegative matrices* M_j *with respective positive ranks* m_j*, where* N *is a nonnegative integer. In case* $N = 0$*, it is understood that* $\mathbf{S}$ *consists only of* $S(k)$ *for* $k \in \mathbf{R}$ *and that* $\mathcal{N}$ *appearing in* (3.8) *is zero. The scattering data set* $\mathbf{S}$ *belongs to the Marchenko class if and only if* $\mathbf{S}$ *satisfies the five conditions* (1)*,* (2)*,* (**III**)*,* (**4**)*, and* (**V**)*, where* (**III**) *represents any one of the three conditions* $(\mathbf{III}_a)$*,* $(\mathbf{III}_b)$*,* $(\mathbf{III}_c)$*;* (**4**) *represents any one of the five conditions* $(\mathbf{4}_a)$*,* $(\mathbf{4}_b)$*,* $(\mathbf{4}_c)$*,* $(\mathbf{4}_d)$*,* $(\mathbf{4}_e)$*; and* (**V**) *represents any one of the eight conditions* $(\mathbf{V}_a)$*,* $(\mathbf{V}_b)$*,* $(\mathbf{V}_c)$*,* $(\mathbf{V}_d)$*,* $(\mathbf{V}_e)$*,* $(\mathbf{V}_f)$*,* $(\mathbf{V}_g)$*,* $(\mathbf{V}_h)$*. We remark that* (1)*,* (2)*, and* $(\mathbf{4}_a)$ *are listed in Definition 4.1;* $(\mathbf{III}_a)$ *and* $(\mathbf{V}_c)$ *are listed in Theorem 4.2;* $(\mathbf{4}_b)$*,* $(\mathbf{4}_c)$*,* $(\mathbf{4}_d)$*,* $(\mathbf{4}_e)$ *are listed in Theorem 4.4; and the remaining conditions are listed below:*

(III$_b$) *For the matrix-valued function* $F_s(y)$ *given in* (4.3)*, the derivative* $F_s'(y)$ *for* $y \in \mathbf{R}^-$ *can be written as a sum of two matrix-valued functions, one of which is integrable and the other is square integrable in* $y \in \mathbf{R}^-$*. Furthermore, the only solution* $\hat{X}(k)$ *to the homogeneous Riemann–Hilbert problem*

$$-\hat{X}(-k) + \hat{X}(k)\, S(k) = 0, \qquad k \in \mathbf{R},$$

where $\hat{X}(k)$ *is a row vector with* n *components belonging to the class* $\mathbf{H}^2(\mathbf{C}^-)$*, is the trivial solution* $\hat{X}(k) \equiv 0$*.*

(III$_c$) *For the matrix-valued function* $F_s(y)$ *given in* (4.3)*, the derivative* $F_s'(y)$ *for* $y \in \mathbf{R}^-$ *can be written as a sum of two matrix-valued functions, one of which is integrable and the other is square integrable in* $y \in \mathbf{R}^-$*. Furthermore, the only solution* $h(k)$ *to the homogeneous Riemann–Hilbert problem*

$$-h(-k) + S(k)\, h(k) = 0, \qquad k \in \mathbf{R}, \tag{4.17}$$

where $h(k)$ is a column vector with n components belonging to the class $\mathbf{H}^2(\mathbf{C}^-)$, is the trivial solution $h(k) \equiv 0$.

($\mathbf{V}_a$) *Each of the N normalized bound-state matrix solutions $\Psi_j(x)$ constructed as in (3.11) satisfies the boundary condition (2.5), i.e.*

$$-B^\dagger \Psi_j(0) + A^\dagger \Psi_j'(0) = 0, \qquad j = 1, \ldots, N. \tag{4.18}$$

We clarify this statement as follows. The scattering matrix $S(k)$ and the bound-state data $\{\kappa_j, M_j\}_{j=1}^N$ given in $\mathbf{S}$ yield $n \times n$ matrices $\Psi_j(x)$ as in (3.11), where each $\Psi_j(x)$ is a solution to (2.2) at $k = i\kappa_j$. As stated in ($\mathbf{3}_a$) of Definition 4.1, the scattering matrix given in $\mathbf{S}$ yields a pair of matrices A and B (modulo an invertible matrix) satisfying (2.6) and (2.7). The statement ($\mathbf{V}_a$) is equivalent to saying that (2.5) is satisfied if we use in (2.5) the quantities $\Psi_j(x)$, A, and B constructed from the quantities appearing in $\mathbf{S}$. If $N = 0$, then the condition (4.18) is redundant.

($\mathbf{V}_b$) *The normalization matrices M_j appearing in $\mathbf{S}$ satisfy*

$$J(i\kappa_j)^\dagger M_j = 0, \qquad j = 1, \ldots, N. \tag{4.19}$$

We clarify this condition as follows. As indicated in ($\mathbf{3}_b$) of Theorem 4.3, the scattering matrix $S(k)$ given in $\mathbf{S}$ yields a Jost matrix $J(k)$. Using in (4.19) the matrix M_j given in $\mathbf{S}$ and the Jost matrix constructed from $S(k)$, at each κ_j-value listed in $\mathbf{S}$ the matrix equation (4.19) holds. If $N = 0$, then the condition (4.19) is redundant.

($\mathbf{V}_d$) *The homogeneous Riemann–Hilbert problem given by*

$$\hat{X}(-k) + \hat{X}(k)\, S(k) = 0, \qquad k \in \mathbf{R}, \tag{4.20}$$

has precisely $\mathcal{N}$ linearly independent row-vector solutions with n components in $\hat{L}^1(\mathbf{C}^+)$. Here, $\mathcal{N}$ is the nonnegative integer given in (3.8). If $\mathcal{N} = 0$, it is understood that the only solution in $\hat{L}^1(\mathbf{C}^+)$ to (4.20) is the trivial solution $\hat{X}(k) \equiv 0$.

($\mathbf{V}_e$) *The homogeneous Riemann–Hilbert problem given by*

$$h(-k) + S(k)\, h(k) = 0, \qquad k \in \mathbf{R}, \tag{4.21}$$

has precisely $\mathcal{N}$ linearly independent column-vector solutions with n components in $\hat{L}^1(\mathbf{C}^+)$, where $\mathcal{N}$ is the nonnegative integer given in (3.8). If $\mathcal{N} = 0$, it is understood that the only solution in $\hat{L}^1(\mathbf{C}^+)$ to (4.21) is the trivial solution $h(k) \equiv 0$.

($\mathbf{V}_f$) *The integral equation (4.11) has precisely $\mathcal{N}$ linearly independent row-vector solutions $X(y)$ with n components in $L^2(\mathbf{R}^+)$, where $\mathcal{N}$ is the nonnegative integer given in (3.8). If $\mathcal{N} = 0$, it is understood that the only solution in $L^2(\mathbf{R}^+)$ to (4.11) is the trivial solution $X(y) \equiv 0$. We remark that the matrix $F_s(y)$ appearing in the kernel of (4.11) is defined in (4.3).*

(V$_g$) *The homogeneous Riemann–Hilbert problem given in* (4.20) *has precisely* $\mathcal{N}$ *linearly independent row-vector solutions* $\hat{X}(k)$ *with* n *components in* $\mathbf{H}^2(\mathbf{C}^+)$. *Here,* $\mathcal{N}$ *is the nonnegative integer given in* (3.8). *If* $\mathcal{N} = 0$, *it is understood that the only solution in* $\mathbf{H}^2(\mathbf{C}^+)$ *to* (4.20) *is the trivial solution* $\hat{X}(k) \equiv 0$.

(V$_h$) *The homogeneous Riemann–Hilbert problem given in* (4.21) *has precisely* $\mathcal{N}$ *linearly independent row-vector solutions with* n *components in* $\mathbf{H}^2(\mathbf{C}^+)$. *Here,* $\mathcal{N}$ *is the nonnegative integer given in* (3.8). *If* $\mathcal{N} = 0$, *it is understood that the only solution in* $\mathbf{H}^2(\mathbf{C}^+)$ *to* (4.21) *is the trivial solution* $h(k) \equiv 0$.

