

# Construction of KdV Flow I. $\tau$ -Function via Weyl Function

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*Dedicated to Professor V.A. Marchenko  
on his 95th birthday*

Sato introduced the  $\tau$ -function to describe solutions to a wide class of completely integrable differential equations. Later Segal–Wilson represented it in terms of the relevant integral operators on Hardy space of the unit disc. This paper gives another representation of the  $\tau$ -functions by the Weyl functions for 1d Schrödinger operators with real valued potentials, which will make it possible to extend the class of initial data for the KdV equation to more general one.

*Key words:* KdV equation, Sato theory, Schrödinger operator.

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## 1. Introduction

The discovery of spectral invariants by [2] was a trigger of the succeeding rapid development of the study of the KdV equation

$$\partial_t f = 6f\partial_x f - \partial_x^3 f.$$

Since then, most of the works have been done by using the scattering data for decaying solutions, and the discriminant for periodic solutions. On the other hand, the algebraic structure of the KdV equation was revealed by [13] based on the results by [4, 8], and provided a unified approach to a wide class of integrable systems. Since his argument was algebraic, so obtained solutions were rational, multi-solitons and algebro-geometric ones, and all these solutions were described by  $\tau$ -functions. It has been a problem to what extent this method is effective to obtain general solutions to the KdV equation such as solutions starting from almost periodic functions. [14] considered a kind of closure of Sato's framework to obtain a certain class of transcendental solutions. However, their solutions still remain in a meromorphic class on the entire complex plane  $\mathbb{C}$ . It should be noted that [10] proposed an algorithm to construct solutions to the KdV equation, although it seems that his method also has difficulty to go beyond the class investigated by [14].

Since Sato's theory gives a unified way to solve the KdV equation at least algebraically, there is some hope to exceed the already existing frameworks. In the following the outline of Sato's method developed by [14] is described. Let  $q$  be a function (for the present  $q$  can be complex valued) on  $\mathbb{R}$ . The basic assumption on  $q$  is that the associated Schrödinger equation

$$-\partial_x^2 f + qf = -z^2 f \quad (1.1)$$

has a Baker–Akhiezer function  $f$  for all  $z \in \mathbb{C}$  satisfying  $|z| > R$  for some  $R > 0$ , where  $f$  is called a *Baker–Akhiezer function* if  $f$  has an expression

$$f(x, z) = e^{-xz} \left( 1 + \sum_{n=1}^{\infty} a_n(x) z^{-n} \right)$$

converging on  $\{|z| > R\}$ . This condition on  $q$  is equivalent to the reflectionless property on  $(R^2, \infty)$  if the potential is real valued, which will be seen later. For  $r > R$  let  $W$  be the closure of the linear span of  $\{f(x, z)\}_{x \in \mathbb{R}}$  in the Hilbert space  $H = L^2(|z| = r)$  ( $\mathbb{D}_r = \{z \in \mathbb{C}; |z| < r\}$ ,  $\partial\mathbb{D}_r = \{z \in \mathbb{C}; |z| = r\}$ ). From (1.1) we have easily

**(W.1)** If  $f \in W$ , then  $z^2 f \in W$ .

Let  $H_+ = \text{span}\{z^n : n = 0, 1, 2, \dots\}$ ,  $H_- = \text{span}\{z^n : n = -1, -2, \dots\}$  in  $H$ , and  $\mathfrak{p}_{\pm}$  be the orthogonal projections to  $H_{\pm}$  respectively. Define the second property by

**(W.2)**  $\mathfrak{p}_+ : W \rightarrow H_+$  is bijective.

The totality of closed subspaces of  $H$  satisfying **(W.1)**, **(W.2)** is denoted by  $Gr^{(2)}$ , which will be replaced by  $Gr^{(2)}(\mathbb{D}_r)$  if it is necessary. In a general setting of  $Gr^{(2)}$  the property **(W.2)** is replaced by the Fredholm condition of  $\mathfrak{p}_+$ . Set

$$\Gamma = \left\{ g = e^h : h \text{ is holomorphic on } \{|z| < r'\} \text{ for some } r' > r \right\}.$$

Then, for  $W \in Gr^{(2)}$ ,  $g \in \Gamma$  a new subspace  $gW$  satisfies **(W.1)**, however the property **(W.2)** is not always valid. For a given  $W \in Gr^{(2)}$  let  $\delta > 0$  be such that  $e^{xz}W \in Gr^{(2)}$  holds for any  $x \in \mathbb{R}$  such that  $|x| < \delta$ . To find such a  $\delta$  is always possible because  $e^{xz}W \in Gr^{(2)}$  for  $x = 0$ . Then, an adhoc derivation of the potential  $q$  and the Baker–Akhiezer function from  $W$  is as follows.

The property **(W.2)** for  $e^{xz}W$  implies that there exists a unique element  $f$  of  $W$  satisfying

$$e^{xz} f(z) \in 1 + H_-,$$

which is denoted by  $f(x, z)$ . Since any element of  $H_-$  has Taylor expansion at  $z = \infty$ , we have

$$u(x, z) \equiv e^{xz} f(x, z) - 1 = \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \frac{a_3(x)}{z^3} + \dots$$

Taking derivative with respect to  $x$  yields

$$\begin{cases} f'(x, z) = -ze^{-xz}(1 + u(x, z)) + e^{-xz}u'(x, z), \\ f''(x, z) = z^2 e^{-xz}(1 + u(x, z)) - 2ze^{-xz}u'(x, z) + e^{-xz}u''(x, z), \end{cases}$$

which shows

$$\begin{aligned} e^{xz} &(-f''(x, z) - 2a'_1(x) f(x, z) + z^2 f(x, z)) \\ &= 2zu'(x, z) - u''(x, z) - 2a'_1(x) (1 + u(x, z)) \\ &= \sum_{k=1}^{\infty} (2a'_{k+1}(x) - a''_k(x) - 2a'_1(x) a_k(x)) z^{-k}. \end{aligned}$$

Since the first term belongs to  $e^{xz}W$  due to (W,1) and the last term is an element of  $H_-$ , we have

$$\begin{cases} -f''(x, z) - 2a'_1(x) f(x, z) + z^2 f(x, z) = 0, \\ 2a'_{k+1}(x) - a''_k(x) - 2a'_1(x) a_k(x) = 0, \quad k = 1, 2, \dots, \end{cases} \tag{1.2}$$

where we have used the property (W.2) again for the space  $e^{xz}W$ . Therefore, if we define

$$q_W(x) = -2a'_1(x), \tag{1.3}$$

then  $f(x, z)$  is the Baker–Akhiezer function for  $q_W$ .

A solution to the KdV equation starting from  $q_W$  is obtained similarly. Assume  $e^{xz+tz^3}W \in Gr^{(2)}$  for any  $x, t \in \mathbb{R}$ . Let  $f(x, t, z)$  be a unique element  $f$  of  $W$  satisfying

$$e^{xz+tz^3} f(z) \in 1 + H_-,$$

and set

$$\begin{cases} e^{xz+tz^3} f(x, t, z) = 1 + u(x, t, z) \quad \text{with } u(x, t, z) \in H_-, \\ u(x, t, z) = \frac{a_1(x, t)}{z} + \frac{a_2(x, t)}{z^2} + \frac{a_3(x, t)}{z^3} + \dots, \\ q_W(x, t) = a'_1(x, t), \end{cases}$$

where  $'$  denotes the derivative with respect to  $x$ . Taking derivatives of  $e^{xz+tz^3} f$  with respect to  $t, x$  yields

$$\begin{cases} e^{xz+tz^3} \partial_t f = -z^3 (1 + u) + \partial_t u, \\ e^{xz+tz^3} f' = -z (1 + u) + u', \\ e^{xz+tz^3} f''' = -z^3 (1 + u) + 3z^2 u' - 3zu'' + u''', \end{cases}$$

and hence we obtain

$$e^{xz+tz^3} (\partial_t f - f''' - 3a'_1 f') = \partial_t u - 3z^2 u' + 3zu'' - u''' + 3za'_1 (1 + u) - 3u' a'_1.$$

Then, the right side function is

$$-3a'_2 + 3a''_1 + 3a'_1 a_1 \quad \text{modulo } H_-.$$

Applying the second identities of (1.2) to  $e^{tz^3} f(x, t, z)$  and  $e^{tz^3}W$  yields

$$-3a'_2 + 3a''_1 + 3a'_1 a_1 = \frac{3}{2} (-a''_1 - 2a'_1 a_1 + 2a''_1 + 2a'_1 a_1) = \frac{3}{2} a''_1,$$

hence

$$\begin{aligned} e^{xz+tz^3} \left( \partial_t f - f''' - 3a_1' f' - \frac{3}{2} a_1'' f \right) \\ = \partial_t u - 3z^2 u' + 3z u'' - u''' + 3z a_1' (1+u) - 3u' a_1' - \frac{3}{2} a_1'' (1+u), \end{aligned}$$

which is an element of  $H_-$ . Therefore, the property **(W.2)** for  $e^{xz+tz^3} W$  implies

$$\partial_t f - f''' - 3a_1' f' - \frac{3}{2} a_1'' f = 0,$$

thus

$$\partial_t u - 3z^2 u' + 3z u'' - u''' + 3z a_1' (1+u) - 3u' a_1' - \frac{3}{2} a_1'' (1+u) = 0.$$

The coefficient of  $z^{-1}$  of the above left side function is

$$\partial_t a_1 - 3a_3' + 3a_2'' - a_1''' + 3a_1' a_2 - 3(a_1')^2 - \frac{3}{2} a_1'' a_1 = 0.$$

The second identity of **(1.2)** for  $e^{tz^3} W$  yields

$$2a_2' - a_1'' - 2a_1' a_1 = 0, \quad 2a_3' - a_2'' - 2a_1' a_2 = 0,$$

hence

$$\begin{aligned} -3a_3' + 3a_2'' - a_1''' + 3a_1' a_2 - 3(a_1')^2 - \frac{3}{2} a_1'' a_1 \\ = -\frac{3}{2} (a_2'' + 2a_1' a_2) + 3a_2'' - a_1''' + 3a_1' a_2 - 3(a_1')^2 - \frac{3}{2} a_1'' a_1 \\ = -\frac{1}{4} a_1''' - \frac{3}{2} (a_1')^2. \end{aligned}$$

Consequently, we have

$$\partial_t a_1 - \frac{1}{4} a_1''' - \frac{3}{2} (a_1')^2 = 0.$$

Taking derivative with respect to  $x$  shows

$$-\frac{1}{2} \partial_t q_W + \frac{1}{8} q_W''' - \frac{3}{4} q_W q_W' = 0,$$

hence

$$\partial_t q_W - \frac{1}{4} q_W''' + \frac{3}{2} q_W q_W' = 0,$$

which shows that  $q_W(x, -4t)$  satisfies the KdV equation with initial data  $q_W(x)$ . The idea behind this calculation is an effective use of algebra of pseudo differential operators, and a more systematic argument can be found in [14].

The  $\tau$ -function was introduced by Sato to describe the  $\Gamma$ -action on  $Gr^{(2)}$ . For  $W \in Gr^{(2)}$  and  $f_+ \in H_+$  let  $f_-$  be the unique element of  $H_-$  such that

$$f_+ + f_- \in W$$

is valid, which is possible due to **(W.2)**. An operator  $A_W$  from  $H_+$  to  $H_-$  is defined by  $A_W f_+ = f_-$ . Then, for  $g \in \Gamma$  the *tau-function* is defined by

$$\tau_W(g) = \det(I + g^{-1} \mathbf{p}_+ g A_W), \tag{1.4}$$

and the functions  $q_W(x)$ ,  $q_W(x, t)$  are given by the  $\tau$ -functions in Lemma 4.4 as follows:

$$q_W(x) = -2\partial_x^2 \log \tau_W(e^{xz}), \quad q_W(x, t) = -2\partial_x^2 \log \tau_W(e^{xz+tz^3}).$$

If we use  $e^{xz+tz^{2n+1}}$  in place of  $e^{xz+tz^3}$ , then we obtain solutions to the higher order KdV equations. It is known that  $gW \in Gr^{(2)}$  if and only if  $\tau_W(g) \neq 0$ , and  $\tau_W(g)$  has a cocycle property

$$\tau_W(g_1 g_2) = \tau_W(g_1) \tau_{g_1 W}(g_2)$$

(see Proposition 4.3), which will leads us to the definition of the KdV flow.

To treat real valued potentials some notions for  $Gr^{(2)}$  are necessary. For a function  $f$  on a domain of  $\mathbb{C}$  set  $\bar{f}(z) = \overline{f(\bar{z})}$ . Define  $\bar{W} = \{\bar{f}; f \in W\}$  and set

$$Gr_{\text{real}}^{(2)} = \left\{ W \in Gr^{(2)} : W = \bar{W} \right\}, \quad \Gamma_{\text{real}} = \{g \in \Gamma : g = \bar{g}\},$$

and

$$Gr_+^{(2)} = \left\{ W \in Gr_{\text{real}}^{(2)} : \tau_W(g) \geq 0 \text{ for any } g \in \Gamma_{\text{real}} \right\}.$$

For  $W \in Gr_{\text{real}}^{(2)}$  the corresponding potential  $q_W$  takes real values. The first theorem is

**Theorem 1.1.** *An identity*

$$Gr_+^{(2)} = \left\{ W \in Gr_{\text{real}}^{(2)} : gW \in Gr^{(2)} \text{ for any } g \in \Gamma_{\text{real}} \right\}$$

holds.

A sufficient condition for  $W$  to be an element of  $Gr_+^{(2)}$  will be given by using the  $m$ -function. Although the potential  $q_W$  is obtained from  $W \in Gr^{(2)}$  through  $\tau_W$ , this correspondence is not one to one. The quantity determining  $q_W$  is called in this paper  $m$ -function given by

$$m_W(z) = -\frac{f'(0, z)}{f(0, z)}$$

with the Baker–Akhiezer function  $f$ . Section 5 is devoted to investigation of  $m$ -functions, especially  $\tau_W$  is decomposed into two components, one of which is expressed by  $\tau_{m_W}$  and depends only on  $m_W$ .

Usually integrable Hamiltonian systems have been linearized through action-angle variables. In order to apply this point of view to the KdV equation we need at least integrability of the solutions, namely  $\int_{-\infty}^{\infty} |u(x)| dx < \infty$  in decaying case and  $\int_0^1 |u(x)| dx < \infty$  in periodic case, and one can define conserved quantities suitably. For almost periodic case [6, 12] considered the KdV equation in the framework of Hamiltonian systems. However, the lack of compactness and connectedness of the invariant leaves prohibits to develop the argument further. Sato did not take this approach and constructed directly the flow by the  $\tau$ -functions. Therefore, in his theory the  $\tau$ -function is the crucial quantity. In the context of Sato's theory the Weyl functions were first used by [5] to define an element of Sato's Grassmann manifold. The purpose of the present paper is to give a representation of the  $\tau$ -functions by Weyl functions for 1d Schrödinger operators with real potentials, so that one can obtain more general solutions.

To state the main results we need more terminologies. Suppose a Schrödinger operator  $L_q = -\partial_x^2 + q$  with real valued  $q \in L_{loc}^1(\mathbb{R})$  is essentially self-adjoint on  $L^2(\mathbb{R})$  (the boundedness of  $q$  is sufficient for this). Then it is known that  $\dim \{f \in L^2(\mathbb{R}_{\pm}) : L_q f = z f\} = 1$  for every  $z \in \mathbb{C} \setminus \mathbb{R}$ . The Weyl functions  $m_{\pm}$  are defined by

$$m_{\pm}(z) = \pm \frac{f'_{\pm}(0, z)}{f_{\pm}(0, z)}$$

with two non-trivial  $f_{\pm} \in \{f \in L^2(\mathbb{R}_{\pm}) : L_q f = z f\}$ .  $m_{\pm}$  are holomorphic on  $\mathbb{C} \setminus \mathbb{R}$  and have positive imaginary parts on  $\mathbb{C}_+$ . Gelfand–Levitán, Marchenko showed that  $q$  can be recovered from  $m_{\pm}$  uniquely (see [10]). A potential  $q$  is called *reflectionless* on  $F \in \mathcal{B}(\mathbb{R})$  if its Weyl functions  $m_{\pm}$  satisfy

$$m_+(\xi + i0) = -\overline{m_-(\xi + i0)} (= -m_-(\xi - i0)) \quad \text{a.e. } \xi \in F. \quad (1.5)$$

Set

$$m(z) = \begin{cases} -m_+(-z^2) & \text{if } \operatorname{Re} z > 0, \\ m_-(-z^2) & \text{if } \operatorname{Re} z < 0, \end{cases} \quad (1.6)$$

and assume that there exist  $\lambda_0 < 0 < \lambda_1$  such that

$$\inf \operatorname{sp} L_q > \lambda_0 \quad \text{and} \quad q \text{ is reflectionless on } (\lambda_1, \infty).$$

Then,  $m$  is holomorphic on  $\mathbb{C} \setminus ([-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}] \cup i[-\sqrt{\lambda_1}, \sqrt{\lambda_1}])$ , and has an expansion at  $z = \infty$  like

$$m(z) = z + m_1 z^{-1} + m_2 z^{-2} + \dots$$

It will be seen later that this property of  $m$  implies the existence of the Baker–Akhiezer function with  $r > \sqrt{(-\lambda_0) \vee \lambda_1}$ , and

$$W_m = \{\varphi(z^2) + \psi(z^2) m(z) : \varphi, \psi \in H_+\}$$

is an element of  $Gr_+^{(2)}$  (see Proposition 6.7).

