Journal of Mathematical Physics, Analysis, Geometry
2018, Vol. 14, No. 3, pp. 336–361
doi: https://doi.org/10.15407/mag14.03.336

The Extended Leibniz Rule and Related Equations in the Space of Rapidly Decreasing Functions

Hermann König and Vitali Milman

Dedicated to the 95th birthday
of the great mathematician Vladimir Marchenko
and to the 80th birthday
of our friend and great mathematical physicist Leonid Pastur

We solve the extended Leibniz rule $T(f \cdot g) = Tf \cdot Ag + Af \cdot Tg$ for operators T and A in the space of rapidly decreasing functions in both cases of complex and real-valued functions. We find that Tf may be a linear combination of logarithmic derivatives of f and its complex conjugate $\overline{f}$ with smooth coefficients up to some finite orders m and n respectively and $Af = f^m \cdot \overline{f}^{\,n}$. In other cases Tf and Af may include separately the real and the imaginary part of f. In some way the equation yields a joint characterization of the derivative and the Fourier transform of f. We discuss conditions when T is the derivative and A is the identity. We also consider differentiable solutions of related functional equations reminiscent of those for the sine and cosine functions.

Key words: rapidly decreasing functions, extended Leibniz rule, Fourier transform.

Mathematical Subject Classification 2010: 39B42, 47A62, 26A24.

1. Motivation and results

Often important operations and transformations in Analysis are nearly characterized by some simple properties like operator functional equations on classical function spaces X. To give some examples, let us introduce the following function spaces X.

For an open set $I \subset \mathbb{R}$, $k \in \mathbb{N}_0$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $C^k(I, \mathbb{K})$ denote the space of k-times continuously differentiable $\mathbb{K}$-valued functions on $f : I \to \mathbb{K}$, $C^\infty(I, \mathbb{K}) := \cap_{k \in \mathbb{N}} C^k(I, \mathbb{K})$ and $C(I, \mathbb{K})$ denote the continuous functions $f : I \to \mathbb{K}$. The space of rapidly decreasing functions is given by

$$\mathcal{S}(\mathbb{R}, \mathbb{K}) := \{f \in C^\infty(\mathbb{R}, \mathbb{K}) \mid \sup_{x \in \mathbb{R}} (1 + |x|)^l |D^k f(x)| =: c_{k,l} < \infty \text{ for all } k, l \in \mathbb{N}_0\}.$$

As a first example, let us consider the chain rule equation $T(f \circ g) = Tf \circ g \cdot Tg$ for all $f, g \in C^k(\mathbb{R}, \mathbb{R})$, satisfied by an operator $T : C^k(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ for some fixed $k \in \mathbb{N}$: this nearly characterizes powers of derivatives, cf. Artstein-Avidan, König, Milman [5]. Here the composition is mapped to some "compound" product. Even simpler are product-preserving maps $T : X \to X$, satisfying $T(f \cdot g) = Tf \cdot Tg$ for all $f, g \in X$. These have been studied for a while: for $X = C(I, \mathbb{K})$ this was done by Milgram [8], for $X = C^k(I, \mathbb{K})$ by Mrčun, Šemrl [9, 10] and for $X = \mathcal{S}(\mathbb{R}, \mathbb{C})$ by Artstein-Avidan, Faifman, Milman [4], see also the earlier joint paper with Alesker [2, 3]:

Theorem 1.1. *Let $T : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ be a bijective multiplicative transformation, i.e.,*

$$T(f \cdot g) = Tf \cdot Tg, \quad f, g \in \mathcal{S}(\mathbb{R}, \mathbb{C}).$$

Then there exists a C^∞-diffeomorphism $u : \mathbb{R} \to \mathbb{R}$ such that either $Tf = f \circ u$ for all $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ or $Tf = \overline{f \circ u}$ for all $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$.

This leads directly to a characterization of the Fourier transform $\mathcal{F}$

$$\mathcal{F}F(x) = \int_{\mathbb{R}} F(t) \exp(-2\pi i x t)\, dt, \quad F \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

which satisfies the *"product" formula*

$$\mathcal{F}(F * G) = \mathcal{F}(F) \cdot \mathcal{F}(G), \quad F, G \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

mapping convolutions to products (and vice-versa). Recall that $\mathcal{F}$ is a bijective map from $\mathcal{S}(\mathbb{R}, \mathbb{C})$ to itself, and in this context it is natural to consider complex-valued functions on $\mathbb{R}$. As a consequence of Theorem 1.1, these properties characterize the Fourier transform and its complex conjugate up to C^∞-diffeomorphisms, cf. [2–4]:

Theorem 1.2. *Let $S : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ be a bijective map satisfying*

$$S(F * G) = SF \cdot SG, \quad F, G \in \mathcal{S}(\mathbb{R}, \mathbb{C}). \tag{1.1}$$

Then there is a C^∞-diffeomorphism $u : \mathbb{R} \to \mathbb{R}$ such that either $SF = (\mathcal{F}F) \circ u$ for all $F \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ or $SF = \overline{(\mathcal{F}F) \circ u}$ for all $F \in \mathcal{S}(\mathbb{R}, \mathbb{C})$.

In this paper we consider two related operator equations, the *extended* Leibniz rule for the derivative in $\mathcal{S}(\mathbb{R}, \mathbb{C})$

$$T(f \cdot g) = Tf \cdot Ag + Af \cdot Tg, \quad f, g \in \mathcal{S}(\mathbb{R}, \mathbb{C}), \tag{1.2}$$

and its convolution counter-part

$$R(F * G) = RF \cdot SG + SF \cdot RG, \quad F, G \in \mathcal{S}(\mathbb{R}, \mathbb{C}). \tag{1.3}$$

Interestingly, equation (1.3) provides a joint characterization of the Fourier transform and the derivative, cf. Theorem 1.12 below, and its solutions are directly related to those of equation (1.2) by taking Fourier transforms. In fact, our main task will be to determine the solutions of the extended Leibniz rule (1.2) and then receive those of (1.3) as a consequence. Of course, the standard Leibniz rule

$$B(f \cdot g) = B(f) \cdot g + f \cdot B(g), \quad f, g \in \mathcal{S}(\mathbb{R}, \mathbb{C}), \tag{1.4}$$

is just equation (1.2) for $B = T$ and $A = \mathrm{Id}$. In König, Milman [7], we determined the solutions of this equation when B is an operator on the C^k-spaces, $B : C^k(I, \mathbb{R}) \to C(I, \mathbb{R})$, $I \subset \mathbb{R}$ open. They have the form $Bf = a_0 f \ln |f| + a_1 \, f'$, $f \in C^k(I, \mathbb{R})$, where $a_0, a_1 \in C(I, \mathbb{R})$ are suitable continuous functions on I. In general, for $f \in \mathcal{S}(\mathbb{R}, \mathbb{R})$, $f \ln |f|$ will not be in $\mathcal{S}(\mathbb{R}, \mathbb{R})$. Thus a natural question is whether equation (1.4) admits only the solution $Bf = a_1 f'$ when considered in $\mathcal{S}$, i.e., when $B : \mathcal{S}(\mathbb{R}, \mathbb{K}) \to \mathcal{S}(\mathbb{R}, \mathbb{K})$. We investigate this question in this paper. In fact, we will do so in a more general context, motivated by applying B satisfying the Leibniz rule (1.4) to the Fourier transform functional equation (1.1). We then find, putting $R := BS : \mathcal{S}(\mathbb{R}, \mathbb{K}) \to \mathcal{S}(\mathbb{R}, \mathbb{K})$, the above mentioned Leibniz type equation (1.3) for convolutions

$$R(F * G) = RF \cdot SG + SF \cdot RG, \quad F, G \in \mathcal{S}(\mathbb{R}, \mathbb{C}).$$

After taking the inverse Fourier transform, it becomes a multiplicative Leibniz type equation: Let $f := \mathcal{F}F$ and $g := \mathcal{F}G$. Then $F * G = \mathcal{F}^{-1}(f \cdot g)$ and hence

$$R\mathcal{F}^{-1}(f \cdot g) = (R\mathcal{F}^{-1})f \cdot (S\mathcal{F}^{-1})g + (S\mathcal{F}^{-1})f \cdot (R\mathcal{F}^{-1})g, \quad f, g \in \mathcal{S}(\mathbb{R}, \mathbb{C}).$$

Thus the operators $T := R\mathcal{F}^{-1}$, $A := S\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ satisfy the operator equation

$$T(f \cdot g) = Tf \cdot Ag + Af \cdot Tg, \quad f, g \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

which is (1.2) and which we call the *extended Leibniz rule* operator equation since it extends the ordinary Leibniz rule (1.4) where $A = \mathrm{Id}$. We are interested in finding all solutions of (1.2) in the space $\mathcal{S}(\mathbb{R}, \mathbb{C})$ under some non-degeneration condition and a weak continuity assumption on the operators T and A. As we have already mentioned, finding all solutions of (1.2) is equivalent to finding all solutions of (1.3) since taking the Fourier transform of a solution of (1.2) yields one of (1.3) and vice-versa.

We will show that under some additional assumptions, e.g., initial or surjectivity conditions, that the operators T and A satisfying (1.2) turn out to have the form $Tf = af'$ and $Af = f$ or $Tf = a\bar{f}'$ and $Af = \bar{f}$ or $Tf = a \operatorname{Im} f$ and $Af = \operatorname{Re} f$ for some C^∞-function a. Then we conclude from the definition of T and A that $SF = \mathcal{F}F$ and $RF = a(\mathcal{F}F)'$ or $SF = \overline{\mathcal{F}F}$ and $RF = a(\overline{\mathcal{F}F})'$ or $SF = \operatorname{Re}(\mathcal{F}F)$ and $RF = a \operatorname{Im}(\mathcal{F}F)$, jointly characterizing the Fourier transform and the derivative, up to complex conjugation or taking Re- and Im-parts.

In general, however, more solutions are to be expected for equation (1.2) than just $Tf = af'$, $Af = f$ or $Tf = a\bar{f}'$, $Af = \bar{f}$ or $Tf = a \operatorname{Im}(f)$, $Af = \operatorname{Re}(f)$, since the operators T, A are arbitrary, just intertwined by the operator equation (1.2).

Our method of investigation of the solutions of (1.2) rests on the localization of the operators, i.e., that $Tf(x)$ and $Af(x)$ depend only on x, $f(x)$ and the derivative values of f at x. To assure this, we have to assume that T and A are not homothetic on functions with small support, excluding a "resonance" situation between T and A. We want to avoid examples like

$$Tf(x) = f(x) - f(x+1), \quad Af(x) = \frac{1}{2}(f(x) + f(x+1)), \quad f \in \mathcal{S}(\mathbb{R}, \mathbb{C}), \quad x \in \mathbb{R},$$

where T and A satisfy (1.2) but are not localized. Here for functions f with support in $(-\frac{1}{2}, \frac{1}{2})$, we have that $Tf(x) = 2Af(x) = f(x)$ for all $x \in (-\frac{1}{2}, \frac{1}{2})$. To avoid examples of this form, we will make an assumption of non-degeneration, which means that T and A should not be proportional on functions with small support.

Definition 1.3. Let $T, A : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ be operators. The pair (T, A) is C^k-*non-degenerate* if, for every open interval $J \subset \mathbb{R}$ and any $x \in J$, there are functions $g_1, g_2 \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ with support in J such that $z_i := (Tg_i(x), Ag_i(x)) \in \mathbb{C}^2$ are linearly independent vectors in $\mathbb{C}^2$ for $i = 1, 2$.

We also consider the following weak continuity condition.

Definition 1.4. An operator $T : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ is *pointwise continuous* provided that, for any sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{S}(\mathbb{R}, \mathbb{C})$-functions and $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ such that $f_n^{(j)} \to f^{(j)}$ converge uniformly on all compact subsets of $\mathbb{R}$ for all $j \in \mathbb{N}_0$, we have pointwise convergence $\lim_{n\to\infty} Tf_n(x) = Tf(x)$ for every $x \in \mathbb{R}$.

We now state the main result for the extended Leibniz rule equation on $\mathcal{S}(\mathbb{R}, \mathbb{C})$.