5. The characterization of the scattering data

In this section we consider the characterization of the scattering data. In the next theorem we present one of our main characterization results. It shows that the four conditions given in Definition 4.1 for the Marchenko class form a characterization of the scattering data sets $\mathbf{S}$ so that there exists a one-to-one correspondence between a scattering data set in the Marchenko class and an input data set $\mathbf{D}$ in the Faddeev class specified in Definition 2.1. From Section 4 we know that the Marchenko class can be described in various equivalent ways, and hence it is possible to present the characterization in various different ways.

Theorem 5.1. *Consider a scattering data set* $\mathbf{S}$ *as in* (3.12), *which consists of an* $n \times n$ *scattering matrix* $S(k)$ *for* $k \in \mathbf{R}$, *a set of* N *distinct positive constants* κ_j, *and a set of* N *constant* $n \times n$ *Hermitian and nonnegative matrices* M_j *with respective positive ranks* m_j, *where* N *is a nonnegative integer. In case* $N = 0$, *it is understood that* $\mathbf{S}$ *consists only of* $S(k)$ *for* $k \in \mathbf{R}$ *and that* $\mathcal{N}$ *appearing in* (3.8) *is zero. Consider also an input data set* $\mathbf{D}$ *as in* (2.1) *consisting of an* $n \times n$ *matrix potential* $V(x)$ *satisfying* (2.3) *and* (2.4) *and a pair of constant* $n \times n$ *matrices* A *and* B *satisfying* (2.6) *and* (2.7). *Then, we have the following:*

(a) *For each input data set* $\mathbf{D}$ *in the Faddeev class specified in Definition 2.1, there exists and uniquely exists a scattering data set* $\mathbf{S}$ *in the Marchenko class specified in Definition 4.1.*

(b) *Conversely, for each* $\mathbf{S}$ *in the Marchenko class, there exists and uniquely exists an input data set* $\mathbf{D}$ *in the Faddeev class, where the boundary matrices* A *and* B *are uniquely determined up to a postmultiplication by an invertible* $n \times n$ *matrix* T.

(c) *Let* $\tilde{\mathbf{S}}$ *be the scattering data set corresponding to* $\mathbf{D}$ *given in the previous step* (b), *where* $\mathbf{D}$ *is constructed from the scattering data set* $\mathbf{S}$. *Then, we have* $\tilde{\mathbf{S}} = \mathbf{S}$, *i.e. the scattering data set constructed from* $\mathbf{D}$ *is equal to the scattering data set used to construct* $\mathbf{D}$.

(d) *The characterization outlined in the steps* (a)–(c) *given above can equivalently be stated as follows. A set* $\mathbf{S}$ *as in* (3.12) *is the scattering data set correspond-*

ing to an input data set **D** *in the Faddeev class if and only if* **S** *satisfies* (**1**), (**2**), ($\mathbf{3}_a$), *and* ($\mathbf{4}_a$) *stated in Definition 4.1.*

(e) *The characterization outlined in the steps* (a)–(c) *given above can equivalently be stated as follows. A set* **S** *as in* (3.12) *is the scattering data set corresponding to an input data set* **D** *in the Faddeev class if and only if* **S** *satisfies* (**1**), (**2**), ($\mathbf{4}_a$) *of Definition 4.1 and* ($\mathbf{III}_a$) *and* ($\mathbf{V}_c$) *of Theorem 4.2.*

(f) *The characterization outlined in the steps* (a)–(c) *given above can equivalently be stated as follows. A set* **S** *as in* (3.12) *is the scattering data set corresponding to an input data set* **D** *in the Faddeev class if and only if* **S** *satisfies* (**1**), (**2**), (**3**), *and* (**4**), *where* (**3**) *can be either one of* ($\mathbf{3}_a$) *and* ($\mathbf{3}_b$)*; and* (**4**) *can be any one of* ($\mathbf{4}_a$), ($\mathbf{4}_b$), ($\mathbf{4}_c$), ($\mathbf{4}_d$), ($\mathbf{4}_e$). *We recall that* (**1**), (**2**), ($\mathbf{3}_a$), ($\mathbf{4}_a$) *are listed in Definition 4.1;* ($\mathbf{3}_b$) *is listed in Theorem 4.3; and* ($\mathbf{4}_b$), ($\mathbf{4}_c$), ($\mathbf{4}_d$), ($\mathbf{4}_e$) *are listed in Theorem 4.4.*

(g) *The characterization outlined in the steps* (a)–(c) *given above can equivalently be stated as follows. A set* **S** *as in* (3.12) *is the scattering data set corresponding to an input data set* **D** *in the Faddeev class if and only if* **S** *satisfies the five conditions* (**1**), (**2**), (**III**), (**4**), *and* (**V**), *where* (**III**) *can be any one of* ($\mathbf{III}_a$), ($\mathbf{III}_b$), ($\mathbf{III}_c$)*;* (**4**) *can be any one of* ($\mathbf{4}_a$), ($\mathbf{4}_b$), ($\mathbf{4}_c$), ($\mathbf{4}_d$), ($\mathbf{4}_e$)*; and* (**V**) *can be any one of* ($\mathbf{V}_a$), ($\mathbf{V}_b$), ($\mathbf{V}_c$), ($\mathbf{V}_d$), ($\mathbf{V}_e$), ($\mathbf{V}_f$), ($\mathbf{V}_g$), ($\mathbf{V}_h$). *We recall that* (**1**), (**2**), ($\mathbf{4}_a$) *are listed in Definition 4.1;* ($\mathbf{III}_a$) *and* ($\mathbf{V}_c$) *are listed in Theorem 4.2;* ($\mathbf{4}_b$), ($\mathbf{4}_c$), ($\mathbf{4}_d$), ($\mathbf{4}_e$) *are listed in Theorem 4.4; and* ($\mathbf{III}_b$), ($\mathbf{III}_c$), ($\mathbf{V}_a$), ($\mathbf{V}_b$), ($\mathbf{V}_d$), ($\mathbf{V}_e$), ($\mathbf{V}_f$), ($\mathbf{V}_g$), ($\mathbf{V}_h$) *are listed in Theorem 4.5.*

We have the following remarks on the results presented in Theorem 5.1. The characterization result stated in Theorem 5.1(e) follows from Theorem 4.2. The result stated in Theorem 5.1(f) is a consequence of Theorem 4.4. The result in Theorem 5.1(e) is a particular case of the result in Theorem 5.1(g), but we prefer to state it separately because it resembles the characterization result stated by Agranovich and Marchenko [1] in the Dirichlet case. Finally we remark that Theorem 5.1(g) is a direct consequence of Theorem 4.5.

6. An alternate characterization of the scattering data

It is possible to present an alternate characterization of the scattering data using Levinson's theorem. This characterization again establishes a one-to-one correspondence between the Faddeev class of input data sets **D** and the Marchenko class of scattering data sets **S**. Hence, such an alternate characterization can also be viewed as an alternate description of the Marchenko class of scattering data sets with the help of Levinson's theorem.

In general, the bound-state data $\{\kappa_j, M_j\}_{j=1}^N$ appearing in the scattering data set **S** of (3.12) and the scattering matrix $S(k)$ are independent, and they need to be specified separately. On the other hand, the determinant of the scattering matrix contains the information of the number of bound states including the multiplicities, which is the nonnegative integer $\mathcal{N}$ appearing in (3.8). The change

in the argument of the determinant of $S(k)$ as k changes from $k = 0^+$ to $k = +\infty$ in the k-interval $(0, +\infty)$ is related to the number of bound states including multiplicities. This general fact is usually known as Levinson's theorem.

When the input data set **D** belongs to the Faddeev class, we have [8] Levinson's theorem stated in the following.