The second theorem is related to the construction of KdV flow by using  $m$ . Set

$$\begin{cases} \mathcal{Q}_\infty &= \{q : q \text{ is reflectionless on } (r^2, \infty) \text{ and} \\ &\quad \text{sp } L_q \subset [-r^2, \infty) \text{ for some } r > 0\}, \\ \Gamma_{\text{real}}^\infty &= \{g = e^h : h \text{ is an entire function with } \bar{h} = h\}, \end{cases} \quad (1.7)$$

and for  $q \in \mathcal{Q}_\infty$  define

$$(K(g)q)(x) = -2\partial_x^2 \log \tau_{W_m}(ge_x),$$

where  $m$  is the  $m$ -function defined by (1.6). Then, we have

**Theorem 1.2.**  $\{K(g)\}_{g \in \Gamma_{\text{real}}^\infty}$  defines a flow on  $\mathcal{Q}_\infty$  such that

$$\begin{cases} (K(e^{tz})q)(x) = q(x+t), \\ (K(e^{-4tz^3})q)(x) \text{ satisfies the KdV equation.} \end{cases}$$

The next task is to represent the  $\tau$ -function  $\tau_{W_m}$  more concretely. For a function  $f$  set

$$f_e(z) = \frac{1}{2}(f(\sqrt{z}) + f(-\sqrt{z})), \quad f_o(z) = \frac{1}{2\sqrt{z}}(f(\sqrt{z}) - f(-\sqrt{z})).$$

Let  $D, D'$  be simply connected bounded domains in  $\mathbb{C}$  containing the interval  $[-\lambda_1, -\lambda_0]$ , and satisfy  $D \subset D'$ . Set  $C = \partial D, C' = \partial D'$  the boundaries of  $D, D'$  respectively. We assume  $C, C'$  are smooth curves and surround  $[-\lambda_1, -\lambda_0]$  counterclockwise (see Fig. 1.1).

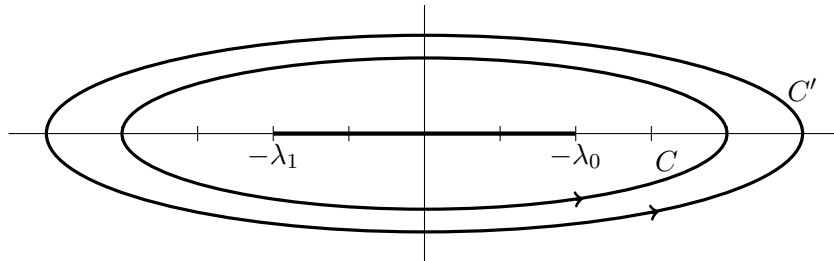


Fig. 1.1: Curves  $C$  and  $C'$ .

For  $\delta$  whose  $\delta_e, \delta_o$  are holomorphic in a simply connected domain including  $C'$ , set  $\tilde{m}(z) = m(z) - \delta(z)$ , and define

$$\begin{cases} M_g(z, \lambda) = \frac{\hat{g}_o(z)(g\tilde{m})_e(\lambda) + \hat{g}_e(z)(g\tilde{m})_o(\lambda)}{\lambda - z}, \\ N_g(z, \lambda) = \frac{1}{2\pi i} \int_{C'} \frac{M_g(\lambda', \lambda)}{\lambda' - z} m_o(\lambda')^{-1} d\lambda', \\ (N_m(g)f)(z) = \frac{1}{2\pi i} \int_C N_g(z, \lambda) f(\lambda) d\lambda, \end{cases} \quad (1.8)$$

where  $\widehat{g}(z) = g(z)^{-1}$ . The operator  $N_m(g)$  defines a trace class operator on  $L^2(C)$ .

**Theorem 1.3.** For  $q \in \mathcal{Q}_\infty$  and  $g \in \Gamma_{\text{real}}^\infty$  we have

$$\tau_{W_m}(g) = \det(I + N_m(g)).$$

*Remark 1.4.* The class  $\mathcal{Q}_\infty$  contains multi-solitons, algebro-geometric solutions and they are dense in  $\mathcal{Q}_\infty$ . Especially all solutions considered in [3] are included if  $\inf \text{sp } L_q > -\infty$ . A necessary condition was obtained by [9], namely she proved that if  $\lambda_0 = \inf \text{sp } L_q$ , and  $q$  is reflectionless on  $(\lambda_1, \infty)$ , then  $q$  is holomorphic on  $\left\{ |\text{Im } z| < (\lambda_1 - \lambda_0)^{-1/2} \right\}$  and has bound

$$|q(z) - \lambda_1| \leq 2(\lambda_1 - \lambda_0) \left(1 - \sqrt{\lambda_1 - \lambda_0} |\text{Im } z|\right)^{-2}.$$

Since  $\tau_W(e_x)$  is entire as a function of  $x$ , we know from Theorem 1.2 that  $q(x)$  is meromorphic on the entire complex plane  $\mathbb{C}$ .

This paper is intended to be self-contained, so many results are overlapping with those of [14]. The author already published one paper [7] on the property  $\tau_W(g) \neq 0$ . However, it contains several mistakes in the proofs, moreover, the whole story was not well organized. The present paper tries to improve these points by employing several basic notions from the theory of Toeplitz operators.

Throughout the paper we use the following notations.  $\mathbb{R}$  denotes the real line and  $\mathbb{C}$  denotes the whole complex plane, and

$$\begin{cases} \mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}, & \mathbb{R}_- = \{x \in \mathbb{R} : x < 0\}, \\ \mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}, & \mathbb{C}_- = \{z \in \mathbb{C} : \text{Im } z < 0\}, \\ \mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}, & \partial\mathbb{D}_r = \{z \in \mathbb{C} : |z| = r\}. \end{cases}$$

Moreover,  $\sqrt{z}$  is defined as a holomorphic function on  $\mathbb{C} \setminus \mathbb{R}_-$  so that  $\sqrt{1} = 1$ , hence  $\sqrt{z}$  satisfies  $\sqrt{z} = \overline{\sqrt{\bar{z}}}$ .

## 2. Grassmann manifold $Gr^{(2)}$

For completeness sake we define the relevant spaces again. Let  $H = L^2(\partial\mathbb{D}_r)$  and set

$$H_+ = \text{the span of } \{z^n\}_{n \geq 0}, \quad H_- = \text{the span of } \{z^n\}_{n \leq -1}$$

in  $H$ . Then,  $H = H_+ \oplus H_-$  (orthogonal sum) holds and  $\mathbf{p}_\pm$  are the orthogonal projections to  $H_\pm$  respectively. A closed subspace  $W$  of  $H$  is  $z^2$ -invariant if it satisfies the condition:

$$z^2 W \subset W. \tag{2.1}$$



Any  $z^2$ -invariant subspace can be identified with a shift invariant subspace in the product space as follows. Product spaces of  $H, H_+, H_-$  are denoted by the bold  $\mathbf{H}, \mathbf{H}_+, \mathbf{H}_-$ , namely

$$\mathbf{H} = H \times H, \quad \mathbf{H}_+ = H_+ \times H_+, \quad \mathbf{H}_- = H_- \times H_-,$$

respectively. An identity  $\mathbf{H} = \mathbf{H}_+ \oplus \mathbf{H}_-$  holds and the associated orthogonal projections are denoted by  $\mathbf{p}_\pm$  again. For  $\mathbf{z} = {}^t(z_1, z_2), \mathbf{w} = {}^t(w_1, w_2) \in \mathbb{C}^2$  denote

$$\mathbf{z} \cdot \mathbf{w} = z_1 w_1 + z_2 w_2, \quad \|\mathbf{z}\| = \sqrt{\mathbf{z} \cdot \bar{\mathbf{z}}}, \quad \bar{\mathbf{z}} = {}^t(\bar{z}_1, \bar{z}_2).$$

For a function  $f(z) = \sum_n f_n z^n$  on  $\partial\mathbb{D}_r$  set

$$f_e(z) = \sum_n f_{2n} z^n, \quad f_o(z) = \sum_n f_{2n+1} z^n.$$

Then, we have an isomorphism

$$H \ni f \rightarrow \phi(f) = \begin{pmatrix} f_e \\ f_o \end{pmatrix} \in \mathbf{H}. \tag{2.2}$$

Then,  $\mathbf{W} \equiv \phi(W)$  for a  $z^2$ -invariant subspace  $W \subset H$  satisfies

$$z\mathbf{W} \subset \mathbf{W}. \tag{2.3}$$

A closed subspace  $\mathbf{W}$  of  $\mathbf{H}$  is called *shift invariant* if it satisfies (2.3). This  $\phi$  clearly defines an isomorphism between  $z^2$ -invariant subspaces  $W$  and shift invariant subspaces  $\mathbf{W}$ . We identify  $W$  with  $\mathbf{W}$  from now on.

An example of shift invariant subspace in  $\mathbf{H}$  is given by a  $2 \times 2$  non-singular matrix function  $A(z)$  on  $\partial\mathbb{D}_r$ . Assume every entry of  $A(z), A(z)^{-1}$  belongs to  $L^\infty(\partial\mathbb{D}_r)$ , and define

$$\mathbf{W} = A(z)\mathbf{H}_+ = \{\mathbf{f}(z) = A(z)\mathbf{u}(z) : \mathbf{u} \in \mathbf{H}_+\}. \tag{2.4}$$

Then, this  $\mathbf{W}$  is a shift invariant closed subspace of  $\mathbf{H}$ , since so is  $\mathbf{H}_+$ . Denote by  $Gr^{(2)}$  the set of all  $z^2$ -invariant closed subspaces  $W$  of  $H$  satisfying

$$\mathbf{p}_+ : W \rightarrow H_+ \text{ is bijective.} \tag{2.5}$$

$W \in Gr^{(2)}$  if and only if  $\mathbf{W} = \phi(W)$  is a shift invariant closed subspace of  $\mathbf{H}$  satisfying

$$\mathbf{p}_+ : \mathbf{W} \rightarrow \mathbf{H}_+ \text{ is bijective.} \tag{2.6}$$

The property (2.6) is equivalent to the invertibility of the associated Toeplitz operator. A Toeplitz operator  $T(a)$  with  $a \in L^\infty(\partial\mathbb{D}_r)$  is a bounded operator on  $H_+$  defined by

$$T(a)f = \mathbf{p}_+(af) \quad \text{for } f \in H_+.$$

For a bounded matrix function  $A(z)$  the associated (matrix) Toeplitz operator  $\mathbf{T}(A)$  is defined by

$$\mathbf{T}(A)f = \mathbf{p}_+(Af) \quad \text{for } \mathbf{f} \in \mathbf{H}_+.$$

In (2.4)  $\mathbf{W}$  is a shift invariant closed subspace of  $\mathbf{H}$ , and  $\mathbf{W}$  satisfies (2.6) if and only if  $\mathbf{T}(A)$  is invertible. Generally the invertibility of  $\mathbf{T}(A)$  is not easy to verify. However, there is a case where one can reduce the problem to the invertibility of a scalar Toeplitz operator as follows. For a bounded function  $m$  on  $\partial\mathbb{D}_r$  define

$$A(z) = \begin{pmatrix} 1 & m_e(z) \\ 0 & m_o(z) \end{pmatrix},$$

then

$$\mathbf{T}(A) = \begin{pmatrix} I & T(m_e) \\ 0 & T(m_o) \end{pmatrix}. \quad (2.7)$$

Since

$$\begin{pmatrix} I & T(m_e) \\ 0 & T(m_o) \end{pmatrix} \begin{pmatrix} I & -T(m_e)T(m_o)^{-1} \\ 0 & T(m_o)^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

$$\begin{pmatrix} I & -T(m_e)T(m_o)^{-1} \\ 0 & T(m_o)^{-1} \end{pmatrix} \begin{pmatrix} I & T(m_e) \\ 0 & T(m_o) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

$\mathbf{T}(A)$  is invertible if and only if so is  $T(m_o)$ . A sufficient condition for the invertibility of  $T(m_o)$  is given in the Lemma:

**Lemma 2.1.** *Let  $a$  be continuous on  $\mathbb{C} \setminus \mathbb{D}_r$  satisfying*

(i)  $a(z) \neq 0$  for any  $z \in \mathbb{C} \setminus \mathbb{D}_r$ .

(ii)  $a(z) - 1 \in H_-$ .

Then,  $T(a)$  is invertible on  $H_+$  and  $T(a)^{-1} = T(a^{-1})$  is valid.

*Proof.* Under the conditions (i), (ii)  $a$  maps  $H_-$  to  $H_-$  bijectively. For  $u \in H_+$  let

$$a(z)u(z) = f_+(z) + f_-(z) \quad \text{with } f_{\pm} \in H_{\pm}.$$

Then,  $f_+ = \mathbf{p}_+(au) = T(a)u$ , and

$$u = a^{-1}f_+ + a^{-1}f_- = \mathbf{p}_+(a^{-1}f_+) = T(a^{-1})f_+ = T(a^{-1})T(a)u,$$

hence  $T(a^{-1})T(a)u = u$  holds. Similarly one can prove  $T(a)T(a^{-1})u = u$ .  $\square$

An example of  $a$  is given by  $m$  defined in (1.6) by two Weyl functions  $m_{\pm}$ , and assume  $m$  is holomorphic on  $\mathbb{C} \setminus ([-r, r] \cup i[-r, r])$ . Then,  $T(m_o)$  defined on  $\{|z| = s^2\}$  is invertible for any  $s > r$ . This is because

$$m_o(z) = -\frac{m_+(-z) + m_-(-z)}{2\sqrt{z}},$$

and  $m_+ + m_-$  has positive imaginary part on  $\mathbb{C}_+$ .

### 3. Characteristic matrix

In this section we investigate the condition (2.5) (equivalently (2.6)) by introducing characteristic matrix.

Let  $H = L^2(\partial\mathbb{D}_r)$  with inner product

$$(f, g) = \frac{r}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} d\theta.$$

Set

$$(Jf)(z) = \bar{z}f(\bar{z}) \quad \text{for } f \in \mathbf{H} (= H \times H).$$

Then,  $J$  maps  $\mathbf{H}_+$  onto  $\mathbf{H}_-$  and satisfies

$$J^* = J, \quad J^2 = r^2I \quad \text{and} \quad zJ = J\bar{z}.$$

Define a dual object of a closed subspace  $\mathbf{W}$  of  $\mathbf{H}$  by

$$\widetilde{\mathbf{W}} = J\mathbf{W}^\perp.$$

For a closed subspace  $\mathbf{W} \in Gr^{(2)}$  (the correct notation is  $W \in Gr^{(2)}$ , however we abuse the notation) the operator associating  $\mathbf{f}_+ \in \mathbf{H}_+$  to a unique  $\mathbf{f}_- \in \mathbf{H}_-$  such that  $\mathbf{f}_+ + \mathbf{f}_- \in \mathbf{W}$  is denoted by  $A_{\mathbf{W}}$ .

**Lemma 3.1.** *The following assertions hold.*

- (i) *If  $\mathbf{W}$  satisfies  $z\mathbf{W} \subset \mathbf{W}$ , then so does  $\widetilde{\mathbf{W}}$ .*
- (ii) *If  $\mathbf{W}$  satisfies the condition of (2.6), so does  $\widetilde{\mathbf{W}}$ , and*

$$A_{\widetilde{\mathbf{W}}} = -r^{-2}JA_{\mathbf{W}}^*J.$$

*Proof.* For  $\mathbf{f} = J\mathbf{u} \in \widetilde{\mathbf{W}}$  with  $\mathbf{u} \in \mathbf{W}^\perp$

$$z\mathbf{f} = zJ\mathbf{u} = J(\bar{z}\mathbf{u})$$

and for  $\mathbf{v} \in \mathbf{W}$

$$(\bar{z}\mathbf{u}, \mathbf{v}) = (\mathbf{u}, z\mathbf{v}) = 0 \rightarrow \bar{z}\mathbf{u} \in \mathbf{W}^\perp$$

hence  $J(\bar{z}\mathbf{u}) \in J\mathbf{W}^\perp$ , and  $z\mathbf{f} \in \widetilde{\mathbf{W}}$ , which proves (i).