Theorem 1.5. *Suppose that $T, A : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ are operators satisfying the extended Leibniz rule equation*

$$T(f \cdot g) = Tf \cdot Ag + Af \cdot Tg, \quad f, g \in \mathcal{S}(\mathbb{R}, \mathbb{C}). \tag{1.5}$$

Assume, in addition, that T and A are pointwise continuous and that the pair (T, A) is non-degenerate. Then there are non-negative integers $m, n, M, N \in \mathbb{N}_0$ with $m + n \geq 1$, $M + N \geq 1$ and functions $a_1, \ldots, a_m, b_1, \ldots b_n \in C^\infty(\mathbb{R}, \mathbb{C})$ such that either

$$\begin{aligned} Tf &= \left(\sum_{j=1}^m a_j \left(\frac{f'}{f}\right)^{(j-1)} + \sum_{j=1}^n b_j \left(\frac{\bar{f}'}{\bar{f}}\right)^{(j-1)} \right) f^m \bar{f}^n, && f \in \mathcal{S}(\mathbb{R}, \mathbb{C}), \\ Af &= f^m \bar{f}^n, && f \in \mathcal{S}(\mathbb{R}, \mathbb{C}), \end{aligned} \tag{1.6}$$

or

$$Tf = \frac{a_1}{2}\left(f^m \bar{f}^n - f^M \bar{f}^N\right), \qquad f \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

$$Af = \frac{1}{2}\left(f^m \bar{f}^n + f^M \bar{f}^N\right), \qquad f \in \mathcal{S}(\mathbb{R}, \mathbb{C}). \tag{1.7}$$

Conversely, the operators (T, A) given by formulas (1.6) or (1.7) satisfy equation (1.5) if the functions a_j and b_j are bounded on $\mathbb{R}$.

The real-valued analogue of Theorem 1.5 is:

Theorem 1.6. *Suppose that $T, A : \mathcal{S}(\mathbb{R}, \mathbb{R}) \to \mathcal{S}(\mathbb{R}, \mathbb{R})$ are operators satisfying*

$$T(f \cdot g) = Tf \cdot Ag + Af \cdot Tg, \quad f, g \in \mathcal{S}(\mathbb{R}, \mathbb{C}).$$

Assume, in addition, that T and A are pointwise continuous and that the pair (T, A) is non-degenerate. Then there are integers $m, n \in \mathbb{N}$ and functions $a_1, \ldots, a_m \in C^\infty(\mathbb{R}, \mathbb{R})$ such that either

$$Tf = \left(\sum_{j=0}^{m} a_j (\ln|f|)^{(j)}\right) f^m, \qquad Af = f^m, \qquad f \in \mathcal{S}(\mathbb{R}, \mathbb{R}),$$

or

$$Tf = \frac{a_1}{2}\left(f^m - f^n\right), \qquad Af = \frac{1}{2}\left(f^m + f^n\right), \qquad f \in \mathcal{S}(\mathbb{R}, \mathbb{R}).$$

Remark 1.7.

(a) In Theorem 1.5, for $(m, n) = (1, 0)$ or $(m, n) = (0, 1)$ in (1.6), we get the solutions $Tf = a_1 f'$, $Af = f$ or $Tf = b_1 \bar{f}'$, $Af = \bar{f}$. Locally, if $f \neq 0$, $\ln f$ is defined and one has $(\ln f)' = \frac{f'}{f}$, so that $(\ln f)^{(j)} = (\frac{f'}{f})^{(j-1)}$. However, since there is no continuous branch of ln on all of $\mathbb{C}$ with $\ln(f \cdot g) = \ln f + \ln g$, we have that the formulas $Tf = a_0(\ln f) \cdot f^m$, $Af = f^m$, $m \in \mathbb{N}$ give no valid solution of (1.5) from $\mathcal{S}(\mathbb{R}, \mathbb{C})$ to itself. Therefore the sums in (1.6) start at $j = 1$ and not at $j = 0$. However, in the real-valued case of Theorem 1.6, the sum starts at $j = 0$.

Note that $(\frac{f'}{f})^{(j-1)}$ has a singularity of order j as $f \to 0$, which is cancelled by the multiplication with f^m when $j \leq m$.

For $(m, n) = (2, 0)$ or $(m, n) = (0, 2)$, the solutions in (1.6) are given more explicitly by $Tf = a_1 f' f + a_2(f'' f - f'^2)$, $Af = f^2$ or $Tf = b_1 \bar{f}' \bar{f} + b_2(\bar{f}'' \bar{f} - \bar{f}'^2)$, $Af = \bar{f}^2$.

(b) Choose $M = n$ and $N = m$ in (1.7) and $a = \frac{1}{i}$ to see that $Tf = \operatorname{Im}(f^m \bar{f}^n)$ and $Af = \operatorname{Re}(f^m \bar{f}^n)$ are solutions of (1.5). In particular, $Tf = \operatorname{Im} f$ and $Af = \operatorname{Re} f$ are solutions.

(c) In the situation that real-valued functions are considered, and that the image is just required to consist of continuous functions, i.e., when $T, A : \mathcal{S}(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ satisfy (1.5), there are additional solutions such as $Tf = \sin(\sum_{j=0}^{k} a_j (\ln|f|)^{(j)}) \cdot f^n$, $Af = \cos(\sum_{j=0}^{k} a_j (\ln|f|)^{(j)}) \cdot f^n$ for $k \in \mathbb{N}_0, n \in$

$\mathbb{N}$. Note again that $(\ln|f|)^{(j)} = (\frac{f'}{f})^{(j-1)}$ for $j \in \mathbb{N}$. Further, f^n in both formulas may be replaced by $|f|^p$ or $|f|^p(\operatorname{sgn} f)$ for any continuous function $p : \mathbb{R} \to \mathbb{R}_{>0}$, and this will still give a solution of (1.5). If $f(x) = 0$, the formulas should be interpreted as $Tf(x) = 0$ and $Af(x) = 0$. These solutions are not surprising since (1.5) reminds of the addition theorem of the sine function, although in a multiplicative setup, so that taking logarithms first is needed.

We now look for additional conditions guaranteeing that — up to complex conjugation — A is the identity and T a multiple of the derivative in Theorem 1.5. For this, it is not sufficient that A is surjective or even bijective in $\mathcal{S}(\mathbb{R}, \mathbb{C})$, as the following example due to E. Shustin shows. In the real-valued case of $\mathcal{S}(\mathbb{R}, \mathbb{R})$, a corresponding example was given by M. Sodin.

Example 1.8 (E. Shustin [11]). Equation (1.7) may provide a bijective map $A : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ without being linear or antilinear. Choose $m = 1$, $n = 0$ and $M = l + 1$, $N = l$, $l \in \mathbb{N}$ in (1.7) so that $Af = f(1 + |f|^{2l})$. For any $g \in \mathcal{S}(\mathbb{R}, \mathbb{C})$, $Af = g$ has a unique solution $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$. Indeed, let $G := g\bar{g}$. We first look for $F := f\bar{f}$ such that $F(1 + F^l)^2 = G$. The left-hand side is a strictly increasing real-analytic function K of F vanishing at zero and having derivative one at zero, namely $K(x) = x(1 + x^l)^2$. Thus K is invertible on the non-negative ray $\mathbb{R}_{\geq 0}$, i.e., $F = H(G)$, where the inverse H of K is real-analytic on $\mathbb{R}_{\geq 0}$. Therefore,

$$f = \frac{g}{1 + F^l} = \frac{g}{1 + H(G)^l} \in \mathcal{S}(\mathbb{R}, \mathbb{C})$$

uniquely solves $Af = g$ in $\mathcal{S}(\mathbb{R}, \mathbb{C})$. Hence by (1.7),

$$Tf = \frac{a_1}{2} f(1 - |f|^{2l}), \quad Af = \frac{1}{2} f(1 + |f|^{2l}), \qquad f \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

is a pointwise continuous, non-degenerate solution of the extended Leibniz rule (1.5) in $\mathcal{S}(\mathbb{R}, \mathbb{C})$ with A being non-linear and bijective. However, T does not locally vanish on functions which are locally constant.

On the other hand, M. Sodin [12] proved that if P is a *real* polynomial with $P(0) = 0$ and non-real critical points and if $A : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ given by $Af = P(f)$ is surjective, then P is linear. Note that in the above example $Af = P(f)$, P is not a polynomial of a real variable, but a polynomial in z and $\bar{z}$.

Definition 1.9. An operator $T : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ *vanishes locally on constants* if there is a bounded open set $I \subset \mathbb{R}$ and a point $x \in I$ such that for any $c \in \mathbb{C}$ there is $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ with $f|_I = c$ and $Tf(x) = 0$.

Note that any solution of (1.5) satisfies for every open interval I that $f|_I = 1$ implies $Tf|_I = 0$ and $Af|_I = 1$.

Theorem 1.10. *Suppose that T and A satisfy the conditions of Theorem 1.5. Assume, in addition, that A is surjective and that T vanishes locally on constants. Then there is $a \in C^\infty(\mathbb{R}, \mathbb{C})$ such that either*

$$Tf = af', \qquad Af = f, \qquad f \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

or

$$Tf = a\bar{f}', \qquad Af = \bar{f}, \qquad f \in \mathcal{S}(\mathbb{R}, \mathbb{C}).$$

An initial condition on A yields the following characterization of T and A without assuming that A is surjective.

Theorem 1.11. *Suppose that T and A satisfy the conditions of Theorem 1.5. Assume, in addition, that there are $x, c \in \mathbb{R}$ with $1 \neq c > 0$ and that there is $g \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ with $g(x) = c$ and $Ag(x) = c$. Then there is $a \in C^\infty(\mathbb{R}, \mathbb{C})$ such that either*

$$Tf = af', \qquad Af = f, \qquad f \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

or

$$Tf = a\bar{f}', \qquad Af = \bar{f}, \qquad f \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

or

$$Tf = a\,\mathrm{Im}\, f, \qquad Af = \mathrm{Re}\, f, \qquad f \in \mathcal{S}(\mathbb{R}, \mathbb{C}).$$

Theorem 1.10 yields the following joint characterization of the Fourier transform and the derivative

Theorem 1.12. *Suppose that $R, S : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ are pointwise continuous and satisfy*

$$R(F * G) = RF \cdot SG + SF \cdot RG \quad ; \quad F, G \in \mathcal{S}(\mathbb{R}, \mathbb{C}) \ .$$

Assume, in addition, that S is surjective, that $R \circ \mathcal{F}^{-1}$ vanishes locally on constants and that the pair $(R \circ \mathcal{F}^{-1}, S \circ \mathcal{F}^{-1})$ is non-degenerate. Then there is $a \in C^\infty(\mathbb{R}, \mathbb{C})$ such that either

$$RF = a(\mathcal{F}F)', \qquad SF = \mathcal{F}F, \qquad F \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

or

$$RF = a(\overline{\mathcal{F}F})', \qquad SF = \overline{\mathcal{F}F}, \qquad F \in \mathcal{S}(\mathbb{R}, \mathbb{C}).$$

Although the operators R and S are intertwined by the convolution functional equation, their actions are separated: the Fourier transform shows up in the "tuning" operator S and the derivative in the operator R.

As a consequence of Theorem 1.11, we get

Theorem 1.13. *Suppose that $R, S : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ are pointwise continuous and satisfy*

$$R(F * G) = RF \cdot SG + SF \cdot RG, \quad F, G \in \mathcal{S}(\mathbb{R}, \mathbb{C}).$$

Assume, in addition, that the pair $(R\circ\mathcal{F}^{-1}, S\circ\mathcal{F}^{-1})$ *is non-degenerate and that there exists* $0\neq G\in\mathcal{S}(\mathbb{R},\mathbb{C})$ *such that* $SG=\mathcal{F}G\in\mathcal{S}(\mathbb{R},\mathbb{R})$*. Then there is* $a\in C^\infty(\mathbb{R},\mathbb{C})$ *such that either*

$$RF=a(\mathcal{F}F)',\qquad SF=\mathcal{F}F,\qquad F\in\mathcal{S}(\mathbb{R},\mathbb{C}),$$

or

$$RF=a(\overline{\mathcal{F}F})',\qquad SF=\overline{\mathcal{F}F},\qquad F\in\mathcal{S}(\mathbb{R},\mathbb{C}),$$

or

$$RF=a\,\mathrm{Im}(\mathcal{F}F),\qquad SF=\mathrm{Re}(\mathcal{F}F),\qquad F\in\mathcal{S}(\mathbb{R},\mathbb{C}).$$

Remark 1.14. If $\mathcal{F}G$ is not real-valued, but properly complex-valued, it may be used to distinguish between the possible three solutions: In this case, if $SG=\mathcal{F}G$ holds, $SF=\mathcal{F}F$ and $RF=a(\mathcal{F}F)'$ holds for all $F\in\mathcal{S}(\mathbb{R},\mathbb{C})$. Similarly, $SG=\overline{\mathcal{F}G}$ implies $SF=\overline{\mathcal{F}F}$ and $RF=a(\overline{\mathcal{F}F})'$ for all $F\in\mathcal{S}(\mathbb{R},\mathbb{C})$, and $SG=\mathrm{Re}(\mathcal{F}G)$ yields $SF=\mathrm{Re}(\mathcal{F}F)$ and $RF=a\,\mathrm{Im}(\mathcal{F}F)$ for all $F\in\mathcal{S}(\mathbb{R},\mathbb{C})$. Therefore the image of one function determines the images of all functions.