Theorem 6.1. *Consider the matrix Schrödinger equation* (2.2) *with the self-adjoint boundary condition* (2.5). *Assume that the corresponding input data set* **D** *given in* (2.1) *belongs to the Faddeev class. Let* **S** *appearing in* (3.12) *be the scattering data set corresponding to* **D**. *Then, the number* $\mathcal{N}$ *of bound states including the multiplicities appearing in* (3.8) *is related to the argument of the determinant of the scattering matrix* $S(k)$ *as*

$$\arg\left[\det[S(0^+)]\right] - \arg\left[\det[S(+\infty)]\right] = \pi\left(2\mathcal{N} + \mu + n_D - n\right), \tag{6.1}$$

where μ *is the (algebraic and geometric) multiplicity of the eigenvalue* $+1$ *of the zero-energy scattering matrix* $S(0)$, n *is the positive integer appearing in the matrix size* $n \times n$ *of the scattering matrix* $S(k)$, *and* n_D *is the number of Dirichlet boundary conditions in the diagonal representation* (2.8) *of the boundary matrices* A *and* B. *We remark that* n_D *is the same as the nonzero integer which is equal to the multiplicity of the eigenvalue* -1 *of the constant* $n \times n$ *matrix* S_∞ *appearing in* (4.1).

In some cases, by using Levinson's theorem we may be able to quickly determine if a given scattering data set **S** does not belong to the Marchenko class. Using the scattering matrix $S(k)$, we readily know the positive integer n appearing in the matrix size $n \times n$ of the matrix $S(k)$. The zero-energy scattering matrix $S(0)$ has eigenvalues equal to either -1 or $+1$. Thus, we can identify μ as the multiplicity of the eigenvalue $+1$ of $S(0)$. From the large-k limit of $S(k)$ given in (4.1) we can easily construct the constant matrix S_∞ and we already know that S_∞ has eigenvalues equal to either -1 or $+1$. Thus, we can identify n_D as the multiplicity of the eigenvalue -1 of S_∞. Then, from the scattering matrix $S(k)$ we can evaluate the change in the argument of $\det[S(k)]$ given on the left-hand side of (6.1). We can then use (6.1) to determine the value of $\mathcal{N}$ predicted by Levinson's theorem. If that value of $\mathcal{N}$ evaluated from (6.1) does not turn out to be a nonnegative integer, we know that the corresponding **S** does not belong to the Marchenko class.

The next theorem shows that we can obtain an equivalent description of the Marchenko class specified in Definition 4.1, by replacing **(3$_a$)** by a set of two conditions one of which is related to Levinson's theorem, and at the same time by replacing **(4$_a$)** by any one of three other conditions.

Theorem 6.2. *Consider a scattering data set* **S** *as in* (3.12), *which consists of an* $n \times n$ *scattering matrix* $S(k)$ *for* $k \in \mathbf{R}$, *a set of* N *distinct positive constants* κ_j, *and a set of* N *constant* $n \times n$ *Hermitian and nonnegative matrices* M_j *with respective positive ranks* m_j, *where* N *is a nonnegative integer. In case* $N = 0$, *it is understood that* **S** *consists only of* $S(k)$ *for* $k \in \mathbf{R}$ *and that* $\mathcal{N}$ *appearing*

in (3.8) *is zero. The scattering data set* $\mathbf{S}$ *belongs to the Marchenko class if and only if* $\mathbf{S}$ *satisfies the five conditions, two of which are listed as* (**1**) *and* (**2**) *in Definition 4.1, the third and the fourth are the respective conditions listed as* (**L**) *and* (**5**°) *below, and the fifth is any one of the three conditions listed as* ($\mathbf{4}_c^\circ$), ($\mathbf{4}_d^\circ$), *and* ($\mathbf{4}_e^\circ$) *below:*

(**L**) *The scattering matrix* $S(k)$ *appearing in* $\mathbf{S}$ *is continuous for* $k \in \mathbf{R}$, *and the equality* (6.1) *of Levinson's theorem is satisfied with* μ, n_D, *and* $\mathcal{N}$ *coming from* $\mathbf{S}$. *Here,* μ *is the (algebraic and geometric) multiplicity of the eigenvalue* $+1$ *of the zero-energy scattering matrix* $S(0)$, n_D *is the (algebraic and geometric) multiplicity of the eigenvalue* -1 *of the Hermitian matrix* S_∞ *appearing in* (4.1), *and* $\mathcal{N}$ *is the nonnegative integer in* (3.8) *which is equal to the sum of the ranks* m_j *of the matrices* M_j *appearing in* $\mathbf{S}$.

($\mathbf{4}_c^\circ$) *The only square-integrable solution* $X(y)$, *which is a row vector with* n *square-integrable components in* $y \in \mathbf{R}^+$, *to the linear homogeneous integral equation*

$$X(y) + \int_0^\infty dz\, X(z)\, F(z+y) = 0, \qquad y \in \mathbf{R}^+, \tag{6.2}$$

is the trivial solution $X(y) \equiv 0$. *Here,* $F(y)$ *is the quantity defined in* (4.4).

($\mathbf{4}_d^\circ$) *The only solution* $\hat{X}(k)$ *to the system*

$$\begin{cases} \hat{X}(i\kappa_j)\, M_j = 0\,, & j = 1, \dots, N, \\ \hat{X}(-k) + \hat{X}(k)\, S(k) = 0\,, & k \in \mathbf{R}, \end{cases} \tag{6.3}$$

where $\hat{X}(k)$ *is a row vector with* n *components belonging to the Hardy space* $\mathbf{H}^2(\mathbf{C}^+)$, *is the trivial solution* $\hat{X}(k) \equiv 0$.

($\mathbf{4}_e^\circ$) *The only solution* $h(k)$ *to the system*

$$\begin{cases} M_j\, h(i\kappa_j) = 0\,, & j = 1, \dots, N, \\ h(-k) + S(k)\, h(k) = 0\,, & k \in \mathbf{R}, \end{cases} \tag{6.4}$$

where $h(k)$ *is a column vector with* n *components belonging to the Hardy space* $\mathbf{H}^2(\mathbf{C}^+)$, *is the trivial solution* $h(k) \equiv 0$.

(**5**°) *For the matrix-valued function* $F_s(y)$ *given in* (4.3), *the derivative* $F_s'(y)$ *for* $y \in \mathbf{R}^-$ *can be written as a sum of two matrix-valued functions, one of which is integrable and the other is square integrable in* $y \in \mathbf{R}^-$.

We remark that the conditions ($\mathbf{4}_c^\circ$), ($\mathbf{4}_d^\circ$), ($\mathbf{4}_e^\circ$) listed in Theorem 6.2 are somehow similar to the respective conditions ($\mathbf{4}_c$), ($\mathbf{4}_d$), ($\mathbf{4}_e$) of Theorem 4.4. However, there are also some differences; for example, $X(y)$ appearing in (4.14) belongs to $L^1(\mathbf{R}^+)$ whereas $X(y)$ appearing in (6.2) belongs to $L^2(\mathbf{R}^+)$, $\hat{X}(k)$ of (4.15) belongs to $\hat{L}^1(\mathbf{C}^+)$ whereas $\hat{X}(k)$ of (6.3) belongs to $\mathbf{H}^2(\mathbf{C}^+)$, and $h(k)$ of (4.16) belongs to $\hat{L}^1(\mathbf{C}^+)$ whereas $h(k)$ of (6.4) belongs to $\mathbf{H}^2(\mathbf{C}^+)$.