It follows from  $\mathbf{W} = \{\mathbf{f} + A_{\mathbf{W}}\mathbf{f} : \mathbf{f} \in \mathbf{H}_+\}$  that

$$\mathbf{W}^\perp = \{\mathbf{u} - A_{\mathbf{W}}^*\mathbf{u} : \mathbf{u} \in \mathbf{H}_-\}.$$

Hence

$$\widetilde{\mathbf{W}} \ni J\mathbf{u} - JA_{\mathbf{W}}^*\mathbf{u} = J\mathbf{u} - r^{-2}JA_{\mathbf{W}}^*JJu \quad \text{for } \mathbf{u} \in \mathbf{H}_-.$$

Since  $J : \mathbf{H}_- \rightarrow \mathbf{H}_+$  (bijective), the above identity shows that  $\widetilde{\mathbf{W}}$  satisfies (2.6) and simultaneously  $A_{\widetilde{\mathbf{W}}} = -r^{-2}JA_{\mathbf{W}}^*J$  holds.  $\square$

Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbf{H}_+.$$

For  $\mathbf{W} \in Gr^{(2)}$  set

$$\boldsymbol{\varphi}_{\mathbf{W}} = A\mathbf{W}\mathbf{e}_1, \quad \boldsymbol{\psi}_{\mathbf{W}} = A\mathbf{W}\mathbf{e}_2 \in \mathbf{H}_-,$$

and define a  $2 \times 2$  matrix by

$$\Pi_{\mathbf{W}}(z) = [\mathbf{e}_1 + \boldsymbol{\varphi}_{\mathbf{W}}(z), \mathbf{e}_2 + \boldsymbol{\psi}_{\mathbf{W}}(z)] = I + [\boldsymbol{\varphi}_{\mathbf{W}}(z), \boldsymbol{\psi}_{\mathbf{W}}(z)].$$

**Proposition 3.2.** For  $\mathbf{W} \in Gr^{(2)}$  the matrices  $\Pi_{\mathbf{W}}(z)$ ,  $\Pi_{\widetilde{\mathbf{W}}}(z)$  satisfy the following properties:

(i) For  $\mathbf{f} \in \mathbf{H}_+$

$$\begin{aligned} A\mathbf{W}z\mathbf{f} &= zA\mathbf{W}\mathbf{f} + r^{-1}(J\mathbf{f}, \boldsymbol{\varphi}_{\widetilde{\mathbf{W}}})(\mathbf{e}_1 + \boldsymbol{\varphi}_{\mathbf{W}}) \\ &\quad + r^{-1}(J\mathbf{f}, \boldsymbol{\psi}_{\widetilde{\mathbf{W}}})(\mathbf{e}_2 + \boldsymbol{\psi}_{\mathbf{W}}). \end{aligned} \quad (3.1)$$

(ii)  $\Pi_{\mathbf{W}}(z)$  is invertible for any  $z$  such that  $|z| > r$  and for a.e.  $z$  in  $\partial\mathbb{D}_r$ . Moreover, every entry of  $\Pi_{\mathbf{W}}(z)$ ,  $\Pi_{\mathbf{W}}(z)^{-1}$  belongs to  $\overline{\mathbf{H}_+}$ . Additionally  $\widetilde{\mathbf{W}} \in Gr^{(2)}$  is valid and it holds that

$$\Pi_{\mathbf{W}}(z) \overline{\Pi_{\widetilde{\mathbf{W}}}(\bar{z})} = I. \quad (3.2)$$

*Proof.* For  $\mathbf{f} \in \mathbf{H}_+$

$$\mathbf{W} \ni z\mathbf{f} + zA\mathbf{W}\mathbf{f} = (z\mathbf{f} + \mathfrak{p}_+zA\mathbf{W}\mathbf{f}) + \mathfrak{p}_-zA\mathbf{W}\mathbf{f},$$

hence

$$A\mathbf{W}(z\mathbf{f} + \mathfrak{p}_+zA\mathbf{W}\mathbf{f}) = \mathfrak{p}_-zA\mathbf{W}\mathbf{f} = zA\mathbf{W}\mathbf{f} - \mathfrak{p}_+zA\mathbf{W}\mathbf{f},$$

thus

$$A\mathbf{W}z\mathbf{f} = zA\mathbf{W}\mathbf{f} - \mathfrak{p}_+zA\mathbf{W}\mathbf{f} - A\mathbf{W}\mathfrak{p}_+zA\mathbf{W}\mathbf{f}. \quad (3.3)$$

Since, for  $\mathbf{u} = A\mathbf{W}\mathbf{f} \in \mathbf{H}_-$

$$\mathbf{u} = \sum_{n \geq 1} \mathbf{u}_n z^{-n} \rightarrow \mathfrak{p}_+(z\mathbf{u}) = \mathbf{u}_1 = r^{-1}(z\mathbf{u}, \mathbf{e}_1)\mathbf{e}_1 + r^{-1}(z\mathbf{u}, \mathbf{e}_2)\mathbf{e}_2,$$

$$\begin{aligned} \mathfrak{p}_+(z\mathbf{u}) &= r^{-1}(\mathbf{u}, \bar{z}\mathbf{e}_1)\mathbf{e}_1 + r^{-1}(\mathbf{u}, \bar{z}\mathbf{e}_2)\mathbf{e}_2 \\ &= r^{-1}(A\mathbf{W}\mathbf{f}, \bar{z}\mathbf{e}_1)\mathbf{e}_1 + r^{-1}(A\mathbf{W}\mathbf{f}, \bar{z}\mathbf{e}_2)\mathbf{e}_2 \\ &= r^{-3}(J\mathbf{f}, JA_{\widetilde{\mathbf{W}}}^*J\mathbf{e}_1)\mathbf{e}_1 + r^{-3}(J\mathbf{f}, JA_{\widetilde{\mathbf{W}}}^*J\mathbf{e}_2)\mathbf{e}_2 \\ &= -r^{-1}(J\mathbf{f}, A_{\widetilde{\mathbf{W}}}\mathbf{e}_1)\mathbf{e}_1 - r^{-1}(J\mathbf{f}, A_{\widetilde{\mathbf{W}}}\mathbf{e}_2)\mathbf{e}_2 \\ &= -r^{-1}(J\mathbf{f}, \boldsymbol{\varphi}_{\widetilde{\mathbf{W}}})\mathbf{e}_1 - r^{-1}(J\mathbf{f}, \boldsymbol{\psi}_{\widetilde{\mathbf{W}}})\mathbf{e}_2, \end{aligned}$$

thus (3.1) follows from (3.3). Since  $\widetilde{\mathbf{W}} \in Gr^{(2)}$  follows from Lemma 3.1, we have only to show (3.2). Applying (3.3) to  $\mathbf{f} = z^n\mathbf{e}_1$  ( $n \geq 0$ ) yields

$$A\mathbf{W}z^{n+1}\mathbf{e}_1$$

$$\begin{aligned}
 &= zA_{\mathbf{W}}z^n\mathbf{e}_1 + r^{-1} (Jz^n\mathbf{e}_1, \varphi_{\widetilde{\mathbf{W}}}) (\mathbf{e}_1 + \varphi_{\mathbf{W}}) + r^{-1} (Jz^n\mathbf{e}_1, \psi_{\widetilde{\mathbf{W}}}) (\mathbf{e}_2 + \psi_{\mathbf{W}}) \\
 &= zA_{\mathbf{W}}z^n\mathbf{e}_1 + r^{-1} (\bar{z}^{n+1}\mathbf{e}_1, \varphi_{\widetilde{\mathbf{W}}}) (\mathbf{e}_1 + \varphi_{\mathbf{W}}) + r^{-1} (\bar{z}^{n+1}\mathbf{e}_1, \psi_{\widetilde{\mathbf{W}}}) (\mathbf{e}_2 + \psi_{\mathbf{W}}),
 \end{aligned}$$

similarly

$$\begin{aligned}
 &A_{\mathbf{W}}z^{n+1}\mathbf{e}_2 \\
 &= zA_{\mathbf{W}}z^n\mathbf{e}_2 + r^{-1} (\bar{z}^{n+1}\mathbf{e}_2, \varphi_{\widetilde{\mathbf{W}}}) (\mathbf{e}_1 + \varphi_{\mathbf{W}}) + r^{-1} (\bar{z}^{n+1}\mathbf{e}_2, \psi_{\widetilde{\mathbf{W}}}) (\mathbf{e}_2 + \psi_{\mathbf{W}}),
 \end{aligned}$$

hence, for  $j = 1, 2$

$$A_{\mathbf{W}}z^n\mathbf{e}_j = z^n \left( A_{\mathbf{W}}\mathbf{e}_j + (\mathbf{e}_1 + \varphi_{\mathbf{W}}) \overline{\varphi_{\widetilde{\mathbf{W}},j,n}} + (\mathbf{e}_2 + \psi_{\mathbf{W}}) \overline{\psi_{\widetilde{\mathbf{W}},j,n}} \right), \tag{3.4}$$

holds, where

$$\varphi_{\widetilde{\mathbf{W}},j,n}(z) = r^{-1} \sum_{k=1}^n z^{-k} \left( \varphi_{\widetilde{\mathbf{W}}}, \bar{z}^k \mathbf{e}_j \right), \psi_{\widetilde{\mathbf{W}},j,n}(z) = r^{-1} \sum_{k=1}^n z^{-k} \left( \psi_{\widetilde{\mathbf{W}}}, \bar{z}^k \mathbf{e}_j \right),$$

and generally  $\overline{f}(z) = \overline{f(\bar{z})}$ . In a matrix form the identity (3.4) turns out to be

$$z^{-n} [A_{\mathbf{W}}z^n\mathbf{e}_1, A_{\mathbf{W}}z^n\mathbf{e}_2] = \Pi_{\mathbf{W}}(z) \overline{{}^t\Pi_{\widetilde{\mathbf{W}},n}(\bar{z})} - I$$

for any  $n \geq 0$ , where

$$\Pi_{\widetilde{\mathbf{W}},n}(z) = \begin{bmatrix} 1 + \varphi_{\widetilde{\mathbf{W}},1,n}(z) & \psi_{\widetilde{\mathbf{W}},1,n}(z) \\ \varphi_{\widetilde{\mathbf{W}},2,n}(z) & 1 + \psi_{\widetilde{\mathbf{W}},2,n}(z) \end{bmatrix}.$$

Noting

$$\begin{pmatrix} \varphi_{\widetilde{\mathbf{W}},1,n}(z) \\ \varphi_{\widetilde{\mathbf{W}},2,n}(z) \end{pmatrix} \rightarrow \varphi_{\widetilde{\mathbf{W}}}(z), \quad \begin{pmatrix} \psi_{\widetilde{\mathbf{W}},1,n}(z) \\ \psi_{\widetilde{\mathbf{W}},2,n}(z) \end{pmatrix} \rightarrow \psi_{\widetilde{\mathbf{W}}}(z) \text{ in } \mathbf{H}_-$$

as  $n \rightarrow \infty$  and

$$\|A_{\mathbf{W}}z^n\mathbf{e}_1\| \leq \|A_{\mathbf{W}}\| \|z^n\mathbf{e}_1\| = r^{n+1/2} \|A_{\mathbf{W}}\|,$$

we see

$$\Pi_{\mathbf{W}}(z) \overline{{}^t\Pi_{\widetilde{\mathbf{W}}}(\bar{z})} - I = 0 \text{ for } z \text{ such that } |z| > r,$$

by letting  $n \rightarrow \infty$  in (3.4), which completes the proof of (3.2) by letting  $|z| \rightarrow r$ . □

The identities (3.1), (3.2) show that the operator  $A_{\mathbf{W}}$  is uniquely determined by  $\{\varphi_{\mathbf{W}}, \psi_{\mathbf{W}}\}$ , since  $\mathbf{H}_+$  is generated by  $\{z^m\mathbf{e}_1, z^n\mathbf{e}_2\}_{m,n \geq 0}$ . This implies that  $\mathbf{W} \in Gr^{(2)}$  is uniquely determined by  $\Pi_{\mathbf{W}}$ , and we call  $\Pi_{\mathbf{W}}$  as the *characteristic matrix* of  $\mathbf{W}$  (or  $W$ ).

#### 4. Group action on $\overline{Gr}^{(2)}$ and $\tau$ -function

In this section we consider an commutative action on  $Gr^{(2)}$ . Set

$$\Gamma = \left\{ g = e^h : h \text{ holomorphic on } \mathbb{D}_s \text{ for some } s > r \text{ and } h(0) = 0 \right\}. \quad (4.1)$$

Then  $\Gamma$  is commutative and we can consider a closed subspace  $gW$  for  $W \in Gr^{(2)}$ . To investigate this action we define an operator  $R_W$  on  $H_+$  for  $W \in Gr^{(2)}$  by

$$R_W(g) = g^{-1} \mathfrak{p}_+ g A_W.$$

Then, the  $\tau$ -function is  $\tau_W(g) = \det(I + R_W(g))$ . To define this determinant the traceability of  $R_W(g)$  is required.

**Lemma 4.1.** *Suppose  $g_j \in \Gamma$  for  $j = 1, 2$ . Then*

$$\begin{aligned} & \|g_1^{-1} \mathfrak{p}_+ g_1 - g_2^{-1} \mathfrak{p}_+ g_2\|_{\text{trace}} \\ & \leq 3r^{-1/2} (\|g_1^{-1} - g_2^{-1}\| (\|g_1 - 1\| + r^2 \|g_1''\|) \\ & \quad + \|g_2^{-1}\| (\|g_1 - g_2\| + r^2 \|g_1'' - g_2''\|)). \end{aligned}$$

*Proof.* For  $f \in H_-$  and  $g_j \in H^2(\partial\mathbb{D}_r)$  let

$$f(z) = \sum_{n \geq 1} f_n z^{-n}, \quad g_j(z) = \sum_{n \geq 0} g_{jn} z^n.$$

Then

$$\begin{aligned} & \partial_\theta (\mathfrak{p}_+ g_1 f - \mathfrak{p}_+ g_2 f) (r e^{i\theta}) \\ & = i \sum_{m \geq 1, n-m \geq 0} (n-m) r^{n-m} (g_{1,n} - g_{2,n}) f_m e^{i(n-m)\theta} \\ & = i \sum_{m \geq 1} f_m \sum_{k \geq 0} k r^k e^{ik\theta} (g_{1,k+m} - g_{2,k+m}), \end{aligned}$$

hence

$$\begin{aligned} & \|(1 + \partial_\theta) (\mathfrak{p}_+ g_1 - \mathfrak{p}_+ g_2)\|_{HS}^2 \\ & = \sum_{m \geq 1} \frac{r}{2\pi} \int_0^{2\pi} \left| r^{m-1/2} \sum_{k \geq 0} (1 + ik) r^k e^{ik\theta} (g_{1,k+m} - g_{2,k+m}) \right|^2 d\theta \\ & = \sum_{m \geq 1} \sum_{k \geq m} r^{2k} (1 + (k-m)^2) |g_{1,k} - g_{2,k}|^2 \leq \sum_{k \geq 1} r^{2k} k^3 |g_{1,k} - g_{2,k}|^2, \end{aligned}$$

since  $\sum_{m=1}^k (1 + (k-m)^2) \leq k^3$  holds if  $k \geq 1$ . Note

$$\sum_{k \geq 1} r^{2k} k^3 |g_{1,k} - g_{2,k}|^2$$

$$\begin{aligned} &\leq |rg_{1,1} - rg_{2,1}|^2 + 2 \sum_{k \geq 2} k^2 (k-1)^2 \left| r^k g_{1,k} - r^k g_{2,k} \right|^2 \\ &\leq 2r^{-1} \left( \|g_1 - g_2\|^2 + r^4 \|g_1'' - g_2''\|^2 \right). \end{aligned}$$

Since

$$\left\| (1 + \partial_\theta)^{-1} \right\|_{HS}^2 = \sum_{k \geq 0} |1 + ik|^{-2} \leq 1 + \frac{\pi^2}{6} < 4 < \infty,$$

thus

$$\begin{aligned} &\| \mathfrak{p}_+ g_1 - \mathfrak{p}_+ g_2 \|_{\text{trace}} \\ &\leq \left\| (1 + \partial_\theta)^{-1} \right\|_{HS} \| (1 + \partial_\theta) (\mathfrak{p}_+ g_1 - \mathfrak{p}_+ g_2) \|_{HS} \\ &\leq 2\sqrt{2}r^{-1/2} (\|g_1 - g_2\| + r^2 \|g_1'' - g_2''\|). \end{aligned}$$

Consequently

$$\begin{aligned} &\| g_1^{-1} \mathfrak{p}_+ g_1 - g_2^{-1} \mathfrak{p}_+ g_2 \|_{\text{trace}} \\ &\leq \| g_1^{-1} - g_2^{-1} \| \| \mathfrak{p}_+ g_1 \|_{\text{trace}} + \| g_2^{-1} \| \| \mathfrak{p}_+ g_1 - \mathfrak{p}_+ g_2 \|_{\text{trace}} \\ &\leq 3r^{-1/2} (\|g_1^{-1} - g_2^{-1}\| (\|g_1 - 1\| + r^2 \|g_1''\|) \\ &\quad + \|g_2^{-1}\| (\|g_1 - g_2\| + r^2 \|g_1'' - g_2''\|)) \end{aligned}$$

which completes the proof. □

This Lemma shows that  $R_W(g)$  is of trace class and  $\tau_W(g)$  can be defined if  $g \in \Gamma$ . This  $\tau_W(g)$  is called as  $\tau$ -function and plays a crucial role in Sato's theory.

**Lemma 4.2.**  $gW \in Gr^{(2)}$  holds if and only if  $\ker(I + R_W(g)) = \{0\}$  is valid. In this case, the  $A$ -operator corresponding to  $gW$  is given by

$$A_{gW} = \mathfrak{p}_- g^{-1} A_W (I + R_W(g))^{-1} g.$$

*Proof.* Since  $R_W(g)$  is a compact operator,  $\ker(I + R_W(g)) = \{0\}$  implies the existence of  $(I + R_W(g))^{-1}$  as a bounded operator on  $H_+$ . Set

$$B = \mathfrak{p}_- g^{-1} A_W (I + R_W(g))^{-1} g.$$

For  $f \in H_+$ , an identity

$$\begin{aligned} g^{-1}(f + Bf) &= g^{-1}f + g^{-1} \mathfrak{p}_- g A_W (I + R_W(g))^{-1} g^{-1} f \\ &= g^{-1}f + A_W (I + R_W(g))^{-1} g^{-1} f - R_W(g) (I + R_W(g))^{-1} g^{-1} f \\ &= (I + R_W(g))^{-1} g^{-1} f + A_W (I + R_W(g))^{-1} g^{-1} f \end{aligned} \tag{4.2}$$

is valid, hence  $g^{-1}(f + Bf) \in W$  and  $gW \supset \{f + Bf : f \in H_+\}$  holds. Conversely,  $f = g(I + R_W(g))u \in H_+$  for  $u \in H_+$  satisfies  $g^{-1}(f + Bf) = u + A_W u \in W$  due to (4.2). Hence we have

$$gW = \{f + Bf; f \in H_+\},$$

which implies  $gW \in Gr^{(2)}$  and  $B = A_{gW}$ .