Theorems 1.10 and 1.11 also yield the following two characterizations of the Fourier transform.

Theorem 1.15. *Suppose that* $S:\mathcal{S}(\mathbb{R},\mathbb{C})\to\mathcal{S}(\mathbb{R},\mathbb{C})$ *is surjective, pointwise continuous and satisfies*

$$S(f*g)=Sf\cdot Sg,\quad f,g\in\mathcal{S}(\mathbb{R},\mathbb{C}).$$

If the pair $(D\circ S\circ\mathcal{F}^{-1}, S\circ\mathcal{F}^{-1})$ *is non-degenerate and* $D\circ S\circ\mathcal{F}^{-1}$ *locally vanishes on constants, we have that either* $Sf=\mathcal{F}f$, $f\in\mathcal{S}(\mathbb{R},\mathbb{C})$ *or* $Sf=\overline{\mathcal{F}f}$, $f\in\mathcal{S}(\mathbb{R},\mathbb{C})$*, i.e., S is the Fourier transform or its conjugate.*

Theorem 1.16. *Suppose that* $S:\mathcal{S}(\mathbb{R},\mathbb{C})\to\mathcal{S}(\mathbb{R},\mathbb{C})$ *is pointwise continuous and satisfies*

$$S(f*g)=Sf\cdot Sg,\quad f,g\in\mathcal{S}(\mathbb{R},\mathbb{C}).$$

If the pair $(D\circ S\circ\mathcal{F}^{-1}, S\circ\mathcal{F}^{-1})$ *is non-degenerate and there is* $0\neq g\in\mathcal{S}(\mathbb{R},\mathbb{C})$ *with* $Sg=\mathcal{F}g\in\mathcal{S}(\mathbb{R},\mathbb{R})$*, we have that either* $Sf=\mathcal{F}f$*, or* $Sf=\overline{\mathcal{F}f}$*, or* $Sf=\mathrm{Re}\,\mathcal{F}f$, $f\in\mathcal{S}(\mathbb{R},\mathbb{C})$*, i.e., S is the Fourier transform, its conjugate or the real part of the Fourier transform.*

A similar remark as the one after Theorem 1.13 applies here as well.

Comparing Theorems 1.15 and 1.16 with Theorem 1.2, we do not assume that S is bijective, but only that S is surjective or satisfies some initial condition. On the other hand, we have a non-degeneration condition which implies localization so that the diffeomorphism $u:\mathbb{R}\to\mathbb{R}$ in Theorem 1.2 does not occur in Theorems 1.15 and 1.16.

As for the Leibniz rule itself, when $A=\mathrm{Id}$, the only possibility in Theorem 1.5 is $m=1$, $n=0$ in (1.6) so that we get

Theorem 1.17. *Suppose that $T : \mathcal{S}(\mathbb{R},\mathbb{C}) \to \mathcal{S}(\mathbb{R},\mathbb{C})$ satisfies*

$$T(f\cdot g) = Tf\cdot g + f\cdot Tg, \quad f,g\in\mathcal{S}(\mathbb{R},\mathbb{C}),$$

that T is pointwise continuous and that (T,Id) is non-degenerate. Then there is $a\in C^\infty(\mathbb{R},\mathbb{C})$ such that

$$Tf = af', \quad f\in\mathcal{S}(\mathbb{R},\mathbb{C}).$$

Theorem 1.5 is reminiscent of the addition formula for the sine function if one were to replace the multiplication $f\cdot g$ by the sum $f+g$. A corresponding result may be shown for the analogous cosine-type operator equation

$$T(f\cdot g) = Tf\cdot Tg - Af\cdot Ag, \quad f,g\in\mathcal{S}(\mathbb{R},\mathbb{C}).$$

Theorem 1.18. *Suppose that $T, A : \mathcal{S}(\mathbb{R},\mathbb{C}) \to \mathcal{S}(\mathbb{R},\mathbb{C})$ are operators satisfying the cosine-type operator equation*

$$T(f\cdot g) = Tf\cdot Tg - Af\cdot Ag, \quad f,g\in\mathcal{S}(\mathbb{R},\mathbb{C}). \tag{1.8}$$

Assume, in addition, that T and A are pointwise continuous and that the pair (T,A) is non-degenerate. Then there are non-negative integers $m,n,M,N\in\mathbb{N}_0$ with $m+n\geq 1$, $M+N\geq 1$, $k\notin\{1,-1\}$ and functions $a_1,\dots,a_m,b_1,\dots b_n\in C^\infty(\mathbb{R},\mathbb{C})$ such that

$$\begin{aligned} Tf &= \left(1-\sum_{j=1}^m a_j\left(\frac{f'}{f}\right)^{(j-1)} - \sum_{j=1}^n b_j\left(\frac{\bar f'}{\bar f}\right)^{(j-1)}\right) f^m\bar f^n, && f\in\mathcal{S}(\mathbb{R},\mathbb{C}),\\ Af &= \pm\left(\sum_{j=1}^m a_j\left(\frac{f'}{f}\right)^{(j-1)} + \sum_{j=1}^n b_j\left(\frac{\bar f'}{\bar f}\right)^{(j-1)}\right), && f\in\mathcal{S}(\mathbb{R},\mathbb{C}), \end{aligned} \tag{1.9}$$

or

$$\begin{aligned} Tf &= \frac{a_1}{2}\big((1-k)f^m\bar f^n + (1+k)f^M\bar f^N\big), && f\in\mathcal{S}(\mathbb{R},\mathbb{C}),\\ Af &= \frac{1}{2}\sqrt{k^2-1}\big(f^m\bar f^n - f^M\bar f^N\big), && f\in\mathcal{S}(\mathbb{R},\mathbb{C}). \end{aligned} \tag{1.10}$$

Conversely, the operators (T,A) given by formulas (1.9) and (1.10) satisfy equation (1.8) if the functions a_j and b_j are bounded on $\mathbb{R}$.

The simplest solutions for $(m,n)=(1,0)$ or $(m,n)=(0,1)$ are given by

$$Tf = (f-a_1f'),\quad Af = a_1f' \quad\text{and}\quad Tf=(\bar f - b_1\bar f'),\quad Af = b_1\bar f'.$$

For $M=n$, $N=m$ and $k=0$, we find the solutions $Tf = a_1\operatorname{Re}(f^n\bar f^m)$, $Af = \operatorname{Im}(f^n\bar f^m)$. In particular, the pair of operators $Tf=\operatorname{Re} f$ and $Af=\operatorname{Im} f$ solves (1.8).

There are, of course, consequences of Theorem 1.18 similar to those of Theorem 1.5.

2. Functional equations

We will later show that the operators T and A in equations (1.5) and (1.8) are localized, i.e., that $Tf(x)$ and $Af(x)$ depend only on x, $f(x)$ and all derivatives $f^{(j)}(x)$, $j \in \mathbb{N}$. The operator equations then turn into functional equations for two unknown functions in many variables. We start by solving these functional equations on $\mathbb{C}^n$. They are of the sine and cosine type. To solve them, we first need the form of the solutions of a standard functional equation for one function.

For $c = (c_j)_{j=1}^n, z = (z_j)_{j=1}^n \in \mathbb{C}^n$, let $\bar{z} := (\bar{z}_j)_{j=1}^n$ and $< c, z >:= \sum_{j=1}^n c_j z_j$.

Proposition 2.1. *Let $n \in \mathbb{N}$ and suppose that $F : \mathbb{C}^n \to \mathbb{C}\setminus\{0\}$ is continuous and satisfies*

$$F(z+w) = F(z) \cdot F(w), \quad z, w \in \mathbb{C}^n. \tag{2.1}$$

Then there are $c, d \in \mathbb{C}^n$ such that

$$F(z) = \exp(\langle c, z\rangle + \langle d, \bar{z}\rangle), \quad z \in \mathbb{C}^n.$$

Conversely, functions of this form satisfy equation (2.1).

For $n = 1$, this is found in Aczél's book [1] in Section 5.1.1, Theorem 3. The generalization to the case $n > 1$ is straightforward and thus will not be done here.

Proposition 2.2. *Let $n \in \mathbb{N}$ and $F, B : \mathbb{C}^n \to \mathbb{C}$ be continuous functions satisfying*

$$F(z+w) = F(z) \cdot B(w) + F(w) \cdot B(z), \qquad z, w \in \mathbb{C}^n. \tag{2.2}$$

Suppose that F is not identically zero. Then there are vectors $c_1, c_2, d_1, d_2 \in \mathbb{C}^n$ and there are $k \in \mathbb{C} \setminus \{0\}$ and $\epsilon_1, \epsilon_2 \in \{0, 1\}$, with ϵ_1, ϵ_2 not both zero, such that F and B have one of the following two forms:

(a) $F(z) = (\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle) \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)$,
$B(z) = \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)$;

(b) $F(z) = \dfrac{1}{2k}\big(\epsilon_1 \exp(\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle) - \epsilon_2 \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)\big)$,
$B(z) = \dfrac{1}{2}\big(\epsilon_1 \exp(\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle) + \epsilon_2 \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)\big), \qquad z \in \mathbb{C}^n$.

Conversely, these functions satisfy equation (2.2).

For $n = 1$, this is in Aczél [1], Section 4.2.5, Theorem 2 and Corollary.

Proposition 2.3. *Let $n \in \mathbb{N}$ and $F, B : \mathbb{C}^n \to \mathbb{C}$ be continuous functions satisfying*

$$F(z+w) = F(z) \cdot F(w) - B(w) \cdot B(z), \quad z, w \in \mathbb{C}^n. \tag{2.3}$$

Suppose that F is not identically zero. Then there are vectors $c_1, c_2, d_1, d_2 \in \mathbb{C}^n$ and there is $k \in \mathbb{C}$ such that F and B have one of the following three forms:

(a) $F(z) = \dfrac{1}{1-k^2} \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)$,
$B(z) = \dfrac{k}{1-k^2} \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle), \quad k \neq 1;$

(b) $F(z) = \big(1 + \langle c_1, z\rangle + \langle c_2, \bar z\rangle\big) \exp(\langle d_1, z\rangle + \langle d_2, \bar z\rangle)$,
$B(z) = \pm\big(\langle c_1, z\rangle + \langle c_2, \bar z\rangle\big) \exp(\langle d_1, z\rangle + \langle d_2, \bar z\rangle)$;

(c) $F(z) = \dfrac{1}{2}\big((1-k)\exp(\langle c_1, z\rangle + \langle c_2, \bar z\rangle) + (1+k)\exp(\langle d_1, z\rangle + \langle d_2, \bar z\rangle)\big)$,
$B(z) = \dfrac{1}{2}\sqrt{k^2-1}\big(\exp(\langle c_1, z\rangle + \langle c_2, \bar z\rangle) - \exp(\langle d_1, z\rangle + \langle d_2, \bar z\rangle)\big), \quad z \in \mathbb{C}^n$.

Conversely, these functions satisfy equation (2.3).