Let us also remark that the condition (**5°**) in Theorem 6.2 is the same as the first sentence given in (**III**$_a$) of Theorem 4.2. We note that Theorem 6.2 is the generalization of a characterization result by Agranovich and Marchenko presented in [1, Theorem 2, p. 281], which utilizes Levinson's theorem in the purely Dirichlet case. That characterization result by Agranovich and Marchenko is valid only in the case of the Dirichlet boundary condition and does not include the condition stated in (**5°**) in Theorem 6.2. In the special case of the purely Dirichlet boundary condition, it turns out that (**5°**) in Theorem 6.2 is not needed. This has something to do with the fact that in the purely Dirichlet case the Marchenko integral equation (4.5) alone plays a key role in the solution to the inverse problem whereas in the non-Dirichlet case not only the Marchenko integral equation but also the derivative Marchenko integral equation both play a key role in the solution to the inverse problem, in particular in the satisfaction of the selfadjoint boundary condition given in (3.5). The derivative Marchenko integral equation is obtained by taking the x-derivative of (4.5), and hence the quantity $F_s'(y)$ appears in the nonhomogeneous term of the derivative Marchenko integral equation. That presence of $F_s'(y)$ somehow results in the condition stated in (**5°**) of Theorem 6.2. The presence of (**5°**) in Theorem 6.2 also has something to do with the fact that the boundary condition stated in (3.5) must hold for all $k \in \mathbf{R}$. By taking the Fourier transform of both sides of (3.5), we end up with the requirement that the Fourier transform of the left-hand side of (3.5) must identically vanish. For this, one needs the necessity of the satisfaction of (**5°**) in Theorem 6.2, unless $A = 0$ in (3.5). Since the case $A = 0$ is the same as having the purely Dirichlet boundary condition, (**5°**) in Theorem 6.2 is relevant only in the non-Dirichlet case. For the mathematical elaboration on (**5°**) we refer the reader to [10].

The presence of (**5°**) in Theorem 6.2 is an indication of one of several reasons why the characterization of scattering data sets with the general selfadjoint boundary condition is more involved than the characterization with the Dirichlet boundary condition.

We conclude that the result presented in Theorem 6.2, compared to Theorem 5.1, constitutes an alternate characterization of the scattering data sets **S**. Recall that Theorem 5.1 characterizes the scattering data sets that are in a one-to-one correspondence with the input data sets **D** in the Faddeev class. With the help of Theorem 6.2 we have the following alternate characterization of the scattering data sets.

Theorem 6.3. *Consider a scattering data set* **S** *as in* (3.12)*, which consists of an* $n \times n$ *scattering matrix* $S(k)$ *for* $k \in \mathbf{R}$*, a set of* N *distinct positive constants* κ_j*, and a set of* N *constant* $n \times n$ *Hermitian and nonnegative matrices* M_j *with respective positive ranks* m_j*, where* N *is a nonnegative integer. In case* $N = 0$*, it is understood that* **S** *consists only of* $S(k)$ *for* $k \in \mathbf{R}$*. Consider also an input data set* **D** *as in* (2.1) *consisting of an* $n \times n$ *matrix potential* $V(x)$ *satisfying* (2.3) *and* (2.4) *and a pair of constant* $n \times n$ *matrices* A *and* B *satisfying* (2.6) *and* (2.7)*, where it is understood that the boundary matrices* A *and* B *are unique up to a postmultiplication by an invertible* $n \times n$ *matrix* T*. Then, we have the following*

characterization of the scattering data sets. A set **S** *as in* (3.12) *is the scattering data set corresponding to an input data set* **D** *in the Faddeev class if and only if* **S** *satisfies* (**1**) *and* (**2**) *of Definition 4.1, both* (**L**) *and* (**5**°) *of Theorem 6.2, and any one of the three conditions listed as* ($\mathbf{4}_c^\circ$), ($\mathbf{4}_d^\circ$), ($\mathbf{4}_e^\circ$) *in Theorem 6.2.*

7. Another characterization of the scattering data

In this section we provide yet another description of the Marchenko class of scattering data sets **S** so that there exists a one-to-one correspondence between an input data set **D** in the Faddeev class and a scattering data set **S** in the Marchenko class. Such a description allows us to have yet another characterization of the scattering data sets **S** in a one-to-one correspondence with the input data sets **D** in the Faddeev class.

The characterization stated in this section resulting from a new description of the Marchenko class has some similarities and differences compared to the first characterization presented in Theorem 5.1 and the alternate characterization presented in Theorem 6.3. Related to this new characterization, the construction of the potential in the solution to the inverse problem is the same as in the previous characterizations; namely, one constructs the potential by solving the Marchenko equation. Hence, the conditions (**1**), (**2**), ($\mathbf{4}_a$) in the first characterization, the conditions (**1**), (**2**), ($\mathbf{4}_c^\circ$) in the alternate characterization, and the conditions (**I**), (**2**), ($\mathbf{4}_c$) in this new characterization are essentially used to construct the potential. This new characterization differs from the two earlier ones in regard to the satisfaction of the boundary condition by the physical solution $\Psi(k,x)$ and by the normalized bound-state matrix solutions $\Psi_j(x)$. It is based on the alternate solution to the inverse problem by using the generalized Fourier map [22]. This new characterization uses six conditions, indicated as (**I**), (**2**), (**A**), ($\mathbf{4}_c$), either one of ($\mathbf{V}_e$) and ($\mathbf{V}_h$), and (**VI**). Recall that (**2**) is described in Definition 4.1, ($\mathbf{4}_c$) is described in Theorem 4.4, and ($\mathbf{V}_e$) and ($\mathbf{V}_h$) are described in Theorem 4.5. In the following definition we describe the conditions (**I**), (**A**), and (**VI**).

Definition 7.1. The properties (**I**), (**A**), and (**VI**) for the scattering data set **S** in (3.12) are defined as follows:

(I) The scattering matrix $S(k)$ satisfies (4.9), the quantity S_∞ appearing in (4.1) exists, the quantity $S(k)-S_\infty$ is square integrable in $k\in\mathbf{R}$, and the quantity $F_s(y)$ defined in (4.3) is bounded in $y\in\mathbf{R}$ and integrable in $y\in\mathbf{R}^+$.

(A) Consider the nonhomogeneous Riemann–Hilbert problem given by

$$h(k)+S(-k)\,h(-k)=g(k),\qquad k\in\mathbf{R},\tag{7.1}$$

where the nonhomogeneous term $g(k)$ belongs to a dense subset $\mathring{\Upsilon}$ of the vector space Υ of column vectors with n square-integrable components and satisfying $g(-k)=S(k)\,g(k)$ for $k\in\mathbf{R}$. Then, for each such given $g(k)$, the equation (7.1) has a solution $h(k)$ as a column vector with n components belonging to the Hardy space $\mathbf{H}^2(\mathbf{C}^+)$.

(VI) The scattering matrix $S(k)$ is continuous in $k \in \mathbf{R}$.

We remark that (**I**) of Definition 7.1 is weaker than (**1**) of Definition 4.1. The quantity G_1 and hence (4.2) appearing in (**1**) are used to construct the boundary matrices A and B. In order to construct the potential $V(x)$ only, it is enough to use the weaker condition (**I**). The condition (**A**) of Definition 7.1 somehow resembles ($\mathbf{III}_c$) of Theorem 4.5, but there are also some major differences. In ($\mathbf{III}_c$) a solution is sought to the homogeneous Riemann–Hilbert problem (4.17) as a column vector with n components where each of those components belongs to $\mathbf{H}^2(\mathbf{C}^-)$, and the only solution is expected to be the trivial solution $h(k) \equiv 0$. On the other hand, in (**A**) one solves a nonhomogeneous Riemann–Hilbert problem and the solution is sought as a column vector where each of the n components belongs the Hardy space $\mathbf{H}^2(\mathbf{C}^+)$. The solution $h(k)$ to (7.1) is in general nontrivial because the nonhomogeneous term $g(k)$ there is in general nontrivial, and the existence of a solution to (7.1) is more relevant than its uniqueness. The condition (**VI**) of Definition 7.1, which is the continuity of the scattering matrix $S(k)$, is mainly needed to prove that the physical solution $\Psi(k,x)$ satisfies the boundary condition (2.5).