Now suppose  $f \in \ker(I + R_W(g))$ , then, from an identity  $gf + \mathfrak{p}_+(gA_W f) = 0$  it follows that

$$\mathfrak{p}_-(gA_W f) = gA_W f - \mathfrak{p}_+(gA_W f) = g(f + A_W f) \in gW.$$

Therefore  $gW \in Gr^{(2)}$  implies  $\mathfrak{p}_-(gA_W f) = 0$ . Hence,  $f + A_W f = 0$  and  $f = 0$  holds, which completes the proof.  $\square$

**Proposition 4.3.**  $\tau_W(g)$  satisfies the following properties.

- (i)  $gW \in Gr^{(2)}$  holds for  $g \in \Gamma$  and  $W \in Gr^{(2)}$  if and only if  $\tau_W(g) \neq 0$ .
- (ii) For  $g_1, g_2 \in \Gamma$ ,  $W \in Gr^{(2)}$  suppose  $g_1W \in Gr^{(2)}$  (equivalently  $\tau_W(g_1) \neq 0$ ). Then

$$\tau_W(g_1g_2) = \tau_W(g_1)\tau_{g_1W}(g_2) \quad (\text{cocycle property}). \quad (4.3)$$

- (iii) If  $g = e^h \in \Gamma$ , and  $g_1(z) = e^{h_e(z^2)}$ ,  $g_2(z) = e^{zh_o(z^2)}$ , then

$$\tau_W(g) = \tau_W(g_1)\tau_W(g_2).$$

- (iv)  $\tau_W(g)$  is continuous on  $\Gamma$  with Sobolev  $H^2$ -norm.

*Proof.* Since  $\tau_W(g) = 0$  if and only if  $\ker(I + R_W(g)) = \{0\}$ , (i) is valid by Lemma 4.2. To show (ii) note

$$\begin{aligned} (g_1g_2)A_W &= g_2\mathfrak{p}_+g_1A_W + g_2\mathfrak{p}_-g_1A_W \\ &= g_2g_1R_W(g_1) + g_2A_{g_1W}g_1(I + R_W(g_1)), \end{aligned}$$

and

$$\mathfrak{p}_+((g_1g_2)A_W) = g_2g_1R_W(g_1) + \mathfrak{p}_+(g_2A_{g_1W}g_1(I + R_W(g_1))),$$

hence

$$\begin{aligned} I + R_W(g_1g_2) &= I + R_W(g_1) + g_1^{-1}g_2^{-1}\mathfrak{p}_+(g_2A_{g_1W}g_1(I + R_W(g_1))) \\ &= I + R_W(g_1) + g_1^{-1}R_{g_1W}(g_2)g_1(I + R_W(g_1)) \\ &= g_1^{-1}(I + R_{g_1W}(g_2))g_1(I + R_W(g_1)). \end{aligned}$$

Consequently, if  $g_1W \in Gr^{(2)}$ , then we have (ii). (iii) follows immediately from (ii) if we notice  $g_1W = W$ . (iv) is a direct consequence of Lemma 4.1.  $\square$

The entries of characteristic matrices can be obtained from  $\tau_W(g)$  by choosing  $g$  appropriately. Let

$$\varphi_W = A_W 1, \quad \psi_W = A_W z \in H_-, \quad (\text{then } \varphi_W = {}^t(\varphi_e, \varphi_o), \quad \psi_W = {}^t(\psi_e, \psi_o)).$$



An element  $q_\zeta$  of  $\Gamma$  defined by

$$q_\zeta(z) = (1 - z\zeta^{-1})^{-1}$$

for  $\zeta \in \mathbb{C}$  such that  $|\zeta| > r$  plays a crucial role in the  $\Gamma$ -action, since any  $g \in \Gamma$  can be expressed as a limit of  $q_{\zeta_1}q_{\zeta_2} \cdots q_{\zeta_n}$ . For  $f \in H_-$  we have a decomposition of  $q_\zeta f$  into  $H_- \oplus H_+$

$$q_\zeta f(z) = \left(1 - \frac{z}{\zeta}\right)^{-1} f(z) = \left(1 - \frac{z}{\zeta}\right)^{-1} (f(z) - f(\zeta)) + \left(1 - \frac{z}{\zeta}\right)^{-1} f(\zeta),$$

which yields

$$\left(q_\zeta^{-1}\mathfrak{p}_+q_\zeta f\right)(z) = \left(1 - \frac{z}{\zeta}\right) \left(1 - \frac{z}{\zeta}\right)^{-1} f(\zeta) = f(\zeta).$$

Hence, if  $W \in Gr^{(2)}$ , then for  $f \in H_+$

$$\left(q_\zeta^{-1}\mathfrak{p}_+q_\zeta A_W f\right)(z) = (A_W f)(\zeta) = (A_W f)(\zeta) 1$$

holds, which implies that  $q_\zeta^{-1}\mathfrak{p}_+q_\zeta A_W$  is a linear operator of rank 1. Thus

$$\tau_W(q_\zeta) = \det\left(I + q_\zeta^{-1}\mathfrak{p}_+q_\zeta A_W\right) = 1 + (A_W 1)(\zeta) = 1 + \varphi_W(\zeta) \tag{4.4}$$

follows. Further calculations on  $\tau$ -functions can be found in the Appendix.

The next role of the  $\tau$ -function is to express a potential of Schrödinger operator for a given  $W \in Gr^{(2)}$ . Let  $\{e_x\}_{x \in \mathbb{C}}$  be a one parameter group of elements of  $\Gamma$  defined by

$$e_x(z) = e^{xz}.$$

**Lemma 4.4.** *Suppose  $\tau_W(e_x) \neq 0$  for  $x$  in a domain  $D$  of  $\mathbb{C}$ . Then, the function*

$$f_W(x, \zeta) = e^{-x\zeta} (1 + \varphi_{e_x W}(\zeta)) = e^{-x\zeta} \tau_{e_x W}(q_\zeta) \in W \text{ (as a function of } \zeta) \tag{4.5}$$

satisfies

$$-f_W''(x, z) + q_W(x)f_W(x, z) = -z^2 f_W(x, z), \tag{4.6}$$

namely  $f_W$  is a Baker–Akhiezer function for the Schrödinger operator  $L_{q_W}$  in  $D$ . Moreover, if  $\{a_n(x)\}_{n \geq 1}$  is defined as coefficients of an expansion

$$f_W(x, \cdot) = e^{-xz} \left(1 + \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \cdots\right),$$

then

$$a_1(x) = \partial_x \log \tau_W(e_x), \tag{4.7}$$

and  $q_W$  is given by

$$q_W(x) = -2\partial_x^2 \log \tau_W(e_x). \tag{4.8}$$

*Proof.* Since  $e_x W \in Gr^{(2)}$  for  $x \in D$ , there exists uniquely  $u \in e_x W$  such that  $\mathbf{p}_+ u = 1$ . Set  $f_W(x, z) = e_x^{-1} u \in W$ . Then, the calculation in the Introduction shows that  $f_W$  satisfies the equation (4.6). The formula (4.7) is verified as follows. Since

$$e^{x\zeta} f_W(x, \zeta) = 1 + (A_{e_x W} 1)(\zeta) = \tau_{e_x W}(q_\zeta) = \frac{\tau_W(e_x q_\zeta)}{\tau_W(e_x)}$$

is valid, we see

$$\begin{aligned} a_1(x) &= \lim_{\zeta \rightarrow \infty} \zeta \left( e^{x\zeta} f_W(x, \zeta) - 1 \right) = \lim_{\zeta \rightarrow \infty} \zeta \frac{\tau_W(e_x q_\zeta) - \tau_W(e_x)}{\tau_W(e_x)} \\ &= \lim_{\zeta \rightarrow \infty} \zeta \frac{\tau_W(e_{x+\zeta^{-1}}) - \tau_W(e_x)}{\tau_W(e_x)} = \partial_x \log \tau_W(e_x), \end{aligned}$$

which yields (4.7).  $\square$

Since  $\tau_W(e_x)$  is an entire function of  $x$ , we see that  $q_W(x)$  is meromorphic on  $\mathbb{C}$ , and it has poles of degree 2 on  $\{x \in \mathbb{C} : \tau_W(e_x) = 0\}$ . Proposition 4.3 and Lemma 4.4 are rearrangement of the corresponding results obtained by [14].

The formula (4.8) defines a map from  $W \in Gr^{(2)}$  to a space of potentials, however this map is not injective. Later we will see a quantity of  $W$ , which will be called as  $m$ -function, determines  $q_W$ .

## 5. $m$ -function

In this section we define a crucial quantity of  $W \in Gr^{(2)}$  which determines  $q_W$ . Let  $f_W(x, z)$  be the function introduced in Lemma 4.4 and define the  $m$ -function for  $W \in Gr^{(2)}$  by

$$m_W(z) = -\frac{f'_W(0, z)}{f_W(0, z)}. \quad (5.1)$$

$m_W$  can be described by the elements of the characteristic matrix as follows. Setting

$$f_W(x, z) = e^{-xz} \left( 1 + \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \dots \right) \in W,$$

we have

$$\begin{aligned} f_W(0, z) &= 1 + \frac{a_1(0)}{z} + \frac{a_2(0)}{z^2} + \dots = 1 + \varphi_W(z), \\ f'_W(0, z) &= -z - a_1(0) + \frac{a'_1(0) - a_2(0)}{z} + \dots \\ &= -z - \psi_W(z) - a_1(0)(1 + \varphi_W(z)), \end{aligned}$$

due to  $f_W(0, z), f'_W(0, z) \in W$ , hence

$$m_W(z) = \frac{z + \psi_W(z)}{1 + \varphi_W(z)} + a_1(W), \quad (5.2)$$

where we have defined

$$a_1(W) = a_1(0) = \lim_{z \rightarrow \infty} z\varphi_W(z). \tag{5.3}$$

One of the importance of  $m$ -functions lies in its close relationship with  $\tau$ -functions. Namely, one can decompose  $\tau_W(g)$  into the two factors, one of which is a group homomorphism from  $\Gamma$  to  $\mathbb{C}$  and the other part depends only on  $m_W$ .

Since

$$1 + \varphi_W(z) = 1 + \frac{a_1(0)}{z} + \frac{a_2(0)}{z^2} + \dots,$$

there exists  $r_W > r$  such that  $1 + \varphi_W(z) \neq 0$  on  $\{|z| \geq r_W\}$ . Let

$$\log(1 + \varphi_W(z)) = b_1z^{-1} + b_2z^{-2} + \dots.$$

Set

$$\Gamma_W = \left\{ g = e^h; h \text{ holomorphic on } \{|z| < r_W + \epsilon\} \text{ for some } \epsilon > 0 \right\}, \tag{5.4}$$

and define

$$\rho_W(e^h) = \exp\left(\sum_{k=1}^{\infty} kb_k h_k\right) = \exp\left(\frac{1}{2\pi i} \int_{|z|=r_W} h'(z) \log(1 + \varphi_W(z)) dz\right) \tag{5.5}$$

for  $h(z) = \sum_{k=1}^{\infty} h_k z^k$ . Then, (5.5) is convergent for  $g \in \Gamma_W$ . If  $g = q_\zeta$  with  $|\zeta| > r_W$ , then, in view of  $\log q_\zeta(z) = \sum_{k=1}^{\infty} \zeta^{-k} z^k / k$

$$\rho_W(q_\zeta) = \exp\left(\sum_{k=1}^{\infty} \zeta^{-k} b_k\right) = 1 + \varphi_W(\zeta) \tag{5.6}$$

holds. For a holomorphic function  $m$  on  $\{|z| > r\}$  and  $\zeta \in \{|z| > r\}$  define

$$(d_\zeta m)(z) = \frac{z^2 - \zeta^2}{m(z) - m(\zeta)} - m(\zeta).$$

One can see easily that  $d_{\zeta_1} d_{\zeta_2} = d_{\zeta_2} d_{\zeta_1}$ . Then, (8.4) in the Appendix implies that  $m_{q_{\zeta_1} q_{\zeta_2} \dots q_{\zeta_n} W}(z)$  for  $|\zeta_k| > r, k = 1, 2, \dots, n$  is generated from  $m_W$  by

$$m_{q_{\zeta_1} q_{\zeta_2} \dots q_{\zeta_n} W}(z) = (d_{\zeta_1} d_{\zeta_2} \dots d_{\zeta_n} m_W)(z).$$

For  $g = q_{\zeta_1} q_{\zeta_2} \dots q_{\zeta_n}$  with  $|\zeta_k| > r, k = 1, 2, \dots, n$  define  $\tau_m(g)$  inductively by

$$\begin{cases} \tau_m(q_{\zeta_1}) = 1, \\ \tau_m(q_{\zeta_1} q_{\zeta_2} \dots q_{\zeta_n}) \tau_m(q_{\zeta_1} q_{\zeta_2} \dots q_{\zeta_{n-1}})^{-1} \\ \quad = \frac{(d_{\zeta_1} d_{\zeta_2} \dots d_{\zeta_{n-k-1}} m)(\zeta_n) - (d_{\zeta_1} d_{\zeta_2} \dots d_{\zeta_{n-k-1}} m)(\zeta_{n-k})}{\zeta_n - \zeta_{n-k}}. \end{cases} \tag{5.7}$$

**Proposition 5.1.**  $\rho_W$  satisfies

$$\rho_W(g_1 g_2) = \rho_W(g_1) \rho_W(g_2), \quad (5.8)$$

for any  $g_1, g_2 \in \Gamma_W$ . Moreover,  $\tau_m(g)$  is extendable to  $\Gamma_W$  if  $m = m_W$  so that

$$\tau_W(g) = \rho_W(g) \tau_{m_W}(g) \quad (5.9)$$

holds, and  $\tau_{m_W}(g)$  depends on  $W$  only through  $m_W$ , namely if  $m_{W_1} = m_{W_2}$  for  $W_1, W_2 \in Gr^{(2)}$ , then  $\tau_{m_{W_1}}(g) = \tau_{m_{W_2}}(g)$  holds.

*Proof.* (5.8) follows easily from the definition. We show (5.9) for  $g = q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_n}$  with  $|\zeta_k| > r_W$ ,  $k = 1, 2, \dots, n$ . If  $n = 2$ , then, from (8.5) in the Appendix

$$\tau_W(q_{\zeta_1} q_{\zeta_2}) = (1 + \varphi_W(\zeta_1))(1 + \varphi_W(\zeta_2)) \frac{m_W(\zeta_1) - m_W(\zeta_2)}{\zeta_1 - \zeta_2},$$

so that in this case

$$\frac{\tau_W(q_{\zeta_1} q_{\zeta_2})}{\rho_W(q_{\zeta_1} q_{\zeta_2})} = \frac{m_W(\zeta_1) - m_W(\zeta_2)}{\zeta_1 - \zeta_2} = \tau_{m_W}(q_{\zeta_1} q_{\zeta_2}).$$

Assume

$$\tau_W(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-1}}) = \rho_W(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-1}}) \tau_{m_W}(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-1}}).$$

Then, from (4.3) and (4.4)

$$\begin{aligned} & \tau_W(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-1}} q_{\zeta_n}) \\ &= \tau_W(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-1}}) \tau_{q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-1}} W}(q_{\zeta_n}) \\ &= \rho_W(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-1}}) \tau_{m_W}(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-1}}) \left(1 + \varphi_{q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-1}} W}(\zeta_n)\right) \end{aligned}$$

follows. On the other hand, iterated use of (8.1) shows

$$\begin{aligned} & 1 + \varphi_{q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-1}} W}(\zeta_n) \\ &= (1 + \varphi_W(\zeta_n)) \prod_{k=1}^{n-1} \frac{m_{q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-k-1}} W}(\zeta_n) - m_{q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-k-1}} W}(\zeta_{n-k})}{\zeta_n - \zeta_{n-k}}. \end{aligned}$$

Thus, for  $g = q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_n}$  (5.9) is valid. At each step  $q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_{n-k-1}} W \in Gr^{(2)}$  should be examined. However, the final identity (5.9) implies that we have only to take some limit if necessary. For general  $g = e^h \in \Gamma_W$  let  $h_m(z) = \sum_{k=1}^m h_k z^k$ , and  $\{\zeta_k^{(n)}\}_{1 \leq k \leq m}$  be the all zeros of  $1 - h_m(z)/n$ . One can assume  $|\zeta_k^{(n)}| > r_W$ , because  $1 - h_m(z)/n \rightarrow 1$  as  $n \rightarrow \infty$ . Since

$$g_n^{(m)}(z) \equiv \left(1 - \frac{h_m(z)}{n}\right)^{-n} = \left(q_{\zeta_1^{(n)}}(z) q_{\zeta_2^{(n)}}(z) \cdots q_{\zeta_m^{(n)}}(z)\right)^n,$$

the identity (5.9) is valid for  $g_n^{(m)} \in \Gamma_W$ . Then, the continuity of  $\tau_W$  and  $\rho_W$  show that  $\tau_{m_W}$  is extendable by letting  $n \rightarrow \infty$  and  $m \rightarrow \infty$ .  $\square$

Corollary below shows that the non-vanishing property of  $\tau_W(g)$  on  $\Gamma$  is determined by the  $m$ -function.