Proof of Proposition 2.2. **(i)** Suppose that F and B satisfy equation (2.2). Fix $t \in \mathbb{C}^n \setminus \{0\}$. We claim that F, B and $B(\cdot + t)$ are linearly dependent functions. For all $x, y \in \mathbb{C}^n$,

$$F(x+t)B(y)+B(x+t)F(y) = F(x+y+t) = F(x)B(y+t)+B(x)F(y+t). \tag{2.4}$$

Since F is not identically zero, by (2.2), also B is not identically zero. Hence there is $y_1 \in \mathbb{C}^n$ such that $B(y_1) \neq 0$. Choosing $y = y_1$, equation (2.4) shows that $F(\cdot + t)$ is a linear combination of F, B and $B(\cdot + t)$ with coefficients depending on the values $B(y_1)$, $F(y_1)$, $B(y_1+t)$ and $F(y_1+t)$. Inserting this back into (2.4) yields for all $x, y \in \mathbb{C}^n$,

$$\begin{aligned} F(x)\big(B(y)B(y_1+t) - B(y_1)B(y+t)\big)& \\ + B(x)\big(B(y)F(y_1+t) - B(y_1)F(y+t)\big)& \\ + B(x+t)\big(B(y_1)F(y) - B(y)F(y_1)\big) &= 0. \end{aligned} \tag{2.5}$$

Suppose $B(y_1)F(y) - B(y)F(y_1) = 0$ holds for all $y \in \mathbb{C}^n$. Then $F = \frac{F(y_1)}{B(y_1)}B$, and already F and B are linearly dependent. Also there is $y_2 \in \mathbb{C}^n$ such that $B(y_1)F(y_2) - B(y_2)F(y_1) \neq 0$, and equation (2.5) shows that F, B and $B(\cdot + t)$ are linearly dependent.

(ii) Assume that $B = kF$ for some $k \in \mathbb{C}$. Then $F(x+y) = 2kF(x)F(y)$, with $k \neq 0$ since F is not identically zero. Hence $\tilde F := 2kF$ satisfies $\tilde F(x+y) = \tilde F(x)\tilde F(y)$. Thus, by Proposition 2.1, there are $d_1, d_2 \in \mathbb{C}^n$ such that $F(z) = \frac{1}{2k}\exp(\langle d_1, z\rangle + \langle d_2, \bar z\rangle)$, $B(z) = \frac{1}{2}\exp(\langle d_1, z\rangle + \langle d_2, \bar z\rangle)$. This is a solution of type (b) with $\epsilon_1 = 1$, $\epsilon_2 = 0$.

(iii) We may now assume that B and F are linearly independent. Then, by **(i)**, there are functions $a_1, a_2 : \mathbb{C}^n \to \mathbb{C}$ such that

$$B(x+t) = a_1(t)F(x) + a_2(t)B(x), \quad x, t \in \mathbb{C}^n. \tag{2.6}$$

The left-hand side is symmetric in x and t. Applying it to $x + y + t$, we get the equation

$$a_1(x)F(y+t)+a_2(x)B(y+t) = B(x+y+t) = a_1(y+t)F(x)+a_2(y+t)B(x) \tag{2.7}$$

which is similar to (2.4). Choosing $y_2 \in \mathbb{C}^n$ with $F(y_2 + t) \neq 0$, we may express a_1 as a linear combination of F, B and a_2, with coefficients depending on $a_1(y_2 + t)$, $a_2(y_2 + t)$, $F(y_2 + t)$ and $B(y_2 + t)$. Inserting this back into (2.7) yields for all $x, y \in \mathbb{C}^n$,

$$\begin{aligned}\big(a_1(y+t)F(y_2+t) - a_1(y_2+t)F(y+t)\big)F(x)\\ + \big(a_2(y+t)F(y_2+t) - a_2(y_2+t)F(y+t)\big)B(x)\\ + \big(B(y_2+t)F(y+t) - B(y+t)F(y_2+t)\big)a_2(x) = 0,\end{aligned}$$

which means that a_2, B and F are linearly dependent functions. Therefore there are $b_1, b_2 \in \mathbb{C}$ such that

$$a_2(x) = b_1 B(x) + b_2 F(x).$$

Inserting this back into (2.6) and using the symmetry in (x, t), we find

$$\begin{aligned}B(x+t) &= a_1(t)F(x) + \big(b_1 B(t) + b_2 F(t)\big)B(x)\\ &= a_1(x)F(t) + \big(b_1 B(x) + b_2 F(x)\big)B(t),\\ a_1(x) - b_2 B(x) &= \frac{a_1(t) - b_2 B(t)}{F(t)} F(x) =: b_3 F(x)\end{aligned}$$

for any fixed t with $F(t) \neq 0$. Hence $a_1(x) = b_2 B(x) + b_3 F(x)$, and again by (2.6),

$$B(x+t) = \big(b_2 B(t) + b_3 F(t)\big)F(x) + \big(b_1 B(t) + b_2 F(t)\big)B(x).$$

Insert this and formula (2.2) for $F(x+t)$ into (2.4) to find, after some calculation,

$$\big((1-b_1)B(t) - b_2 F(t)\big)\big(F(x)B(y) - F(y)B(x)\big) = 0$$

for all $x, y, t \in \mathbb{C}^n$. Since B and F are linearly independent, we first conclude that $(1-b_1)B(t) = b_2 F(t)$ for all t, and then that $b_1 = 1$, $b_2 = 0$. Therefore, $a_1 = b_3 F$, $a_2 = B$ and (2.6) yields

$$B(x+t) = b_3 F(t)F(x) + B(t)B(x), \quad x, t \in \mathbb{C}^n.$$

Take $k \in \mathbb{C}$ with $k^2 = b_3$. Using this and (2.2) again, we find

$$\big(B(x+y) \pm kF(x+y)\big) = \big(B(x) \pm kF(x)\big)\big(B(y) \pm kF(y)\big)\ ,$$

so that $f := B \pm kF$ solves the equation $f(x+y) = f(x)f(y)$. Since $f \not\equiv 0$, by Proposition 2.1, there are $c_1, c_2, d_1, d_2 \in \mathbb{C}^n$ such that

$$\begin{aligned}B(z) + kF(z) &= \exp(\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle),\\ B(z) - kF(z) &= \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle),\end{aligned}$$

which gives solution (b) with $\epsilon_1 = \epsilon_2 = 1$ if $k \neq 0$.

(iv) If $k = 0$, again by Proposition 2.1, $B(z) = \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)$ for suitable $d_1, d_2 \in \mathbb{C}$. Define $G(z) := \frac{F(z)}{B(z)}$. Since $B(z+w) = B(z)B(w)$, equation (2.2) yields

$$G(z+w) = G(z) + G(w), \quad z, w \in \mathbb{C}^n.$$

Hence G satisfies the *Cauchy equation*, i.e., it is additive and continuous on $\mathbb{C}^n$. This implies by Aczél [1] that there are $c_1, c_2 \in \mathbb{C}^n$ such that $G(z) = \langle c_1, z\rangle + \langle c_2, \bar{z}\rangle$, which yields $F(z) = G(z)B(z)$, i.e., the form of F and B given in part (a). $\square$

Proof of Proposition 2.3. **(i)** Suppose that F and B satisfy equation (2.3). Fix $t \in \mathbb{C}^n \setminus \{0\}$. We again claim that F,B and $B(\cdot + t)$ are linearly dependent functions. Since $F \not\equiv 0$, there is $y_1 \in \mathbb{C}^n$ with $F(y_1) \neq 0$. For all $x, y \in \mathbb{C}^n$,

$$F(x+t)F(y) - B(x+t)B(y) = F(x+y+t) = F(x)F(y+t) - B(x)B(y+t). \quad (2.8)$$

Choosing $y = y_1$, equation (2.8) shows that $F(\cdot + t)$ is a linear combination of F, B and $B(\cdot + t)$, with coefficients depending on the values $B(y_1)$, $F(y_1)$, $B(y_1 + t)$ and $F(y_1 + t)$. Inserting this back into (2.8) yields for all $x, y \in \mathbb{C}^n$,

$$\begin{aligned}\big(B(y_1)F(y) - B(y)F(y_1)\big)B(x+t) \\ + \big(F(y_1+t)F(y) - F(y+t)F(y_1)\big)F(x) \\ + \big(B(y+t)F(y_1) - B(y_1+t)F(y)\big)B(x) = 0\ . \quad (2.9)\end{aligned}$$

If for all $y \in \mathbb{C}^n$, $B(y_1)F(y) - B(y)F(y_1) = 0$, already F and B are linearly dependent. Otherwise, equation (2.9) shows that F, B and $B(\cdot + t)$ are linearly dependent.

(ii) Assume that $B = kF$ for some $k \in \mathbb{C}$. Then $F(z+w) = (1-k^2)F(z)F(w)$ for all $z, w \in \mathbb{C}^n$ by (2.3). For $k = 1$, $F \equiv 0$, which was excluded. Hence, $k \neq 1$. Let $\tilde{F} = (1-k^2)F$. Then $\tilde{F}(z+w) = \tilde{F}(z)\tilde{F}(w)$. By Proposition 2.1, there are $d_1, d_2 \in \mathbb{C}^n$ such that $\tilde{F}(z) = \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)$, implying

$$F(z) = \frac{1}{1-k^2}\exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle), B(z) = \frac{k}{1-k^2}\exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle),$$

which is a solution of the first type (a).

(iii) By part **(i)**, there are functions $a_1, a_2 : \mathbb{C}^n \to \mathbb{C}$ such that

$$B(x+t) = a_1(t)F(x) + a_2(t)B(x), \quad x, t \in \mathbb{C}^n. \quad (2.10)$$

Exactly as in part **(i)** of the previous proof, this implies that a_2, B and F are linearly dependent. In view of **(ii)**, we may assume that B and F are linearly independent. Then there are $b_1, b_2 \in \mathbb{C}$ such that

$$a_2(x) = b_1 B(x) + b_2 F(x), \quad x \in \mathbb{C}^n.$$

Insert this into (2.10) to find, using the symmetry in x and t,

$$\begin{aligned}B(x+t) &= a_1(t)F(x) + \big(b_1 B(t) + b_2 F(t)\big)B(x) \\ &= a_1(x)F(t) + \big(b_1 B(x) + b_2 F(x)\big)B(t), \\ a_1(x) - b_2 B(x) &= \frac{a_1(t) - b_2 B(t)}{F(t)}F(x) =: b_3 F(x)\end{aligned}$$

for any fixed t with $F(t) \neq 0$. Hence $a_1(x) = b_2 B(x) + b_3 F(x)$ and again by (2.10),

$$B(x+t) = \big(b_2 B(t) + b_3 F(t)\big)F(x) + \big(b_1 B(t) + b_2 F(t)\big)B(x).$$

Insert this formula for $B(x+t)$ and formula (2.3) for $F(x+t)$ into (2.8) to find, after some calculation,

$$\big((1-b_2)B(t)-b_3F(t)\big)\big(F(x)B(y)-F(y)B(x)\big)=0$$

for all $x,y,t\in\mathbb{C}^n$. Since B and F are linearly independent, this yields $b_2=1$ and $b_3=0$. Therefore,

$$F(x+t)=F(x)F(t)-B(x)B(t),\quad B(x+t)=B(x)F(t)+F(x)B(t)+b_1B(x)B(t).$$

Let $\alpha\in\mathbb{C}$ satisfy $\alpha^2=\alpha b_1-1$. Then

$$\begin{aligned}F(x+t)+\alpha B(x+t)&=F(x)F(t)+\alpha(B(x)F(t)+F(x)B(t))\\&\qquad+(\alpha b_1-1)B(x)B(t)\\&=(F(x)+\alpha B(x))(F(t)+\alpha B(t)),\end{aligned}\tag{2.11}$$

where $\alpha=\frac{b_1}{2}\pm\sqrt{\frac{b_1^2}{4}-1}$.