Let us first describe the solution to the inverse scattering problem related to this new characterization and then present the characterization itself. As already indicated, the part of the solution to the inverse problem involving the construction of the potential is practically the same as the solution outlined in Section 4. However, the part of the solution related to the boundary condition is different than the procedure outlined in Section 4. We summarize below the construction of $\mathbf{D}$ from $\mathbf{S}$ in this new method, where the existence and uniqueness are implicit at each step:

(a) From the large-k asymptotics of the scattering matrix $S(k)$, with the help of (4.1), we determine the $n \times n$ constant matrix S_∞. Contrary to the method of Section 4, we do not deal with the determination of the constant $n \times n$ matrix G_1 appearing in (4.2). It follows from (4.9) that the matrix S_∞ is Hermitian when $\mathbf{S}$ satisfies the condition (**I**) described in Definition 7.1.

(b) In terms of the quantities in $\mathbf{S}$, we uniquely construct the $n \times n$ matrix $F_s(y)$ by using (4.3) and the $n \times n$ matrix $F(y)$ by using (4.4). This step is the same as steps (b) and (c) of the summary of the method outlined in Section 4.

(c) If the condition ($\mathbf{4}_c$) of Theorem 4.4 is also satisfied, then one uses the matrix $F(y)$ as input to the Marchenko integral equation (4.5). If $F(y)$ is integrable in $y \in (x, +\infty)$ for each $x \geq 0$, then for each fixed $x \geq 0$ there exists a solution $K(x,y)$ integrable in $y \in (x, +\infty)$ to (4.5) and such a solution is unique. The solution $K(x,y)$ can be constructed by iterating (4.5). We remark that this step is the same as step (d) of the summary of the method outlined in Section 4. Even though $K(x,y)$ is constructed only for $0 \leq x < y$, one can extend $K(x,y)$ to $y \in \mathbf{R}^+$ by letting $K(x,y) = 0$ for $0 \leq y < x$.

(d) Having obtained $K(x,y)$ uniquely from $\mathbf{S}$, one constructs the potential $V(x)$ via (4.6) and also constructs the Jost solution $f(k,x)$ via (4.8). Then, by using

(**I**), (**2**), and ($\mathbf{4}_c$), one proves that the constructed $V(x)$ satisfies (2.3) and (2.4) and that the constructed $f(k,x)$ satisfies (2.2) used with the constructed potential $V(x)$.

(e) Having constructed the Jost solution $f(k,x)$, one then constructs the physical solution $\Psi(k,x)$ via (3.4) and the normalized bound-state matrix solutions $\Psi_j(x)$ via (3.11). One then proves that the constructed matrix $\Psi(k,x)$ satisfies (2.2) and that the constructed $\Psi_j(x)$ satisfies (2.2) at $k = i\kappa_j$, with the understanding that the constructed potential $V(x)$ is used in (2.2).

(f) Having constructed the potential $V(x)$, one forms a matrix-valued differential operator denoted by $\mathcal{L}_{\min}$, which acts as $(-D_x^2 I + V)$ with $D_x := d/dx$, with a domain that is a dense subset of $L^2(\mathbf{R}^+)$. More precisely, the domain of $\mathcal{L}_{\min}$ consists of the column vectors with n components each of which is a function of x belonging to a dense subset of $L^2(\mathbf{R}^+)$. The constructed operator $\mathcal{L}_{\min}$ is symmetric, i.e. it satisfies $\mathcal{L}_{\min} \subset \mathcal{L}_{\min}^\dagger$, but is not selfadjoint, i.e. it does not satisfy $\mathcal{L}_{\min} = \mathcal{L}_{\min}^\dagger$. The operator inclusion $\mathcal{L}_{\min} \subset \mathcal{L}_{\min}^\dagger$ indicates that the domain of the operator $\mathcal{L}_{\min}$ is a subset of the domain of the operator $\mathcal{L}_{\min}^\dagger$ and these two operators have the same value on the domain of $\mathcal{L}_{\min}$.

(g) One then constructs a selfadjoint realization of $\mathcal{L}_{\min}$, namely an operator $\mathcal{L}$ in such a way that $\mathcal{L}_{\min} \subset \mathcal{L}$ and $\mathcal{L} = \mathcal{L}^\dagger$. The constructed operator $\mathcal{L}$ is a restriction of $\mathcal{L}_{\min}^\dagger$, i.e. we have $\mathcal{L} \subset \mathcal{L}_{\min}^\dagger$ but not $\mathcal{L} = \mathcal{L}_{\min}^\dagger$.

(h) The construction of the operator $\mathcal{L}$ is achieved [9,10,22] by using the so-called generalized Fourier map $\mathbf{F}$ and its adjoint $\mathbf{F}^\dagger$. The generalized Fourier map $\mathbf{F}$ corresponds to a generalization of the Fourier transform between the space of square-integrable functions of x and the space of square-integrable functions of k.

(i) Once the selfadjoint operator $\mathcal{L}$ is constructed, it follows [9,10] that the domain of $\mathcal{L}$ is a maximal isotropic subspace, which is sometimes also called a Lagrange plane. Once we know that the domain of $\mathcal{L}$ is a maximal isotropic subspace, it follows [9,10] that the functions in the domain of $\mathcal{L}$ must satisfy the boundary condition (2.5) for some boundary matrices A and B satisfying (2.6) and (2.7), where A and B are uniquely determined up to a postmultiplication by an invertible matrix T.

(j) Finally, one proves that the constructed physical solution $\Psi(k,x)$ and the constructed normalized bound-state matrix solutions $\Psi_j(x)$ satisfy the boundary condition (2.5) with the boundary matrices A and B specified in the previous step; however, such a proof is different in nature than the proofs for the previous characterizations. For the constructed matrices $\Psi_j(x)$, it is immediate that they satisfy the boundary condition because they belong to the domain of $\mathcal{L}$. Thus, it remains to prove that the constructed $\Psi(k,x)$ satisfies the boundary condition. We note that the matrix $\Psi(k,x)$ does not belong to the domain of $\mathcal{L}$ because its entries do not belong to $L^2(\mathbf{R}^+)$. On the other

hand, $\Psi(k,x)$ is locally square integrable in $x \in [0,+\infty)$, i.e. it is square integrable in every compact subset of $[0,+\infty)$. Hence, it is possible to use a simple limiting argument to prove that $\Psi(k,x)$ satisfies the boundary condition (2.5), and the condition **(VI)** is utilized in the aforementioned limiting argument.

(k) As in the previous characterization given in Theorem 5.1(c), we still need to prove that the input data set **D** of (2.1) constructed from the scattering data set **S** of (3.12) yields **S**. The proof of this step is the same as in the proof of Theorem 5.1(c).

Based on the procedure outlined above, we next present another description of the Marchenko class of scattering data sets **S**. Recall that **(I)**, **(A)**, and **(VI)** are described in Definition 7.1, **(2)** is described in Definition 4.1, $(\mathbf{4}_c)$ is described in Theorem 4.4, and $(\mathbf{V}_e)$ and $(\mathbf{V}_h)$ are described in Theorem 4.5.

Theorem 7.2. *Consider a scattering data set* **S** *as in* (3.12)*, which consists of an* $n\times n$ *scattering matrix* $S(k)$ *for* $k \in \mathbf{R}$*, a set of* N *distinct positive constants* κ_j*, and a set of* N *constant* $n \times n$ *Hermitian and nonnegative matrices* M_j *with respective positive ranks* m_j*, where* N *is a nonnegative integer. In case* $N = 0$*, it is understood that* **S** *consists only of* $S(k)$ *for* $k \in \mathbf{R}$ *and that* $\mathcal{N}$ *appearing in* (3.8) *is zero. The set* **S** *is the scattering data set corresponding to a unique input data set* **D** *as in* (4.2) *in the Faddeev class specified in Definition 2.1 if and only if* **S** *satisfies the six conditions consisting of* **(I)**, **(2)**, **(A)**, $(\mathbf{4}_c)$*, either one of* $(\mathbf{V}_e)$ *and* $(\mathbf{V}_h)$*, and* **(VI)**. *We recall that the uniqueness of the input data set* **D** *is understood in the sense that the boundary matrices* A *and* B *in* (4.2) *are unique up to a postmultiplication by an arbitrary invertible* $n \times n$ *matrix* T.