**Corollary 5.2.** *For  $W_1, W_2 \in Gr^{(2)}$  assume  $m_{W_1}(z) = m_{W_2}(z)$ . Then, for  $g \in \Gamma$  it holds that  $\tau_{W_1}(g) \neq 0$  is valid if and only if  $\tau_{W_2}(g) \neq 0$ .*

*Proof.* Assume  $\tau_{W_1}(g) \neq 0$ . Then,  $\tau_{W_1}(e^{h_n}) \neq 0$  for every sufficiently large  $n$ , where  $h_n(z) = \sum_{k=1}^n h_k z^k$ , and Proposition 5.1 implies

$$\tau_{W_2}(\tilde{g}e^{h_n}) = \frac{\rho_{W_2}(\tilde{g}) \rho_{W_2}(e^{h_n})}{\rho_{W_1}(\tilde{g}) \rho_{W_1}(e^{h_n})} \tau_{W_1}(\tilde{g}e^{h_n})$$

for any  $\tilde{g} \in \Gamma_{W_1} \cap \Gamma_{W_2}$ . Since  $\tau_{W_1}(\tilde{g}e^{h_n}), \tau_{W_2}(\tilde{g}e^{h_n}) \rightarrow \tau_{W_1}(\tilde{g}g), \tau_{W_2}(\tilde{g}g)$  respectively, there exists a  $c \in \mathbb{C}$  such that  $\rho_{W_2}(e^{h_n})/\rho_{W_1}(e^{h_n}) \rightarrow c$ , and

$$\tau_{W_2}(\tilde{g}g) = c \frac{\rho_{W_2}(g_1)}{\rho_{W_1}(g_1)} \tau_{W_1}(\tilde{g}g)$$

holds. Suppose  $c = 0$ . Then,  $\tau_{W_2}(\tilde{g}g) = 0$  for any  $\tilde{g} \in \Gamma_{W_1} \cap \Gamma_{W_2}$ , which contradicts  $\tau_{W_2}(1) = 1$ , if we choose  $\tilde{g} = e^{-h_n}$ . Therefore, we have  $c \neq 0$ , which shows  $\tau_{W_2}(g) \neq 0$ . □

**Corollary 5.3.** *For  $W \in Gr^{(2)}$  it holds that*

$$q_W(x) = -2\partial_x^2 \log \tau_{m_W}(e_x).$$

*Proof.* Since  $\rho_W(e_x) = e^{xb_1}$ , Proposition 5.1 completes the proof. □

To show the continuity of  $m_{gW}$  with respect to  $g$  we need a representation of  $m_W$  by the  $\tau$ -functions.

**Lemma 5.4.** *For  $W \in Gr^{(2)}$  we have*

$$m_W(\zeta) = \zeta + \frac{1}{2\pi i} \int_{|\omega|=R} \left( \tau_W(q_\omega) - \frac{\tau_W(q_\zeta q_\omega)}{\tau_W(q_\zeta)} \right) d\omega$$

for any  $R > r$  and  $\zeta$  such that  $\tau_W(q_\zeta) = 1 + \varphi_W(\zeta) \neq 0$ .

*Proof.* (8.3) reads

$$m_W(\zeta) = \zeta + a_1(W) - a_1(q_\zeta W)$$

with  $a_1(W)$  the first coefficient of the expansion for  $\varphi_W$ , hence

$$a_1(W) = \frac{1}{2\pi i} \int_{|\omega|=R} \varphi_W(\omega) d\omega.$$

Since  $\varphi_W(\zeta) = \tau_W(q_\zeta) - 1$ , we have

$$\begin{aligned} m_W(\zeta) &= \zeta + \frac{1}{2\pi i} \int_{|\omega|=R} (\tau_W(q_\omega) - \tau_{q_\zeta W}(q_\omega)) d\omega \\ &= \zeta + \frac{1}{2\pi i} \int_{|\omega|=R} \left( \tau_W(q_\omega) - \frac{\tau_W(q_\zeta q_\omega)}{\tau_W(q_\zeta)} \right) d\omega. \end{aligned}$$

□

**Proposition 5.5.** *Suppose  $g_n, g \in \Gamma$  and  $\tau_W(g_n) \neq 0, \tau_W(g) \neq 0$ . If  $g_n \rightarrow g$  in  $H^2(\partial\mathbb{D}_r)$ , then  $m_{g_n W}(\zeta) \rightarrow m_{g W}(\zeta)$  for any  $\zeta$  such that  $|\zeta| > r, \tau_W(q_\zeta g) \neq 0$ .*

*Proof.* The integral representation of Lemma 5.4 completes the proof.  $\square$

The following proposition says that  $q_W$  is determined by  $m_W$ .

**Proposition 5.6.** *For  $W_1, W_2 \in Gr^{(2)}$  we have  $q_{W_1} = q_{W_2}$  if and only if  $m_{W_1} = m_{W_2}$ . Moreover, assume  $m_{W_1} = m_{W_2}$ . Then, it holds that*

$$m_{gW_1}(z) = m_{gW_2}(z) \quad (5.10)$$

for any  $g \in \Gamma$  such that  $\tau_{W_1}(g) \neq 0$  (hence  $\tau_{W_2}(g) \neq 0$ ).

*Proof.* Set

$$\frac{f_W(x, z)}{f_W(0, z)} = e^{-xz} \left( 1 + \frac{\tilde{a}_1(x)}{z} + \frac{\tilde{a}_2(x)}{z^2} + \cdots \right).$$

Then, (1.2) implies

$$\begin{cases} q_W(x) = -2\tilde{a}'_1(x) \quad (\text{since } \tilde{a}_1(x) = a_1(x) - a_1(0)), \\ 2\tilde{a}'_{k+1}(x) - \tilde{a}''_k(x) - 2\tilde{a}'_1(x)\tilde{a}_k(x) = 0, \quad k = 1, 2, \dots \end{cases}$$

Due to  $\tilde{a}_k(0) = 0$  for any  $k = 1, 2, \dots$  we see that  $q_W$  determines  $\frac{f_W(x, z)}{f_W(0, z)}$ . Keeping this in mind, suppose  $q_{W_1} = q_{W_2}$ . Then

$$\frac{f_{W_1}(x, z)}{f_{W_1}(0, z)} = \frac{f_{W_2}(x, z)}{f_{W_2}(0, z)}$$

holds, which implies

$$\frac{f'_{W_1}(0, z)}{f_{W_1}(0, z)} = \frac{f'_{W_2}(0, z)}{f_{W_2}(0, z)}$$

and  $m_{W_1}(z) = m_{W_2}(z)$  follows. Conversely, if  $m_{W_1}(z) = m_{W_2}(z)$ , then Corollary 5.3 shows  $q_{W_1} = q_{W_2}$ .

To show the identity (5.10) assume  $m_{W_1}(z) = m_{W_2}(z)$ . Then, (8.4) shows that  $m_{gW} = d_{\zeta_1} d_{\zeta_2} \cdots d_{\zeta_n} m_W$  for  $g = q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_n}$ , hence Proposition 5.5 shows  $m_{gW_1}(z) = m_{gW_2}(z)$  for general  $g \in \Gamma$ .  $\square$

## 6. KdV flow

Let  $\Gamma$  be a commutative group and  $\mathcal{Q}$  be a set. Suppose there exists a set of maps  $\{K(g)\}_{g \in \Gamma}$  on  $\mathcal{Q}$  satisfying  $K(g_1 g_2) = K(g_1)K(g_2)$  for any  $g_1, g_2 \in \Gamma$ , we call  $\{K(g)\}_{g \in \Gamma}$  as a flow on  $\mathcal{Q}$ . The purpose of this section is to construct such a flow on a certain set of potentials  $\mathcal{Q}$  and a subgroup of the previous  $\Gamma$ .

**6.1.  $m$ -function and Weyl functions.** In the last section we defined the  $m$ -function  $m_W$  for any  $W \in Gr^{(2)}$ . On the other hand, for  $W \in Gr^{(2)}$  a potential  $q_W$  was introduced by (4.8), and if  $q_W$  takes real values, then one can define the Weyl functions  $m_{\pm}$ . If the Baker–Akhiezer function  $f_W$  belongs to  $L^2(\mathbb{R}_+)$ , then we have  $m(z) = -m_+(-z^2)$ . In this subsection we investigate this identity by imposing an additional condition on  $W \in Gr^{(2)}$ , namely  $\tau_W(g) \neq 0$  for any real  $g \in \Gamma$ .

Recall  $\bar{f}(z) = \overline{f(\bar{z})}$  for  $f \in H = L^2(\partial\mathbb{D}_r)$ , and set  $\bar{W} = \{f \in H : \bar{f} \in W\}$  for  $W \in Gr^{(2)}$ . Then, clearly  $\bar{W} \in Gr^{(2)}$  holds and an identity  $\Pi_{\bar{W}} = \overline{\Pi_W}$  is straightforward.  $W \in Gr^{(2)}$  is called *real* if  $W = \bar{W}$ , and this is the case if and only if  $\Pi_W = \overline{\Pi_W}$  is valid. Define

$$\begin{cases} \Gamma_{\text{real}} = \{g \in \Gamma : g = \bar{g}\}, \\ Gr_{\text{real}}^{(2)} = \{W \in Gr^{(2)} : W = \bar{W}\}, \\ Gr_+^{(2)} = \{W \in Gr_{\text{real}}^{(2)} : \tau_W(g) \geq 0 \text{ for any } g \in \Gamma_{\text{real}}\}. \end{cases}$$

If  $W \in Gr_{\text{real}}^{(2)}$  and  $g \in \Gamma_{\text{real}}$ , then  $\tau_W(g) \in \mathbb{R}$ . Recall  $q_{\zeta}(z) = (1 - z/\zeta)^{-1}$  and define a dual object

$$p_{\zeta}(z) = 1 + z/\zeta = q_{-\zeta}(z)^{-1}.$$

**Lemma 6.1.** *The followings are valid.*

- (i) *If  $\tau_W(g) \geq 0$  holds for any  $g$  of a form  $g = \prod_{k=1}^n q_{\zeta_k} q_{\bar{\zeta}_k}$  with  $\zeta_k \in \{|z| > r\}$  and  $\text{Im } \zeta_k \neq 0$ , then  $W \in Gr_+^{(2)}$  is valid.*
- (ii) *Suppose  $\tau_W(q_{\zeta} q_{\bar{\zeta}}) \geq 0$  for any  $\zeta \in \{|z| > r\}$  for a  $W \in Gr_{\text{real}}^{(2)}$ . Then  $\tau_W(q_{\zeta} q_{\bar{\zeta}}) > 0$  holds for any  $\zeta \in \{|z| > r\}$ .*
- (iii) *Assume  $W \in Gr_+^{(2)}$ . Then,  $\tau_W(g) > 0$  holds for any  $g \in \Gamma_{\text{real}}$  such that  $g(z) = \prod_{k=1}^n q_{\zeta_k} q_{\bar{\zeta}_k}$  or  $g(z) = \prod_{k=1}^n p_{\zeta_k} p_{\bar{\zeta}_k}$  with  $\zeta_k \in \{|z| > r\}$  and  $\text{Im } \zeta_k \neq 0$ .*

*Proof.* To show (i) let  $g = e^h \in \Gamma_{\text{real}}$ ,  $h(z) = \sum_{k=1}^{\infty} h_k z^k$  and  $h_n(z) = \sum_{k=1}^{2n} h_k z^k$ . Set  $g_n(z) = (1 - h_n(z)/n)^{-n}$ . Then,  $g(z) = \lim_{n \rightarrow \infty} g_n(z)$  holds. We have only to show  $\tau_W(g_n) \geq 0$  for sufficiently large  $n$  due to the continuity of  $\tau_W$ . One can assume  $h_{2n} \neq 0$  and  $1 - h_n(z)/n$  has no real zeros, since, otherwise we have only to deform slightly  $h$ . Then, there exist  $\zeta_k \in \{|z| > r\}$  and  $\text{Im } \zeta_k \neq 0$  for  $k = 1, 2, \dots, n$  such that

$$g_n(z) = \left(1 - \frac{h_n(z)}{n}\right)^{-n} = \left(\prod_{k=1}^n q_{\zeta_k}(z) q_{\bar{\zeta}_k}(z)\right)^n \tag{6.1}$$

holds, which proves (i).

To prove (ii) note that

$$\det \Pi_W(z^2) = (1 + \varphi_{W,e}(z^2))(1 + \psi_{W,o}(z^2)) - \varphi_{W,o}(z^2)\psi_{W,e}(z^2)$$

$$= \frac{(1 + \varphi_W(-z))(z + \psi_W(z)) - (1 + \varphi_W(z))(-z + \psi_W(-z))}{2z}$$

is valid, hence  $1 + \varphi_W(z)$  and  $z + \psi_W(z)$  do not vanish simultaneously due to  $\det H_W(z) \neq 0$ . Suppose

$$\tau_W(q_{\zeta_1} q_{\bar{\zeta}_1}) = 0 \text{ and } 1 + \varphi_W(\zeta_1) \neq 0 \text{ for } \zeta_1 \text{ such that } \text{Im } \zeta_1 > 0.$$

From (8.5)

$$\tau_W(q_\zeta q_{\bar{\zeta}}) = |1 + \varphi_W(\zeta)|^2 \frac{\text{Im } m_W(\zeta)}{\text{Im } \zeta}$$

holds, hence  $\text{Im } m_W(\zeta) \geq 0$  is valid if  $\text{Im } \zeta > 0$ . Therefore, if  $\tau_W(q_{\zeta_1} q_{\bar{\zeta}_1}) = 0$ , then  $\text{Im } m_W(\zeta) = 0$  identically due to the fact that  $\text{Im } m_W(\zeta)$  is harmonic and  $\text{Im } m_W(\zeta_1) = 0$ . However, as  $\zeta \rightarrow \infty$ ,  $\tau_W(q_\zeta q_{\bar{\zeta}}) \rightarrow 1$  holds, which leads us to contradiction. The case  $\zeta_1 + \psi_W(\zeta_1) \neq 0$  can be treated similarly. Thus  $\tau_W(q_{\zeta_1} q_{\bar{\zeta}_1}) > 0$  should hold for any  $\zeta_1$ .

We prove (iii) by induction. For  $n = 1$  (ii) implies the strict positivity of  $\tau_W(q_\zeta q_{\bar{\zeta}})$ . Assume  $\tau_W(g_1) > 0$  is valid for  $g_1 = \prod_{k=1}^{n-1} q_{\zeta_k} q_{\bar{\zeta}_k}$ . For any  $h \in \Gamma_{\text{real}}$

$$\tau_{g_1 W}(h) = \frac{\tau_W(g_1 h)}{\tau_W(g_1)} \geq 0$$

holds, hence  $g_1 W \in Gr_2^+$  and the argument above shows  $\tau_{g_1 W}(q_{\zeta_n} q_{\bar{\zeta}_n}) > 0$ . Now, for  $g = g_1 q_{\zeta_n} q_{\bar{\zeta}_n}$  an identity

$$\tau_W(g) = \tau_{g_1 W}(q_{\zeta_n} q_{\bar{\zeta}_n}) \tau_W(g_1)$$

shows  $\tau_W(g) > 0$ . For  $g(z) = \prod_{k=1}^n p_{\zeta_k} p_{\bar{\zeta}_k}$  the identity (8.7) in the Appendix implies

$$\tau_W(p_{\zeta_1} p_{\zeta_2} \cdots p_{\zeta_n}) = (\tau_W(r_{\zeta_1}) \tau_W(r_{\zeta_2}) \cdots \tau_W(r_{\zeta_n}))^{-1} \tau_W(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_n}).$$

Since  $\tau_W(r_\zeta) \neq 0$ , the proof is complete. □

*Proof of Theorem 1.1.* Assume  $W \in Gr_{\text{real}}^{(2)}$  satisfies  $gW \in Gr^{(2)}$  for any  $g \in \Gamma_{\text{real}}$ . Then,  $\tau_W(q_{\bar{\zeta}} q_\zeta) \neq 0$  holds for any  $\zeta \in \{|z| > r\}$ . Since  $\tau_W(q_{\bar{\zeta}} q_\zeta)$  takes real values and approaches to 1 as  $\zeta \rightarrow \infty$ , we have  $\tau_W(q_{\bar{\zeta}} q_\zeta) > 0$ . Inductively one can see  $\tau_W(g) > 0$  for any  $g = \prod_{k=1}^n q_{\zeta_k} q_{\bar{\zeta}_k}$  with  $\zeta_k \in \{|z| > r\}$ , which shows  $W \in Gr_+^{(2)}$  due to (i) of Lemma 6.1.