(iv) Let $G(x):=F(x)+\frac{b_1}{2}B(x)$, $H(x):=\sqrt{\frac{b_1^2}{4}-1}\;B(x)$. Then by (2.11)

$$G(x+t)\pm H(x+t)=(G(x)\pm H(x))(G(t)\pm H(t)).\tag{2.12}$$

Suppose first that $b_1\notin\{2,-2\}$. Then $H\not\equiv 0$ and, by Proposition 2.1, there are vectors $c_1,c_2,d_1,d_2\in\mathbb{C}^n$ such that for all $z\in\mathbb{C}^n$,

$$\begin{aligned}G(z)+H(z)&=\exp(\langle c_1,z\rangle+\langle c_2,\bar z\rangle),\\G(z)-H(z)&=\exp(\langle d_1,z\rangle+\langle d_2,\bar z\rangle).\end{aligned}$$

Hence,

$$\begin{aligned}G(z)&=\frac{1}{2}\big(\exp(\langle c_1,z\rangle+\langle c_2,\bar z\rangle)+\exp(\langle d_1,z\rangle+\langle d_2,\bar z\rangle)\big),\\H(z)&=\frac{1}{2}\big(\exp(\langle c_1,z\rangle+\langle c_2,\bar z\rangle)-\exp(\langle d_1,z\rangle+\langle d_2,\bar z\rangle)\big)=\sqrt{\frac{b_1^2}{4}-1}\;B(z).\end{aligned}$$

Let $k:=\frac{b_1/2}{\sqrt{(b_1/2)^2-1}}\in\mathbb{C}$. Then $\sqrt{k^2-1}=\frac{1}{\sqrt{(b_1/2)^2-1}}$ and, using $\frac{b_1}{2}\sqrt{k^2-1}=k$,

$$\begin{aligned}B(z)&=\frac{\sqrt{k^2-1}}{2}\big(\exp(\langle c_1,z\rangle+\langle c_2,\bar z\rangle)-\exp(\langle d_1,z\rangle+\langle d_2,\bar z\rangle)\big),\\F(z)&=G(z)-\frac{b_1}{2}\,B(z)\\&=\frac{1}{2}\big((1-k)\exp(\langle c_1,z\rangle+\langle c_2,\bar z\rangle)+(1+k)\exp(\langle d_1,z\rangle+\langle d_2,\bar z\rangle)\big),\end{aligned}$$

which is the third solution (c).

(v) Now suppose that $b=2\epsilon$, $\epsilon\in\{+1,-1\}$. By (2.12), $G(x)=F(x)+\epsilon B(x)$ satisfies $G(x+t)=G(x)G(t)$. Hence, by Proposition 2.1, there are $d_1,d_2\in\mathbb{C}^n$

such that $G(z) = \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)$. Therefore $F(z) = \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle) - \epsilon B(z)$. Inserting this into the functional equation for F, we find

$$\begin{aligned} F(x+t) &= \exp(\langle d_1, x+t\rangle + \langle d_2, \bar{x}+\bar{t}\rangle) - \epsilon B(x+t) \\ &= F(x)F(t) - B(x)B(t) \\ &= \big(\exp(\langle d_1, x\rangle + \langle d_2, \bar{x}\rangle) - \epsilon B(x)\big)\big(\exp(\langle d_1, t\rangle + \langle d_2, \bar{t}\rangle) - \epsilon B(t)\big) \\ &\quad - B(x)B(t) \\ &= \exp(\langle d_1, x+t\rangle + \langle d_2, \bar{x}+\bar{t}\rangle) \\ &\quad - \epsilon B(x)\exp(\langle d_1, t\rangle + \langle d_2, \bar{t}\rangle) - \epsilon B(t)\exp(\langle d_1, x\rangle + \langle d_2, \bar{x}\rangle). \end{aligned}$$

Let $H(x) := \frac{B(x)}{\exp(\langle d_1, x\rangle + \langle d_2, \bar{x}\rangle)}$. Then, by the last chain of equations, H satisfies the Cauchy equation $H(x+t) = H(x) + H(t)$. Since H is also continuous, there are vectors $c_1, c_2 \in \mathbb{C}^n$ such that $H(z) = (\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle)$, i.e.,

$$\begin{aligned} B(z) &= (\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle)\ \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle), \\ F(z) &= G(z) - \epsilon B(z) \\ &= \big(1 - \epsilon(\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle)\big)\ \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle), \end{aligned}$$

which gives the second solution (b).

Calculation shows that, conversely, the formulas (a), (b) and (c) actually yield solutions of the functional equation (2.3). □

If $F, B : \mathbb{R}^n \to \mathbb{R}$ are continuous functions satisfying the same functional equations (2.2) or (2.3) on $\mathbb{R}^n$, respectively, their extensions $F, B : \mathbb{C}^n \to \mathbb{R} \subset \mathbb{C}$ given by $\tilde{F}(z) := F(\mathrm{Re}\ z)$, $\tilde{B}(z) := B(\mathrm{Re}\ z)$ satisfy the same equations on $\mathbb{C}^n$. Checking when the solutions given in Proposition 2.2 and 2.3, respectively, are real-valued, we get the following corollaries

Corollary 2.4. *Let $F, B : \mathbb{R}^n \to \mathbb{R}$ be continuous functions satisfying*

$$F(x+y) = F(x)B(y) + F(y)B(x), \quad x, y \in \mathbb{R}^n.$$

Suppose F is not identically zero. Then there are vectors $c, d \in \mathbb{R}^n$ and there is $a \in \mathbb{R}$ such that F and B have one of the following four forms:

(a) $F(x) = \langle c, x\rangle \exp(\langle d, x\rangle)$, $B(x) = \exp(\langle d, x\rangle)$;

(b) $F(x) = a\exp(\langle c, x\rangle)\sin(\langle d, x\rangle)$, $B(x) = \exp(\langle c, x\rangle)\cos(\langle d, x\rangle)$;

(c) $F(x) = a\exp(\langle c, x\rangle)\sinh(\langle d, x\rangle)$, $B(x) = \exp(\langle c, x\rangle)\cosh(\langle d, x\rangle)$;

(d) $F(x) = a\exp(\langle d, x\rangle)$, $B(x) = \dfrac{1}{2}\exp(\langle d, x\rangle)$, $\quad x \in \mathbb{R}^n$.

Conversely, these functions satisfy the above functional equation.

Corollary 2.5. *Let $F, B : \mathbb{R}^n \to \mathbb{R}$ be continuous functions satisfying*

$$F(x+y) = F(x)F(y) - B(y)B(x), \quad x, y \in \mathbb{R}^n.$$

Suppose F is not identically zero. Then there are vectors $c, d \in \mathbb{R}^n$ and there is $k \in \mathbb{R}$ such that F and B have one of the following four forms:

(a) $F(x) = \frac{1}{1-k^2}\exp(\langle d, x\rangle)$, $B(x) = \frac{k}{1-k^2}\exp(\langle d, x\rangle)$ *with* $|k| \neq 1$;

(b) $F(x) = (1 + \langle c, x\rangle)\exp(\langle d, x\rangle)$, $B(x) = \langle c, x\rangle \exp(\langle d, x\rangle)$;

(c) $F(x) = \exp(\langle c, x\rangle)\big(\cos(\langle d, x\rangle) + k\sin(\langle d, x\rangle)\big)$,
$B(x) = \sqrt{k^2+1}\exp(\langle c, x\rangle)\sin(\langle d, x\rangle)$;

(d) $F(x) = \exp(\langle c, x\rangle)\big(\cosh(\langle d, x\rangle) + k\sinh(\langle d, x\rangle)\big)$,
$B(x) = \sqrt{k^2-1}\exp(\langle c, x\rangle)\sinh(\langle d, x\rangle)$ *with* $|k| \geq 1$

for all $x \in \mathbb{R}^n$.

Conversely, these functions satisfy the above functional equation.

3. Proofs of the theorems and corollaries

We first prove that the operators T and A satisfying the functional equations (1.5) and (1.8) are localized.

Proposition 3.1. *Suppose that* $T, A : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ *satisfy*

$$T(f \cdot g) = Tf \cdot Ag + Af \cdot Tg, \quad f, g \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

and that the pair (T, A) *is non-degenerate. Then* T *and* A *are localized, i.e., there are functions* $F, B : \mathbb{R} \times \mathbb{C}^\infty \to \mathbb{C}$ *such that for all* $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ *and all* $x \in \mathbb{R}$,

$$Tf(x) = F(x, f(x), \dots, f^{(j)}(x), \dots),$$

and

$$Af(x) = B(x, f(x), \dots, f^{(j)}(x), \dots),$$

i.e., $Tf(x)$ *and* $Af(x)$ *depend only on* x *and the jet of* f *at* x. *We also have* $T(\mathbb{1}) = 0$ *and* $A(\mathbb{1}) = \mathbb{1}$.

Proof. **(i)** Choose $f = 1$ in (1.5) to find for all $g \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ and all $x \in \mathbb{R}$,

$$(1 - A(\mathbb{1})(x)) \cdot Tg(x) = T(\mathbb{1})(x) \cdot Ag(x).$$

By non-degeneracy of (T, A), there are functions $g_1, g_2 \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ such that $(Tg_i(x), Ag_i(x)) \in \mathbb{C}^2$ are linearly independent for $i = 1, 2$. Thus, choosing $g = g_1$ and $g = g_2$ in the previous equation yields $A(\mathbb{1}) = \mathbb{1}$, $T(\mathbb{1}) = 0$.

(ii) Let $J \subset \mathbb{R}$ be open and $f_1, f_2 \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ with $f_1|_J = f_2|_J$. Take any $g \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ with support in J. Then $f_1 \cdot g = f_2 \cdot g$. By (1.5),

$$\begin{aligned} Tf_1 \cdot Ag + Af_1 \cdot Tg &= T(f_1 \cdot g) = T(f_2 \cdot g) \\ &= Tf_2 \cdot Ag + Af_2 \cdot Tg, \\ \big(Tf_1(x) - Tf_2(x)\big) \cdot Ag(x) &= \big(Af_2(x) - Af_1(x)\big) \cdot Tg(x), \quad x \in \mathbb{R}\,. \end{aligned}$$

For a given $x \in J$, choose $g_1, g_2 \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ with supports in J such that $(Tg_i(x), Ag_i(x)) \in \mathbb{C}^2$ are linearly independent for $i \in \{1, 2\}$. The previous equation then yields for $g = g_1$ and $g = g_2$ that $Tf_1(x) = Tf_2(x)$, $Af_1(x) = Af_2(x)$, i.e. $Tf_1|_J = Tf_2|_J$, $Af_1|_J = Af_2|_J$.

(iii) We now claim that for any $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ and $x \in \mathbb{R}$, $Tf(x)$ depends only on x, $f(x)$ and all derivative values at x, $f^{(j)}(x)$, $j \in \mathbb{N}$. Suppose $g \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ is another function with the same jet at x, i.e., $f^{(j)}(x) = g^{(j)}(x)$ for all $j \in \mathbb{N}_0$. Let $J_- := (-\infty, x)$ and $J_+ := (x, \infty)$ and define a function $h : \mathbb{R} \to \mathbb{C}$ by

$$h(x) := \begin{cases} f(x), & x \in \overline{J_-}, \\ g(x), & x \in J_+. \end{cases}$$

Then $h \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ and $f|_{J_-} = h|_{J_-}$, $h|_{J_+} = g|_{J_+}$. By part (ii), $Tf|_{J_-} = Th|_{J_-}$ and $Th|_{J_+} = Tg|_{J_+}$. Since Tf, Th and Tg are continuous functions and $\{x\} = \overline{J_-} \cap \overline{J_+}$, we find that $Tf(x) = Th(x) = Tg(x)$. Hence $Tf(x)$ only depends on x and all values $f^{(j)}(x)$, $j \in \mathbb{N}_0$, i.e., there is a function $F : I \times \mathbb{C}^\infty \to \mathbb{C}$ such that

$$Tf(x) = F\big(x, f(x), \dots, f^{(j)}(x), \dots\big) \ ,$$

for all $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$, $x \in \mathbb{R}$. The proof for A is identical. ☐

Proposition 3.2. *Suppose that $T, A : \mathcal{S}(\mathbb{R}, \mathbb{C}) \to \mathcal{S}(\mathbb{R}, \mathbb{C})$ satisfy*

$$T(f \cdot g) = Tf \cdot Tg - Af \cdot Ag, \quad f, g \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

and that the pair (T, A) is non-degenerate. Then there are functions $F, B : \mathbb{R} \times \mathbb{C}^\infty \to \mathbb{C}$ such that for all $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ and all $x \in \mathbb{R}$,

$$Tf(x) = F(x, f(x), \dots, f^{(j)}(x), \dots),$$

and

$$Af(x) = B(x, f(x), \dots, f^{(j)}(x), \dots).$$

We also have $T(\mathbb{1}) = \mathbb{1}$ and $A(\mathbb{1}) = 0$.