8. Some elaborations

In this section we make a comparison with the definitions of the Jost matrix and the scattering matrix in the scalar case appearing in the literature. We also elaborate on the nonuniqueness issue arising if the scattering matrix is defined differently when the Dirichlet boundary condition is used. The reader is referred to Section 4 of [6] and Example 6.3 of [6] for further elaborations on the nonuniqueness issue.

In the scalar case, i.e. when $n = 1$, from (2.8) we see that we can choose

$$A = -\sin\theta, \quad B = \cos\theta, \qquad \theta \in (0,\pi], \tag{8.1}$$

where θ represents the boundary parameter. We can write the boundary condition (2.5) in the equivalent form

$$-A^\dagger\,\psi'(0) + B^\dagger\,\psi(0) = 0. \tag{8.2}$$

Using (8.1) in (8.2) we see that our boundary condition (2.5) in the scalar case is equivalent to

$$(\sin\theta)\,\psi'(0) + (\cos\theta)\,\psi(0) = 0, \qquad \theta \in (0,\pi]. \tag{8.3}$$

We remark that the boundary condition (8.3) agrees with the boundary condition used in the literature [7,19,20] in the scalar case. Since $\theta = \pi$ corresponds to the Dirichlet boundary condition, we can write (8.3) in the equivalent form

$$\begin{cases} \psi(0) = 0\,, & \text{Dirichlet case,} \\ \psi'(0) + (\cot\theta)\,\psi(0) = 0\,, & \text{non-Dirichlet case,} \end{cases} \tag{8.4}$$

where $\theta \in (0,\pi)$ in the non-Dirichlet case. The boundary condition (8.4) is also identical [7,19,20] to that used in the literature in the scalar case. As stated below (2.7), the boundary matrices A and B in (2.5) can be postmultiplied by any invertible matrix T without affecting (2.5)–(2.7). Hence, the constants A and B appearing in (8.1) can be multiplied by any nonzero constant. In any case, the boundary condition (2.5) we use is the same as the boundary condition used in the literature [7,19,20] in the scalar case.

Using (8.1) we see that the Jost matrix defined in (3.2) is given by

$$J(k) = f(-k^*,0)^\dagger\,(\cos\theta) + f'(-k^*,0)^\dagger\,(\sin\theta), \qquad k \in \mathbf{R}, \tag{8.5}$$

where we recall that $\theta = \pi$ in the Dirichlet case and $\theta \in (0,\pi)$ in the non-Dirichlet case. Using (2.2), (2.3), and (3.1), for each fixed $x \geq 0$ one can prove that $f(k,x)$ and $f'(k,x)$ in the scalar case satisfy

$$f(-k^*,x)^* = f(k,x), \quad f'(-k^*,x)^* = f'(k,x), \qquad k \in \overline{\mathbf{C}^+}. \tag{8.6}$$

Informally speaking, $f(k,x)$ and $f'(k,x)$ each contain k as ik, and hence we have (8.6). Using (8.6) in (8.5) we see that the Jost matrix in the scalar case is given by

$$J(k) = f(k,0)\,(\cos\theta) + f'(k,0)\,(\sin\theta), \qquad k \in \mathbf{R}, \tag{8.7}$$

which is equivalent to

$$J(k) = \begin{cases} -f(k,0)\,, & \text{Dirichlet case,} \\ (\sin\theta)\,[f'(k,0) + (\cot\theta)\,f(k,0)]\,, & \text{non-Dirichlet case.} \end{cases} \tag{8.8}$$

In the literature in the scalar case the Jost matrix is usually called the Jost function and is defined [7,19,20] as

$$J(k) = \begin{cases} f(k,0)\,, & \text{Dirichlet case,} \\ -i\,[f'(k,0) + (\cot\theta)\,f(k,0)]\,, & \text{non-Dirichlet case.} \end{cases} \tag{8.9}$$

The primary motivation behind the definition in (8.9) is to define the Jost function $J(k)$ in the scalar case in such a way that as $k \to \infty$ in $\overline{\mathbf{C}^+}$ we have $J(k) = 1 + O(1/k)$ in the Dirichlet case and $J(k) = k + O(1)$ in the non-Dirichlet case. We remark that (8.8) and (8.9) do not agree, and we further elaborate on this disagreement. We know from (b) in Section 3 that the right-hand side of (8.7) can be multiplied by any nonzero constant because the boundary matrices A and B appearing in (3.2) can be postmultiplied by any invertible matrix T without

affecting (2.5)–(2.7). Comparing (8.8) and (8.9) we see that it is impossible to modify the right-hand side of (8.8) through a multiplication by a nonzero scalar so that the right-hand sides of (8.8) and (8.9) agree. In other words, we cannot use the same multiplicative constant both in the Dirichlet case and in the non-Dirichlet case so that (8.8) and (8.9) can agree.

Using (8.8) in (3.3) we obtain the scattering matrix in the scalar case as

$$S(k) = \begin{cases} -\dfrac{f(-k,0)}{f(k,0)}\,, & \text{Dirichlet case,} \\ -\dfrac{f'(-k,0) + (\cot\theta)\, f(-k,0)}{f'(k,0) + (\cot\theta)\, f(k,0)}\,, & \text{non-Dirichlet case.} \end{cases} \tag{8.10}$$

On the other hand, the scattering matrix in the scalar case is defined in the literature [7, 19, 20] as

$$S(k) = \begin{cases} \dfrac{f(-k,0)}{f(k,0)}\,, & \text{Dirichlet case,} \\ -\dfrac{f'(-k,0) + (\cot\theta)\, f(-k,0)}{f'(k,0) + (\cot\theta)\, f(k,0)}\,, & \text{non-Dirichlet case.} \end{cases} \tag{8.11}$$

Thus, the first lines of (8.10) and (8.11) differ by a minus sign and their second lines are identical. Note that (8.9) and (8.11) indicate that the scattering matrix in the literature [7, 19, 20] in the scalar case is related to the Jost matrix as

$$S(k) = \begin{cases} J(-k)\, J(k)^{-1}\,, & \text{Dirichlet case,} \\ -J(-k)\, J(k)^{-1}\,, & \text{non-Dirichlet case.} \end{cases} \tag{8.12}$$

The definition (8.12) of the scattering matrix in the scalar case in the literature is motivated by the fact that (8.12) ensures that S_∞ defined in (4.1) is equal to 1, regardless of the Dirichlet case or the non-Dirichlet case. Comparing (8.12) with (3.3) we see that (3.3) and the first line of (8.12) differ by a minus sign and that (3.3) and the second line of (8.12) agree with each other.

In the previous literature [1, 21], the scattering matrix in the Dirichlet case is defined as in the first line of (8.12) even in the nonscalar case, i.e. when $n \geq 2$. Again, this ensures that $S_\infty = I$, where we recall that I is the $n \times n$ identity matrix. However, defining the scattering matrix in the Dirichlet case as in (8.12) and not as (3.3) makes it impossible to have a unique solution to the inverse scattering problem unless the boundary condition is already known as a part of the scattering data. If the physical problem arises mainly from quantum mechanics and hence the boundary condition is the purely Dirichlet condition, which corresponds to having $A = 0$ in (2.5), this does not present a problem. On the other hand, if the determination of the selfadjoint boundary condition is a part of the solution to the inverse problem, then the definition of the scattering matrix $S(k)$ given in (8.12) is problematic and that is one of the reasons why we use the definition of $S(k)$ given in (3.3) regardless of the boundary condition. Note that we define the scattering matrix as in (3.3) so that the associated Schrödinger operator for the unperturbed problem has the

Neumann boundary condition. This definition is motivated by the theory of quantum graphs, where the Neumann boundary condition is usually used for the unperturbed problem. We refer the reader to [15, 17, 18, 22] for further details.

In the following three examples, we illustrate the drawback of using (8.12) and not (3.3) as the definition of the scattering matrix.