To show the converse direction let  $g$  be in  $\Gamma_{\text{real}}$  and  $g_n$  be the function defined in (6.1) by replacing  $g$  with  $g^{-1}$ , hence  $g_n \rightarrow g^{-1}$  in this case. Then, Lemma 6.1 shows  $\tau_W(g_n) > 0$ . On the other hand, since  $\tau_W$  is continuous and  $\tau_W(1) = 1$ , we see

$$\tau_W(g g_n) > 0$$



for sufficiently large  $n$ . Since  $gg_nW \in Gr_+^{(2)}$  is valid (see the argument in the proof of (iii) of Lemma 6.1), from Lemma 6.1 it follows that  $\tau_{gg_nW}(p) > 0$  for any  $p$  of the form  $\prod_{k=1}^n p_{\zeta'_k} p_{\bar{\zeta}'_k}$ . Therefore

$$\tau_W(gg_n p) = \tau_{gg_nW}(p) \tau_W(gg_n) > 0$$

is valid. Now, taking  $p = g_n^{-1}$ , we see  $\tau_W(g) > 0$ , which shows  $gW \in Gr_+^{(2)}$ . This completes the proof.  $\square$

**Corollary 6.2.** *Suppose  $W \in Gr_+^{(2)}$ . Then, the followings are valid.*

(i)  $a(1 + \varphi_W(z)) + b(z + \psi_W(z))$  has no zeros in  $\{|z| > r\} \cap (\mathbb{C} \setminus \mathbb{R})$  for any  $a, b \in \mathbb{R}$  such that  $|a| + |b| \neq 0$ . Moreover,  $1 + \varphi_W(x) > 0$  holds for any  $x \in \mathbb{R}$  such that  $|x| > r$ .

(ii)  $m_W(z)$  is holomorphic on  $\{|z| > r\}$  and has no zeros in  $\{|z| > r\} \cap (\mathbb{C} \setminus \mathbb{R})$ . Moreover,  $m_W$  satisfies

$$\left\{ \begin{array}{l} \frac{\text{Im } m_W(z)}{\text{Im } z} > 0 \quad \text{for } z \in \{|z| > r\} \text{ with } \text{Im } z \neq 0, \\ \frac{m_W(x) - m_W(-x)}{2x} > 0, \quad m'_W(x) > 0 \quad \text{on } \mathbb{R} \setminus [-r, r]. \end{array} \right.$$

*Proof.* For  $\zeta \in \{|z| > r\} \cap (\mathbb{C} \setminus \mathbb{R})$

$$\begin{aligned} \tau_W(q_\zeta q_{\bar{\zeta}}) &= \frac{1}{\zeta - \bar{\zeta}} \left( (\zeta + \psi_W(\zeta)) \overline{(1 + \varphi_W(\zeta) + b(\zeta + \psi_W(\zeta)))} \right. \\ &\quad \left. - \overline{(\zeta + \psi_W(\zeta))} (1 + \varphi_W(\zeta) + b(\zeta + \psi_W(\zeta))) \right) \end{aligned}$$

is valid. Hence, if  $1 + \varphi_W(\zeta) + b(\zeta + \psi_W(\zeta)) = 0$ , then  $\tau_W(q_\zeta q_{\bar{\zeta}}) = 0$ , which contradicts Theorem 1.1. Similarly we have  $a(1 + \varphi_W(\zeta)) + \zeta + \psi_W(\zeta) \neq 0$ . On the other hand, since  $q_x \in \Gamma_{\text{real}}$  for  $x \in \mathbb{R}$  and  $|x| > r$ , Theorem 1.1 implies

$$1 + \varphi_W(x) = \tau_W(q_x) > 0,$$

which shows (i). The first inequality of (ii) follows from

$$\tau_W(q_\zeta q_{\bar{\zeta}}) = |1 + \varphi_W(\zeta)|^2 \text{Im } m_W(\zeta) / \text{Im } \zeta.$$

The second two inequalities are shown by (8.6) and

$$\tau_W(q_x^2) = (1 + \varphi_W(x))^2 m_W(x). \quad \square$$

Theorem 1.1 shows that the group  $\Gamma_{\text{real}}$  acts on  $Gr_+^{(2)}$ . Corollary 5.2 implies that the non-vanishing property of  $\tau_W(g)$  for  $g \in \Gamma_{\text{real}}$  can be stated only by  $m_W$ , hence the next task is to find some concrete criterion in terms of  $m_W$  for  $W \in Gr_+^{(2)}$  to be an element of  $Gr_+^{(2)}$ .

To proceed further we prepare some results from the spectral theory of one dimensional Schrödinger operators. For a real valued  $q \in L^1_{loc}(\mathbb{R})$  let  $L_q$  be a Schrödinger operator defined by

$$(L_q f)(x) = -f''(x) + q(x)f(x),$$

and for  $\lambda \in \mathbb{C}$  consider a time independent Schrödinger equation

$$(L_q f)(x) = \lambda f(x). \quad (6.2)$$

**Lemma 6.3** (see [1]). *There occur two cases on the behavior of solutions to (6.2) at  $+\infty$ .*

- (i) *Limit circle type:*  $\dim \{f \in L^2(\mathbb{R}_+) : L_q f = \lambda f\} = 2$  for any  $\lambda \in \mathbb{C}$ .
- (ii) *Limit point type:*  $\dim \{f \in L^2(\mathbb{R}_+) : L_q f = \lambda f\} = 1$  for any  $\lambda \in \mathbb{C} \setminus \text{sp} L_+$ .

The boundary  $-\infty$  has also the same classification. If the boundary  $+\infty$  is of limit point type, the operator  $L_+$  is uniquely extendable as a self-adjoint operator in  $L^2(\mathbb{R}_+)$ , where  $L_+$  is the Schrödinger operator  $L_q$  restricted to  $L^2(\mathbb{R}_+)$  with Dirichlet boundary condition at 0.

**Lemma 6.4.** *Suppose the boundaries  $\pm\infty$  are of limit point type. If there exists a positive solution  $f$  to (6.2), then,  $\lambda \leq \inf \text{sp} L_q$  holds.*

*Proof.* Although this is widely known in a more general framework, for completeness sake we give a proof. For a fixed  $a > 0$  let  $\lambda_0$  be the minimum eigenvalue for the operator  $L$  restricted to an interval  $(-a, a)$  with Dirichlet boundary condition at the boundaries  $\pm a$ , and  $u$  be the eigenfunction. One can assume  $u$  takes positive value in  $(-a, a)$ . An integration by parts shows

$$\begin{aligned} \lambda \int_{-a}^a f(x) u(x) dx &= \int_{-a}^a L_q f(x) u(x) dx \\ &= f(a)u'(a) - f(-a)u'(-a) + \lambda_0 \int_{-a}^a f(x) u(x) dx. \end{aligned}$$

Since  $u'(-a) > 0$ ,  $u'(a) < 0$ , we have  $\lambda < \lambda_0$ , which leads us to  $\lambda \leq \inf \text{sp} L_q$  by letting  $a \rightarrow \infty$ .  $\square$

Denote by  $f_{\pm}(x, \lambda)$  the solutions to (6.2) belonging to  $L^2(\mathbb{R}_{\pm})$  respectively when the boundary  $\pm\infty$  are of limit point type, and define

$$m_{\pm}(\lambda) = \pm \frac{f'_{\pm}(0, \lambda)}{f_{\pm}(0, \lambda)}.$$

These two functions  $m_{\pm}$  are known to be of *Herglotz* (a holomorphic function  $m$  on  $\mathbb{C} \setminus \mathbb{R}$  satisfying  $m(z) = \overline{m(\bar{z})}$  and  $\text{Im} m(z) > 0$  on  $\mathbb{C}_+$ ), and called *Weyl functions* (or *Weyl–Titchmarsh function*).  $m_{\pm}$  are holomorphic on  $\mathbb{C} \setminus \text{sp} L_{\pm}$  respectively. The following proposition identifies  $m_W$  with the Weyl functions.

**Proposition 6.5.** *Suppose  $W \in Gr_+^{(2)}$ . Then, the associated  $q_W$  has no singularities on  $\mathbb{R}$  and  $q_W$  is real valued there. Assume  $\{m_{e_x W}(z)\}_{x \in \mathbb{R}}$  forms a normal family on  $\mathbb{C} \setminus \overline{\mathbb{D}}_r$ . Then,  $\pm\infty$  are of limit point type for the associated  $L_{q_W}$ , and  $-r^2 \leq \inf \text{sp } L_{q_W}$ . The  $m$ -function  $m_W$  is related to the Weyl functions  $m_{\pm}$  of  $q_W$  by*

$$m_W(z) = \begin{cases} -m_+(-z^2), & \text{for } \text{Re } z > 0, \\ m_-(-z^2), & \text{for } \text{Re } z < 0. \end{cases} \tag{6.3}$$

Consequently,  $m_W$  known to be holomorphic on  $\mathbb{C} \setminus ([-r, r] \cup i[-r, r])$  and have a property

$$\frac{\text{Im } m_W(z)}{\text{Im } z} > 0 \quad \text{on } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}).$$

*Proof.*  $q_W$  has no singularity on  $\mathbb{R}$  due to Theorem 1.1, since  $e_x \in \Gamma_{\text{real}}$  if  $x \in \mathbb{R}$ . The first assertion is The Baker–Akhiezer function  $f_W(x, z)$  satisfies Schrödinger equation with potential  $q_W$  and  $\lambda = -z^2$ . On the other hand, (4.5) implies

$$f_W(x + y, \zeta) = e^{-xz-y\zeta} \tau_{e_{x+y}W}(q_\zeta) = e^{-x\zeta-y\zeta} \tau_{e_y e_x W}(q_\zeta) = e^{-x\zeta} f_{e_x W}(y, \zeta).$$

This together with (5.1) yields

$$m_{e_x W}(\zeta) = -\frac{f'_{e_x W}(0, \zeta)}{f_{e_x W}(0, \zeta)} = -\frac{f'_W(x, \zeta)}{f_W(x, \zeta)},$$

from which an identity

$$f_W(x, \zeta) = f_W(0, \zeta) \exp\left(-\int_0^x m_{e_y W}(\zeta) dy\right) \tag{6.4}$$

follows. Since  $W \in Gr_+^{(2)}$ , Theorem 1.1 implies  $e_y W \in Gr_+^{(2)}$ . Therefore, Corollary 6.2 shows  $m_{e_y W}(z)$  is holomorphic on  $\mathbb{C} \setminus \overline{\mathbb{D}}_r$ .

Set  $\phi_x(z) = z m_{e_x W}(z^{-1})$ . Then,  $\phi_x$  is holomorphic on  $\mathbb{D}_{r^{-1}}$  satisfying  $\phi_x(0) = 1$ , and  $\{\phi_x(z)\}_{x \in \mathbb{R}}$  forms a normal family on  $\mathbb{D}_{r^{-1}}$ . Denote by  $z(x) \in (-r^{-1}, r^{-1})$  a zero of  $\phi_x$  if it exists. If there exists a sequence  $\{x_n\}_{n \geq 1} \subset \mathbb{R}$  such that  $z(x_n) \rightarrow 0 \in (-r^{-1}, r^{-1})$ , then, one can assume  $\phi_{x_n} \rightarrow \phi$ , and  $\phi(0) = 0$ , which contradicts  $\phi(0) = 1$ . Therefore,  $|z(x)| \geq r_0^{-1}$  holds uniformly for some  $r_0 > r$ , which means that  $m_{e_x W}(z)$  has no zero on  $|z| > r_0$  for any  $x \in \mathbb{R}$ . For  $\zeta = a > r_0$  in (6.4) we see

$$\begin{cases} f_W(x, a) = f_W(0, a) \exp\left(-\int_0^x m_{e_y W}(a) dy\right) & \text{decreasing,} \\ f_W(x, -a) = f_W(0, -a) \exp\left(-\int_0^x m_{e_y W}(-a) dy\right) & \text{increasing,} \end{cases}$$

and Lemma 6.3 implies the boundaries  $\pm\infty$  are of limit point type, which shows

$$f_W(x, a) = f_+(x, -a^2), \quad f_W(x, -a) = f_-(x, -a^2).$$

These identities are valid for any  $a > r_0$ , therefore  $f_W(x, z) = f_+(x, -z^2)$ ,  $f_W(x, -z) = f_-(x, -z^2)$  hold for any  $z \in \mathbb{C} \setminus \overline{\mathbb{D}}_r$ , which implies  $m_+(-z^2) = -m_W(z)$ ,  $m_-(-z^2) = m_W(-z)$ . Lemma 6.4 implies  $-r^2 \leq \inf \text{sp } L_+$ .  $\square$

In the above proof the normality of  $\{m_{e_x W}(z)\}_{x \in \mathbb{R}}$  was crucial. It should be remarked that the converse statement holds. Namely, for  $W \in Gr_+^{(2)}$  assume  $m_W$  is connected with the Weyl functions  $m_{\pm}$  as in (6.3). Then, Lemma 8.2 implies that there exists a measure  $\sigma_y$  on  $[-\sqrt{2r}, \sqrt{2r}]$  such that

$$m_{e_y W}(z) = \sqrt{z^2 + r^2} + \int_{-\sqrt{2r}}^{\sqrt{2r}} \frac{\sigma_y(d\xi)}{\xi - \sqrt{z^2 + r^2}}, \tag{6.5}$$

hence

$$m_{e_y W}(x) - m_{e_y W}(-x) = 2\sqrt{x^2 + r^2} \left( 1 + \int_{-\sqrt{2r}}^{\sqrt{2r}} \frac{1}{\xi^2 - (x^2 + r^2)} \sigma_y(d\xi) \right).$$

On the other hand, Corollary 6.2 shows for  $x > r$

$$m_{e_y W}(x) - m_{e_y W}(-x) > 0,$$

which means

$$\int_{-\sqrt{2r}}^{\sqrt{2r}} \frac{1}{(x^2 + r^2) - \xi^2} \sigma_y(d\xi) < 1 \quad \text{for any } x > r.$$

Consequently, letting  $x \rightarrow r$ , we have

$$\int_{-\sqrt{2r}}^{\sqrt{2r}} \frac{1}{2r^2 - \xi^2} \sigma_y(d\xi) \leq 1 \quad \text{for any } y \in \mathbb{R},$$

which implies the normality of  $\{m_{e_x W}(z)\}_{x \in \mathbb{R}}$ .

Proposition 6.5 asserts that the  $m$ -function is directly related to the Weyl functions. Set

$$\mathcal{H} = \left\{ m : m \text{ is holomorphic on } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \text{ satisfying } m(z) = \overline{m(\bar{z})} \text{ and } \text{Im } m(z) > 0 \text{ on } \mathbb{C}_+ \setminus i\mathbb{R} \right\}.$$

We introduce a subclass  $\mathcal{M}_r$  of  $\mathcal{H}$  in view of the property of Proposition 6.5. Denote by  $\mathcal{M}_r$  the set of all functions  $m$  satisfying the following conditions: Let  $I_r = [-r, r]$  for  $r > 0.0$

$$\left\{ \begin{array}{l} \text{(i)} \quad m \in \mathcal{H}. \\ \text{(ii)} \quad m \text{ is holomorphic on } \mathbb{C} \setminus (I_r \cup iI_r), \text{ continuous on } \partial\mathbb{D}_r \\ \quad \text{and satisfies } m(r) > m(-r). \\ \text{(iii)} \quad m \text{ has a pole at } \infty \text{ of a form } m(z) = z + O(z^{-1}). \end{array} \right. \tag{6.6}$$

The next goal is to show the converse statement. For that purpose recall transformation

$$(d_{\zeta} m)(z) = \frac{z^2 - \zeta^2}{m(z) - m(\zeta)} - m(\zeta),$$

and define

$$(D_{\zeta} m)(z) = \frac{z - \zeta}{m(z) - m(\zeta)} - m(\zeta). \tag{6.7}$$

Then, without difficulty one can show  $D_{\zeta_1} D_{\zeta_2} = D_{\zeta_2} D_{\zeta_1}$ ,  $d_{\zeta_1} d_{\zeta_2} = d_{\zeta_2} d_{\zeta_1}$ .