Proof. Basically the same proof as for the previous Proposition applies. Just replace the central equations in parts (i) and (ii) of that proof by

$$(1 - T(\mathbb{1})(x)) \cdot Tg(x) = -A(\mathbb{1})(x) \cdot Ag(x)$$

and

$$(Tf_1(x) - Tf_2(x)) \cdot Tg(x) = (Af_1(x) - Af_2(x)) \cdot Ag(x).$$

The non-degeneracy of (T, A) then yields $T(\mathbb{1}) = \mathbb{1}$ and $A(\mathbb{1}) = 0$, and that $f_1|_J = f_2|_J$ implies $Tf_1|_J = Tf_2|_J$, $Af_1|_J = Af_2|_J$. Part (iii) then applies directly. ☐

Proof of Theorem 1.5. **(i)** By Proposition 3.1, there are functions $\tilde{F}, \tilde{B} : \mathbb{R} \times \mathbb{C}^\infty \to \mathbb{C}$ such that for any $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ and any $x \in \mathbb{R}$,

$$Tf(x) = \tilde{F}(x, f(x), \dots, f^{(j)}(x), \dots),$$

and

$$Af(x) = \tilde{B}(x, f(x), \dots, f^{(j)}(x), \dots).$$

Since $T(\mathbb{1}) = 0$ and $A(\mathbb{1}) = \mathbb{1}$, we have $\tilde{F}(x, 0, \cdots) = 0$ and $\tilde{B}(x, 1, 0, \dots) = 1$.

Let $I \subset \mathbb{R}$ be a bounded open interval. Take any function $h \in C^\infty(\bar{I}, \mathbb{C})$, i.e., h should be a C^∞-function on I which together with all derivatives may be extended by continuity to $\bar{I}$. Then also $\exp(h) \in C^\infty(\bar{I}, \mathbb{C})$. Functions in $\mathcal{S}(\mathbb{R}, \mathbb{C})$, when restricted to $\bar{I}$, are just $C^\infty(\bar{I}, \mathbb{C})$-functions, and $\exp(h)$ may be extended to a function $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ on $\mathbb{R}$ such that $f|_{\bar{I}} = \exp(h)|_{\bar{I}}$, just making sure that f and its derivatives decay rapidly as $|x| \to \infty$. Define operators $S, R : C^\infty(\bar{I}, \mathbb{C}) \to C^\infty(I, \mathbb{C})$ by

$$Sh(x) := T(\exp(h))(x), \quad Rh(x) := A(\exp(h))(x)$$

for any $h \in C^\infty(\bar{I}, \mathbb{C})$ and $x \in I$. More precisely, $Sh(x) = T(f)(x)$, $x \in I$, where $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ is such that $f|_{\bar{I}} = \exp(h)|_{\bar{I}}$. Note that, by localization, this is well-defined for any $x \in I$, independently of the particular extension of $\exp(h)$ to a function in $\mathcal{S}(\mathbb{R}, \mathbb{C})$. Moreover, since $Tf \in \mathcal{S}(\mathbb{R}, \mathbb{C})$, we know that $Sh = Tf|_I$ is a C^∞-function on I.

Since the derivatives of $\exp(h)$ of order j can be written as the functions of h and its derivatives of order $\leq j$, the operators S and R are localized as well. Thus there exist functions $F, B : I \times \mathbb{C}^\infty \to \mathbb{C}$ such that for any $h \in C^\infty(\bar{I}, \mathbb{C})$ and any $x \in I$,

$$Sh(x) = F(x, h(x), \dots, h^{(j)}(x), \dots), \quad Rh(x) = B(x, h(x), \dots, h^{(j)}(x), \dots),$$

with $F(x, 0) = 0$ and $B(x, 0) = 1$. The extended Leibniz rule equation (1.5) yields for any $h_1, h_2 \in C^\infty(\bar{I}, \mathbb{C})$ and any $x \in I$,

$$\begin{aligned} S(h_1 + h_2)(x) &= T(\exp(h_1)\exp(h_2))(x) \\ &= T(\exp(h_1))(x) \cdot A(\exp(h_2))(x) + A(\exp(h_1))(x) \cdot T(\exp(h_2))(x) \\ &= S(h_1)(x) \cdot R(h_2)(x) + R(h_1)(x) \cdot S(h_2)(x). \end{aligned} \tag{3.1}$$

Let $\alpha = (\alpha_j)_{j \in \mathbb{N}_0}$, $\beta = (\beta_j)_{j \in \mathbb{N}_0} \in \mathbb{C}^\infty$ and $x \in I$ be arbitrary. Then there exist $h_1, h_2 \in C^\infty(\bar{I}, \mathbb{C})$ with $h_1^{(j)} = \alpha_j$ and $h_2^{(j)}(x) = \beta_j$ for all $j \in \mathbb{N}_0$, cf. Hörmander [6], page 16. This may be shown by adding infinitely many small bump functions. Therefore equation (3.1) is equivalent to the functional equation for F and B,

$$F(x, \alpha + \beta) = F(x, \alpha) \cdot B(x, \beta) + B(x, \alpha) \cdot F(x, \beta) \tag{3.2}$$

for all $x \in I$ and $\alpha, \beta \in C^\infty$.

(ii) For $k \in \mathbb{N}_0$, define $F_k, B_k : I \times \mathbb{C}^{k+1} \to \mathbb{C}$ by

$$F_k(x, \alpha_0, \dots, \alpha_k) := F(x, \alpha_0, \dots, \alpha_k, 0, 0, \dots),$$

$$B_k(x, \alpha_0, \dots, \alpha_k) := B(x, \alpha_0, \dots, \alpha_k, 0, 0, \dots),$$

$x \in I$, $\alpha \in \mathbb{C}^{k+1}$. We claim that for fixed $x \in I$, $F_k(x, \cdot)$ and $B_k(x, \cdot)$ are continuous functions from $\mathbb{C}^{k+1}$ to $\mathbb{C}$. To verify this, take any sequence $\alpha_n = (\alpha_{n,j})_{j=0}^k \in \mathbb{C}^{k+1}$ and $\alpha \in \mathbb{C}^{k+1}$ such that $\alpha_n \to \alpha$ in $\mathbb{C}^{k+1}$. Consider the functions $h_n(t) := \sum_{j=0}^k \frac{\alpha_{n,j}}{j!}(t-x)^j$, $h(t) := \sum_{j=0}^k \frac{\alpha_j}{j!}(t-x)^j$. Then $h_n^{(j)} \to h^{(j)}$ converges uniformly on $\bar{I}$ for any $j \in \mathbb{N}_0$, and similarly $\exp(h_n)^{(j)} \to \exp(h)^{(j)}$. A corresponding statement holds for suitable extensions of $\exp(h_n)$ and $\exp(h)$ to functions in $\mathcal{S}(\mathbb{R}, \mathbb{C})$. Therefore the assumption of pointwise continuity of T and A implies that

$$\begin{aligned}
F_k(x, \alpha_{n,0}, \dots, \alpha_{n,k}) &= T(\exp(h_n))(x) \to T(\exp(h))(x) = F_k(x, \alpha_0, \dots, \alpha_k), \\
B_k(x, \alpha_{n,0}, \dots, \alpha_{n,k}) &= A(\exp(h_n))(x) \to A(\exp(h))(x) = B_k(x, \alpha_0, \dots, \alpha_k).
\end{aligned}$$

Hence, for any $k \in \mathbb{N}_0$ and $x \in I$, $F_k(x, \cdot)$ and $B_k(x, \cdot)$ are continuous functions from $\mathbb{C}^{k+1}$ to $\mathbb{C}$. Clearly, by the definition of F_k and B_k,

$$\begin{aligned}
F_{k+1}(x, \alpha_0, \dots, \alpha_k, 0) &= F_k(x, \alpha_0, \dots, \alpha_k), \\
B_{k+1}(x, \alpha_0, \dots, \alpha_k, 0) &= B_k(x, \alpha_0, \dots, \alpha_k).
\end{aligned}$$

(iii) Let $\mathcal{P}_k$ denote the complex valued polynomials on $\mathbb{R}$ of degree $\leq k$. Then for any $h \in \mathcal{P}_k$, any extension f of $\exp(h)|_{\bar{I}}$ to a function in $\mathcal{S}(\mathbb{R}, \mathbb{C})$ and any $x \in I$, by definition of F and B,

$$\begin{aligned}
Tf(x) &= S(h)(x) = F(x, h(x), \dots, h^{(k)}(x), 0, 0, \dots) = F_k(x, h(x), \dots, h^{(k)}(x)), \\
Af(x) &= B_k(x, h(x), \dots, h^{(k)}(x)). \qquad (3.3)
\end{aligned}$$

Since by (3.2) for any $k \in \mathbb{N}_0$,

$$F_k(x, \alpha + \beta) = F_k(x, \alpha) B_k(x, \beta) + B_k(x, \alpha) F_k(x, \beta), \quad x \in I, \ \alpha, \beta \in \mathbb{C}^{k+1}. \quad (3.4)$$

Proposition 2.2 implies that there are functions $c_1, c_2, d_1, d_2 : I \to \mathbb{C}^{k+1}$ and $\gamma : I \to \mathbb{C} \setminus \{0\}$ and $\epsilon_1, \epsilon_2 \in \{0, 1\}$ not both zero such that either

$$\begin{aligned}
F_k(x, \alpha) &= (\langle c_1(x), \alpha \rangle + \langle c_2(x), \bar{\alpha} \rangle) \exp(\langle d_1(x), \alpha \rangle + \langle d_2(x), \bar{\alpha} \rangle), \\
B_k(x, \alpha) &= \exp(\langle d_1(x), \alpha \rangle + \langle d_2(x), \bar{\alpha} \rangle) \qquad (3.5)
\end{aligned}$$

or

$$\begin{aligned}
F_k(x, \alpha) &= \frac{\gamma(x)}{2} \big(\epsilon_1 \exp(\langle c_1(x), \alpha \rangle \\
&\qquad + \langle c_2(x), \bar{\alpha} \rangle) - \epsilon_2 \exp(\langle d_1(x), \alpha \rangle + \langle d_2(x), \bar{\alpha} \rangle) \big), \\
B_k(x, \alpha) &= \frac{1}{2} \big(\epsilon_1 \exp(\langle c_1(x), \alpha \rangle \\
&\qquad + \langle c_2(x), \bar{\alpha} \rangle) + \epsilon_2 \exp(\langle d_1(x), \alpha \rangle + \langle d_2(x), \bar{\alpha} \rangle) \big) \qquad (3.6)
\end{aligned}$$

for all $x \in I$ and $\alpha \in \mathbb{C}^{k+1}$. Due to the definition of F_k and B_k, the functions $c_1, c_2, d_1, d_2 : I \to \mathbb{C}^{k+1}$ depend on $k \in \mathbb{N}$ in a very simple "imbedded" way: Let, e.g.,

$$d_{1,k} = d_1 = (a_0, \dots, a_{k-1}, a_k), \quad d_{2,k} = d_2 = (b_0, \dots, b_{k-1}, b_k)$$

with coordinate functions $a_0, \dots, a_k, b_0, \dots, b_k : I \to \mathbb{C}$. Then $d_{1,k-1} = (a_0, \dots, a_{k-1})$, $d_{2,k-1} = (b_0, \dots, b_{k-1})$: the transfer from $d_{1,k-1}$ to $d_{1,k}$ is just by adding the coordinate function a_k. Moreover, $\gamma : I \to \mathbb{C}$ is independent of k.

(iv) Let us first analyze the solution for (T, A) which originates from the first solution (3.5) of the functional equation (3.4). In this case, for any $h \in \mathcal{P}_k$, $x \in I$,

$$\begin{aligned} A(f) &= B_k(x, h(x), \dots, h^{(k)}(x)) \\ &= \exp\left(\sum_{j=0}^{k} a_j(x) h^{(j)}(x) + \sum_{j=0}^{k} b_j(x) \overline{h^{(j)}(x)}\right), \end{aligned} \tag{3.7}$$

where f is a suitable extension of $\exp(h)$ to $\mathcal{S}(\mathbb{R}, \mathbb{C})$. Choosing h to be arbitrary complex constants, the continuity of $A(f)$ implies that a_0 and b_0 are continuous functions on I. Next, choosing linear polynomials h, we conclude that also a_1 and b_1 are continuous on I. Continuing with polynomials of successively higher degrees, we get that all coordinate functions $a_0, \dots, a_k, b_0, \dots, b_k$ are continuous functions on I. By the way, a continuity argument of this type also shows that ϵ_1 and ϵ_2 in solution (3.6) do not depend on $x \in I$.