Example 8.1. Let us use (8.12) as the definition of the scattering matrix, instead of using (3.3). Let us assume that we are in the scalar case. Let us consider the input data set $\mathbf{D}$ given in (2.1) and the scattering data set $\mathbf{S}$ given in (3.12). The input data set $\mathbf{D}_1$ corresponding to the trivial potential $V_1(x) \equiv 0$ and the Dirichlet boundary condition with $\theta_1 = \pi$ yields the Jost solution $f_1(k,x) = e^{ikx}$, and hence the corresponding Jost matrix is evaluated by using the first line of (8.9) as $J_1(k) = f_1(k,0) = 1$. There are no bound states because $J_1(k)$ does not vanish on the positive imaginary axis in the complex k-plane. Thus, using the first line of (8.11), we evaluate the scattering matrix as $S_1(k) \equiv 1$, and hence the corresponding scattering data set $\mathbf{S}_1$ consists of $S_1(k) \equiv 1$ without any bound states. On the other hand, the input data set $\mathbf{D}_2$ corresponding to the trivial potential $V_2(x) \equiv 0$ and the Neumann boundary condition with $\theta_2 = \pi/2$ corresponds to the Jost solution $f_2(k,x) = e^{ikx}$ and hence, by using the second line of (8.9), the corresponding Jost matrix is evaluated as $J_2(k) = -if_2'(k,0) = k$. There are no bound states because $J_2(k)$ does not vanish on the positive imaginary axis in the complex k-plane. Thus, using the second line of (8.11) or equivalently using the second line of (8.12), we evaluate the scattering matrix as $S_2(k) \equiv 1$, and hence the corresponding scattering data set $\mathbf{S}_2$ consists of $S_2(k) \equiv 1$ without any bound states. Thus, we have shown that $\mathbf{S}_1 = \mathbf{S}_2$ even though $\mathbf{D}_1 \neq \mathbf{D}_2$. This nonuniqueness would not occur if we used (3.3) as the definition of the scattering matrix $S(k)$. We would then get $S_1(k) \equiv -1$ and $S_2(k) \equiv 1$, and hence $\mathbf{S}_1 \neq \mathbf{S}_2$.

Next, we further illustrate the nonuniqueness encountered in Example 8.1 with a nontrivial example.

Example 8.2. Let us use (8.12) as the definition of the scattering matrix, instead of using (3.3). Let us assume that we are in the scalar case. Let us choose a nontrivial potential $V_1(x)$ so that it is real valued and satisfies (2.4). Let us also view $V_1(x)$ as a full-line potential with support on $x \in \mathbf{R}^+$. We refer the reader to any reference on the scattering theory for the full-line Schrödinger equation such as [2, 11–13, 19, 20] for the description of the corresponding scattering coefficients. As a full-line potential, let us also assume that $V_1(x)$ has no bound states and corresponds to the full-line exceptional case. The assumption of the absence of bound states on the full line is the same as assuming that the transmission coefficient has no poles on the positive imaginary axis in the complex k-plane, and the exceptional case on the full line is equivalent to the assumption that the transmission coefficient does not vanish at $k = 0$. Corresponding to $V_1(x)$ as a full-line potential we have the full-line scattering data $\{T_1(k), R_1(k), L_1(k)\}$, where $T_1(k)$ is the transmission coefficient, $R_1(k)$ is the reflection coefficient from

the right, and $L_1(k)$ is the reflection coefficient from the left. It is known [2, 11–13, 19, 20] that the full-line scattering data $\{T_2(k), R_2(k), L_2(k)\}$, where we have

$$T_2(k) := T_1(k), \quad R_2(k) := -R_1(k), \quad L_2(k) := -L_1(k), \tag{8.13}$$

corresponds to a nontrivial full-line potential $V_2(x)$ so that $V_2(x)$ is real valued, vanishes when $x < 0$, has no bound states on the full line, and corresponds to the full-line exceptional case. Furthermore, $V_2(x)$ satisfies (2.4). Viewing $V_1(x)$ and $V_2(x)$ as half-line potentials, let us now evaluate the corresponding half-line scattering data sets $\mathbf{S}_1$ and $\mathbf{S}_2$ associated with the full-line scattering data sets $\{T_1(k), R_1(k), L_1(k)\}$ and $\{T_2(k), R_2(k), L_2(k)\}$, respectively. Since $V_1(x)$ and $V_2(x)$ both vanish when $x < 0$, the corresponding respective Jost solutions $f_1(k, x)$ and $f_2(k, x)$ yield

$$f_1(k, 0) = \frac{1 + L_1(k)}{T_1(k)}, \qquad f_2(k, 0) = \frac{1 + L_2(k)}{T_2(k)}, \tag{8.14}$$

$$f_1'(k, 0) = ik\,\frac{1 - L_1(k)}{T_1(k)}, \qquad f_2'(k, 0) = ik\,\frac{1 - L_2(k)}{T_2(k)}. \tag{8.15}$$

Let us now view $V_1(x)$ as a half-line potential, associate it with the Dirichlet boundary condition $\theta_1 = \pi$, and use $\mathbf{D}_1$ to denote the resulting input data set. Similarly, let us view $V_2(x)$ as a half-line potential, associate it with the Neumann boundary condition $\theta_2 = \pi/2$, and use $\mathbf{D}_2$ to denote the resulting input data set. Clearly, we have $\mathbf{D}_1 \neq \mathbf{D}_2$ because $\theta_1 \neq \theta_2$. Using (8.14) in the first lines of (8.9) and (8.11) we obtain the Jost matrix $J_1(k)$ and the scattering matrix $S_1(k)$ as

$$\begin{aligned} J_1(k) &= f_1(k, 0) = \frac{1 + L_1(k)}{T_1(k)}, \\ S_1(k) &= \frac{f_1(-k, 0)}{f_1(k, 0)} = \frac{T_1(k)}{T_1(-k)}\,\frac{1 + L_1(-k)}{1 + L_1(k)}. \end{aligned} \tag{8.16}$$

On the other hand, using (8.15) and the second lines of (8.9) and (8.11) with $\theta = \pi/2$, we obtain the Jost matrix $J_2(k)$ and the scattering matrix $S_2(k)$ as

$$\begin{aligned} J_2(k) &= -i\,f_2'(k, 0) = k\,\frac{1 - L_2(k)}{T_2(k)}, \\ S_2(k) &= -\frac{f_2'(-k, 0)}{f_2'(k, 0)} = \frac{T_2(k)}{T_2(-k)}\,\frac{1 - L_2(-k)}{1 - L_2(k)}. \end{aligned} \tag{8.17}$$

Using (8.13) in (8.16) and (8.17) we see that $S_1(k) \equiv S_2(k)$, and hence the corresponding scattering data sets $\mathbf{S}_1$ and $\mathbf{S}_2$ coincide. Thus, we get $\mathbf{D}_1 \neq \mathbf{D}_2$ and $\mathbf{S}_1 = \mathbf{S}_2$. This nonuniqueness can be fixed by using (3.3) and not (8.12) as the definition of the scattering matrix $S(k)$.

The nonuniqueness problem encountered in the previous two examples can also occur in the nonscalar case, as shown in the following example. This new example is the generalization of Example 8.2 from the scalar case to the $n \times n$ matrix case for any positive integer n. For the relevant scattering theory for the matrix Schrödinger equation on the full line, we refer the reader to [3].