**Lemma 6.6.** *It holds that  $d_{\bar{\zeta}}d_{\zeta}m \in \mathcal{H}$  for  $\zeta \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$  and  $m \in \mathcal{H}$ .*

*Proof.* For  $m \in \mathcal{H}$  define

$$m_+(z) = m(\sqrt{z}), \tag{6.8}$$

where  $\sqrt{z}$  is defined on  $\mathbb{C} \setminus \mathbb{R}_-$  so that  $\sqrt{1} = 1$ . Then,  $m_+$  turns out to be an irrational Herglotz function, and Lemma 8.1 in the Appendix implies  $D_{\bar{\zeta}}D_{\zeta}m_+$  is of Herglotz, hence for  $\zeta, z$  satisfying  $\zeta^2 \in \mathbb{C} \setminus \mathbb{R}$  and  $\operatorname{Re} z > 0, \operatorname{Im} z > 0$  we see

$$\operatorname{Im} \left( d_{\bar{\zeta}}d_{\zeta}m \right) (z) = \operatorname{Im} \left( D_{\bar{\zeta}^2}D_{\zeta^2}m_+ \right) (z^2) > 0,$$

because  $z^2 \in \mathbb{C}_+$ . To obtain the result for  $z$  satisfying  $\operatorname{Re} z < 0, \operatorname{Im} z > 0$  we define

$$m_-(z) = -m(-\sqrt{z}).$$

Then,  $m_-$  is again of Herglotz, hence

$$\operatorname{Im} \left( d_{\bar{\zeta}}d_{\zeta}m \right) (z) = -\operatorname{Im} \left( D_{\bar{\zeta}^2}D_{\zeta^2}m_- \right) (z^2) > 0$$

due to  $z^2 \in \mathbb{C}_-$  and  $m(z) = -m_-(z^2)$ , which completes the proof. □

**Proposition 6.7.** *Let  $m \in \mathcal{M}_r$  and  $s > r$ . Set*

$$W_m = \left\{ \varphi(z^2) + \psi(z^2)m(z) : \varphi, \psi \in H_+(\partial\mathbb{D}_s) \right\}.$$

*Then,  $W_m \in Gr_+^{(2)}(\mathbb{D}_s)$  and  $m_W = m, \tau_W(g) = \tau_m(g)$  hold. Moreover,  $m_{gW} \in \mathcal{M}_s$  is valid for any  $g \in \Gamma_{\text{real}}(\mathbb{D}_s)$ .*

*Proof.* Since

$$\mathbf{W}_m = \left\{ \mathbf{f}(z) = \begin{pmatrix} 1 & m_e(z) \\ 0 & m_o(z) \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \psi(z) \end{pmatrix} : \varphi, \psi \in H_+(\partial\mathbb{D}_s) \right\},$$

Applying Lemma 2.1 to  $a(z) = m_o(z)$  on  $\partial\mathbb{D}_s$  shows  $T(m_o)$  is invertible on  $H_+$ , hence  $\mathbf{W}_m \in Gr^{(2)}(\mathbb{D}_s)$ , and  $W_m \in Gr^{(2)}(\mathbb{D}_s)$ . In this case

$$\varphi_{W_m}(z) = 0, \quad \psi_{W_m}(z) = m(z) - z$$

hold, hence  $m_{W_m} = m, \tau_{W_m}(g) = \tau_m(g)$  follow. Then, the rest of the proof is to show the property  $\tau_{W_m}(g) \geq 0$  for  $g = \prod_{k=1}^n q_{\zeta_k}q_{\bar{\zeta}_k}$  with  $\zeta_k \in \{|z| > s\}$  and  $\operatorname{Im} \zeta_k \neq 0$ . We show  $\tau_{W_m}(g) > 0$  by induction. For  $n = 1$  (8.5) implies

$$\tau_{W_m}(q_{\zeta}q_{\bar{\zeta}}) = |1 + \varphi_{W_m}(\zeta)|^2 \frac{m(\zeta) - \overline{m(\zeta)}}{\zeta - \bar{\zeta}} = \frac{m(\zeta) - \overline{m(\zeta)}}{\zeta - \bar{\zeta}},$$

which is a positive quantity because of  $m \in \mathcal{M}_r$ . Suppose  $\tau_W(g_{n-1}) > 0$  for  $g_{n-1} = \prod_{k=1}^{n-1} q_{\zeta_k}q_{\bar{\zeta}_k}$ . Then,  $g_{n-1}W_m \in Gr^{(2)}(\mathbb{D}_s)$  and is real, hence

$$\tau_{g_{n-1}W_m}(q_{\zeta_n}q_{\bar{\zeta}_n}) = \left| 1 + \varphi_{g_{n-1}W_m}(\zeta_n) \right|^2 \frac{m_{g_{n-1}W_m}(\zeta_n) - \overline{m_{g_{n-1}W_m}(\zeta_n)}}{\zeta_n - \bar{\zeta}_n}.$$

Since (8.4) implies  $m_{q_\zeta W_m}(z) = (d_\zeta m)(z)$ , we have

$$m_{g_{n-1}W_m}(z) = \left(d_{\bar{\zeta}_{n-1}} d_{\zeta_{n-1}} \cdots d_{\bar{\zeta}_1} d_{\zeta_1} m\right)(z). \tag{6.9}$$

Therefore, Lemma 6.6 implies  $m_{g_{n-1}W_m} \in \mathcal{H}$  due to  $m \in \mathcal{H}$ , hence  $m_{g_{n-1}W_m}(z) \in \mathbb{C}_+$  for any  $z \in \mathbb{C}_+ \setminus (\mathbb{R} \cup i\mathbb{R})$ . Consequently, we see  $\tau_{g_{n-1}W_m}(q_{\zeta_n} q_{\bar{\zeta}_n}) \geq 0$  for any  $\zeta_n \in \mathbb{C}_+ \setminus (I_s \cup iI_s)$ , and  $\tau_{g_{n-1}W_m}(q_{\zeta_n} q_{\bar{\zeta}_n}) > 0$  there from (ii) of Lemma 6.1. This completes the induction and we have

$$\tau_{W_m}(g) = \tau_{W_m}(g_{n-1}) \tau_{g_{n-1}W_m}(q_{\zeta_n} q_{\bar{\zeta}_n}) > 0.$$

The last statement is easily verified by starting from  $g = g_n$  and noting  $m_{g_n W_m} \in \mathcal{H}$ . The property (ii) in (6.6) follows from Corollary 6.2, since  $gW_m \in Gr_+^{(2)}$ .  $\square$

**6.2. Reflectionless property of underlying potentials.** Reflectionless property was originally introduced for decaying potentials with vanishing reflection coefficients. However, for our purpose it is better to define this property for more general potentials.

For a real valued  $q \in L^1_{loc}(\mathbb{R})$  with  $\pm\infty$  boundaries of limit point type let  $m_\pm$  be the Weyl functions. Let  $F$  be a Borel set in  $\mathbb{R}$  with positive Lebesgue measure. Then,  $q$  is called *reflectionless* on  $F$  if

$$m_+(\xi + i0) = -\overline{m_-(\xi + i0)} \quad \text{for a.e. } \xi \in F$$

holds. It can be shown without difficulty that  $\overline{F} \subset \text{sp } L_q$ , where  $L_q$  is the Schrödinger operator with potential  $q$ . In particular, any periodic potential is reflectionless on the spectrum.

Define

$$m(z) = \begin{cases} -m_+(-z^2) & \text{for } \text{Re } z > 0, \\ m_-(-z^2) & \text{for } \text{Re } z < 0. \end{cases} \tag{6.10}$$

If  $F \in \mathcal{B}(\mathbb{R}_+)$ , then  $q$  is reflectionless on  $F$  if and only if

$$m(i\xi + 0) = m(i\xi - 0) \quad \text{for a.e. } \xi \in \sqrt{F} \tag{6.11}$$

holds.

Define

$$\mathcal{Q}_r = \left\{ q = q_W : W \in Gr_+^{(2)}(\mathbb{D}_r) \text{ and } m_W \in \mathcal{M}_r \right\}.$$

**Proposition 6.8.** *If  $q \in \mathcal{Q}_r$ , then,  $q$  is reflectionless on  $(r^2, \infty)$  and  $\text{sp } L_q \subset [-r^2, \infty)$ . Conversely, if  $q$  is reflectionless on  $(r^2, \infty)$  and  $\text{sp } L_q \subset [-r^2, \infty)$ , then  $q \in \mathcal{Q}_r$ .*

*Proof.* The first assertion follows from Lemma 6.4 and Proposition 6.5. Due to Proposition 6.7 it is sufficient to show  $m \in \mathcal{M}_r$  for the proof of the second assertion. Since (6.11) for  $(r, \infty)$  implies  $m$  is holomorphic outside of  $[-r, r] \cup i[-r, r]$ , and the other properties of  $\mathcal{M}_r$  are clearly satisfied by  $m$ , we have  $m \in \mathcal{M}_r$ .  $\square$

**6.3. Proof of Theorem 1.2.** Now we construct the KdV flow. Set

$$\mathcal{M}_\infty = \bigcup_{r>0} \mathcal{M}_r, \quad \Gamma_{\text{real}}^\infty = \left\{ g = e^h : h \text{ is an entire function with } \bar{h} = h \right\}.$$

Then, Proposition 6.8 shows

$$\mathcal{Q}_\infty = \bigcup_{r>0} \mathcal{Q}_r,$$

where the definition of  $\mathcal{Q}_\infty$  is given (1.7). On the other hand, Proposition 6.7 shows  $\mathcal{M}_\infty$  corresponds to  $\mathcal{Q}_\infty$  through (6.10) one to one. For  $q \in \mathcal{Q}_\infty$  define  $m \in \mathcal{M}_\infty$  by (6.10). Then,  $gW_m \in Gr_+^{(2)}$  for  $g \in \Gamma_{\text{real}}^\infty$  holds for some  $r > 0$  due to Proposition 6.7. Then, one can define  $\tilde{q}(x) = -2\partial_x^2 \log \tau_{gW_m}(e_x) \in \mathcal{Q}_\infty$ . The property of  $\tau_W$  and Proposition 6.7 show

$$\tau_{gW_m}(e_x) = \tau_{W_m}(ge_x) / \tau_{W_m}(g) = \tau_m(ge_x) / \tau_m(g),$$

hence

$$\tilde{q}(x) = -2\partial_x^2 \log \tau_m(ge_x),$$

which is denoted by  $(K(g)q)(x)$ . The flow property of  $K(g)$  is verified as follows. Since the potential  $K(g_2)q$  is associated with  $g_2W_m \in Gr_+^{(2)}$ , we see

$$\begin{aligned} (K(g_1g_2)q)(x) &= -2\partial_x^2 \log \tau_{g_1g_2W_m}(e_x) \\ &= -2\partial_x^2 \log \tau_{g_1(g_2W_m)}(e_x) = (K(g_1)K(g_2)q)(x). \end{aligned}$$

### 7. Proof of Theorem 1.3

In this section we give a more concrete representation of  $\tau_m(g) = \tau_{W_m}(g)$ . For  $m \in \mathcal{M}_r$  set

$$\Pi_m = \begin{pmatrix} 1 & m_e \\ 0 & m_o \end{pmatrix}, \quad \mathbf{W}_m = \Pi_m \mathbf{H}_+.$$

Then,  $\mathbf{W}_m \in Gr^{(2)}(\mathbb{D}_s)$  for  $s > r$  and its characteristic matrix is  $\Pi_m$ . Since  $g^{-1}\mathbf{p}_+gA_{W_m}$  is unitarily equivalent to  $G^{-1}\mathbf{p}_+GA_{W_m}$ , where

$$\begin{cases} G = \begin{pmatrix} g_e(z) & zg_o(z) \\ g_o(z) & g_e(z) \end{pmatrix}, \\ A_{W_m} \mathbf{u} = \Pi_m \mathbf{T} (\Pi_m)^{-1} \mathbf{u} - \mathbf{u}, \end{cases}$$

its  $\tau$ -function is

$$\tau_{W_m}(g) = \det (I + G^{-1}\mathbf{p}_+GA_{W_m}).$$

In this case everything is discussed in the Hilbert space  $H = L^2(\partial\mathbb{D}_s)$  and  $\mathbf{H} = H \times H$ . As we have seen in the first section,  $\mathbf{T}(\Pi_m)$  is invertible if and only if so is  $T(m_o)$ . The present  $m_o$  satisfies the condition of Lemma 2.1, hence we have

$$\mathbf{T}(\Pi_m)^{-1} = \begin{pmatrix} I & -T(m_e)T(m_o^{-1}) \\ 0 & T(m_o^{-1}) \end{pmatrix}.$$

Therefore, for  $\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 \in \mathbf{H}_+$

$$\begin{aligned} A_{\mathbf{W}_m} \mathbf{u} &= (u_1 - T(m_e) T(m_o^{-1}) u_2 + m_e T(m_o^{-1}) u_2) \mathbf{e}_1 + m_o T(m_o^{-1}) u_2 \mathbf{e}_2 - \mathbf{u} \\ &= \mathbf{p}_- m_e T(m_o^{-1}) u_2 \mathbf{e}_1 + \mathbf{p}_- m_o T(m_o^{-1}) u_2 \mathbf{e}_2, \end{aligned}$$

thus

$$\begin{aligned} G^{-1} \mathbf{p}_+ G A_{\mathbf{W}_m} \mathbf{u} &= G^{-1} \mathbf{p}_+ G \mathbf{p}_- m_e T(m_o^{-1}) u_2 \mathbf{e}_1 + G^{-1} \mathbf{p}_+ G \mathbf{p}_- m_o T(m_o^{-1}) u_2 \mathbf{e}_2. \quad (7.1) \end{aligned}$$

Then, denoting by  $\pi_1, \pi_2$  the projections

$$\pi_1 \mathbf{u} = (\mathbf{u} \cdot \mathbf{e}_1) \mathbf{e}_1, \quad \pi_2 \mathbf{u} = (\mathbf{u} \cdot \mathbf{e}_2) \mathbf{e}_2,$$

from (7.1) we see  $G^{-1} \mathbf{p}_+ G A_{\mathbf{W}_m} \pi_1 = 0$  and

$$I + G^{-1} \mathbf{p}_+ G A_{\mathbf{W}_m} = \begin{pmatrix} I_1 & \pi_1 G^{-1} \mathbf{p}_+ G A_{\mathbf{W}_m} \pi_2 \\ 0 & I_2 + \pi_2 G^{-1} \mathbf{p}_+ G A_{\mathbf{W}_m} \pi_2 \end{pmatrix},$$

hence

$$\tau_{\mathbf{W}_m}(g) = \det(I_2 + \pi_2 G^{-1} \mathbf{p}_+ G A_{\mathbf{W}_m} \pi_2).$$

On the other hand, (7.1) implies also

$$\begin{aligned} \pi_2 G^{-1} \mathbf{p}_+ G A_{\mathbf{W}_m} \pi_2 &= ((\widehat{g}_o \mathbf{p}_+ g_e + \widehat{g}_e \mathbf{p}_+ g_o) \mathbf{p}_- m_e + (\widehat{g}_o \mathbf{p}_+ z g_o + \widehat{g}_e \mathbf{p}_+ g_e) \mathbf{p}_- m_o) T(m_o^{-1}) \\ &= ((\widehat{g}_o \mathbf{p}_+ g_e + \widehat{g}_e \mathbf{p}_+ g_o) m_e + (\widehat{g}_o \mathbf{p}_+ z g_o + \widehat{g}_e \mathbf{p}_+ g_e) m_o) T(m_o^{-1}) \\ &\quad - ((\widehat{g}_o \mathbf{p}_+ g_e + \widehat{g}_e \mathbf{p}_+ g_o) T(m_e) + (\widehat{g}_o \mathbf{p}_+ z g_o + \widehat{g}_e \mathbf{p}_+ g_e) T(m_o)) T(m_o^{-1}) \\ &= ((\widehat{g}_o \mathbf{p}_+ g_e + \widehat{g}_e \mathbf{p}_+ g_o) m_e + (\widehat{g}_o \mathbf{p}_+ z g_o + \widehat{g}_e \mathbf{p}_+ g_e) m_o) T(m_o^{-1}) - I \\ &= (\widehat{g}_o T((gm)_e) + \widehat{g}_e T((gm)_o) - T(m_o)) T(m_o^{-1}) \end{aligned}$$

with  $\widehat{g} = g^{-1}$ . Therefore, we have

$$\tau_{\mathbf{W}_m}(g) = \det(I + (\widehat{g}_o T((gm)_e) + \widehat{g}_e T((gm)_o) - T(m_o)) T(m_o^{-1})). \quad (7.2)$$

Recall  $r < s$  and  $m \in \mathcal{M}_r$ . Let  $C, C'$  be simple closed curves of Fig. 1.1 in the introduction surrounding the interval  $[-r, r]$  and contained in  $\mathbb{D}_s$ . Then, it holds that for  $f \in H_+$  and  $z$  located inside of  $C'$

$$\begin{aligned} T(m_o^{-1}) f(z) &= \frac{1}{2\pi i} \int_{|\xi|=s} \frac{m_o(\lambda')^{-1} f(\lambda')}{\lambda' - z} d\lambda' \\ &= \frac{1}{2\pi i} \int_{C'} \frac{m_o(\lambda')^{-1} f(\lambda')}{\lambda' - z} d\lambda' \\ &\equiv T f(z), \end{aligned}$$

and for  $z$  satisfying  $|z| \leq s$

$$(\widehat{g}_o T((gm)_e) + \widehat{g}_e T((gm)_o) - T(m_o)) f(z)$$



$$\begin{aligned} &= \frac{1}{2\pi i} \int_{|\lambda|=s} \frac{\widehat{g}_o(z)(gm)_e(\lambda) + \widehat{g}_o(z)(gm)_o(\lambda) - m_o(\lambda)}{\lambda - z} f(\lambda) d\lambda \\ &= \frac{1}{2\pi i} \int_C \frac{\widehat{g}_o(z)(gm)_e(\lambda) + \widehat{g}_o(z)(gm)_o(\lambda) - m_o(\lambda)}{\lambda - z} f(\lambda) d\lambda \\ &\equiv Sf(z). \end{aligned}$$

It should be noted that  $S$  does not change if we replace  $m$  by  $\widetilde{m}$  in the integration on the curve  $C$  defined by

$$\widetilde{m}(z) = m(z) - \delta(z),$$

where  $\delta_e, \delta_o$  are holomorphic in a simply connected domain containing  $C$ . We can regard  $S$  and  $T$  as operators from  $L^2(C)$  to  $H_+ (= H_+(\mathbb{D}_s))$  and from  $H_+$  to  $L^2(C)$  respectively. Now (7.2) implies

$$\tau_{W_m}(g) = \det(I + ST) = \det(I + TS).$$

The operator  $TS : L^2(C) \rightarrow L^2(C)$  is

$$(TSu)(z) = \frac{1}{2\pi i} \int_{C'} \frac{m_o(\lambda')^{-1}}{\lambda' - z} d\lambda' \frac{1}{2\pi i} \int_C L_g(\lambda', \lambda) u(\lambda) d\lambda,$$

where

$$L_g(z, \lambda) = M_g(z, \lambda) - \frac{m_o(\lambda)}{\lambda - z}.$$

Refer to (1.8) for the definition of  $M_g$ . Note

$$\frac{1}{2\pi i} \int_{C'} \frac{m_o(\lambda')^{-1} m_o(\lambda)}{\lambda' - z} d\lambda' = 0$$

for  $z, \lambda$  located inside of  $C'$ . Consequently, we obtain

$$(TSu)(z) = (N_m(g)u)(z),$$

which completes the proof of Theorem 1.3 in the Introduction.