We now claim that $a_k = b_k = 0$ on I for all $k \in \mathbb{N}$, allowing only a_0 or b_0 to be non-zero. For simplicity of notation, to prove $a_k(x) = b_k(x) = 0$, we will assume that $x = 0 \in I$ and show $a_k(0) = b_k(0) = 0$ for $k \in \mathbb{N}$. Choose $\epsilon_0 > 0$ with $[-\epsilon_0, \epsilon_0] \subset I$. The argument uses successively better approximations of the zero function by functions $\exp(h_\epsilon)$, where the h_ϵ are polynomials of fixed degree, but only on successively smaller intervals around zero.

We first prove that $a_1(0) = b_1(0) = 0$. Suppose this were false and that, e.g., $|a_1(0)| > 0$, $|a_1(0)| \geq |b_0(0)|$. Choose $\theta \in \mathbb{C}$ with $\theta a_1(0) = |a_1(0)|$. For any $0 < \epsilon \leq \epsilon_0$, define $h_\epsilon(x) := -\frac{1}{\sqrt{\epsilon}} + \frac{\theta}{\epsilon} x$, $x \in \mathbb{R}$. Then $h_\epsilon \in \mathcal{P}_1$ and $\exp(h_\epsilon)(x) = \exp(-\frac{1}{\sqrt{\epsilon}}) \exp(\frac{\theta}{\epsilon} x)$. For $|x| \leq \epsilon$, this is bounded in modulus by $\exp(1 - \frac{1}{\sqrt{\epsilon}})$, quickly tending to zero for $\epsilon \to 0$. As for the j-th derivative, we have

$$\exp(h_\epsilon)^{(j)}(x) = \frac{\theta^j}{\epsilon^j} \exp(-\frac{1}{\sqrt{\epsilon}}) \exp(\frac{\theta}{\epsilon} x).$$

For $|x| \leq \epsilon$, this is bounded in modulus by $\frac{1}{\epsilon^j} \exp(1 - \frac{1}{\sqrt{\epsilon}})$, tending to zero for $\epsilon \to 0$ for all $j \in \mathbb{N}$. We may extend each function $\exp(h_\epsilon) : [-\epsilon, \epsilon] \to \mathbb{C}$ to a function $f_\epsilon \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ such that for any fixed $j \in \mathbb{N}_0$, also $f_\epsilon^{(j)} \to 0$ as $\epsilon \to 0$ uniformly on any compact set. By the assumption of pointwise continuity of A, we know that $A(f_\epsilon)(0) \to A(0)(0)$ converges.

Since the operators A and R are localized, we know that for all $|x| \leq \epsilon$, using $h_\epsilon^{(j)} = 0$ for all $j \geq 2$,

$$A(f_\epsilon)(x) = B_k(x, h_\epsilon(x), h'_\epsilon(x), 0, 0, \dots)\ ,$$

$$A(f_\epsilon)(0) = B_k(0, -\frac{1}{\sqrt{\epsilon}}, \frac{\theta}{\epsilon}, 0, \dots)$$

$$= \exp(-\frac{a_0(0)}{\sqrt{\epsilon}} + \frac{|a_1(0)|}{\epsilon} - \frac{b_0(0)}{\sqrt{\epsilon}} + \frac{\theta b_1(0)}{\epsilon}) \to A(0)(0)\ .$$

Since $\exp(\frac{1}{\epsilon})$ grows much faster than $\exp(\frac{1}{\sqrt{\epsilon}})$ as $\epsilon \to 0$, $A(f_\epsilon)(0)$ converges only if $a_1(0) = 0$ and hence also $b_1(0) = 0$ since $|a_1(0)| \geq |b_1(0)|$. Then also $\mathrm{Re}\ a_0(0) \geq 0$, $\mathrm{Re}\ b_0(0) \geq 0$ is required and $A(0)(0) = 0$ follows, i.e., $A(0) = 0$.

If $a_1(0) = \cdots = a_{k-1}(0) = 0$ and $b_1(0) = \cdots = b_{k-1}(0) = 0$ were shown and $|a_k(0)| > 0$, $|a_k(0)| \geq |b_k(0)|$ would hold, choose $\theta \in \mathbb{C}$ with $\theta^k a_k(0) = |a_k(0)|$ and define $h_\epsilon(x) := -\frac{1}{\sqrt{\epsilon}} + \frac{\theta}{k!}(\frac{x}{\epsilon})^k$. Then $h_\epsilon \in \mathcal{P}_k$ and we have for any $|x| \leq \epsilon$ and $j \in \mathbb{N}_0$ that

$$|\exp(h_\epsilon)^{(j)}(x)| \leq p_j\left(\frac{1}{\epsilon}\right)\exp\left(-\frac{1}{\sqrt{\epsilon}}\right)$$

for a suitable real polynomial p_j of degree j. Therefore $p_j(\frac{1}{\epsilon})\ \exp(-\frac{1}{\sqrt{\epsilon}}) \to 0$ as $\epsilon \to 0$ for all $j \in \mathbb{N}_0$. Again for suitable extensions f_ϵ of $\exp(h_\epsilon)|_{[-\epsilon,\epsilon]}$ to $\mathcal{S}(\mathbb{R}, \mathbb{C})$, for any fixed $j \in \mathbb{N}_0$, $f_\epsilon^{(j)} \to 0$ as $\epsilon \to 0$ uniformly on compacta. We find similarly as before

$$\begin{aligned} A(f_\epsilon)(0) &= B_k\left(0, -\frac{1}{\sqrt{\epsilon}}, 0, \ldots, 0, \frac{\theta^k}{\epsilon^k}\right) \\ &= \exp\left(-\frac{a_0(0)}{\sqrt{\epsilon}} + \frac{|a_k(0)|}{\epsilon^k} - \frac{b_0(0)}{\sqrt{\epsilon}} + \frac{\theta^k b_k(0)}{\epsilon^k}\right) \to A(0)(0) = 0, \end{aligned}$$

which yields $a_k(0) = b_k(0) = 0$.

Formula (3.7) therefore implies that for all complex-valued polynomials $h \in \mathcal{P} := \cup_{k\in\mathbb{N}}\mathcal{P}_k$ and all extensions f of $\exp(h)|_{\bar{I}}$ to $\mathcal{S}(\mathbb{R}, \mathbb{C})$,

$$A(f)(x) = \exp\left(a_0(x)h(x) + b_0(x)\overline{h(x)}\right), \quad x \in I,$$

i.e.,

$$A(f)(x) = f(x)^{a_0(x)} \cdot \overline{f(x)}^{b_0(x)}, \quad x \in I\ , \tag{3.8}$$

with $\mathrm{Re}\ a_0(0) \geq 0$, $\mathrm{Re}\ b_0(0) \geq 0$. Given any fixed function $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ which is never zero on I, we may define $\ln f$ as a continuous function on I which then actually is a C^∞-function on I. Approximating this by polynomials on $\bar{I}$, we conclude that (3.8) holds for all $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ which are nowhere zero on I, and then by further approximation also for those $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ which have zeros in I.

(v) Thus we know that $Af = f^{a_0} \cdot \bar{f}^{b_0}$. To assure that $Af \in C^\infty$ for all $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$, we need $a_0 = m \in \mathbb{N}_0$ and $b_0 = n \in \mathbb{N}_0$ to be constant functions which are non-negative integers since otherwise by differentiating sufficiently many times we would get factors $f^{-\delta}$ or $\bar{f}^{-\delta}$ of derivative terms for some $\delta > 0$, which would yield singularities as $f \to 0$. Therefore, by (3.8),

$$Af = f^m \cdot \bar{f}^n, \quad m, n \in \mathbb{N}_0,\ m + n \geq 1.$$

The condition $m + n \geq 1$ is needed since for $n = m = 0$, $Af = \mathbb{1} \notin \mathcal{S}(\mathbb{R}, \mathbb{C})$. The formula for F_k in (3.5) now yields for T,

$$Tf(x) = \left(\sum_{j=0}^{k} c_j(x)h^{(j)}(x) + \sum_{j=0}^{k} \tilde{c}_j(x)\bar{h}^{(j)}(x)\right) \cdot f^m(x) \cdot \bar{f}^n(x), \tag{3.9}$$

if $f|_I = \exp(h)$, $h \in \mathcal{P}_k$, where $c_j, \tilde{c}_j$ are suitable functions on I which clearly are continuous since T is pointwise continuous. Locally, $h = \ln f$ and $h^{(j)} = (\frac{f'}{f})^{(j-1)}$ has a singularity of order f^{-j} as $f \to 0$ when $f' \neq 0$ and $j \in \mathbb{N}$. Therefore in the first sum $k \leq m$ and in the second sum $k \leq n$ are needed. Moreover, since there are no continuous branches of $\ln f$ satisfying $\ln(fg) = \ln f + \ln g$ for all complex-valued functions $f, g \in \mathcal{S}(\mathbb{R}, \mathbb{C})$, which would be needed for (3.9) to be a solution of (1.5) if $c_0 \neq 0$ or $\tilde{c}_0 \neq 0$, we also need that $c_0 = \tilde{c}_0 = 0$. We finally get in the case of the solution (3.5)

$$Tf = \left(\sum_{j=1}^{m} c_j \left(\frac{f'}{f} \right)^{(j-1)} + \sum_{j=1}^{n} \tilde{c}_j \left(\frac{\bar{f}'}{\bar{f}} \right)^{(j-1)} \right) \cdot f^m \cdot \bar{f}^n, \; Af = f^m \cdot \bar{f}^n, \; f \in \mathcal{S}(\mathbb{R}, \mathbb{C}),$$

with continuous functions c_j, $\tilde{c}_j$ which actually have to be in C^∞ to guarantee that the image of T consists of C^∞-functions.

(vi) The analysis of the solution (3.6) of equation (3.4) is similar. First of all, we need $\epsilon_1 = \epsilon_2 = 1$ since in the cases $\epsilon_1 = 1, \epsilon_2 = 0$ or $\epsilon_1 = 0, \epsilon_2 = 1$, T and A would be homothetic, which is excluded by the condition of non-degeneration of (T, A). Also $\gamma(x) \neq 0$ is required. Therefore,

$$\frac{1}{\gamma} F_k(x, \alpha) + B_k(x, \alpha) = \exp(\langle c_1(x), \alpha \rangle + \langle c_2(x), \bar{\alpha} \rangle).$$

Then for all $x \in I$ and $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ extending $\exp(h)$ from $\bar{I}$ to $\mathbb{R}$, where $h \in \mathcal{P}_k$, we have that

$$\frac{1}{\gamma} T(f)(x) + A(f)(x) = \exp\left(\langle c_1(x), (h^{(j)}(x))_{j=0}^k \rangle + \langle c_2(x), (\bar{h}^{(j)}(x))_{j=0}^k \rangle \right).$$

The same arguments as before then show that $\frac{1}{\gamma} T(f) + A(f) = f^m \cdot \bar{f}^n$ with $m, n \in \mathbb{N}_0$, $m + n \geq 1$. Similarly, $-\frac{1}{\gamma} T(f) + A(f) = f^M \cdot \bar{f}^N$, for some $M, N \in \mathbb{N}_0$, $M + N \geq 1$ so that

$$Tf = \frac{\gamma}{2} (f^m \bar{f}^n - f^M \bar{f}^N) \, , \; Af = \frac{1}{2} (f^m \bar{f}^n + f^M \bar{f}^N).$$

This ends the proof of Theorem 1.5. □

The *Proof of Theorem 1.6* follows by similar arguments, just using Corollary 2.4 instead of Proposition 2.2.

Proof of Theorem 1.10. Suppose that T and A satisfy the conditions of Theorem 1.5 and that A is surjective. Consider the first solution (1.6) of (1.5), when $Af = f^m \bar{f}^n$, $m, n \in \mathbb{N}_0$ with $m+n \geq 1$ for any $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$. Take any real-valued function $g \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ which has a zero of order 1 at 0, $g(x) = x \cdot k(x)$, $k(0) \neq 0$. Since A is surjective, there is $f_0 \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ such that $Af_0 = f_0^m \bar{f_0}^n = g$. Obviously, this requires that f_0 also have a zero of order 1 at 0 so that $f_0(x) = x \cdot h(x)$ with $h(x) \neq 0$. But then for $x \in \mathbb{R}$, $x^{m+n} \cdot h(x)^m \cdot \overline{h(x)^n} = x \cdot k(x)$ so that $m+n=1$, which means that either $m=1, n=0$ or $m=0, n=1$, yielding $Af=f$ and $Tf=a_1 f'$ or $Af=\bar{f}$ and $Tf=b_1\bar{f}'$ for all $f\in\mathcal{S}(\mathbb{R},\mathbb{C})$.