Example 8.3. In this example we assume that n is any positive integer and not necessarily restricted to $n = 1$. Let us use (8.12) as the definition of the scattering matrix, instead of using (3.3). Let us again use (2.1) to describe an input data set $\mathbf{D}$ and use (3.12) to describe a scattering data set $\mathbf{S}$ on the half line. Consider the class of $n \times n$ matrix-valued potentials $V(x)$ on the full line satisfying

$$V(x) = V(x)^\dagger, \qquad x \in \mathbf{R}, \tag{8.18}$$

$$\int_{-\infty}^{\infty} dx\,(1 + |x|)\,|V(x)| < +\infty, \tag{8.19}$$

where we recall that the dagger denotes the matrix adjoint and $|V(x)|$ denotes the matrix operator norm. The reader is referred to [3] for the matrix-valued scattering coefficients for the full-line matrix Schrödinger equation with such potentials. Associated with $V(x)$ we have the full-line scattering coefficients $T_{\rm l}(k)$, $R(k)$, and $L(k)$, each of which is an $n \times n$ matrix. These matrix-valued scattering coefficients are the matrix generalizations of the scalar scattering coefficients $T(k)$, $R(k)$, and $L(k)$ considered in Example 8.2. Let us further restrict the full-line potentials $V(x)$ so that they vanish when $x < 0$, they do not possess any bound states on the full line, and they correspond to the purely exceptional case. We refer the reader to [3] for the details on the bound states and the purely exceptional case on the full line. The absence of bound states on the full line is equivalent to having the determinant of the matrix inverse of $T_{\rm l}(k)$ not vanishing on the positive imaginary axis in the complex k-plane. The purely exceptional case on the full line is equivalent to having the limit of $k\,T_{\rm l}(k)^{-1}$ as $k \to 0$ equal to the $n \times n$ zero matrix. For such potentials $T_{\rm l}(0)^{-1}$ is well defined and we have $\det[I \pm L(0)] \neq 0$, where we recall that I denotes the $n \times n$ identity matrix. Since we only consider the full-line potentials $V(x)$ vanishing when $x < 0$, we can view their restrictions on $x \in \mathbf{R}^+$ as half-line potentials $V(x)$. From (8.18) and (8.19) we see that their restrictions on $x \in \mathbf{R}^+$ belong to the Faddeev class. When $x \in \mathbf{R}^+$, the full-line Jost solution from the left $f_{\rm l}(k,x)$ coincides [3] with the half-line Jost solution $f(k,x)$ appearing in (3.1). Furthermore, we have [3]

$$f_{\rm l}(k,0) = [I + L(k)]\,T_{\rm l}(k)^{-1}, \quad f_{\rm l}'(k,0) = ik\,[I - L(k)]\,T_{\rm l}(k)^{-1}, \qquad k \in \mathbf{R}. \tag{8.20}$$

Let $V_1(x)$ be a specific full-line matrix potential satisfying (8.18) and (8.19) such that it vanishes when $x < 0$, does not contain any bound states on the full line, and corresponds to the purely exceptional case on the full line. Let $\{T_{\rm l}(k), R(k), L(k)\}$ be the corresponding full-line scattering data. Let $V_2(x)$ be the full-line matrix potential corresponding to the full-line scattering data $\{T_{\rm l}(k), -R(k), -L(k)\}$, where the signs of the matrix-valued reflection coefficients are changed. The matrix potential $V_2(x)$ also vanishes for $x < 0$, satisfies (8.18) and (8.19), does not possess any bound states on the full line, and corresponds to a purely exceptional case on the full line. The restrictions of $V_1(x)$ and $V_2(x)$ on $x \in \mathbf{R}^+$ can be viewed as half-line potentials. Let A_1, B_1, A_2, B_2 be four $n \times n$ constant matrices in such a way that $A_1 = 0$, $B_2 = 0$, A_2 is an arbitrary invertible matrix, and B_1 is an arbitrary invertible matrix. Let $\mathbf{D}_1 := \{V_1, A_1, B_1\}$

and $\mathbf{D}_2 := \{V_2, A_2, B_2\}$ be the half-line input data sets as in (2.1), with the understanding that the domains of $V_1(x)$ and $V_2(x)$ are restricted to $x \in \mathbf{R}^+$. Let $f_1(k,x)$ and $f_2(k,x)$ be the half-line Jost solutions corresponding to $\mathbf{D}_1$ and $\mathbf{D}_2$, respectively. From (8.20) we see that

$$f_1(k,0) = [I + L(k)]\, T_l(k)^{-1}, \quad f_1'(k,0) = ik\,[I - L(k)]\, T_l(k)^{-1}, \quad k \in \mathbf{R}, \tag{8.21}$$

$$f_2(k,0) = [I - L(k)]\, T_l(k)^{-1}, \quad f_2'(k,0) = ik\,[I + L(k)]\, T_l(k)^{-1}, \quad k \in \mathbf{R}. \tag{8.22}$$

Using (8.21) and (8.22), because of the purely exceptional case on the full line [3], it follows that neither of the determinants of $f_1(k,0)$ and $f_2'(k,0)$ vanish. Let $J_1(k)$, $S_1(k)$, $\mathbf{S}_1$ be the respective Jost matrix, scattering matrix, and scattering data set corresponding to $\mathbf{D}_1$. Similarly, let $J_2(k)$, $S_2(k)$, $\mathbf{S}_2$ be the respective Jost matrix, scattering matrix, and scattering data set corresponding to $\mathbf{D}_2$. Using (8.21) and (8.22) in (3.2) we obtain

$$J_1(k) = f_1(-k,0)^\dagger\, B_1 = [T_l(-k)^\dagger]^{-1}\,[I + L(-k)^\dagger]\, B_1, \tag{8.23}$$

$$J_2(k) = -f_2'(-k,0)^\dagger A_2 = -ik\,[T_l(-k)^\dagger]^{-1}\,[I + L(-k)^\dagger]\, A_2. \tag{8.24}$$

Using (8.23) in the first line of (8.12) we obtain

$$S_1(k) = J_1(-k)\, J_1(k)^{-1} = f_1(k,0)^\dagger \left[f_1(-k,0)^\dagger\right]^{-1},$$

which yields

$$S_1(k) = [T_l(k)^\dagger]^{-1}\,[I + L(k)^\dagger]\,[I + L(-k)^\dagger]^{-1}\, T_l(-k)^\dagger. \tag{8.25}$$

Using (8.24) in the second line of (8.12) we obtain

$$S_2(k) = -J_2(-k)\, J_2(k)^{-1} = -f_2'(k,0)^\dagger \left[f_2'(-k,0)^\dagger\right]^{-1},$$

yielding

$$S_2(k) = [T_l(k)^\dagger]^{-1}\,[I + L(k)^\dagger]\,[I + L(-k)^\dagger]^{-1}\, T_l(-k)^\dagger. \tag{8.26}$$

There are no half-line bound states associated with either of the scattering data sets corresponding to $S_1(k)$ and $S_2(k)$. Hence we have $\mathbf{S}_1 = \{S_1\}$ and $\mathbf{S}_2 = \{S_2\}$. From (8.25) and (8.26) it follows that $S_1(k) \equiv S_2(k)$ and hence we have $\mathbf{S}_1 = \mathbf{S}_2$ even though $\mathbf{D}_1 \neq \mathbf{D}_2$. This nonuniqueness would not occur if we used (3.3) as the definition of the scattering matrix $S(k)$. We would then get $S_1(k) \equiv -S_2(k)$, and hence $\mathbf{S}_1 \neq \mathbf{S}_2$.

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Обернене розсіювання на півпрямій для матричного рівняння Шредінгера

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На півпрямій розглянуто матричне рівняння Шредінгера із загальною самоспряженою крайовою умовою в нулі, яка задається двома матрицями, що задовольняють певні умови. Вважається, що матричний потенціал є самоспряженим, інтегровним та має скінченний перший момент. Побудовано відповідну множину даних розсіяння. Цю множину даних розсіювання характеризовано набором необхідних і достатніх умов, які гарантують єдиність та взаємно однозначну відповідність між множиною даних розсіяння та множиною вхідних даних, яка містить потенціал та крайові матриці. Ця робота надає узагальнення з крайової умови Діріхле на загальну самоспряжену крайову умову для класичного результату Аграновича та Марченка.

Ключові слова: матричне рівняння Шредингера, самоспряжена гранична умова, метод Марченка, матричний метод Марченка, матриця Йоста, матриця розсіяння, обернене розсіювання, характеризація.