This formula for the  $\tau$ -functions makes it possible to establish a theory in a more general framework, which will be realized in the second paper.

## 8. Appendix

**8.1. Calculation of typical  $\Gamma$ -actions.** (4.4) connects the  $\tau$ -function with  $\varphi_W$ , namely

$$\tau_W(q_\zeta) = 1 + \varphi_W(\zeta)$$

for  $\zeta$  such that  $|\zeta| > r$ , where  $q_\zeta(z) = (1 - z/\zeta)^{-1}$ . To calculate  $\tau_W$  for other  $g \in \Gamma$  we compute  $\varphi_{q_\zeta W}, \psi_{q_\zeta W}$ . Let

$$H_- \ni \varphi_{q_\zeta W}(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

Since  $(1 - z/\zeta) + (1 - z/\zeta) \varphi_{q_\zeta W}(z) \in W$  due to  $1 + \varphi_{q_\zeta W}(z) \in q_\zeta W$ , the decomposition

$$\begin{aligned} \left(1 - \frac{z}{\zeta}\right) + \left(1 - \frac{z}{\zeta}\right) \varphi_{q_\zeta W}(z) &= \left(1 - \frac{z}{\zeta}\right) - \frac{z}{\zeta} \left(\frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right) + \varphi_{q_\zeta W}(z) \\ &= \left(1 - \frac{a_1}{\zeta}\right) - \frac{z}{\zeta} + H_- \end{aligned}$$

shows

$$\left(1 - \frac{z}{\zeta}\right) + \left(1 - \frac{z}{\zeta}\right) \varphi_{q_\zeta W}(z) = \left(1 - \frac{a_1}{\zeta}\right) (1 + \varphi_W(z)) - \frac{1}{\zeta} (z + \psi_W(z)).$$

We have used here the bijectivity of  $\mathfrak{p}_+ : W \rightarrow H_+$ . Setting  $z = \zeta$ , we see

$$\left(1 - \frac{a_1}{\zeta}\right) (1 + \varphi_W(\zeta)) - \frac{1}{\zeta} (\zeta + \psi_W(\zeta)) = 0,$$

which yields

$$a_1 = \zeta - \frac{\zeta + \psi_W(\zeta)}{1 + \varphi_W(\zeta)},$$

hence

$$\begin{aligned} 1 + \varphi_{q_\zeta W}(z) &= \frac{\frac{\zeta + \psi_W(\zeta)}{\zeta(1 + \varphi_W(\zeta))} (1 + \varphi_W(z)) - \frac{1}{\zeta} (z + \psi_W(z))}{1 - \frac{z}{\zeta}} \\ &= (1 + \varphi_W(z)) \frac{m_W(\zeta) - m_W(z)}{\zeta - z}. \end{aligned} \quad (8.1)$$

Similarly, for  $\psi_{q_\zeta W}(z) = b_1/z + b_2/z^2 + \dots$

$$\begin{aligned} \left(1 - \frac{z}{\zeta}\right) z + \left(1 - \frac{z}{\zeta}\right) \psi_{q_\zeta W}(z) \\ = -\frac{b_1}{\zeta} (1 + \varphi_W(z)) + (z + \psi_W(z)) - \frac{1}{\zeta} (z^2 + A_W z^2), \end{aligned}$$

hence, setting  $z = \zeta$ , we have

$$-\frac{b_1}{\zeta} (1 + \varphi_W(\zeta)) + (\zeta + \psi_W(\zeta)) - \frac{1}{\zeta} (\zeta^2 + (A_W z^2)(\zeta)) = 0,$$

and

$$b_1 = \zeta \frac{\zeta + \psi_W(\zeta)}{1 + \varphi_W(\zeta)} - \frac{\zeta^2 + (A_W z^2)(\zeta)}{1 + \varphi_W(\zeta)}.$$

To compute  $A_W z^2$  we note the identity  $A_W z^2 \cdot = \mathfrak{p}_- z^2 A_W \cdot - A_W \mathfrak{p}_+ z^2 A_W \cdot$ , and set  $\cdot = 1$ . Then, expanding  $\varphi_W(z) = a_1/z + a_2/z^2 + \dots$  yields

$$z^2 + A_W z^2 = z^2 + \mathfrak{p}_- z^2 A_W 1 - A_W \mathfrak{p}_+ z^2 A_W 1$$

$$\begin{aligned} &= z^2 + \mathbf{p}_- z^2 \varphi_W - A_W \mathbf{p}_+ z^2 \varphi_W \\ &= (1 + \varphi_W(z)) (z^2 - a_2 - a_1 m_W(z) + a_1^2). \end{aligned}$$

Therefore

$$\begin{aligned} &-\frac{b_1}{\zeta} (1 + \varphi_W(z)) + (z + \psi_W(z)) - \frac{1}{\zeta} (z^2 + A_W z^2) \\ &= (1 + \varphi_W(z)) \left( -\frac{b_1}{\zeta} + m_W(z) - a_1 - \frac{z^2 - a_2 - a_1 m_W(z) + a_1^2}{\zeta} \right) \\ &= \frac{1 + \varphi_W(z)}{\zeta} ((\zeta + a_1) (m_W(z) - m_W(\zeta)) + \zeta^2 - z^2), \end{aligned}$$

which shows

$$\begin{aligned} \frac{z + \psi_{q_\zeta W}(z)}{1 + \varphi_W(z)} &= \frac{\frac{1}{\zeta} ((\zeta + a_1) (m_W(z) - m_W(\zeta)) + \zeta^2 - z^2)}{1 - \frac{z}{\zeta}} \\ &= \frac{(\zeta + a_1) (m_W(z) - m_W(\zeta))}{\zeta - z} + \zeta + z. \end{aligned} \tag{8.2}$$

On the other hand, from (8.1)

$$a_1 (q_\zeta W) = \lim_{z \rightarrow \infty} z \varphi_{q_\zeta W}(z) = -m_W(\zeta) + \zeta + a_1, \tag{8.3}$$

hence, from (8.2), (8.3)

$$\begin{aligned} m_{q_\zeta W}(z) &= \frac{z + \psi_{q_\zeta W}(z)}{1 + \varphi_{q_\zeta W}(z)} + a_1 (q_\zeta W) \\ &= \frac{\frac{(\zeta + a_1) (m_W(z) - m_W(\zeta))}{\zeta - z} + \zeta + z}{\frac{m_W(\zeta) - m_W(z)}{\zeta - z}} - m_W(\zeta) + \zeta + a_1 \\ &= \frac{z^2 - \zeta^2}{m_W(z) - m_W(\zeta)} - m_W(\zeta). \end{aligned} \tag{8.4}$$

In the next step we compute  $\tau_W(q_{\zeta_1} q_{\zeta_2})$ . From (4.3) it follows that

$$\tau_W(q_{\zeta_1} q_{\zeta_2}) = \tau_W(q_{\zeta_1}) \tau_{q_{\zeta_1} W}(q_{\zeta_2}) = (1 + \varphi_W(\zeta_1)) (1 + \varphi_{q_{\zeta_1} W}(\zeta_2)),$$

hence, from (8.1)

$$\tau_W(q_{\zeta_1} q_{\zeta_2}) = (1 + \varphi_W(\zeta_1)) (1 + \varphi_W(\zeta_2)) \frac{m_W(\zeta_1) - m_W(\zeta_2)}{\zeta_1 - \zeta_2}. \tag{8.5}$$

In the last step we calculate  $\tau_W(p_\zeta)$  where

$$p_\zeta(z) = 1 + z/\zeta = q_{-\zeta}(z)^{-1}$$

with  $\zeta$  such that  $|\zeta| > r$ . The key observation is

$$r_\zeta(z) = q_\zeta(z) p_\zeta(z)^{-1} = q_\zeta(z) q_{-\zeta}(z) = (1 - z^2/\zeta^2)^{-1},$$

and (iii) of Proposition 4.3 implies

$$\tau_W(q_\zeta p_{\zeta'}) = \tau_W(q_\zeta q_{\zeta'} r_{\zeta'}^{-1}) = \tau_W(r_{\zeta'})^{-1} \tau_W(q_\zeta q_{\zeta'}).$$

We have only to apply (8.5) to  $r_\zeta = q_\zeta q_{-\zeta}$  to compute  $\tau_W(r_\zeta)$ . Thus

$$\begin{aligned} \tau_W(r_\zeta) &= (1 + \varphi_W(\zeta))(1 + \varphi_W(-\zeta)) \frac{m_W(\zeta) - m_W(-\zeta)}{2\zeta} \\ &= \det \Pi_W(\zeta^2). \end{aligned} \quad (8.6)$$

Hence, letting  $\zeta \rightarrow \infty$ , we have

$$\begin{aligned} \tau_W(p_{\zeta'}) &= \tau_W(r_{\zeta'})^{-1} \tau_W(q_{\zeta'}) \\ &= \frac{2\zeta'(1 + \varphi_W(\zeta'))}{(1 + \varphi_W(\zeta'))(1 + \varphi_W(-\zeta'))(m_W(\zeta') - m_W(-\zeta'))}. \end{aligned}$$

Similarly

$$\begin{aligned} &\tau_W(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_n} p_{\zeta'_1} p_{\zeta'_2} \cdots p_{\zeta'_n}) \\ &= \tau_W(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_n} q_{\zeta'_1} q_{\zeta'_2} \cdots q_{\zeta'_n} r_{\zeta'_1}^{-1} r_{\zeta'_2}^{-1} \cdots r_{\zeta'_n}^{-1}) \\ &= \tau_W(r_{\zeta'_1}^{-1} r_{\zeta'_2}^{-1} \cdots r_{\zeta'_n}^{-1}) \tau_W(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_n} q_{\zeta'_1} q_{\zeta'_2} \cdots q_{\zeta'_n}) \\ &= \left( \tau_W(r_{\zeta'_1}) \tau_W(r_{\zeta'_2}) \cdots \tau_W(r_{\zeta'_n}) \right)^{-1} \tau_W(q_{\zeta_1} q_{\zeta_2} \cdots q_{\zeta_n} q_{\zeta'_1} q_{\zeta'_2} \cdots q_{\zeta'_n}). \end{aligned} \quad (8.7)$$

holds.

**8.2. Herglotz function.** As we have defined it in the previous section, a holomorphic function  $m$  on  $\mathbb{C} \setminus \mathbb{R}$  is called a Herglotz function if  $m$  satisfies

$$m(z) = \overline{m(\bar{z})} \quad \text{and} \quad \frac{\operatorname{Im} m(z)}{\operatorname{Im} z} \geq 0 \quad \text{for any } z \in \mathbb{C} \setminus \mathbb{R}.$$

A necessary and sufficient condition for  $m$  to be a Herglotz function is that  $m$  has a representation

$$m(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left( \frac{1}{\xi - z} - \frac{\xi}{\xi^2 + 1} \right) \sigma(d\xi) \quad (8.8)$$

with a real  $\alpha$ , non-negative  $\beta$  and measure  $\sigma$  on  $\mathbb{R}$  satisfying

$$\int_{-\infty}^{\infty} \frac{1}{\xi^2 + 1} \sigma(d\xi) < \infty.$$

**Lemma 8.1.** *Suppose  $m$  is an irrational Herglotz function. Then, so is  $D_{\bar{\zeta}}D_{\zeta}m$  for any  $\zeta \in \mathbb{C} \setminus \mathbb{R}$  (refer to (6.7) for the definition of  $D_{\zeta}$ ).*

*Proof.* Assume  $m$  has a representation (8.8). Then

$$m(z) - m(\zeta) = \beta(z - \zeta) + (z - \zeta) \int_{-\infty}^{\infty} \frac{\xi - \bar{\zeta}}{\xi - z} \sigma_{\zeta}(d\xi),$$

with  $\sigma_{\zeta}(d\xi) = |\xi - \zeta|^{-2} \sigma(d\xi)$ . Note  $\sigma_{\zeta} \neq 0$  since  $m$  is irrational. Hence

$$\frac{z - \zeta}{m(z) - m(\zeta)} = \frac{1}{\beta + \int_{-\infty}^{\infty} \frac{\xi - \bar{\zeta}}{\xi - z} \sigma_{\zeta}(d\xi)} = \frac{1}{\gamma + (z - \bar{\zeta}) \int_{-\infty}^{\infty} \frac{\sigma_{\zeta}(d\xi)}{\xi - z}}$$

where  $\gamma = \beta + \int_{-\infty}^{\infty} \sigma_{\zeta}(d\xi)$ . Then

$$\begin{aligned} \frac{z - \zeta}{m(z) - m(\zeta)} - \frac{\bar{\zeta} - \zeta}{m(\bar{\zeta}) - m(\zeta)} &= \frac{1}{\gamma + (z - \bar{\zeta}) \int_{-\infty}^{\infty} \frac{\sigma_{\zeta}(d\xi)}{\xi - z}} - \frac{1}{\gamma} = \frac{-(z - \bar{\zeta}) \int_{-\infty}^{\infty} \frac{\sigma_{\zeta}(d\xi)}{\xi - z}}{\gamma \left( \gamma + (z - \bar{\zeta}) \int_{-\infty}^{\infty} \frac{\sigma_{\zeta}(d\xi)}{\xi - z} \right)} \end{aligned}$$

and

$$\begin{aligned} (D_{\bar{\zeta}}D_{\zeta}m)(z) + (D_{\zeta}m)(\bar{\zeta}) &= \frac{z - \bar{\zeta}}{(D_{\zeta}m)(z) - (D_{\zeta}m)(\bar{\zeta})} \\ &= \frac{\gamma^2}{-\int_{-\infty}^{\infty} \frac{\sigma_{\zeta}(d\xi)}{\xi - z}} - \gamma z + \gamma \bar{\zeta}. \end{aligned}$$

Since the first term of the right side is a Herglotz function, let its representation be

$$\frac{\gamma^2}{-\int_{-\infty}^{\infty} \frac{\sigma_{\zeta}(d\xi)}{\xi - z}} = \alpha_1 + \beta_1 z + \int_{-\infty}^{\infty} \left( \frac{1}{\xi - z} - \frac{\xi}{\xi^2 + 1} \right) \sigma_1(d\xi).$$

Then

$$\beta_1 = \lim_{z \rightarrow \infty} \frac{\gamma^2 z^{-1}}{-\int_{-\infty}^{\infty} \frac{\sigma_{\zeta}(d\xi)}{\xi - z}} = \frac{\gamma^2}{\int_{-\infty}^{\infty} \sigma_{\zeta}(d\xi)} = \frac{\gamma^2}{\gamma - \beta},$$

hence  $\beta_1 - \gamma = \gamma^2(\gamma - \beta)^{-1} - \gamma = \beta(\gamma - \beta)^{-1} \geq 0$ . On the other hand,

$$\text{Im}(\gamma \bar{\zeta} - (D_{\zeta}m)(\bar{\zeta})) = \text{Im}(\gamma \bar{\zeta} + m(\zeta)) = (\beta - \gamma) \text{Im} \zeta + \gamma \text{Im} \zeta = \beta \text{Im} \zeta \geq 0,$$

which completes the proof. □

**Lemma 8.2.** For  $r > 0$  assume  $m$  is holomorphic on  $\mathbb{C} \setminus ([-r, r] \cup i[-r, r])$  and satisfies

$$\frac{\operatorname{Im} m(z)}{\operatorname{Im} z} > 0 \quad \text{on } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}), \quad m(z) = \overline{m(\bar{z})},$$

and

$$m(z) = z + O(z^{-1})$$

as  $z \rightarrow \infty$ . Then, there exists a measure  $\sigma$  on  $[-\sqrt{2}r, \sqrt{2}r]$  such that

$$m(z) = \sqrt{z^2 + r^2} + \int_{-\sqrt{2}r}^{\sqrt{2}r} \frac{\sigma(d\xi)}{\xi - \sqrt{z^2 + r^2}}.$$

*Proof.* Set

$$\tilde{m}(z) = m\left(\sqrt{z^2 - r^2}\right).$$

Then,  $\tilde{m}$  is holomorphic on  $\mathbb{C} \setminus [-\sqrt{2}r, \sqrt{2}r]$  and satisfies

$$\frac{\operatorname{Im} \tilde{m}(z)}{\operatorname{Im} z} > 0, \quad \tilde{m}(z) = \overline{\tilde{m}(\bar{z})}, \quad \text{and} \quad \tilde{m}(z) = z + O(z^{-1}).$$

Therefore, there exists a measure  $\sigma$  on  $[-\sqrt{2}r, \sqrt{2}r]$  such that

$$\tilde{m}(z) = z + \int_{-\sqrt{2}r}^{\sqrt{2}r} \frac{\sigma(d\xi)}{\xi - z},$$

which gives the representation for  $m$ . □

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## Побудова потоку КдФ I. Подання $\tau$ -функції через функцію Вейля

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Для опису розв'язків широкого класу цілком інтегровних диференціальних операторів Сато запровадив  $\tau$ -функцію. Пізніше Сегал та Вілсон зобразили її в термінах відповідних інтегральних операторів на просторі Харді на одиничному диску. У цій роботі дано інше подання  $\tau$ -функції через функції Вейля для одновимірних операторів Шредінгера з дійсними потенціалами, яке дає можливість розширити клас початкових даних рівняння КдФ до більш загального класу.

*Ключові слова:* рівняння КдФ, теорія Сато, оператор Шредінгера.