In the case of the second solution (1.7) of (1.5), we use the additional assumption that T vanishes locally on constants. Thus, for any $c\in\mathbb{C}$ and $f\in\mathcal{S}(\mathbb{R},\mathbb{C})$ with $f|_I=c$ and $Tf(x)=0$, where I is open and $x\in I$, we get that $0=\frac{2}{a_1(x)}Tf(x)=c^m\bar{c}^n-c^M\bar{c}^N$, which implies that $m=M$ and $n=N$ so that $Af=f^m\bar{f}^n$. This again is surjective only if either $m=1, n=0$ or $m=0, n=1$. But then $T=0$ and the pair (T,A) is not non-degenerate so that only the first solution is relevant. □

Proof of Theorem 1.11. In the case of the first solution (1.6) of (1.5), we have $c=Ag(x)=g(x)^m\overline{g(x)}^n=c^{m+n}$ so that either $(m,n)=(1,0)$ or $(m,n)=(0,1)$ which means that either $Af=f$, $Tf=a_1f'$ or that $Af=\bar{f}$, $Tf=b_1\bar{f}'$. In the case of the second solution (1.7) of (1.5), we have $c=\frac{1}{2}(c^{m+n}+c^{M+N})$ which in view of $c>0, c\neq 1$ implies that $m+n=1$ and $M+N=1$. For $(m,n)=(M,N)=(1,0)$ or $=(0,1)$ we have $T=0$ which is a degenerate case. Only for $(m,n)=(1,0)$ and $(M,N)=(0,1)$ we get a non-degenerate solution, namely $Af=\operatorname{Re} f$ and $Tf=a_1\operatorname{Im} f$. □

Theorems 1.12 and 1.15 follow immediately from Theorem 1.10 and the remarks following formulas (1.2) and (1.3).

Proof of Theorems 1.13 and 1.16. In the case of Theorem 1.13, $T:=R\mathcal{F}^{-1}$ and $A:=S\mathcal{F}^{-1}$ satisfy (1.5) and the assumptions of Theorem 1.5. By assumption, there is $0\neq G\in\mathcal{S}(\mathbb{R},\mathbb{C})$ with $SG=\mathcal{F}G$. Let $g:=\mathcal{F}G\in\mathcal{S}(\mathbb{R},\mathbb{C})$, $g\neq 0$. Since by Theorem 1.5, in the case of solution (1.6) $Ag=g^m\bar{g}^n$ and in case of solution (1.7) $Ag=\frac{1}{2}(g^m\bar{g}^n+g^M\bar{g}^N)$. Now $Ag=g$ means that this holds for a smooth continuum of values of $g(x)$, which in the case of a real-valued $g=\mathcal{F}G$ implies that $m+n=1$ and in the case of (1.7) also $M+N=1$. This yields the solutions stated in Theorem 1.13. If g is properly complex-valued, $Ag=g$ implies that $m=1, n=0$ and $M=1, N=0$, so $Af=f$ for all $f\in\mathcal{S}(\mathbb{R},\mathbb{C})$, $SF=\mathcal{F}F$ for all $F\in\mathcal{S}(\mathbb{R},\mathbb{C})$. Similarly, in that situation $Ag=\bar{g}$ gives $m=0, n=1$ and $M=0, N=1$, hence $SF=\overline{\mathcal{F}F}$. Thirdly, $Ag=\operatorname{Re}(g)$ yields the solution (1.7) with $m=1, n=0$ and $M=0, N=1$, i.e., $SF=\operatorname{Re}(\mathcal{F}F)$ for all F, which proves the Remark after Theorem 1.13. The proof of Theorem 1.16 is similar, with $R=DS$. □

Proof of Theorem 1.18. The proof is very similar to that of Theorem 1.5, just using Proposition 2.3 instead of 2.2. We just give a sketch of the changes which are required. If I is again a bounded open interval and f an extension of $\exp(h)$ from $\bar{I}$ to a function on $\mathbb{R}$ in $\mathcal{S}(\mathbb{R},\mathbb{C})$, where $h\in\mathcal{P}_k$ is a complex-valued polynomial of degree $\leq k$, then

$$Tf(x)=F_k(x,h(x),\dots,h^{(k)}(x)),\quad Af(x)=B_k(x,h(x),\dots,h^{(k)}(x)),$$

where F_k and B_k satisfy the functional equation

$$F_k(x,\alpha+\beta)=F_k(x,\alpha)\cdot F_k(x,\beta)-B_k(x,\alpha)\cdot B_k(x,\beta). \tag{3.10}$$

By Proposition 2.3, equation (3.10) has three types of solutions. The first one is not applicable in our situation since it leads to T and A being proportional, which is excluded by the condition of non-degeneration of (T, A). We are left with the two possibilities:

$$\begin{aligned} F(x,\alpha) &= \big(1 + \langle c_1(x), \alpha\rangle + \langle c_2(x), \bar\alpha\rangle\big) \exp(\langle d_1(x), \alpha\rangle + \langle d_2(x), \bar\alpha\rangle), \\ B(x,\alpha) &= \pm\big(\langle c_1(x), \alpha\rangle + \langle c_2(x), \bar\alpha\rangle\big) \exp(\langle d_1(x), \alpha\rangle + \langle d_2(x), \bar\alpha\rangle), \end{aligned} \tag{3.11}$$

or for $k \notin \{1, -1\}$

$$\begin{aligned} F(x,\alpha) &= \frac{1}{2}\big((1-k)\exp(\langle c_1(x), \alpha\rangle \\ &\qquad\qquad + \langle c_2(x), \bar\alpha\rangle) + (1+k)\exp(\langle d_1(x), \alpha\rangle + \langle d_2(x), \bar\alpha\rangle)\big), \\ B(x,\alpha) &= \frac{1}{2}\sqrt{k^2-1}\big(\exp(\langle c_1(x), \alpha\rangle \\ &\qquad\qquad + \langle c_2(x), \bar\alpha\rangle) - \exp(\langle d_1(x), \alpha\rangle + \langle d_2(x), \bar\alpha\rangle)\big). \end{aligned} \tag{3.12}$$

For extensions f of $\exp(h)$, $h \in \mathcal{P}_k$, from $C^\infty(I, \mathbb{C})$ to $\mathcal{S}(\mathbb{R}, \mathbb{C})$ in the case of (3.11),

$$Tf(x) \mp Af(x) = \exp\left(\langle d_1(x), (h^{(j)}(x))_{j=0}^k\rangle + \langle d_2(x), \overline{(h^{(j)}(x))}_{j=0}^k\rangle\right),$$

and in the second case,

$$Tf(x) + \sqrt{\frac{k-1}{k+1}}Af(x) = \exp\left(\langle d_1(x), (h^{(j)}(x))_{j=0}^k\rangle + \langle d_2(x), \overline{(h^{(j)}(x))}_{j=0}^k\rangle\right).$$

Since $Tf, Af \in C^\infty$, these sums or differences also have C^∞-regularity. The arguments in the proof of Theorem 1.5 then show that the functions $d_1, d_2 : I \to \mathbb{C}^{k+1}$ are continuous and that $d_1 = (a_0, 0, \ldots, 0)$ and $d_2 = (b_0, 0, \ldots, 0)$ only allow for (possibly) non-zero coordinate functions a_0 and/or b_0 with Re $a_0 \geq 0$ and Re $b_0 \geq 0$. To guarantee that the image is in C^∞, we again need that $a_0 = m$ and $b_0 = n$ are constant functions with non-negative integer values. Then in the first case $Tf \mp Af = f^m \bar f^n$ and in the second case $Tf + \sqrt{\frac{k-1}{k+1}}Af = f^m \bar f^n$. In the second case we also have

$$Tf(x) + \sqrt{\frac{k+1}{k-1}}Af(x) = \exp\left(\langle c_1(x), (h^{(j)}(x))_{j=0}^k\rangle + \langle c_2(x), \overline{(h^{(j)}(x))}_{j=0}^k\rangle\right),$$

implying $Tf + \sqrt{\frac{k+1}{k-1}}Af = f^M \bar f^N$, $M, N \in \mathbb{N}_0$ with $M + N \geq 1$, yielding together with $Tf + \sqrt{\frac{k-1}{k+1}}Af = f^m \bar f^n$ the formulas for Tf and Af given in Theorem 1.18 in the second case. In the first case, we use the same arguments as in part **(v)** of the proof of Theorem 1.5 to finish the proof. □

Acknowledgments. We would like to thank M. Sodin and E. Shustin for discussions concerning the example preceding Theorem 1.10, and also for allowing us to include the example here.

Supports. The first author is supported by Minerva. The second author is supported in part by the Alexander von Humboldt Foundation, by ISF grant 387/09 and by BSF grant 200 6079.

References

[1] J. Aczél, *Lectures on Functional Equations and Their Applications,* Academic Press, 1966.

[2] S. Alesker, S. Artstein-Avidan, D. Faifman, and V. Milman, *A characterization of product preserving maps with applications to a characterization of the Fourier transform,* Illinois J. Math. **54** (2010), 1115–1132.

[3] S. Alesker, S. Artstein-Avidan, and V. Milman, *A characterization of the Fourier transform and related topics,* in: A. Alexandrov et al. (eds), Linear and complex analysis, Dedicated to V.P. Havin, Amer. Math. Soc. Transl. **226**, Advances in the Math. Sciences **63** (2009), 11–26.

[4] S. Artstein-Avidan, D. Faifman, and V. Milman, *On multiplicative maps of continuous and smooth functions,* Geometric Aspects of Functional Analysis, Lecture Notes in Math., **2050**, Springer, Heidelberg, 2012, 35–59.

[5] S. Artstein-Avidan, H. König, and V. Milman, *The chain rule as a functional equation,* J. Funct. Anal. **259** (2010), 2999–3024.

[6] L. Hörmander, *The Analysis of Linear Partial Differential Operators,* ***I****. Distribution Theory and Fourier Analysis,* Springer-Verlag, Berlin, 1983.

[7] H. König and V. Milman, *Characterizing the derivative and the entropy function by the Leibniz rule, with an appendix by D. Faifman,* J. Funct. Anal. **261** (2011), 1325–1344.

[8] A.N. Milgram, *Multiplicative semigroups of continuous functions,* Duke Math. J. **16** (1949), 377–383.

[9] J. Mrčun, *On isomorphisms of algebras of smooth functions,* Proc. Amer. Math. Soc. **133** (2005), 3109–3113.

[10] J. Mrčun and P. Šemrl, *Multplicative bijections between algebras of differentiable functions,* Ann. Acad. Sci. Fenn. Math. **32** (2007), 471–480.

[11] E. Shustin, *private communication.*

[12] M. Sodin, *private communication.*

Received February 8, 2018.

Hermann König,
Mathematisches Seminar, Universität Kiel, 24098 Kiel, Germany,
E-mail: hkoenig@math.uni-kiel.de

Vitali Milman,
School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel,
E-mail: milman@post.tau.ac.il

Узагальнене правило Лейбниця та пов'язані з ним рівняння у просторі швидко спадних функцій

Hermann König and Vitali Milman

Ми розв'язуємо узагальнене правило Лейбниця

$$T(f \cdot g) = Tf \cdot Ag + Af \cdot Tg$$

для операторів T та A у просторі швидко спадних функцій, як у випадку комплекснозначних функцій, так і у випадку дійснозначних функцій. Ми встановлюємо, що T може бути лінійною комбінацією логарифмічних похідних f та її комплексного спряження $\overline{f}$ до порядків m і n відповідно з гладкими коефіцієнтами та $Af = f^m \cdot \overline{f}^n$. В інших випадках Tf та Af можуть містити окремо дійсну та уявну частину f. У деякому сенсі з цього рівняння випливає сукупна характерізація похідних та перетворення Фур'є f. Ми обговорюємо умови, за яких T є похідною, а A є тотожністю. Ми також розглядаємо диференційовні розв'язки функціональних рівнянь, які нагадують рівняння для синуса та косинуса.

Key words: швидко спадні функції, узагальнене правило Лейбниця, перетворення Фур'є.